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ONE REMARK CONCERNING DOUBLE-INFINITE TODA LATTICE

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Abstract. We propose the power moment approach to investigation of double-infinite Toda lattices, which was contained in author's article [6]. As a result, we give the main theorem from [6] in a more effective form.

1. Introduction

In the articles [3, 4], the author has proposed an approach for finding a solution of the Cauchy problem for a semi-infinite Toda equation, which is a difference analog of the classical approach to a similar problem for the KdV equation. Author's articles [5, 6] containing a similar approach to a double-infinite Toda lattice ([5] is a previous version, on the "physical level" of rigor, but it can be applied to more general objects than the Toda lattice). In the exact version [6], we could not find a general solution of a first order three-dimensional linear differential system for a spectral matrix of the corresponding block Jacobi operator. Therefore, [6] can be regarded only as a direction of investigation (in the semi-infinite case [3, 4], the corresponding differential equation can be solved).

In this article, we show that, instead of a differential system for spectral the matrix from [6], it is convenient to consider its power matrix moment [7, 2] interpretation, which is equivalent to the initial system. The corresponding moments can be found successively and, therefore, we get a more effective way of finding a solution of our Cauchy problem for the double-infinite Toda lattice. More exactly, let $\alpha_n(t), \beta_n(t)$, where $n = \ldots, -1, 0, 1, \ldots$, $t \in [0, T)$, be our solution. Then we have a procedure of finding this solution for every n and all t if we know the initial data and the solutions $\beta_0(t)$, $\beta_1(t)$ for every $t \in [0, T)$ (it seems that it is impossible to give a general formula for $\alpha_n(t), \beta_n(t)$ that would express this solution in terms of the initial conditions).

This article consists of two sections, — in Section 2 we repeat the main results from [6] in a form convenient for the subsequent exposition; in the Section 3, we present the moment approach.

2. The linearization of the Cauchy problem for a double-infinite Toda lattice

Consider the double-infinite Toda lattice

$$
\begin{aligned}
&\dot{\alpha}_n(t) = \frac{1}{2}\alpha_n(t)(\beta_{n+1}(t) - \beta_n(t)), \\
&\dot{\beta}_n(t) = \alpha_n^2(t) - \alpha_{n-1}^2(t), \quad n \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}, \quad t \in [0, T), \quad T > 0,
\end{aligned} \tag{1}
$$

where $\alpha_n(t) > 0, \beta_n(t)$ are real continuously differentiable functions uniformly bounded w.r.t. $n \in \mathbb{Z}$. For (1) we set the Cauchy problem: to find the solutions $\alpha_n(t), \beta_n(t)$ from the initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{Z}$.

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1^0. In [6], Sections 3,4, we have proposed the following procedure of linearization of this problem using the spectral theory of Jacobi-type block matrices in the space

$$\mathbf{l}_2 = \mathbb{C}^1 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \dots; \quad \mathbf{l}_2 \ni f = (f_0, (f_{1,0}, f_{1,1}), (f_{2,0}, f_{2,1}), \dots). \tag{2}$$

Use $\alpha_n(t), \beta_n(t)$ from (1) to construct the following block Jacobi matrix: $\forall t \in [0, T)$

(3)

$$J(t) = \left|\begin{array}{c|cc|cc|cc|cc|c}
\beta_0(t) & \alpha_0(t) & \alpha_{-1}(t) & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
\alpha_0(t) & \beta_1(t) & 0 & \alpha_1(t) & 0 & 0 & 0 & 0 & 0 & \dots \\
\alpha_{-1}(t) & 0 & \beta_{-1}(t) & 0 & \alpha_{-2}(t) & 0 & 0 & 0 & 0 & \dots \\
\hline
0 & \alpha_1(t) & 0 & \beta_2(t) & 0 & \alpha_2(t) & 0 & 0 & 0 & \dots \\
0 & 0 & \alpha_{-2}(t) & 0 & \beta_{-2}(t) & 0 & \alpha_{-3}(t) & 0 & 0 & \dots \\
\hline
0 & 0 & 0 & \alpha_2(t) & 0 & \beta_3(t) & 0 & \alpha_3(t) & 0 & \dots \\
0 & 0 & 0 & 0 & \alpha_{-3}(t) & 0 & \beta_{-3}(t) & 0 & \alpha_{-4}(t) & \dots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right.$$

$$=: \left|\begin{array}{cccccc}
b_0(t) & a_0^*(t) & 0 & 0 & 0 & \dots \\
a_0(t) & b_1(t) & a_1(t) & 0 & 0 & \dots \\
0 & a_1(t) & b_2(t) & a_2(t) & 0 & \dots \\
0 & 0 & a_2(t) & b_3(t) & a_3(t) & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right., \quad \text{i.e.} \quad \begin{array}{l} b_0(t) = \begin{bmatrix}\beta_0(t)\end{bmatrix} : \mathbb{C}^1 \to \mathbb{C}^1, \\ a_0(t) = \begin{bmatrix}\alpha_0(t) \\ \alpha_{-1}(t)\end{bmatrix} : \mathbb{C}^1 \to \mathbb{C}^2, \end{array}$$

$$a_0^*(t) = \begin{bmatrix}\alpha_0(t) & \alpha_{-1}(t)\end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^1, \quad a_n(t) = a_n^*(t) : \mathbb{C}^2 \to \mathbb{C}^2,$$
$$b_n(t) = b_n^*(t) : \mathbb{C}^2 \to \mathbb{C}^2, \quad n \in \mathbb{N} = \{1, 2, \dots\}.$$

Every such a matrix generates, in the space $\mathbf{l}_2$, a Hermitian operator the domain of which is the linear space $\mathbf{l}_{\text{fin}}$ of all finite sequences from $\mathbf{l}_2$. The closure $\mathbf{J}(t)$ of this operator is a bounded selfadjoint operator in $\mathbf{l}_2$; $t \in [0, T)$.

