

## THE NEVANLINNA-TYPE FORMULA FOR THE MATRIX HAMBURGER MOMENT PROBLEM

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ABSTRACT. In this paper we obtain a Nevanlinna-type formula for the matrix Hamburger moment problem. We only assume that the problem is solvable and has more than one solution. We express the matrix coefficients of the corresponding linear fractional transformation in terms of the prescribed moments. Necessary and sufficient conditions for the determinacy of the moment problem in terms of the given moments are obtained.

### 1. INTRODUCTION

Recall that the matrix Hamburger moment problem consists of finding a left-continuous non-decreasing matrix function  $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$  on  $\mathbb{R}$ ,  $M(-\infty) = 0$ , such that

$$(1) \quad \int_{\mathbb{R}} x^n dM(x) = S_n, \quad n \in \mathbb{Z}_+,$$

where  $\{S_n\}_{n=0}^{\infty}$  is a prescribed sequence of Hermitian  $(N \times N)$  complex matrices (moments),  $N \in \mathbb{N}$ . The moment problem (1) is said to be *determinate* if it has a unique solution and *indeterminate* in the opposite case.

Set

$$(2) \quad \Gamma_n = \begin{pmatrix} S_0 & S_1 & \dots & S_n \\ S_1 & S_2 & \dots & S_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_n & S_{n+1} & \dots & S_{2n} \end{pmatrix}, \quad n \in \mathbb{Z}_+.$$

It is well known that the following condition

$$(3) \quad \Gamma_n \geq 0, \quad n \in \mathbb{Z}_+,$$

is necessary and sufficient for the solvability of the moment problem (1).

In the scalar case, a description of all solutions of the moment problem (1) can be found, e.g., in [1], [2] for the nondegenerate case, and in [3] for the degenerate case.

The matrix moment problem (1) was introduced in 1949 by Krein [4], and he described all solutions in the case when the corresponding J-matrix defines a symmetric operator with maximal defect numbers. This result appeared without proof in [5]. Berezansky in 1965 proved the main fact in this theory of Krein: the convergence of the series from the polynomials of the first kind, even for the operator moment problem [2, Ch. 7, Section 2]. Under similar conditions, some descriptions of solutions were obtained by Kovalishina [6], by Lopez-Rodriguez [7], and by Dyukarev [8]. The above-mentioned descriptions were stated in terms of matrix linear fractional transformations of parameters from some sets of analytic functions (Nevanlinna-type formulas).

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Under a condition of strict positivity  $\Gamma_n \geq \delta_n I_{(n+1)N}$ ,  $\delta_n > 0$ ,  $\forall n \in \mathbb{Z}_+$ , the matrix Hamburger moment problem (1) was studied by Ilmushkin and Aleksandrov [9], [10] (even for the operator Hamburger moment problem). A Nevanlinna-type formula was stated there. However the proof was not complete, since the formula connecting symmetric functions of the corresponding symmetric operator and solutions of the moment problem was stated without proof in [10, formula (2.3)] and referred papers [9, Theorem 2], [11, Theorem 1]. Moreover, it was stated without proof that the latter formula holds true for the general case [9, pp. 77–78].

Probably a first description of all solutions of the operator Hamburger moment problem was given by Kheifets [12]. No additional assumptions were assumed. The well-known abstract interpolation problem (AIP), introduced by Katsnelson, Kheifets and Yuditskii, was used. This description was not given by a Nevanlinna-type formula and it had a more complicated structure

$$(4) \quad \int_{-\infty}^{\infty} \frac{\sigma(dx)}{x-z} = i \frac{1+w(\zeta)}{1-w(\zeta)}, \quad z = i \frac{1+\zeta}{1-\zeta},$$

$$(5) \quad w = s_0 + s_2(1_{N_2} - \omega s)^{-1} \omega s_1,$$

where  $\sigma(dx)$  is a solution of the moment problem,  $S = \begin{pmatrix} s & s_1 \\ s_2 & s_0 \end{pmatrix}$  is the scattering function of the AIP, and  $\omega(\zeta)$  is an arbitrary analytic operator-valued contractive function on  $\mathbb{D}$ ,  $\omega(\zeta) : N_1 \rightarrow N_2$  ( $N_1, N_2$  be some Hilbert spaces). For details we refer to the paper [12]. We do not see an easy way to simplify the description (4)–(5) to get a Nevanlinna-type formula.

