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REPRESENTATIONS OF RELATIONS WITH ORTHOGONALITY CONDITION AND THEIR DEFORMATIONS

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Abstract. Irreducible representations of $*$-algebras A_q generated by relations of the form $a_i^*a_i + a_ia_i^* = 1$, $i = 1, 2$, $a_1^*a_2 = qa_2a_1^*$, where $q \in (0, 1)$ is fixed, are classified up to the unitary equivalence. The case $q = 0$ is considered separately. It is shown that the C^*-algebras $\mathcal{A}_q^F$ and $\mathcal{A}_0^F$ generated by operators of Fock representations of A_q and A_0 are isomorphic for any $q \in (0, 1)$. A realisation of the universal C^*-algebra $\mathcal{A}_0$ generated by A_0 as an algebra of continuous operator-valued functions is given.

Introduction

In this note we study irreducible representations of a subclass of the so-called q_{ij}-CCR introduced by M. Bozejko and R. Speicher, see [1]. Namely, q_{ij}-CCR with d degrees of freedom is a $*$-algebra generated by a_i, $i = 1, \dots, d$, satisfying commutation relations of the following form:

$$a_i^*a_j = 1 + q_{ij}a_ja_i^*, \quad q_{ji} = \overline{q}_{ij} \in \mathbb{C}, \quad |q_{ij}| \le 1, \quad i, j = 1, \dots, d. \tag{1}$$

If all of $q_{ij} = 0$, we get the Cuntz-Toeplitz algebra $\mathcal{O}_d^{(0)}$, see [2]. The case $|q_{ij}| = 1$ if $i \ne j$ corresponds to an algebra of generalized quons introduced by W. Marcinek and M. Ralowski, see [5], [6]. If we put all of $q_{ij} = 1$ we get the Wick algebra associated with canonical commutation relations and the case $q_{ij} = -1$, $i, j = 1, \dots, d$ determines the Wick version of canonical anti-commutation relations, see [3].

We study representations of $*$-algebras A_q generated by pairs of elements, a_1, a_2, satisfying, for fixed $q \in (0, 1)$, the following commutation relations:

$$a_1^*a_1 + a_1a_1^* = 1, \quad a_2^*a_2 + a_2a_2^* = 1, \quad a_1^*a_2 = qa_2a_1^*. \tag{2}$$

Namely, in Section 2 we give a complete classification, up to the unitary equivalence, of irreducible representations of (2). In Section 3, the case $q = 0$ is considered separately. In particular it follows that in both cases the Fock representation is positive. In Section 4 we prove that the C^*-algebras $\mathcal{A}_q^F$ and $\mathcal{A}_0^F$ generated by operators of Fock representation of A_q and A_0 are, respectively, isomorphic.

1. Preliminaries

In this section we collect some results on representation theory of canonical anti-commutation and q-canonical commutation relations with one degree of freedom, which will be useful for us below. For details see the book [8] and the references therein.

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First we recall the description of irreducible representations of the one-dimensional Wick version of CAR, i.e., of the $*$-algebra, generated by a, a^* subject to the relation

$$a^*a = 1 - aa^*. \tag{3}$$

Obviously, any Hilbert space representation of (3) is bounded, namely in any representation one has $||a|| \leq 1$. The C^*-algebra, generated by (3) is called the quantum analog of the algebra of continuous functions on the unit circle, see for example [7].

The proof of the following statement can be found in [8].

Theorem 1. *Any irreducible representation of (3) is unitary equivalent to some of the presented below.*

1. *The* **Fock representation:** π_F *acting on* $\mathbb{C}^2$,
$$\pi_F(a) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
2. *The* **regular representations:** $\pi_{x,\phi}$ *acting on* $\mathbb{C}^2$,
$$\pi_{x,\phi}(a) = \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{1-x} \\ \sqrt{x} & 0 \end{pmatrix},$$
where $\phi \in [0, 2\pi)$ *and* $0 < x < \frac{1}{2}$ *are fixed.*
3. *The* **one-dimensional representations:** ρ_ϕ *acting on* $\mathbb{C}$,
$$\rho_\phi(a) = \frac{e^{i\phi}}{\sqrt{2}}, \quad \phi \in [0, 2\pi).$$

Representations corresponding to different types or to different values of the parameters are non-equivalent.

Using the description of irreducible representations one can get an analog of the Wold decomposition for the operator $A\colon \mathcal{H} \to \mathcal{H}$ satisfying (3). Namely, let $A = UC$, where $C = (A^*A)^{\frac{1}{2}}$, U is a partial isometry with $\ker U = \ker C = \ker A$, be the polar decomposition. Then one can decompose

$$\mathcal{H} = \mathcal{H}_F \oplus \mathcal{H}_u$$

such that $\mathcal{H}_F$ and $\mathcal{H}_u$ are invariant with respect to A, A^*, and the restriction of A onto $\mathcal{H}_F$ is a multiple of the Fock representation and the phase operator of restriction of A onto $\mathcal{H}_u$ is unitary.

Below we will also use a description of irreducible bounded representations of the quantum disk D_q, $0 < q < 1$, see [7, 8]. Recall that the $*$-algebra D_q of polynomials on a non-commutative unit disk is generated by elements b, b^* satisfying the following relation:

$$b^*b = (1 - q^2)1 + q^2 bb^*. \tag{4}$$

Theorem 2. *Any bounded irreducible representation of (4) is unitary equivalent to one of the described below*

1. *The* **Fock representation** π_F *acting on* $l_2(\mathbb{Z}_+)$
$$\pi_F(b) = T_q, \quad T_q e_n = \sqrt{1 - q^{2(n+1)}}\, e_{n+1}, \quad n \in \mathbb{Z}_+, \tag{5}$$
where $\{e_n,\ n \in \mathbb{Z}_+\}$ *is the standard orthonormal basis of* $l_2(\mathbb{Z})$.
2. *The* **one-dimensional representations** π_ϕ, $\phi \in [0, 2\pi)$
$$\pi_\phi(b) = e^{i\phi}.$$

Representations corresponding to different values of ϕ are non-equivalent.

As in the case of a non-commutative circle, one can formulate an analog of the Wold decomposition for an operator $B\colon \mathcal{H} \to \mathcal{H}$ satisfying (4). Namely, in this case we can decompose $\mathcal{H}$ into an orthogonal sum of subspaces that are invariant with respect to B and B^*,

$$\mathcal{H} = \mathcal{H}_F \oplus \mathcal{H}_u,$$

such that $\mathcal{H}_F \simeq l_2(\mathbb{Z}_+) \otimes \mathcal{K}$ for some Hilbert space $\mathcal{K}$ and $B_{|\mathcal{H}_F}$ is unitary equivalent to $T_q \otimes \mathbf{1}_{\mathcal{K}}$ and $B_{|\mathcal{H}_u} = U$ for some unitary U.