First, we give a construction of a generalized eigenvector expansion for the operator $\mathbf{J}(t)$ with fixed t. It has the following form. Every such generalized eigenvector, $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}$, with the eigenvalue $\lambda \in \mathbb{R}$, is a solution, from the linear space $\mathbf{l}$ of all sequences $f = (f_n)_{n=0}^{\infty}$ ($f_0 \in \mathbb{C}^1, f_n = (f_{n,0}, f_{n,1}) \in \mathbb{C}^2, n \in \mathbb{N}$), of the difference equation

$$\begin{aligned}
&J(t)\varphi(\lambda) = \lambda\varphi(\lambda), \quad \text{i.e.,} \\
&b_0(t)\varphi_0(\lambda) + a_0^*(t)\varphi_1(t) = \lambda\varphi_0(\lambda), \\
&a_{n-1}(t)\varphi_{n-1}(\lambda) + b_n(t)\varphi_n(\lambda) + a_n(t)\varphi_{n+1}(\lambda) = \lambda\varphi_n(\lambda), \quad n \in \mathbb{N}.
\end{aligned} \tag{4}$$

Note that, in fact, $\varphi(\lambda) \in \mathbf{l}_2(p)$, where the latter space is the space $\mathbf{l}_2$ with some weight $p = (p_n)_{n=0}^{\infty}$, $p_n \geq 1$, common for all $\lambda \in \mathbb{R}$ and $t \in [0, T)$.

Every solution $\varphi(\lambda) = (\varphi_n(\lambda))_{n=0}^{\infty}$ of difference equation (4) is real valued and defined by the two initial condition. It is convenient to take, for such solutions, the following two type of conditions:

$$\varphi_0(\lambda) = 1, \varphi_{1,0}(\lambda) = 0 \quad \text{and} \quad \varphi_0(\lambda) = 0, \varphi_{1,0}(\lambda) = 1. \tag{5}$$

A solution with the first pair of conditions in (5) is denoted by $\theta^{(0)}(\lambda)$, with second pair by $\theta^{(1)}(\lambda)$. Linearity of system (4) gives that an arbitrary generalized eigenvector has the form: $\forall \lambda \in \mathbb{R}$

$$\varphi(\lambda) = \varphi_0(\lambda)\theta^{(0)}(\lambda) + \varphi_{1,0}(\lambda)\theta^{(1)}(\lambda). \tag{6}$$

It is convenient to introduce some matrix solutions of equation (4) using $\theta^{(0)}(\lambda)$ and $\theta^{(1)}(\lambda)$. So, we put

$$
\begin{aligned}
P_0(\lambda) &= \begin{bmatrix}\theta_0^{(0)}(\lambda) & \theta_0^{(1)}(\lambda)\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix} =: \begin{bmatrix}P_{0;0,0}(\lambda) & P_{0;0,1}(\lambda)\end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^1,\\
P_n(\lambda) &= \begin{bmatrix}\theta_{n,0}^{(0)}(\lambda) & \theta_{n,0}^{(1)}(\lambda)\\ \theta_{n,1}^{(0)}(\lambda) & \theta_{n,1}^{(1)}(\lambda)\end{bmatrix} =: \begin{bmatrix}P_{n;0,0}(\lambda) & P_{n;0,1}(\lambda)\\ P_{n;1,0}(\lambda) & P_{n;1,1}(\lambda)\end{bmatrix} : \mathbb{C}^2 \to \mathbb{C}^2, \quad n \in \mathbb{N}.
\end{aligned} \tag{7}
$$

Then the following equalities for these matrix solutions ("polynomials of the first kind") follow from (4): $\forall \lambda \in \mathbb{R}, t \in [0,T)$

$$
\begin{aligned}
&b_0(t)P_0(\lambda) + a_0^*(t)P_1(\lambda) = \lambda P_0(\lambda),\\
&a_{n-1}(t)P_{n-1}(\lambda) + b_n(t)P_n(\lambda) + a_n(t)P_{n+1}(\lambda) = \lambda P_n(\lambda), \quad n \in \mathbb{N}.
\end{aligned} \tag{8}
$$

The generalized eigenvectors of operator $\mathbf{J}(t)$ introduced above, of course, depend on t. Further, we will stress this dependence by writing $P_n(\lambda;t)$, $P_{n;\alpha,\beta}(\lambda;t)$ etc.

Introduce now the Fourier transform $\widehat{\ }$ generated by the operator $\mathbf{J}(t)$ with fixed $t \in [0,T)$. Namely, we put for $f \in \mathbf{l}_{\text{fin}} : \forall \lambda \in \mathbb{R}$

$$
\widehat{f}(\lambda;t) = \sum_{n=0}^{\infty} P_n^*(\lambda;t) f_n = (\widehat{f}_0(\lambda;t), \widehat{f}_1(\lambda;t)) \in \mathbb{C}^2. \tag{9}
$$

This Fourier transform acts from the space $\mathbf{l}_2$ into an L^2-Hilbert space that depends on t, $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda;t))$, of functions $\mathbb{R} \ni \lambda \mapsto F(\lambda) \in \mathbb{C}^2$, and is constructed by using the scalar product

$$
(F,G)_{L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))} = \int_{\mathbb{R}} (d\rho(\lambda;t)F(\lambda), G(\lambda))_{\mathbb{C}^2}\,, \tag{10}
$$

where $d\rho(\lambda;t)$ is the spectral 2×2-matrix measure of the operator $\mathbf{J}(t)$. Integral (10) is defined in a standard way starting with simple functions. For the Fourier transform, the Parseval equality takes place,

$$
(f,g)_{\mathbf{l}_2} = \int_{\mathbb{R}} \left(d\rho(\lambda;t)\widehat{f}(\lambda;t), \widehat{g}(\lambda;t)\right)_{\mathbb{C}^2}, \quad f,g \in \mathbf{l}_{\text{fin}}. \tag{11}
$$

Using definition (9) and this equality we can continuously extend the Fourier transform from $\mathbf{l}_{\text{fin}}$ to the whole $\mathbf{l}_2$. Then such an extended transform $f \mapsto \widehat{f}(\lambda;t)$ will make a unitary operator between the spaces $\mathbf{l}_2$ and $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda;t))$. It follows from (9), (8), and (3) that the image of $\mathbf{J}(t)$ is an operator of multiplication by λ in the space $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda;t))$.