The operator Hamburger moment problem was studied by Derkach [13]. He considered another version of the abstract interpolation problem, different from the AIP of Katsnelson, Kheifets and Yuditskii, and used generalized resolvents to obtain the Nevanlinna-type formula in the completely indeterminate case.

In [14] we presented an analytic description of all solutions of the matrix Hamburger moment problem (1) (under condition (3)) in terms of the generalized resolvents.

For a recent discussion on the truncated matrix Hamburger moment problems we refer to the paper [15] and references therein. It is worth mentioning that for the truncated moment problems much is done for the degenerate case, as well.

The main aim of our present investigation is to obtain a Nevanlinna-type formula for the moment problem (1). We only assume that condition (3) holds and the moment problem (1) is indeterminate. We express the matrix coefficients of the corresponding linear fractional transformation in terms of the given moments.

The method of proof is a continuation of our ideas in [14]. Our constructions are performed in an abstract Hilbert space, close to the ideas in [16], [17]. It turned out that it is more convenient to use the Cayley transformation  $V$  of the corresponding symmetric operator  $A$ , rather than the operator  $A$  itself. We used important results of Chumakin on the generalized resolvents of isometric operators [18]. Also we used the Gram-Schmidt orthogonalization procedure several times to obtain some orthonormal bases. Observe that the orthogonalization procedures do not correspond to the usual orthogonalization of powers  $1, x, x^2, \dots$ , even in the scalar case ( $N = 1$ ). The Frobenius formula allowed us to obtain the desired Nevanlinna-type formula.

Some necessary and sufficient conditions for the determinacy of the moment problem (1) in terms of the prescribed moments are given, as well (Theorem 1). The equivalence (A) $\Leftrightarrow$ (B) in Theorem 1, was obtained by Krein in [4, p. 58], see also [5, Theorem 3]. A method for calculation of the defect numbers and the corresponding necessary and sufficient conditions for the determinacy of the moment problem (1) were also given by Krein

in [4, p. 58]. Some sufficient conditions for the determinacy of the matrix Hamburger moment problem were given by Berg in [19, Theorem 3.6, Corollary 3.7].

**Notations.** As usual, we denote by  $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$  the sets of real numbers, complex numbers, positive integers, integers, non-negative integers, respectively;  $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$ ,  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ ,  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ . The notation  $k \in \overline{0, \rho}$  means that  $k \in \mathbb{Z}_+$ ,  $k \leq \rho$ , if  $\rho < \infty$ ; or  $k \in \mathbb{Z}_+$ , if  $\rho = \infty$ . The set of all complex matrices of size  $(m \times n)$  we denote by  $\mathbb{C}_{m \times n}$ ,  $m, n \in \mathbb{N}$ . If  $M \in \mathbb{C}_{m \times n}$  then  $M^T$  denotes the transpose of  $M$ , and  $M^*$  denotes the complex conjugate of  $M$ . The identity matrix from  $\mathbb{C}_{n \times n}$  we denote by  $I_n$ ,  $n \in \mathbb{N}$ ;  $I_\infty = (\delta_{k,l})_{k,l=0}^\infty$ ,  $\delta_{k,l}$  is Kronecker's delta. If a set  $S$  has a finite number of elements, then its number of elements we denote by  $\text{card}(S)$ . If a set  $S$  has an infinite number of elements, then  $\text{card}(S) := \infty$ .

For a separable Hilbert space  $H$  we denote by  $(\cdot, \cdot)_H$  and  $\|\cdot\|_H$  the scalar product and the norm in  $H$ , respectively. The indices may be omitted in obvious cases.

For a linear operator  $A$  in  $H$  we denote by  $D(A)$  its domain, by  $R(A)$  its range, and by  $A^*$  we denote its adjoint if it exists. If  $A$  is invertible, then  $A^{-1}$  means its inverse. If  $A$  is closable, then  $\overline{A}$  means its closure. If  $A$  is bounded, then  $\|A\|$  stands for its operator norm. For a set  $M \subseteq H$ , we denote by  $\overline{M}$  the closure of  $M$  in the norm of  $H$ . By  $\text{Lin } M$  and  $\overline{\text{span}} M$  we denote the linear span of  $M$  and the closed linear span of  $M$  in the norm of  $H$ , respectively.

By  $E_H$  we denote the identity operator in  $H$ , i.e.  $E_H x = x$ ,  $x \in H$ . Let  $H_1$  be a subspace of  $H$ . By  $P_{H_1} = P_{H_1}^H$  we denote the operator of the orthogonal projection on  $H_1$  in  $H$ .