Recall the definition of the universal C^*-algebra generated by a $*$-algebra.

Definition 1. *Let A be a $*$-algebra. The C^*-algebra $\mathcal{A}$ is called the universal C^*-algebra generated by A or the universal bounded representation of A if there exists a $*$-homomorphism $\psi\colon A \to \mathcal{A}$ such that for any representation $\pi\colon A \to B(\mathcal{H})$ one can construct a unique $\widetilde{\pi}\colon \mathcal{A} \to B(\mathcal{H})$ such that*

$$\pi = \widetilde{\pi} \circ \psi.$$

Recall also that $\mathcal{A}$ exists iff the set $\mathbf{Rep}(A)$ of bounded representations of A is non-empty and, for any $a \in A$,

$$\sup_{\pi \in \mathbf{Rep}(A)} ||\pi(a)|| = C_a < \infty.$$

In this case $\mathcal{A}$ is a completion of the quotient of A by $Rad\ A$, where

$$Rad\ A = \{a \in A \mid \pi(a) = 0 \ \text{ for any } \ \pi \in \mathbf{Rep}(A)\},$$

with respect to the norm defined as follows:

$$||a + Rad\ A|| = \sup_{\pi \in \mathbf{Rep}(A)} ||\pi(a)||.$$

Below we will sometimes use the same notations for the generators a_i, $i = 1, 2$, of the algebras under consideration and their images in the representations.

2. Representations of A_q

Let the operators a_1, a_2 acting on a Hilbert space $\mathcal{H}$ determine an irreducible representation of A_q. Construct $C_1 = a_1^2$, $C_2 = a_2^2$.

Proposition 1. *The operators C_1, C_2 are normal and $C_1C_2 = C_2C_1 = 0$. The kernel of each of them is invariant with respect to the action of a_i, a_i^*, $i = 1, 2$.*

Proof. Indeed,

$$\begin{aligned} C_i^*C_i &= (a_i^2)^*a_i^2 = a_i^*(\mathbf{1} - a_ia_i^*)a_i = \mathbf{1} - a_ia_i^* - (\mathbf{1} - a_ia_i^*)(\mathbf{1} - a_ia_i^*) \\ &= \mathbf{1} - a_ia_i^* - 1 + 2a_ia_i^* - a_ia_i^*a_ia_i^* = a_ia_i^* - a_i(\mathbf{1} - a_ia_i^*)a_i^* = a_i^2(a_i^2)^* \\ &= C_iC_i^*. \end{aligned}$$

Further, $a_1^*a_2 = qa_2a_1^*$ implies $C_1^*C_2 = q^4C_2C_1^*$. Put $A = C_1C_2$, then

$$A^*A = C_2^*C_1^*C_1C_2 = C_2^*C_1C_1^*C_2 = q^8C_1C_2^*C_2C_1^* = q^8C_1C_2C_2^*C_1^* = q^8AA^*.$$

Since A is bounded, the relation above is satisfied if and only if $A = C_1C_2 = 0$.

Let us show that $\ker C_i$, $i = 1, 2$, are invariant with respect to a_j, a_j^*, $j = 1, 2$. Indeed,

$$C_1a_1 = a_1C_1, \quad C_1a_1^* = a_1^*C_1$$

imply invariance of $\ker C_1$ w.r.t. a_1, a_1^*. Since $a_2^*C_1 = q^2C_1a_2^*$, applying the Fuglede-Putnam theorem, see [11], we get $a_2^*C_1^* = q^2C_1^*a_2^*$, and taking the adjoints we obtain $C_1a_2 = q^2a_2C_1$. Therefore, $\ker C_1$ is invariant w.r.t. a_2, a_2^* too. □

According to the proposition above, at least one of the operators C_1, C_2 has nonzero kernel that is an invariant subspace.

Assume that $\ker C_1$ is nonzero. Then, for an irreducible representation, $\mathcal{H} = \ker C_1$, i.e. $a_1^2 = 0$ and a_1 is unitary equivalent to a multiple of the operator defining the Fock representation of (3). Then $\mathcal{H}$ can be decomposed as $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_1$, so that

$$a_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{\mathcal{H}_1}.$$

Consider the corresponding block-matrix form of a_2,

$$a_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then $a_{21} = 0$ and, for $A = a_{11}$ and $B = a_{12}$, one has

$$B^*B + q^2AA^* = \mathbf{1} - q^2AA^*, \tag{6}$$

$$A^*A = \mathbf{1} - AA^* - BB^*, \tag{7}$$

$$B^*A = -qAB^*. \tag{8}$$

Using (7) one can present (6) in the following form:

$$B^*B = (1-q^2)\mathbf{1} + q^2BB^*. \tag{9}$$

Proposition 2. *The representation of (2) given on $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_1$ by*

$$a_1 = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} A & B \\ 0 & qA \end{pmatrix}$$

is irreducible iff the family $\{A, A^*, B, B^*\}$ is irreducible on $\mathcal{H}_1$. Representations of (2) corresponding to the families $\{A_i, A_i^*, B_i, B_i^*\}$, $i = 1, 2$, are unitary equivalent iff these families are unitary equivalent.

Proof. To prove the statement on irreducibility we use the Schur lemma. Indeed, it is easy to check that $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ commutes with a_i, a_i^*, $i = 1, 2$, iff $C_{12} = C_{21} = 0$, $C_{11} = C_{22} = \widetilde{C}$, and $\widetilde{C}$ commutes with A, A^*, B, B^*. Therefore, C is scalar iff $\widetilde{C}$ is scalar.

The statement on unitary equivalence can be proved analogously. □

Let us classify the irreducible representations of (9), (7), (8).

Lemma 1. *In any representation, we also have that $AB = -qBA$.*

Proof. Indeed, let $Q = BA + qAB$. Then

$$\begin{aligned} A^*Q &= A^*(AB + qBA) = A^*AB + qA^*BA \\ &= (\mathbf{1} - AA^* - BB^*)B - q^2BA^*A \\ &= B - AA^*B - BB^*B - q^2B(\mathbf{1} - BB^* - AA^*) \\ &= B + qABA^* - B((1-q^2)\mathbf{1} + q^2BB^*) - q^2B + q^2B^2B^* + q^2BAA^* \\ &= B + qABA^* - B + q^2B - q^2B^2B^* - q^2B + q^2B^2B^* + q^2BAA^* \\ &= q(AB + qBA)A^* = qQA^* \end{aligned}$$

and

$$\begin{aligned}
B^*Q &= B^*(AB+qBA) = B^*AB + qB^*BA \\
&= -qAB^*B + q((1-q^2)\mathbf{1} + q^2BB^*)A \\
&= -qA((1-q^2)\mathbf{1} + q^2BB^*) + q(1-q^2)A + q^3BB^*A \\
&= -q(1-q^2)A - q^3ABB^* + q(1-q^2)A - q^4BAB^* \\
&= -q^3(AB+qBA)B^* = -q^3QB^*.
\end{aligned}$$