The spectral matrix measure $d\rho(\lambda;t)$ is an analog of the spectral measure for classical Jacobi matrices. It is a Borel and probability real 2×2-matrix valued measure: $\forall t \in [0,T)$ $\mathfrak{B}(\mathbb{R}) \ni \triangle \mapsto \rho(\triangle;t) = (\rho_{\alpha,\beta}(\triangle;t))_{\alpha,\beta=0}^1, \rho(\mathbb{R},t) = 1$. It corresponds in a one-to-one way to a Weyl matrix-valued function $m(z;t)$ via the classical equality: $\forall t \in [0,T)$

$$
m(z;t) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda;t), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{12}
$$

Here

$$
\begin{aligned}
m(z;t) &= \begin{bmatrix} m_{0,0}(z;t) & m_{0,1}(z;t)\\ m_{1,0}(z;t) & m_{1,1}(z;t)\end{bmatrix}\\
&= \begin{bmatrix} (\mathbf{R}_z(t)\varepsilon_{0,0}, \varepsilon_{0,0})_{\mathbf{l}_2} & (\mathbf{R}_z(t)\varepsilon_{1,0}, \varepsilon_{0,0})_{\mathbf{l}_2}\\ (\mathbf{R}_z(t)\varepsilon_{0,0}, \varepsilon_{1,0})_{\mathbf{l}_2} & (\mathbf{R}_z(t)\varepsilon_{1,0}, \varepsilon_{1,0})_{\mathbf{l}_2}\end{bmatrix},\\
\mathbf{R}_z(t) &= (\mathbf{J}(t) - z1)^{-1}, \quad z \in \mathbb{C}\setminus\mathbb{R}, t \in [0,T),\\
\varepsilon_{0,0} &= (1,(0,0),(0,0),\ldots), \quad \varepsilon_{1,0} = (0,(1,0),(0,0),\ldots) \in \mathbf{l}_2.
\end{aligned} \tag{13}
$$

Since the matrix $J(t)$ is real, it follows that $m_{1,0}(z;t) = m_{0,1}(z;t)$ and $\rho_{1,0}(\triangle;t) = \rho_{0,1}(\triangle;t)$, $\triangle \in \mathfrak{B}(\mathbb{R}), t \in [0,T)$.

As in the case of the classical Jacobi matrices, elements of the matrix $J(t)$ (3) can be reconstructed from its matrix spectral measure $d\rho(\lambda;t)$ and its polynomials of the first kind. Namely, from (9) and (11) it is easy to get: $\forall t\in[0,T)$

$$
\begin{aligned}
&a_n^*(t)=\int_{\mathbb{R}}\lambda P_n(\lambda;t)d\rho(\lambda;t)P_{n+1}^*(\lambda;t),\\
&b_n(t)=\int_{\mathbb{R}}\lambda P_n(\lambda;t)d\rho(\lambda;t)P_n^*(\lambda;t),\quad n\in\mathbb{N}_0:=\{0,1,2,\dots\},\\
&a_n^*(t)=a_n(t),\quad n\in\mathbb{N}
\end{aligned}
\tag{14}
$$

(the integrals (14) of 2×2-matrix valued functions are defined similar to (10)).

We stress that the system of all polynomials of the first kind is orthonormal (and complete) in the space $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$: using (9), (11) we find $\forall t\in[0,T)$ and $j,k\in\mathbb{N}_0$

$$
\int_{\mathbb{R}}P_j(\lambda;t)d\rho(\lambda;t)P_k^*(\lambda;t)=\delta_{j,k}1
\tag{15}
$$

(if $j=k=0$ this integral is equal to number 1).

The orthogonality condition (15) gives a possibility to construct the polynomials of the first kind $P_n(\lambda;t)$ directly using only the given measure $d\rho(\lambda;t)$. At first we note that the support of $d\rho(\lambda;t)$ ($t\in[0,T)$ fixed) is bounded. Therefore, the functions $\mathbb{R}\ni\lambda\mapsto(p(\lambda),q(\lambda))\in\mathbb{R}^2$, where $p(\lambda),q(\lambda)$ are arbitrary polynomials with real coefficients, are dense in the real part $L^2(\mathbb{R}^2,\mathbb{R},d\rho(\lambda;t))$ of the space $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$.

From those facts and formulas (7), (15), it follows that the sequence of rows of the polynomials of the first kind,

$$
\begin{aligned}
&(P_{0;0,0}(\lambda;t),P_{0;0,1}(\lambda;t)),(P_{1;0,0}(\lambda;t),P_{1;0,1}(\lambda;t)),(P_{1;1,0}(\lambda;t),P_{1;1,1}(\lambda;t)),\\
&(P_{2;0,0}(\lambda;t),P_{2;0,1}(\lambda;t)),(P_{2;1,0}(\lambda;t),P_{2;1,1}(\lambda;t)),\\
&(P_{3;0,0}(\lambda;t),P_{3;0,1}(\lambda;t)),(P_{3;1,0}(\lambda;t),P_{3;1,1}(\lambda;t)),\dots
\end{aligned}
\tag{16}
$$

can be found by applying the classical Gramm-Schmidt orthogonalization procedure to the following sequence of $\mathbb{R}^2$-valued functions of the variable $\lambda\in\mathbb{R}$:

$$
(1,0),(0,1),(\lambda,0),(0,\lambda),(\lambda^2,0),(0,\lambda^2),(\lambda^3,0),\dots.
\tag{17}
$$

The knowledge of (16) is equivalent, according to (7), to the knowledge of

$$
P_{n;\alpha,\beta}(\lambda;t),\alpha,\beta=0,1,\lambda\in\mathbb{R},\qquad t\in[0,T).
$$

2^0. In this Subsection we explain the connection between the above stated spectral theory of block Jacobi-type matrices (3) and ordinary double-infinite Jacobi matrices.

So, instead of the space $\mathbf{l}_2$ (2), we will use the usual space

$$
\ell_2(\mathbb{Z})=\ldots\oplus\mathbb{C}^1\oplus\mathbb{C}^1\oplus\mathbb{C}^1\oplus\ldots,\quad \ell_2(\mathbb{Z})\ni u=(u_n)_{n=-\infty}^{\infty}.
\tag{18}
$$

Consider the double-infinite Jacobi matrix $L(t)$ ($t\in[0,T)$ is fixed) with elements $\alpha_n(t)$, $\beta_n(t)$ from (3); it acts on a sequence $u=(u_n)_{n=-\infty}^{\infty};u_n\in\mathbb{C}$, as follows:

$$
(L(t)u)_n=\alpha_{n-1}(t)u_{n-1}+\beta_n(t)u_n+\alpha_n(t)u_{n+1},\quad n\in\mathbb{Z}.
\tag{19}
$$

This expression generates, in the space $\ell_2(\mathbb{Z})$, a bounded selfadjoint operators $\mathbf{L}(t)$: it is necessary to consider (19) on finite sequences $u\in\ell_{\mathrm{fin}}(\mathbb{Z})$ and then to take its closure in $\ell_2(\mathbb{Z})$.

I do not know a systematic account of spectral theory of such operators $\mathbf{L}(t)$, some corresponding facts are contained e.g. in [2, 8]. But it is possible to say that its construction is similar to the spectral theory of the Sturm-Liouville operator $\mathcal{L}$ on the whole axis $\mathbb{R}\ni x$: instead of the spectral measure $d\rho(\lambda;t)$ of the corresponding operator $\mathbf{L}$, we have a spectral 2×2-matrix measure $d\rho(\lambda)$ the construction of which is connected with two solutions of the equation $\mathcal{L}\varphi(x;\lambda)=\lambda\varphi(x;\lambda)$ with the initial data $\varphi(0;\lambda)=1,\varphi'(0;\lambda)=0$ and $\varphi(0;\lambda)=0,\varphi'(0;\lambda)=1,\lambda\in\mathbb{R}$.

In our case (19), it is also necessary to take two solutions of the equation $(L(t)\varphi)_n=\lambda\varphi_n(\lambda;t)$, $n\in\mathbb{Z}$, with the initial data in the points $n=0$ and $n=1$: $\varphi_0(\lambda;t)=1,\varphi_1(\lambda;t)=0$ and $\varphi_0(\lambda;t)=0,\varphi_1(\lambda;t)=1$, construct the corresponding Fourier transform, values of which are two-dimensional etc.

The spaces $\ell_2(\mathbb{Z})$ (18) and $\mathbf{l}_2$ (2) are isometric. For us, it is convenient to use the following isometry $\mathbf{U}$. Namely, in $\mathbf{l}_2$ we have a natural orthonormal basis,

$$\varepsilon_{0,0}=(1,(0,0),(0,0),\ldots),\quad \forall n\in\mathbb{N}\,\varepsilon_{n,\alpha}=(0,(0,0),\ldots,(0,0),\underbrace{(1,0)}_{n^{\text{th}}\text{ place}},(0,0),\ldots) \tag{20}$$

if $\alpha=0$, and $(0,1)$ at the place with index n if $\alpha=1$ (notations of f and $J(t)$ in (2), (3) are given with respect to this basis). In $\ell_2(\mathbb{Z})$, we have the standard basis

$$\delta_n=(\ldots,0,\underbrace{1}_{n\text{ place}},0,\ldots),\quad n\in\mathbb{Z}. \tag{21}$$

We put

$$\mathbf{U}:\ell_2(\mathbb{Z})\to\mathbf{l}_2,\quad \mathbf{U}\delta_0=\varepsilon_{0,0},\quad \forall n\in\mathbb{N}\quad \mathbf{U}\delta_n=\varepsilon_{n,0},\quad \mathbf{U}\delta_{-n}=\varepsilon_{n,1}. \tag{22}$$

It is easy to understand that in terms of such isometry operators, $\mathbf{J}(t)$ and $\mathbf{L}(t)$ ($t\in[0,T)$ is fixed) are unitary equivalent, $\mathbf{L}(t)=\mathbf{U}^{-1}\mathbf{J}(t)\mathbf{U}$. So, we can formulate the following

Remark 1. The corresponding parts of articles [5, 6] (Sections 7 and 4, respectively) and Subsection 1 above, according to (20)–(22), give in fact an account of the spectral theory of double-infinite Jacobi matrices of type (19).

3^0. Let us consider the Cauchy problem for the double-infinite Toda lattice (1). In [6], Section 5, Theorem 1, it was proved that the elements $m_{\alpha,\beta}(z;t)$ of Weyl matrix function $m(z;t)$ (12) are solutions of the following system of three linear differential equations w.r.t. $m_{0,0}(z;t),m_{0,1}(z;t)$ and $m_{1,1}(z;t)$:

$$\begin{aligned}
&\dot m_{0,0}(z;t)=(\beta_0(t)-z)m_{0,0}(z;t)+2\alpha_0(t)m_{0,1}(z;t)-1,\\
&\dot m_{0,1}(z;t)=-\alpha_0(t)m_{0,0}(z;t)+\frac12(\beta_0(t)-\beta_1(t))m_{0,1}(z;t)+\alpha_0(t)m_{1,1}(z;t),\\
&\dot m_{1,1}(z;t)=-2\alpha_0(t)m_{0,1}(z;t)-(\beta_1(t)-z)m_{1,1}(z;t)+1,\\
&m_{1,0}(z;t)=m_{0,1}(z;t),\quad z\in\mathbb{C}\setminus\mathbb{R},\quad t\in[0,T).
\end{aligned} \tag{23}$$

Here $\alpha_0(t),\beta_0(t),\beta_1(t)$ are real smooth coefficients connected by the equality (see (1))

$$\frac{\dot\alpha_0(t)}{\alpha_0(t)}=\frac12(\beta_1(t)-\beta_0(t)),\quad t\in[0,T). \tag{24}$$

It is necessary to find a solution of system (23) with initial Cauchy data $m_{0,0}(z;0)$, $m_{0,1}(z;0)$, $m_{1,1}(z;0)$ for arbitrary $z\in\mathbb{C}\setminus\mathbb{R}$. The coefficients $\alpha_0(t),\beta_0(t)$ and $\beta_1(t)$ must be defined from the conditions:

$$\lim_{|z|\to\infty} zm(z;t)=-1 \tag{25}$$

(which is equivalent to the condition $\rho(\mathbb{R};t)=1,t\in[0,T)$).