Since $A^*Q = qQA^*$, $B^*Q = -q^3QB^*$, one has

$$Q^*Q = (B^*A^* + qA^*B^*)Q = -q^4Q(B^*A^* + qA^*B^*) = -q^4QQ^* \tag{10}$$

implying $Q = 0$. □

Remark 1. *In fact, we have shown above that the element $AB + qBA$ generates a quadratic Wick ideal in the Wick algebra generated by A, B, see* [3, 9] *for details.*

Using the q-Wold decomposition, see Preliminaries, we decompose

$$\mathcal{H}_1 = \mathcal{H}_1^F \oplus \mathcal{H}_1^u$$

so that each summand is invariant w.r.t. B and B^*, the restriction of B onto $\mathcal{H}_1^u$ is unitary, and $\mathcal{H}_1^F = l_2(\mathbb{Z}_+) \otimes \mathcal{K}$, $B_{|\mathcal{H}_1^F} = T_q \otimes \mathbf{1}_{\mathcal{K}}$, where $T_q \colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+)$ is defined by (5).

Proposition 3. *The subspaces $\mathcal{H}_1^u$ and $\mathcal{H}_1^F$ are invariant with respect to the operators A and A^*.*

Proof. Evidently, it is enough to show that A, A^* leave $\mathcal{H}_1^u$ invariant. In fact we show that $A^*x = Ax = 0$ for any $x \in \mathcal{H}_1^u$.

Indeed let $x \in \mathcal{H}_1^u$, $||x|| = 1$. For any $n \in \mathbb{N}$ there exists $y_n \in \mathcal{H}_1^u$, $||y_n|| = 1$, such that $x = B^n y$. Then

$$||A^*x|| = ||A^*B^ny_n|| = q^n||B^nA^*y_n||.$$

Let us stress that relations (9), (7) imply that $||A|| \le 1$ and $||B|| \le 1$. Therefore, for any $n \in \mathbb{N}$, one has

$$||A^*x|| \le q^n \text{ and } A^*x = 0.$$

Let us show that $Ax = 0$. Indeed, since for $x \in \mathcal{H}_1^u$, one has $BB^*x = x$ and $A^*x = 0$, and we get

$$A^*Ax = (\mathbf{1} - BB^* - AA^*)x = 0.$$

□

Corollary 1. *Let $A, B \colon \mathcal{H}_1 \to \mathcal{H}_1$ determine an irreducible representation of* (9), (7), (8). *Then either $\mathcal{H}_1 = \mathcal{H}_1^u$ or $\mathcal{H}_1 = \mathcal{H}_1^F$.*

Proposition 4. *Let $\{A, B, A^*, B^*\}$ be an irreducible family satisfying* (9),(7), (8) *on a Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_1 = \mathcal{H}_1^u$. Then $\dim \mathcal{H}_1 = 1$, $A = 0$, and $B = e^{i\phi}$ for some $\phi \in [0, 2\pi)$. The representations corresponding to different ϕ are non-equivalent.*

Proof. Above we have shown that $A = 0$ on $\mathcal{H}_1^u$. Since B is irreducible and unitary on $\mathcal{H}_1$, we get $\dim \mathcal{H}_1 = 1$. □

It remains to consider the case $\mathcal{H}_1 = \mathcal{H}_1^F$.

Proposition 5. *Let, in an irreducible representation of (9), (7), (8), the representation space be $\mathcal{H}_1 = \mathcal{H}_1^F$. Then, up to the unitary equivalence, one has $\mathcal{H}_1 = l_2(\mathbb{Z}_+) \otimes \mathcal{K}$ and*

$$B = T_q \otimes \mathbf{1}_{\mathcal{K}}, \quad A = d(-q) \otimes \widetilde{A}, \tag{11}$$

where $d(-q) \colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+)$,

$$d(-q)e_n = (-1)^n q^n e_n, \quad n \in \mathbb{Z}_+,$$

and $\widetilde{A}\colon \mathcal{K} \to \mathcal{K}$ *determines an irreducible representation of (3). Representations corresponding to the families* $\{\widetilde{A}_i, \widetilde{A}_i^*\}$, $i = 1, 2$, *are equivalent iff these families are unitary equivalent.*

Proof. The relations $B^*A = -qAB^*$, $AB = -qBA$ imply $B^*BA = AB^*B$. Since $\mathcal{H}_1 = \mathcal{H}_1^F$, we get $\mathcal{H}_1 = l_2(\mathbb{Z}_+) \otimes \mathcal{K}$ and

$$B = T_q \otimes \mathbf{1}_{\mathcal{K}}, \quad B^*B = (T_q^*T_q) \otimes \mathbf{1}_{\mathcal{K}},$$

$T_q^*T_q e_n = (1 - q^{2(n+1)})e_n$, $n \in \mathbb{Z}_+$. Then $\mathcal{K}_n = e_n \otimes \mathcal{K}$ are eigenspaces for $T_q^*T_q$ corresponding to $\lambda_n = 1 - q^{2(n+1)}$ and A, A^* leave any of $\mathcal{K}_n$, $n \in \mathbb{Z}_+$, invariant.

Denote by A_n the restriction of A onto $\mathcal{K}_n$. Then $(AB)_{|\mathcal{K}_n} = -q(BA)_{|\mathcal{K}_n}$ implies

$$A_{n+1} = -qA_n, \quad n \in \mathbb{Z}_+.$$

Put $A_0 = \widetilde{A}$. Then $A_n = (-1)^n q^n \widetilde{A}$, $n \in \mathbb{Z}_+$, and the relation

$$A^*A = \mathbf{1} - BB^* - AA^*$$

is equivalent to $\widetilde{A}^*\widetilde{A} = \mathbf{1} - \widetilde{A}\widetilde{A}^*$ on $\mathcal{K}$.

The proof of the statement on irreducibility and unitary equivalence is the same as in Proposition 2. □

Now we can formulate the result of this section.