System (23) with conditions (25) can be rewritten as a system for elements of the derivatives of the corresponding spectral matrix measure $d\rho(\lambda;t)$ ([6], Theorem 10). More exactly, introduce the following finite Borel measure (a joint spectral scalar measure):

$$\mathfrak{B}(\mathbb{R})\ni\triangle\mapsto\int_0^T\operatorname{Tr}\rho(\triangle;t)dt=:\sigma(\triangle)\geq 0. \tag{26}$$

For arbitrary $t \in [0,T)$, the spectral matrix measure $d\rho(\lambda;t)$ is absolutely continuous w.r.t. $d\sigma(\lambda)$, therefore we can introduce $\forall t \in [0,T)$ the matrix derivative $d\rho(\lambda;t)/d\sigma(\lambda) =: r(\lambda;t) = (r_{\alpha,\beta}(\lambda;t))_{\alpha,\beta=0}^{1}$. The later functions are defined for $d\sigma(\lambda)$-almost all $\lambda \in \mathbb{R}$ and we have the equalities

$$\rho(\triangle;t) = \int_{\triangle} r(\lambda;t)d\sigma(\lambda), \quad \rho_{\alpha,\beta}(\triangle;t) = \int_{\triangle} r_{\alpha,\beta}(\lambda;t)d\sigma(\lambda), \qquad \triangle \in \mathfrak{B}(\mathbb{R}), \quad t \in [0,T), \quad \alpha,\beta = 0,1. \tag{27}$$

System (23), in terms of $r_{\alpha,\beta}(\lambda;t)$, has the following form:

$$\begin{aligned}
&\dot r_{0,0}(\lambda;t) = (\beta_0(t) - \lambda)r_{0,0}(\lambda;t) + 2\alpha_0(t)r_{0,1}(\lambda;t),\\
&\dot r_{0,1}(\lambda;t) = -\alpha_0(t)r_{0,0}(\lambda;t) + \frac{1}{2}(\beta_0(t) - \beta_1(t))r_{0,1}(\lambda;t) + \alpha_0(t)r_{1,1}(\lambda;t),\\
&\dot r_{1,1}(\lambda;t) = -2\alpha_0(t)r_{0,1}(\lambda;t) - (\beta_1(t) - \lambda)r_{1,1}(\lambda;t),\\
&r_{1,0}(\lambda;t) = r_{0,1}(\lambda;t), \quad t \in [0,T).
\end{aligned} \tag{28}$$

Functions $r_{\alpha,\beta}(\lambda;t)$ from (28) are one time continuously differentiable on $[0,T)$ for $d\sigma(\lambda)$-almost all $\lambda \in \mathbb{R}$, the equalities (28) are fulfilled also for such λ. For this system (28), it is also necessary to find a solution of the Cauchy problem if $r_{0,0}(\lambda;0), r_{0,1}(\lambda;0)$, and $r_{1,1}(\lambda;0)$ are given. From (27), it follows that this measure, for every $\triangle \in \mathfrak{B}(\mathbb{R})$, have the same smoothness.

Such problems for (23) and (28) are equivalent, — the connection is given by formulas (12) and (27), (26).

Unfortunately, we cannot find, in the general case, solutions of these Cauchy problems for (23) or (28) (it is possible to do [3, 4, 6] in the case where the double-infinite Toda lattice is replaced with a semi-infinite one, i.e. if $\mathbb{Z}$ is replaced with $\mathbb{N}_0$). In the next Section we will explain in what way it is possible to overcome this difficulty. But now we can only formulate the assertion about the following linearization of the Cauchy problem for (1) [6], Theorem 11.

Theorem 1. *Consider the Cauchy problem for lattice* (1) *with given initial data:* $\alpha_n(0)$, $\beta_n(0), n \in \mathbb{Z}$. *It is possible to find its solution by applying the following linearization procedure.*

1) Using the initial data consider the matrix $J(0)$ (3) *and find its matrix spectral measure* $d\rho(\lambda;0)$ *and Weyl function* $m(z;0), \lambda \in \mathbb{R}, z \in \mathbb{C} \setminus \mathbb{R}$.

2) Consider the linear system (23) *w.r.t. the unknowns* $m_{0,0}(z;t), m_{0,1}(z;t), m_{1,1}(z;t)$, *and find its solution using the initial data* $m_{\alpha,\beta}(z;0)$; $\alpha,\beta = 0,1, z \in \mathbb{C} \setminus \mathbb{R}$.

3) In 2) the coefficients $\alpha_0(t) > 0, \beta_0(t)\ \beta_1(t), t \in [0,T)$, *are real smooth arbitrary functions. We find these functions using the conditions* (24), (25) *and the found above solutions of* (23).

4) Using $m(z;t), z \in \mathbb{C}\setminus\mathbb{R}, t \in [0,T)$ *we find the corresponding matrix spectral measure* $d\rho(\lambda;t), \lambda \in \mathbb{R}, t \in [0,T)$ *(see* (12)*).*

5) For fixed $t \in [0,T)$, *consider the space* $L^2(\mathbb{C}^2, \mathbb{R}, d\rho(\lambda;t))$ *and apply the Gramm-Schmidt orthogonalization to sequence* (17) *in this space. As a result, we get the sequence* (16) *of the polynomials of the first kind.*

6) The solution of our Cauchy problem for (1) *is obtained by using formulas* (14), (3).

Note that, in this approach, a realization of items 2), 3) are problematical but, in some cases, a similar realization is possible. For example, if we assumed that all $\alpha_n(t) = 0, n = \dots, -2, -1; t \in [0,T)$ are arbitrary (in this case we have, actually, the semi-infinite Toda lattice (1) with $\mathbb{N}_0$ instead $\mathbb{Z}$).

In the above scheme, we can use system (28) instead of system (23) in items 2), 3), but in doing so it is necessary to know (from some additional data) the joint scalar spectral measure (26). Now for the knowledge of $d\rho(\lambda;t)$ we use (27).

3. An approach based on the theory of moments

It is not always possible to succeed in finding a solution of the Cauchy problem for a general system of three linear equations (23) (or (28)). Therefore, the results of Section 2 can be considered only as a direction for searching a solution of the Cauchy problem for the Toda lattice (1). In this Section we show that it is possible to pass from system (23) to corresponding moments and, in this way, to get more effective results.