Theorem 3. *Let* π *be an irreducible representation of (2) acting on a Hilbert space* $\mathcal{H}$. *Then* π *is unitary equivalent to one of the representations listed below.*

1. $\pi_F^{(q)}\colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2$,
$$\pi_F^{(q)}(a_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \mathbf{1}_2,$$
$$\pi_F^{(q)}(a_2) = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \otimes d(-q) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes T_q \otimes \mathbf{1}_2.$$
2. $\pi_{\phi_2,x_2}^{(q)}\colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2$,
$$\pi_{\phi_2,x_2}^{(q)}(a_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \mathbf{1}_2,$$
$$\pi_{\phi_2,x_2}^{(q)}(a_2) = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \otimes d(-q) \otimes \begin{pmatrix} 0 & e^{i\phi_2}\sqrt{x_2} \\ \sqrt{1-x_2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes T_q \otimes \mathbf{1}_2,$$
where $\phi_2 \in [0, 2\pi)$, $x_2 \in (0, 1/2)$.
3. $\rho_{2,\phi_2}^{(q)}\colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \to \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+)$ *and*
$$\rho_{2,\phi_2}^{(q)}(a_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)},$$
$$\rho_{2,\phi_2}^{(q)}(a_2) = \frac{e^{i\phi_2}}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \otimes d(-q) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes T_q, \quad \phi_2 \in [0, 2\pi).$$
4. $\theta_{\phi_1,x_1}^{(q)}\colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2$,
$$\theta_{\phi_1,x_1}^{(q)}(a_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \mathbf{1}_2,$$
$$\theta_{\phi_1,x_1}^{(q)}(a_1) = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \otimes d(-q) \otimes \begin{pmatrix} 0 & e^{i\phi_1}\sqrt{x_1} \\ \sqrt{1-x_1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes T_q \otimes \mathbf{1}_2,$$
where $\phi_1 \in [0, 2\pi)$, $x_1 \in (0, 1/2)$.

5. $\nu^{(q)}_{2,\phi_1}\colon \mathbb{C}^2\otimes l_2(\mathbb{Z}_+)\to\mathbb{C}^2\otimes l_2(\mathbb{Z}_+)$,

$$\nu^{(q)}_{2,\phi_1}(a_2)=\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)},$$

$$\nu^{(q)}_{2,\phi_1}(a_1)=\frac{e^{i\phi_1}}{\sqrt{2}}\begin{pmatrix}1&0\\0&q\end{pmatrix}\otimes d(-q)+\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes T_q,\quad \phi_1\in[0,2\pi).$$

6. $\rho^{(q)}_{\phi_2}\colon\mathbb{C}^2\to\mathbb{C}^2$,

$$\rho^{(q)}_{\phi_2}(a_1)=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \rho^{(q)}_{\phi_2}(a_2)=\begin{pmatrix}0&e^{i\phi_2}\\0&0\end{pmatrix},\quad \phi_2\in[0,2\pi).$$

Proof. To get the proof one has to apply Theorem 1 and Propositions 4,5. In particular, these statements imply that representations from the family $\mathcal{R}_1=\{\pi_F,\ \pi_{\phi_2,x_2},\ \rho_{2,\phi_2}\}$ with any admissible values of the parameters are pairwise non-equivalent as also the representations from $\mathcal{R}_2=\{\theta_{\phi_1,x_1},\ \nu_{2,\phi_1}\}$ are non-equivalent. Evidently, representations from $\mathcal{R}_1$ are not equivalent to representations containing in $\mathcal{R}_2$, since one has $a_1^2=0$ in any representation from the first family, while $a_1^2\neq 0$ in any representations from the second one. Finally, any of the representations from $\mathcal{R}_1\cup\mathcal{R}_2$ is infinite-dimensional, hence it is not equivalent to ρ_{ϕ_2}. It is obvious that ρ_{ϕ_2} are non-euqivalent for different $\phi\in[0,2\pi)$. ☐

Remark 2. *The unique irreducible representation π_F of (2), where $a_1^2=0$, $a_2^2=0$ and $\ker a_1^*\cap\ker a_2^*\neq\{0\}$, is called the Fock representation, see* [3].

3. Representations of A_0

The situation with $q=0$ requires a bit more different analysis. So consider operators satisfying, on a Hilbert space $\mathcal{H}$, commutation relations of the following form:

$$a_i^*a_i=1-a_ia_i^*,\quad i=1,2,\qquad a_1^*a_2=0. \tag{12}$$

Lemma 2. *Let $a_i=u_ic_i$, where u_i is a partial isometry, $c_i^2=a_i^*a_i$, and $\ker u_i=\ker c_i$, $i=1,2$, be polar decompositions. Then $u_1^*u_2=u_2^*u_1=0$.*

Proof. Indeed $a_1^*a_2=0$ takes the form $c_1u_1^*u_2c_2=0$. Since c_2 is self-adjoint,

$$\mathcal{H}=\ker c_2\oplus\overline{\mathbf{range}(c_2)}.$$

For any $x\in\ker c_2=\ker u_2$, we have $c_1u_1^*u_2x=0$. For $y\in\mathbf{range}(c_2)$, $y=c_2z$ and

$$c_1u_1^*u_2y=c_1u_1^*u_2c_2z=0.$$

Therefore $c_1u_1^*u_2=0$. Taking the adjoint we get $u_2^*u_1c_1=0$. Then the arguments presented above imply $u_2^*u_1=0$. ☐

Our next aim is to show that in an irreducible representation of (12) at least one of a_i^2, $i=1,2$, is equal to zero.

Proposition 6. *Let a_i, $i=1,2$, determine an irreducible representation of (12). Suppose that the unitary part $\mathcal{H}_u$ of the generalized Wold decomposition of a_1 is non-zero. Then $a_2^2=0$.*

Proof. So, let $\mathcal{H}=\mathcal{H}_u\oplus\mathcal{H}_F$ such that the restriction of u_1 onto $\mathcal{H}_u$ is unitary and the restriction of a_1 onto $\mathcal{H}_F$ is a multiple of the Fock representation of (3). In particular on $\mathcal{H}_F$ one has $a_1^2=0$.

Let $x\in\mathcal{H}_u$. Since $u_2^*u_1=0$ and u_1 is unitary on $\mathcal{H}_u$, one has $x=u_1u_1^*x$ and $u_2^*x=u_2^*u_1u_1^*x=0$. Thus $a_2^*x=c_2u_2^*x=0$ for any $x\in\mathcal{H}_u$.

Let us show that for any $x \in \mathcal{H}_u$, one has $a_{\alpha_1} \cdots a_{\alpha_k} x = 0$, where $\alpha_k \neq 1$, $\alpha_s \in \{1,2\}$, if there exists s such that $\alpha_s = \alpha_{s+1}$. Recall that a_i^2, $i = 1,2$, are normal. Then, for any $x \in \mathcal{H}_u$,

$$\langle a_2^2 x, a_2^2 x \rangle = \langle (a_2^2)^* a_2^2 x, x \rangle = \langle a_2^2 (a_2^2)^* x, x \rangle = 0.$$

Further we use induction on the length of $\alpha = (\alpha_1, \dots, \alpha_k)$. Namely, consider the product $a_{\alpha_1} \cdots a_{\alpha_k} x$, where $k \geq 1$, $\alpha_s \neq \alpha_{s+1}$, $s = 1, \dots, k-1$, $\alpha_k \neq 1$. Let for example $\alpha_1 = 1$, then $k \geq 2$, $\alpha_2 = 2$ and

$$\begin{aligned}
&\langle a_1^2 a_{\alpha_2} \cdots a_{\alpha_k} x, a_1^2 a_{\alpha_2} \cdots a_{\alpha_k} x \rangle = \langle (a_1^2)^* a_1^2 a_{\alpha_2} \cdots a_{\alpha_k} x, a_{\alpha_2} \cdots a_{\alpha_k} x \rangle \\
&\qquad = \langle a_1^2 (a_1^2)^* a_{\alpha_2} \cdots a_{\alpha_k} x, a_{\alpha_2} \cdots a_{\alpha_k} x \rangle = 0, \\
&\langle a_2^2 a_1 a_{\alpha_2} \cdots a_{\alpha_k} x, a_2^2 a_1 a_{\alpha_2} \cdots a_{\alpha_k} x \rangle = \langle (a_2^2)^* a_2^2 a_1 a_{\alpha_2} \cdots a_{\alpha_k} x, a_1 a_{\alpha_2} \cdots a_{\alpha_k} x \rangle \\
&\qquad = \langle a_2^2 (a_2^2)^* a_1 a_{\alpha_2} \cdots a_{\alpha_k} x, a_1 a_{\alpha_2} \cdots a_{\alpha_k} x \rangle = 0.
\end{aligned}$$

If $\alpha_1 = 2$, then $k \geq 1$ and the rest of the verification is the same.

Put $\Lambda_1 = \{\emptyset,\ (\alpha_1, \dots, \alpha_k),\ k \in \mathbb{N},\ \alpha_s = 1,2,\ \alpha_k \neq 1,\ \alpha_s \neq \alpha_{s+1}\}$. For any $\alpha \in \Lambda_1$ denote by a_α the product $a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_k}$, $a_\emptyset = \mathbf{1}$. Let us show that

$$\mathcal{H}_1 = \overline{\langle\ a_\alpha x,\ x \in \mathcal{H}_u,\ \alpha \in \Lambda_1\ \rangle}$$

is invariant with respect to a_i, a_i^*, $i = 1,2$.

The invariance w.r.t. the action of a_i is evident. Furthermore,

$$\begin{aligned}
&a_2^* x = 0, \quad a_1^* x \in \mathcal{H}_u, \quad x \in \mathcal{H}_u, \\
&a_i^* a_i a_{\alpha_2} \cdots a_{\alpha_k} x = (1 - a_i a_i^*) a_{\alpha_2} \cdots a_{\alpha_k} x = a_{\alpha_2} \cdots a_{\alpha_k} x \quad (\text{since } \alpha_2 \neq i), \quad i = 1,2, \\
&a_i^* a_j a_{\alpha_2} \cdots a_{\alpha_k} x = 0, \quad i \neq j, \quad i,j = 1,2.
\end{aligned}$$

Since the representation is irreducible we get $\mathcal{H} = \mathcal{H}_1$. Above we have shown that $a_2^2 z = 0$ for any $z \in \mathcal{H}_1$. So $a_2^2 = 0$. □

The rest of considerations are the same as in the case $q \in (0,1)$. Indeed, suppose that $a_1^2 = 0$ and write the representation space $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{H}_1$. Then

$$a_1 = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix},$$

here $\mathbf{1} = \mathbf{1}_{\mathcal{H}_1}$. Further, $a_1^* a_2 = 0$ is equivalent to $a_2 = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ and $a_2^* a_2 = 1 - a_2 a_2^*$ is equivalent to the following relations:

$$\begin{aligned}
A^*A &= \mathbf{1} - AA^* - BB^*, \\
A^*B &= 0, \\
B^*B &= \mathbf{1}.
\end{aligned} \tag{13}$$

The representation determined by a_i, $i = 1,2$, is irreducible iff the corresponding representation determined by A, B is irreducible on $\mathcal{H}_1$. The statement on unitary equivalence holds also.

Construct $Q = AB$, then it is easy to see that $Q^*Q = 0$,

$$Q^*Q = B^*A^*AB = B^*(\mathbf{1} - AA^* - BB^*)B = B^*B - B^*BB^*B = 0.$$

Hence, we additionally have $AB = 0$.

Proposition 7. *Let A, B determine an irreducible representation of (13) on $\mathcal{H}_1$. Then either B is unitary, $\dim \mathcal{H}_1 = 1$ and $A = 0$ or $\mathcal{H}_1 \simeq l_2(\mathbb{Z}_+) \otimes \mathcal{K}$ for some Hilbert space $\mathcal{K}$ and B is unitary equivalent to a multiple of a unilateral shift operator.*

Proof. As in the q-deformed case, we consider the Wold decomposition of B,

$$\mathcal{H}_1 = \mathcal{H}_1^F \oplus \mathcal{H}_1^u,$$

where the restriction of B onto $\mathcal{H}_1^u$ is unitary and the restriction on $\mathcal{H}_1^F$ is a multiple of a unilateral shift.

We show that $\mathcal{H}_1^u$ is invariant with respect to A, A^*. Indeed, for $x \in \mathcal{H}_1^u$ we get

$$A^*x = A^*BB^*x = 0, \quad A^*Ax = x - AA^*x - BB^*x = 0,$$

so $A^*x = 0$ and $Ax = 0$. Hence if $\mathcal{H}_1^u \neq \{0\}$ in the irreducible case we get $\mathcal{H}_1 = \mathcal{H}_1^u$, $\dim \mathcal{H}_1 = 1$ and $A = 0$, $B = e^{i\phi}$.

To complete the proof it remains to point out that if $\mathcal{H}_1^u = \{0\}$, then $\mathcal{H}_1 = \mathcal{H}_1^F$. □

Below we denote by $S\colon l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+)$ the operator of a unilateral shift.

Proposition 8. *Let A, B determine an irreducible representation of (13) on $\mathcal{H}_1$ and $\mathcal{H}_1 = \mathcal{H}_1^F$. Then $\mathcal{H}_1 = l_2(\mathbb{Z}_+) \otimes \mathcal{K}$ for some Hilbert space $\mathcal{K}$ and*

$$B = S \otimes \mathbf{1}_{\mathcal{K}}, \quad A = (\mathbf{1} - SS^*) \otimes \widetilde{A},$$

where $\widetilde{A}$ determine an irreducible representation of (3) on $\mathcal{K}$. Representations corresponding to families $\{\widetilde{A}_i, \widetilde{A}_i^*\}$, $i = 1, 2$, are unitary equivalent iff these families are unitary equivalent.

Proof. If $\mathcal{H}_1 = \mathcal{H}_1^F$ then, by the definition of $\mathcal{H}_1^F$, we have $\mathcal{H}_1 = l_2(\mathbb{Z}_+) \otimes \mathcal{K}$ and $B = S \otimes \mathbf{1}_{\mathcal{K}}$. Further, it is easy to verify that $AB = 0$, $B^*A = 0$ imply that $A = (1 - SS^*) \otimes \mathcal{A}$ and the relation

$$A^*A = \mathbf{1} - AA^* - BB^*$$

is equivalent to $\widetilde{A}^*\widetilde{A} = \mathbf{1}_{\mathcal{K}} - \widetilde{A}\widetilde{A}^*$.