Consider the classical power moments connected with the Weyl functions $m_{\alpha,\beta}(z;t)$ from (12), (13), and the corresponding spectral measure $d\rho_{\alpha,\beta}(\lambda;t) : \forall t \in [0,T)$

$$
\begin{aligned}
&s_n(\alpha,\beta;t) = \int_{\mathbb{R}} \lambda^n d\rho_{\alpha,\beta}(\lambda;t), \quad \alpha,\beta = 0,1,\\
&\text{i.e.} \quad s_n(t) = \int_{\mathbb{R}} \lambda^n d\rho(\lambda;t) : \mathbb{C}^2 \to \mathbb{C}^2, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{29}
$$

These matrix moments appear also in the following expressions that follow from (12): $\forall t \in [0,T)$

$$
\begin{aligned}
m_{\alpha,\beta}(z;t) &= \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho_{\alpha,\beta}(\lambda;t) = -\frac{1}{z}\int_{\mathbb{R}} (1-\lambda/z)^{-1} d\rho_{\alpha,\beta}(\lambda;t)\\
&= -\frac{1}{z}\int_{\mathbb{R}} \sum_{n=0}^{\infty} \left(\frac{\lambda}{z}\right)^n d\rho_{\alpha,\beta}(\lambda;t) = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} s_n(\alpha,\beta;t), \quad |z| > R, \quad \alpha,\beta = 0,1.
\end{aligned} \tag{30}
$$

Here $R > 0$ is so large that the spectrums of all the operators $\mathbf{J}(t), t \in [0,T)$, are located in the ball $\{z \in \mathbb{C} || z| \le \mathbb{R}\}$. The convergence of the series in (30) is uniform for $|z| \ge R+\varepsilon$, where $\varepsilon > 0$ is arbitrary fixed.

According to (29), (30) we have the following for the first identity in (23) (note that $\forall t \in [0,T)$ $\rho_{0,0}(\mathbb{R};t) = 1$): $\forall |z| > R, t \in [0,T)$

$$
\begin{aligned}
-\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \dot{s}_n(0,0;t) &= \dot{m}_{0,0}(z;t) = (\beta_0(t) - z) m_{0,0}(z;t) + 2\alpha_0(t) m_{0,1}(z;t) - 1\\
&= \int_{\mathbb{R}} \frac{\beta_0(t) - \lambda}{\lambda - z} d\rho_{0,0}(\lambda;t) + \int_{\mathbb{R}} \frac{2\alpha_0(t)}{\lambda - z} d\rho_{0,1}(\lambda;t)\\
&= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} (\beta_0(t) s_n(0,0;t) - s_{n+1}(0,0;t) + 2\alpha_0(t) s_n(0,1;t)).
\end{aligned} \tag{31}
$$

Identity (31) means that

$$
\dot{s}_n(0,0;t) = \beta_0(t) s_n(0,0;t) - s_{n+1}(0,0;t) + 2\alpha_0(t) s_n(0,1;t), \quad n \in \mathbb{N}_0, \quad t \in [0,T). \tag{32}
$$

Similarly to (31), (32) we conclude from the second and the third identities in (23) that $\forall n \in \mathbb{N}_0, t \in [0,T)$

$$
\begin{aligned}
&\dot{s}_n(0,1;t) = -\alpha_0(t) s_n(0,0;t) + \frac{1}{2}(\beta_0(t) - \beta_1(t)) s_n(0,1;t) + \alpha_0(t) s_n(1,1;t),\\
&\dot{s}_n(1,1;t) = -2\alpha_0(t) s_n(0,1;t) - \beta_1(t) s_n(1,1;t) + s_{n+1}(1,1;t).
\end{aligned}
$$

As a result, we have proved the following.

Lemma 1. *The system* (23) *for* $m_{\alpha,\beta}(z;t)$ *is equivalent to the following system for moments* (29)*:*

$$\begin{aligned}
&\dot{s}_n(0,0;t) = \beta_0(t)s_n(0,0;t) - s_{n+1}(0,0;t) + 2\alpha_0(t)s_n(0,1;t),\\
&\dot{s}_n(0,1;t) = \frac{1}{2}(\beta_0(t) - \beta_1(t))s_n(0,1;t) + \alpha_0(t)(s_n(1,1;t) - s_n(0,0;t)),\\
&\dot{s}_n(1,1;t) = -\beta_1(t)s_n(1,1;t) + s_{n+1}(1,1;t) - 2\alpha_0(t)s_n(0,1;t),\\
&s_n(1,0;t) = s_n(0,1;t), \quad n \in \mathbb{N}_0, \quad t \in [0,T).
\end{aligned} \tag{33}$$

For the proof it is necessary to note that the moments (29) for $n \in \mathbb{N}_0$ uniquely defined $d\rho_{\alpha,\beta}(\lambda;t)$ and therefore $m_{\alpha,\beta}(z;t)$, since these measures have bounded support ($\alpha,\beta = 0,1, t \in [0,T)$). Note also that the identities (33), of course, follow from (28), too.

Consider the second identity in (33) as a differential equation in t with respect to $s_n(0,1;t)$ and using (24) we can find its solution and rewrite this identity in the form: $\forall n \in \mathbb{N}_0, t \in [0,T)$

$$s_n(0,1;t) = \alpha_0^{-1}(t)\Big(\alpha_0(0)s_n(0,1;0) + \int_0^t \alpha_0^2(\tau)\big(s_n(1,1;\tau) - s_n(0,0;\tau)\big)\,d\tau\Big). \tag{34}$$

So, we have from (33), (34) that

$$\begin{aligned}
&s_{n+1}(0,0;t) = \beta_0(t)s_n(0,0;t) - \dot{s}_n(0,0;t) + 2\alpha_0(t)s_n(0,1;t),\\
&s_{n+1}(1,1;t) = \beta_1(t)s_n(1,1;t) + \dot{s}_n(1,1;t) + 2\alpha_0(t)s_n(0,1;t),\\
&s_n(0,1;t) = \alpha_0^{-1}(t)\Big(\alpha_0(0)s_n(0,1;0) + \int_0^t \alpha_0^2(\tau)\big(s_n(1,1;\tau) - s_n(0,0;\tau)\big)d\tau\Big),\\
&s_n(1,0;t) = s_n(0,1;t), \quad n \in \mathbb{N}_0, \quad t \in [0,T).
\end{aligned} \tag{35}$$