Application of the Schur lemma shows that the family $\{A, A^*, B, B^*\}$ is irreducible iff $\{\widetilde{A}, \widetilde{A}^*\}$ is irreducible. The statement about unitary equivalence is also obvious. □

Combining the results of Propositions 7, 8 and Theorem 1 we immediately get the following result.

Theorem 4. *Let π be an irreducible representation of (12) on a Hilbert space $\mathcal{H}$. Then π is unitary equivalent to one of the constructed below:*

1. π_F *acting on* $\mathcal{H} = \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2$:

$$\pi_F(a_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \mathbf{1}_2,$$

$$\pi_F(a_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (\mathbf{1} - SS^*) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes S \otimes \mathbf{1}_2.$$

2. $\pi_{\phi_2,x_2}\colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2 \to\colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2$,

$$\pi_{\phi_2,x_2}(a_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \mathbf{1}_2,$$

$$\pi_{\phi_2,x_2}(a_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (\mathbf{1} - SS^*) \otimes \begin{pmatrix} 0 & e^{i\phi_2}\sqrt{x_2} \\ \sqrt{1-x_2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes S \otimes \mathbf{1}_2,$$

where $\phi_2 \in [0, 2\pi)$, $x_2 \in (0, 1/2)$.

3. $\rho_{2,\phi_2}\colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \to \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+)$,

$$\rho_{2,\phi_2}(a_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)},$$

$$\rho_{2,\phi_2}(a_2) = \frac{e^{i\phi_2}}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (\mathbf{1} - SS^*) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes S, \quad \phi_2 \in [0, 2\pi).$$

4. $\theta_{\phi_1,x_1}\colon \mathbb{C}^2\otimes l_2(\mathbb{Z}_+)\otimes\mathbb{C}^2\to\mathbb{C}^2\otimes l_2(\mathbb{Z}_+)\otimes\mathbb{C}^2$,
$$\theta_{\phi_1,x_1}(a_2)=\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes\mathbf{1}_2,$$
$$\theta_{\phi_1,x_1}(a_1)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes(\mathbf{1}-SS^*)\otimes\begin{pmatrix}0&e^{i\phi_1}\sqrt{x_1}\\\sqrt{1-x_1}&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes S\otimes\mathbf{1}_2,$$
where $\phi_1\in[0,2\pi)$, $x_1\in(0,1/2)$.
5. $\nu_{2,\phi_1}\colon \mathbb{C}^2\otimes l_2(\mathbb{Z}_+)\to\mathbb{C}^2\otimes l_2(\mathbb{Z}_+)$ *and*
$$\nu_{2,\phi_1}(a_2)=\begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)},$$
$$\nu_{2,\phi_1}(a_1)=\frac{e^{i\phi_1}}{\sqrt{2}}\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes(\mathbf{1}-SS^*)+\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes S,\quad \phi_1\in[0,2\pi).$$
6. $\rho_{\phi_2}\colon\mathbb{C}^2\to\mathbb{C}^2$,
$$\rho_{\phi_2}(a_1)=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \rho_{\phi_2}(a_2)=\begin{pmatrix}0&e^{i\phi_2}\\0&0\end{pmatrix},\quad \phi_2\in[0,2\pi).$$

Representations from different classes as well as representations from the same class corresponding to different values of the parameters are non-equivalent.

4. C^*-algebras generated by A_q and A_0

In this section, we study the C^*-algebras generated by operators of Fock representations of A_0 and A_q and give a description of the universal C^*-algebra $\mathcal{A}_0$ generated by (12) as algebras of continuous operator-valued functions.

4.1. **The C^*-algebras $\mathcal{A}_q^F$ and $\mathcal{A}_0^F$.** Let C^*-algebras $\mathcal{A}_q^F$ and $\mathcal{A}_0^F$ be generated by Fock representations of (2), (12) respectively.

Theorem 5. *For any $q\in(0,1)$ one has $\mathcal{A}_q^F\simeq\mathcal{A}_0^F$.*

Proof. Denote by E_{ij}, $i,j=1,2$, the matrix units of $M_2(\mathbb{C})$.

First we consider the C^*-algebra $\mathcal{A}_0^F=C^*(a_1,a_2)$, where
$$a_1=E_{21}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes\mathbf{1}_2,$$
and
$$a_2=E_{11}\otimes(\mathbf{1}-SS^*)\otimes E_{21}+E_{12}\otimes S\otimes\mathbf{1}_2.$$
Since E_{21} generates $M_2(\mathbb{C})$ as a $*$-algebra, we conclude that
$$M_2(\mathbb{C})\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes\mathbf{1}_2\subset\mathcal{A}_0^F$$
and
$$a_2\cdot\big(E_{12}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes\mathbf{1}_2\big)=E_{12}\otimes(\mathbf{1}-SS^*)\otimes E_{21}\in\mathcal{A}_0^F,\tag{14}$$
$$a_2\cdot\big(E_{22}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes\mathbf{1}_2\big)=E_{12}\otimes S\otimes\mathbf{1}_2\in\mathcal{A}_0^F.\tag{15}$$
Further, since $M_2(\mathbb{C})$ is simple it follows that
$$A_2=\mathbf{1}_2\otimes(\mathbf{1}-SS^*)\otimes E_{21}\in\mathcal{A}_0^F,\quad A_3=\mathbf{1}_2\otimes S\otimes\mathbf{1}_2\in\mathcal{A}_0^F.\tag{16}$$
Then it is evident now that $\mathcal{A}_0^F$ is generated as a C^*-algebra by operators A_1, A_2, A_3, where
$$A_1=E_{21}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes\mathbf{1}_2.$$
Let us study the C^*-algebra $\mathcal{A}_q^F$. It is generated by
$$a_1^q=E_{21}\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes\mathbf{1}_2$$

and

$$a_2^q = \big(E_{11} + qE_{12}\big) \otimes d(-q) \otimes E_{21} + E_{12} \otimes T_q \otimes \mathbf{1}_2.$$

As above we conclude that $M_2(\mathbb{C}) \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \mathbf{1}_2 \subset \mathcal{A}_q^F$ and

$$A_2^q = \mathbf{1}_2 \otimes d(-q) \otimes E_{21} \in \mathcal{A}_q^F, \quad A_3^q = \mathbf{1}_2 \otimes T_q \otimes \mathbf{1}_2 \in \mathcal{A}_q^F.$$

Moreover $\mathcal{A}_q = C^*(A_1, A_2^q, A_3^q)$.

Since $T_q^* T_q e_n = (1 - q^{2(n+1)}) e_n$, $n \in \mathbb{Z}_+$, one has

$$S = T_q (T_q^* T_q)^{-\frac{1}{2}} \quad \text{and} \quad T_q^* T_q = (1 - q^2) \sum_{n=0}^{\infty} q^{2n} S^n (S^*)^n.$$

I.e., $C^*(S) = C^*(T_q)$, see for example [4]. So $A_3 \in \mathcal{A}_q^F$ and $A_3^q \in C^*(A_3)$. Then since $(\mathbf{1} - SS^*) d(-q) = (\mathbf{1} - SS^*)$,

$$(\mathbf{1} - A_3 A_3^*) A_2^q = \mathbf{1}_2 \otimes (\mathbf{1} - SS^*) d(-q) \otimes E_{21} = A_2$$

and $A_2 \in \mathcal{A}_q^F$. Finally

$$d(-q) = \sum_{n=0}^{\infty} (-q)^n \big(S^n (S^n)^* - S^{n+1} (S^{n+1})^*\big) = \sum_{n=0}^{\infty} (-q)^n S^n \big(\mathbf{1} - SS^*\big) (S^n)^*$$

and

$$A_2^q = \sum_{n=0}^{\infty} (-q)^n A_3^n A_2 (A_3^n)^* \in C^*(A_3, A_2).$$

Therefore $\mathcal{A}_q^F = C^*(A_1, A_2^q, A_3^q) = C^*(A_1, A_2, A_3) = \mathcal{A}_0^F$. $\square$

4.2. **The C^*-algebra $\mathcal{A}_0$.** Consider the C^*-algebra $\mathcal{A}_0$. First of all we present the operators of representations $\pi_{\phi_2, x_2}(a_i)$ and $\theta_{\phi_1, x_1}(a_i)$, $i = 1, 2$, in the form different from that of presented in Theorem 4.

Namely using the isomorphism $l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2 \simeq l_2(\mathbb{Z}_+)$ one can present

(17) $$\pi_{\phi_2, x_2}(a_1) = E_{21} \otimes \mathbf{1}_{l_2(\mathbb{Z}_+)} \otimes \mathbf{1}_2,$$

(18) $$\pi_{\phi_2, x_2}(a_2) = E_{11} \otimes (\mathbf{1} - SS^*) \otimes \begin{pmatrix} 0 & e^{i\phi_2} \sqrt{x_2} \\ \sqrt{1 - x_2} & 0 \end{pmatrix} + E_{12} \otimes S \otimes \mathbf{1}_2$$

as

(19) $$\pi_{\phi_2, x_2}(a_1) = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix},$$

(20) $$\pi_{\phi_2, x_2}(a_2) = \begin{pmatrix} \sqrt{1 - x_2} S (\mathbf{1} - SS^*) + e^{i\phi_2} \sqrt{x_2} (\mathbf{1} - SS^*) S^* & S^2 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{1} = \mathbf{1}_{l_2(\mathbb{Z}_+)}$. Similarly,

$$\theta_{\phi_1, x_1}(a_2) = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix},$$

$$\theta_{\phi_1, x_1}(a_1) = \begin{pmatrix} \sqrt{1 - x_1} S (\mathbf{1} - SS^*) + e^{i\phi_1} \sqrt{x_1} (\mathbf{1} - SS^*) S^* & S^2 \\ 0 & 0 \end{pmatrix}.$$

In particular,

$$\pi_{\phi_2, 0}(a_1) = \theta_{\phi_1, 0}(a_2) = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}$$

and

$$\pi_{\phi_2, 0}(a_2) = \theta_{\phi_1, 0}(a_1) = \begin{pmatrix} S(\mathbf{1} - SS^*) & S^2 \\ 0 & 0 \end{pmatrix}$$

for any $\phi_1, \phi_2 \in [0, 2\pi)$. Note also that $\pi_{\phi_2, 0}(a_i) = \pi_F(a_i)$, $i = 1, 2$.

Consider $T \in M_2(\mathcal{T}(C(\mathbf{T})))$, where $\mathcal{T}(C(\mathbf{T})) = C^*(S)$ is the Toeplitz C^*-algebra generated by a single isometry,

$$T = \begin{pmatrix} \mathbf{1} - SS^* & S \\ S^* & 0 \end{pmatrix}.$$

It is easy to verify that $T = T^* = T^{-1}$ and for any $phi_j \in [0, 2\pi]$, $j = 1, 2$, one has

$$T\theta_{\phi_1,0}(a_1)T = \pi_{\phi_2,0}(a_1), \quad T\theta_{\phi_1,0}(a_2)T = \pi_{\phi_2,0}(a_2). \tag{21}$$

Let $X = \{0\} \times [0, 1/2] \cup [0, 1/2] \times \{0\}$ and $\mathbf{T}^2$ be the two-dimensional torus. Consider the C^*-algebra

$$\mathcal{A} \subset M_2\big(C(X \times \mathbf{T}^2 \to \mathcal{T}(C(\mathbf{T})))\big) \simeq M_2(\mathbb{C}) \otimes C(X) \otimes \mathcal{T}(C(\mathbf{T}))$$

generated by the pair of functions defined as follows:

$$\begin{aligned}
f_1(0, x_2, \phi_1, \phi_2) &= \pi_{\phi_2,x_2}(a_1) = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}, \\
f_1(x_1, 0, \phi_1, \phi_2) &= T\theta_{\phi_1,x_1}(a_1)T \\
&= T \begin{pmatrix} \sqrt{1-x_1}S(\mathbf{1} - SS^*) + e^{i\phi_1}\sqrt{x_1}(\mathbf{1} - SS^*)S^* & S^2 \\ 0 & 0 \end{pmatrix} T, \\
f_2(0, x_2, \phi_1, \phi_2) &= \pi_{\phi_2,x_2}(a_2) \\
&= \begin{pmatrix} \sqrt{1-x_2}S(\mathbf{1} - SS^*) + e^{i\phi_2}\sqrt{x_2}(\mathbf{1} - SS^*)S^* & S^2 \\ 0 & 0 \end{pmatrix}, \\
f_2(x_1, 0, \phi_1, \phi_2) &= T\theta_{\phi_1,x_1}(a_2)T = T \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix} T.
\end{aligned}$$

Note that continuity of f_1, f_2 at $(0, 0, \phi_1, \phi_2)$ follows from (21).

We show that $\mathcal{A}_0 \simeq \mathcal{A}$. Since the functions f_1, f_2 satisfy relations (12), by the universal property of $\mathcal{A}_0$ there exists a homomorphism $\psi \colon \mathcal{A}_0 \to \mathcal{A}$ taking a_i to f_i, $i = 1, 2$.