These identities make it possible by using the functions $\beta_0(t)$, $\beta_1(t)$, $t \in [0,T)$ to recursively find $s_n(\alpha,\beta;t)$ $\forall \alpha,\beta = 0,1, t \in [0,T)$, and $n \in \mathbb{N}_0$. So, using the solution

$$\alpha_0(t) = \alpha_0(0)e^{\frac{1}{2}\int_0^t(\beta_1(\tau)-\beta_0(\tau))\,d\tau}, \quad t \in [0,T), \tag{36}$$

of equation (24) w.r.t. the unknown $\alpha_0(t)$ and the identities $\rho_{\alpha,\beta}(\mathbb{R};t) = 1$ for $\alpha = \beta = 0,1$, or $\rho_{\alpha,\beta}(\mathbb{R};t) = 0$ for $\alpha = 0, \beta = 1$ (for every $t \in [0,T)$) we get: $\forall t \in [0,T)$

$$\begin{aligned}
&s_0(0,0;t) = 1, \quad s_0(1,1;t) = 1, \quad s_0(0,1;t) = 0,\\
&s_1(0,0;t) = \beta_0(t), \quad s_1(1,1;t) = \beta_1(t),\\
&s_1(0,1;t) = \alpha_0^{-1}(t)\big(\alpha_0(0)s_1(0,1;0) + \alpha_0^2(t) - \alpha_0^2(0)\big),\\
&\ldots;\\
&s_n(1,0;t) = s_n(0,1;t), \quad n \in \mathbb{N}_0.
\end{aligned} \tag{37}$$

The identities (35), (36), (37) permit to find recursively all the moments $s_n(\alpha,\beta;t)$, (29), $n \in \mathbb{N}_0$, and therefore the matrix spectral measure $d\rho(\lambda;t), t \in [0,T)$. To carrying out this procedure, it is necessary only to assume that the functions $\beta_0(t)$ and $\beta_1(t)$ are smooth. We will assume that these functions are infinitely differentiable on $[0,T)$.

Thus, we recursively find the functions $s_n(\alpha,\beta;t), \alpha,\beta = 0,1, t \in [0,T)$ from (35), (37) using $\beta_0(t)$, $\beta_1(t)$, $t \in [0,T)$, (36) and the following initial conditions:

$$\begin{aligned}
&s_n(0,0;0) = \int_{\mathbb{R}} \lambda^n d\rho_{0,0}(\lambda;0), \quad s_n(1,1;0) = \int_{\mathbb{R}} \lambda^n d\rho_{1,1}(\lambda;0),\\
&s_n(0,1;0) = \int_{\mathbb{R}} \lambda^n d\rho_{0,1}(\lambda;0), \quad n \in \mathbb{N}_0.
\end{aligned} \tag{38}$$

Note that we know the initial data of our Cauchy problem and, therefore, we know the matrix $J(0)$ from (3). Then we can calculate its spectral measure $d\rho(\lambda;0)$ and the moments of this measure, i.e., the integrals in (38).

It is also necessary to note that there also exists the classical formulas for moments which are calculated directly from elements of the Jacobi matrix (see [9]).

As a result, we know all the matrix moments $s_n(t)$, $n \in \mathbb{N}_0$, of our spectral measure $d\rho(\lambda;t), t \in [0,T)$. It is easy to understand that the knowledge of all the moments gives the possibility to find the solution $\alpha_n(t), \beta_n(t)$ of our Cauchy problem for (1).

Namely, first we note that the knowledge of $s_n(t)$, i.e., $s_n(\alpha,\beta;t)$ (29), permits to find the scalar products in the space $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$ of $\mathbb{R}^2$-values functions of $\lambda \in \mathbb{R}$, which are some linear combinations of the functions

$$\mathbb{R} \ni \lambda \mapsto (\lambda^j,\lambda^k) \in \mathbb{R}^2 \subset \mathbb{C}^2, \quad j,k \in \mathbb{N}_0. \tag{39}$$

So, we have

$$\begin{aligned} ((\lambda^j,\lambda^k),(\lambda^m,\lambda^n))_{L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))} &= \int_{\mathbb{R}} (d\rho(\lambda;t)(\lambda^j,\lambda^k),(\lambda^m,\lambda^n))_{\mathbb{C}^2} \\ &= s_{j+m}(0,0;t) + s_{k+m}(0,1;t) + s_{j+n}(0,1;t) + s_{k+n}(1,1;t), \quad j,k,m,n \in \mathbb{N}_0. \end{aligned} \tag{40}$$

From identity (40) and the Gramm-Schmidt orthogonalization procedure applied to the sequence of $\mathbb{C}^2-$values functions (17) we easily conclude that if we know the all the matrix moments $s_n(t)$ for $n = 1,\dots,2r,\ r \in \mathbb{N}$, then we can find the orthonormal functions (16) up to $(P_{r;1,0}(\lambda;t), P_{r;1,1}(\lambda;t))$, i.e., we can find all polynomials of the first kind, $P_0(\lambda;t),\dots,P_r(\lambda;t)$.

Using formulas (14) (given in terms of the scalar product in the space $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t)))$, the identity (40) and latter calculations we conclude that the knowledge of the matrix moments allows to find $b_q(t)$ (or $a_q(t)$) if we know the moments $s_n(t)$ for $n = 1,\dots,4q+1$ (or for $n = 1,\dots,4q+3$); $t \in [0,T)$.

As a result of the above considerations, we can formulate the following main theorem.

Theorem 2. *Consider the Cauchy problem for Toda lattice* (1) *with the initial data* $\alpha_n(0), \beta_n(0), n \in \mathbb{Z}$. *Assume that we also know the solutions* $\beta_0(t), \beta_1(t)$ *for all* $t \in [0,T)$ *of this problem; we assume that these two functions are infinitely differentiable. To find its complete solution, one can apply the following procedure.*

1) *Using the initial data, consider the matrix* $J(0)$ (3) *and find its matrix spectral measure* $d\rho(\lambda;0)$ *and the corresponding initial moments*

$$s_n(\alpha,\beta;0) = \int_{\mathbb{R}} \lambda^n d\rho_{\alpha,\beta}(\lambda;0), \quad \alpha,\beta = 0,1; \quad s_n(0) = \int_{\mathbb{R}} \lambda^n d\rho(\lambda;0), \quad n \in \mathbb{N}_0. \tag{41}$$

2) *Using formulas* (35), (36), (37) *we recursively find the moments* $s_n(\alpha,\beta;t)$ *for* $t > 0$ *using the initial identities* (41).
3) *For fixed* $t \in (0,T)$, *consider the space* $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$ *and apply the Gramm-Schmidt orthogonalization procedure to sequence* (17) *in this space. We calculate the scalar product by means of identity* (40) *with the constructed above moments* $s_n(\alpha,\beta;t)$. *As a results, we get for* $n \in \mathbb{N}_0$ *the polynomials of the first kind,* $P_n(\lambda;t), t \in [0,T)$.
4) *Using the obtained polynomials of the first kind,* $P_n(\lambda;t)$, *and expressions* (40) *for the scalar product in* $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$ *by means of formulas* (14) *we get the* $a_n^*(t), b_n(t)$, *i.e., the sought solution* $\alpha_n(t),\beta_n(t), n \in \mathbb{Z}, t \in [0,T)$ *(formulas* (14) *should be rewritten with the use of the scalar products in* $L^2(\mathbb{C}^2,\mathbb{R},d\rho(\lambda;t))$*).*

If we want to find $a_n^*(t)$ *(i.e.,* $\alpha_n(t),\alpha_{-n-1}(t)$*) or* $b_n(t)$ *(i.e.,* $\beta_n(t),\beta_{-n}(t),$*)* $n \in \mathbb{N}_0$, *it is necessary to know the first* $4n+3$ *(or* $4n+1$*) matrix moments.*

Remark 2. Of course, it is possible to give some formulas for the expression of $\alpha_n(t), \beta_n(t)$ in terms of the matrix moments (similar to the classical determinant formulas for elements of the Jacobi matrix in terms of the ordinary power moments, see e.g. [1], [9], Supplements).

It is useful to illustrate this moment approach to Cauchy problem for semi-infinite Toda lattice when the equations of type (23), (28) are solvable (see [6], Section 2).

So, we have the Toda lattice (1), but now $n \in \mathbb{N}_0$ and we assume that $\alpha_{-1}(t) = 0$, $t \in [0,T)$; the Cauchy problem is standard: the initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{N}_0$, is given. It is necessary to find the solution $\alpha_n(t), \beta_n(t), n \in \mathbb{N}_0$ for $t \in [0,T)$.

For such a problem, the role of the block Jacobi matrix $J(t)$ (3) plays the classical Jacobi matrix $J(t)$ with $(\beta_n(t))_{n=0}^{\infty}$ on the main diagonal and $(\alpha_n(t))_{n=0}^{\infty}$ on two neighboring diagonals. Instead of the space $\mathbf{l}_2$, we have the usual space ℓ_2 of sequences $f = (f_n)_{n=0}^{\infty}, f_n \in \mathbb{C}$; instead of the matrix spectral measure there appears the scalar spectral measure $d\rho(\lambda;t)$.

Instead of the matrix Weyl function, we have ordinary Weyl function $m(z;t)$ which is a solution of the following differential equation (see [6], (15)):

$$\begin{aligned} &\dot{m}(z;t) = (z - \beta_0(t))m(z;t) + 1, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad t \in [0,T), \\ &m(z;t) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\rho(\lambda;t) \quad \text{(see also [6], Section 6).} \end{aligned} \tag{42}$$

It is easy to calculate that the equation (42), in terms of the moments of the measure $d\rho(\lambda;t)$, is equivalent to the following recurrence equalities of type (35):

$$s_{n+1}(t) = \dot{s}_n(t) + \beta_0(t)s_n(t), \quad n \in \mathbb{N}_0, t \in [0,T); \quad s_n(t) = \int_{\mathbb{R}} \lambda^n d\rho(\lambda;t). \tag{43}$$

We get from (43): $\forall t \in [0,T)$ that

$$s_0(t) = 1, s_1(t) = \beta_0(t), s_2(t) = \dot{\beta}_0(t) + \beta_0^2(t), s_3(t) = \ddot{\beta}_0(t) + \ldots, \ldots \tag{44}$$

We can apply, in this case, the approach of Theorem 2: it is necessary to find $s_n(t)$ recursively from (43), (44) with the "initial conditions"

$$s_n(0) = \int_{\mathbb{R}} \lambda^n d\rho(\lambda;0), \quad n \in \mathbb{N}_0, \tag{45}$$

where $d\rho(\lambda;0)$ is the initial spectral measure of the Jacobi matrix $J(0)$ constructed from the initial data $\alpha_n(0), \beta_n(0), n \in \mathbb{N}_0$. To satisfy conditions (45), it is necessary to take the corresponding function $\beta_0(t)$ with proper values of the derivatives $(\frac{d^m}{dt^m}\beta_0)(0), m \in \mathbb{N}_0)$.

On the other hand, in our case the spectral measure $d\rho(\lambda;t)$ and the function $\beta_0(t)$ can be calculated as follows: according to [6], Section 2, formulas (35), (36), we have

$$\begin{aligned} &d\rho(\lambda;t) = e^{-\int_0^t \beta_0(s)\,ds} e^{\lambda t} d\rho(\lambda;0), \\ &\beta_0(t) = \Big(\int_{\mathbb{R}} e^{\lambda t} d\rho(\lambda;0)\Big)^{-1} \int_{\mathbb{R}} \lambda e^{\lambda t} d\rho(\lambda;0), \quad \lambda \in \mathbb{R}, \quad t \in [0,T). \end{aligned}$$

One can check that this spectral measure is defined uniquely by the recurrence procedure described above and the initial conditions (45).

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