Theorem 6. *The homomorphism $\psi \colon \mathcal{A}_0 \to \mathcal{A}$ is an isomorphism.*

Proof. To prove the statement it is enough to show that for any irreducible representation $\pi \colon \mathcal{A}_0 \to B(\mathcal{H})$ there exists a representation $\widetilde{\pi} \colon \mathcal{A} \to B(\mathcal{H})$ such that $\pi = \widetilde{\pi} \circ \psi$. Indeed in this case $\psi(x) = 0$ implies $\pi(x) = \widetilde{\pi}(\psi(x)) = 0$ for any irreducible representation π of $\mathcal{A}$. Then $x = 0$ and ψ is injective. Since surjectivity of ψ is obvious we conclude that it is an isomorphism.

So, let us construct, for any irreducible representation π of $\mathcal{A}_0$, a corresponding representation $\widetilde{\pi}$ of $\mathcal{A}$.

1. Evidently $\widetilde{\pi}_F(f_i) = f_i(0, 0, \phi_1, \phi_2)$, $i = 1, 2$, for arbitrary fixed $\phi_j \in [0, 2\pi]$, $j = 1, 2$.
2. For any $x_2 \in (0, 1/2)$ and $\phi_2 \in [0, 2\pi)$ one has

$$\widetilde{\pi}_{\phi_2,x_2}(f_i) = f_i(0, x_2, \phi_1, \phi_2), \quad i = 1, 2,$$

where ϕ_1 is arbitrary fixed in $[0, 2\pi]$.

3. Analogously, for any $x_1 \in (0, 1/2)$, $\phi_1 \in [0, 2\pi)$ one has

$$\widetilde{\theta}_{\phi_1,x_1}(f_i) = Tf_i(x_1, 0, \phi_1, \phi_2)T, \quad i = 1, 2,$$

for arbitrary fixed $\phi_2 \in [0, 2\pi]$. Here we use the property $T = T^* = T^{-1}$.

4. Let us construct $\widetilde{\rho}_{2,\phi_2}$. In this case, it will be more convenient for us to consider $f_i(0, x_2, \phi_1, \phi_2)$, $i = 1, 2$, as tensor products of the form (17),(18). Indeed there exists unitary $U \colon \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \to \mathbb{C}^2 \otimes l_2(\mathbb{Z}_+) \otimes \mathbb{C}^2$ such that for any $x_2 \in [0, 1/2]$ and $\phi_2 \in [0, 2\pi]$ one has

$$Uf_i(0, x_2, \phi_1, \phi_2)U^* = \pi_{\phi_2,x_2}(a_i),$$

where $\pi_{\phi_2,x_2}(a_i)$, $i = 1, 2$, are presented as in (17),(18).

Denote by $a(x,\phi)\colon [0,1/2]\times[0,2\pi]\to M_2(\mathbb{C})$ a continuous function of the form

$$a(x,\phi)=\begin{pmatrix}0 & e^{i\phi}\sqrt{1-x}\\ \sqrt{x} & 0\end{pmatrix}. \tag{22}$$

It is easy to verify, see [10], that

$$V^*(\phi)a(1/2,\phi)V(\phi)=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{\frac{i\phi}{2}} & 0\\ 0 & e^{\pi+\frac{i\phi}{2}}\end{pmatrix}$$

for unitary

$$V(\phi)=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{\frac{i\phi}{2}} & -e^{\frac{i\phi}{2}}\\ 1 & 1\end{pmatrix}\in M_2(\mathbb{C}).$$

Put $W_i\colon \mathbb{C}^2\otimes l_2(\mathbb{Z}_+)\otimes\mathbb{C}^2\to\mathbb{C}^2\otimes l_2(\mathbb{Z}_+)\otimes\mathbb{C}^2$, $i=1,2$,

$$W_i(\phi)=(\mathbf{1}_2\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes E_{ii}V(\phi)).$$

Then, if $\phi_2\in[0,\pi)$, one has

$$\widetilde{\rho}_{2,\phi_2}(f_i)=(\mathbf{1}_2\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes p_1)\circ\big(W_1^*(2\phi_2)Uf_i(0,1/2,\phi_1,\phi_2))U^*W_1(2\phi_2)\big)$$

and, for $\phi_2\in[\pi,2\pi)$,

$$\widetilde{\rho}_{2,\phi_2}(f_i)=(\mathbf{1}_2\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes p_2)\circ\big(W_2^*(2\phi_2-2\pi)Uf_i(0,1/2,\phi_1,\phi_2))U^*W_2(2\phi_2-2\pi)\big),$$

where $p_i\colon M_2(\mathbb{C})\to\mathbb{C}$, $i=1,2$, are defined by

$$p_i(\sum_{r,s=1}^{2}\alpha_{rs}E_{rs})=\alpha_{ii}$$

and ϕ_1 is arbitrary fixed in $[0,2\pi]$.

5. Applying the considerations above to $\widetilde{\nu}_{2,\phi_1}$ we get, for $\phi_1\in[0,\pi)$,

$$\widetilde{\nu}_{2,\phi_1}(f_i)=(\mathbf{1}_2\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes p_1)\circ\big(W_1^*(2\phi_1)UTf_i(1/2,0,\phi_1,\phi_2))TU^*W_1(2\phi_1)\big)$$

and, for $\phi_1\in[\pi,2\pi)$,

$$\widetilde{\nu}_{2,\phi_1}(f_i)=(\mathbf{1}_2\otimes\mathbf{1}_{l_2(\mathbb{Z}_+)}\otimes p_2)\circ\big(W_2^*(2\phi_2-2\pi)UTf_i(1/2,0,\phi_1,\phi_2))TU^*W_2(2\phi_2-2\pi)\big)$$

for arbitrary fixed $\phi_2\in[0,2\pi]$.

6. Finally, to construct $\widetilde{\rho}_{\psi_2}$, we consider $\tau_{\psi_2}\colon\mathcal{T}(C(\mathbf{T}))\to\mathbb{C}$, $\tau_{\psi_2}(S)=e^{\frac{i\psi_2}{2}}$ and identify $M_2(\mathcal{T}(C(\mathbf{T})))$ with $M_2(\mathbb{C})\otimes\mathcal{T}(C(\mathbf{T}))$. Then

$$\widetilde{\rho}_{\psi_2}(a_i)=(id_{M_2(\mathbb{C})}\otimes\tau_{\psi_2})(f_i(0,x_2,\phi_1,\phi_2))$$

for arbitrary fixed $x_2\in[0,1/2]$ and $\phi_j\in[0,2\pi]$, $j=1,2$.

Note that representations equivalent to $\widetilde{\rho}_{\psi_2}$ can be also obtained as

$$(id_{M_2(\mathbb{C})}\otimes\tau_\psi)(Tf_i(x_1,0,\phi_1,\phi_2)T)$$

for certain ψ and arbitrary fixed $x_1\in[0,1/2]$ and $\phi_j\in[0,2\pi]$, $j=1,2$. □

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