If  $A$  is symmetric, we set  $R_z = R_z(A) = (A - zE_H)^{-1}$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ . If  $V$  is isometric, we set  $\mathcal{R}_\zeta = \mathcal{R}_\zeta(V) = (E_H - \zeta V)^{-1}$ ,  $\zeta \in \mathbb{C} \setminus \mathbb{T}$ .

2. THE MATRIX HAMBURGER MOMENT PROBLEM: THE DETERMINACY AND A NEVANLINNA-TYPE FORMULA

Let the matrix Hamburger moment problem (1) be given and condition (3) hold. Set

$$(6) \quad \Gamma = (S_{k+l})_{k,l=0}^\infty = \begin{pmatrix} S_0 & S_1 & \dots & S_n & \dots \\ S_1 & S_2 & \dots & S_{n+1} & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ S_n & S_{n+1} & \dots & S_{2n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix  $\Gamma$  is a semi-infinite block matrix. It may be viewed as a usual semi-infinite matrix, as well. Let

$$(7) \quad \Gamma = (\Gamma_{n,m})_{n,m=0}^\infty, \quad \Gamma_{n,m} \in \mathbb{C},$$

and

$$S_n = (s_n^{k,l})_{k,l=0}^{N-1}, \quad s_n^{k,l} \in \mathbb{C}, \quad n \in \mathbb{Z}_+.$$

Notice that

$$(8) \quad \Gamma_{rN+j,tN+n} = s_{r+t}^{j,n}, \quad 0 \leq j, \quad n \leq N-1; \quad r, t \in \mathbb{Z}_+.$$

We need here some constructions from [14]. By Theorem 1 in [14] (and this construction is well known), there exist a Hilbert space  $H$ , and a sequence  $\{x_n\}_{n=0}^\infty$  in  $H$ , such that  $\overline{\text{span}}\{x_n\}_{n=0}^\infty = H$ , and

$$(9) \quad (x_n, x_m)_H = \Gamma_{n,m}, \quad n, m \in \mathbb{Z}_+.$$

We emphasize that such a space  $H$  and a sequence  $\{x_n\}_{n=0}^\infty$  in  $H$  are not unique. We choose an arbitrary such a space  $H$  and a sequence  $\{x_n\}_{n=0}^\infty$  in  $H$ , and fix them in the rest of the paper.

Set  $L := \text{Lin}\{x_n\}_{n=0}^\infty$ , and consider the following operator with the domain  $L$ :

$$(10) \quad Ax = \sum_{k=0}^\infty \alpha_k x_{k+N}, \quad x \in L, \quad x = \sum_{k=0}^\infty \alpha_k x_k, \quad \alpha_k \in \mathbb{C}.$$

This operator is correctly defined and symmetric.

Let  $\widehat{A}$  be an arbitrary self-adjoint extension of  $A$  in a Hilbert space  $\widehat{H} \supseteq H$ . Let  $R_z(\widehat{A}) = (\widehat{A} - zE_{\widehat{H}})^{-1}$  be the resolvent of  $\widehat{A}$  and  $\{\widehat{E}_\lambda\}_{\lambda \in \mathbb{R}}$  be the orthogonal left-continuous resolution of the identity of  $\widehat{A}$ . Recall that the operator-valued function  $\mathbf{R}_z = P_H^{\widehat{H}} R_z(\widehat{A})$  is called a *generalized resolvent* of  $A$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ . The function  $\mathbf{E}_\lambda = P_H^{\widehat{H}} \widehat{E}_\lambda$ ,  $\lambda \in \mathbb{R}$ , is a *spectral function* of a symmetric operator  $A$ . The following relation holds [20, p. 377–378]:

$$(11) \quad (\mathbf{R}_z f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d(\mathbf{E}_\lambda f, g)_H, \quad f, g \in H, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

By the Stieltjes-Perron inversion formula there exists a one-to-one correspondence between the set of (left-continuous) spectral functions of  $A$  and the set of its generalized resolvents. The generalized resolvent and the spectral function related by (11) are said to *belong to each other* [20, p. 378].

By Theorem 2 in [14], all solutions of the moment problem (1) have the following form:

$$(12) \quad M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (\mathbf{E}_\lambda x_k, x_j)_H,$$

where  $\mathbf{E}_\lambda$  is a spectral function of the operator  $A$ . Moreover, the correspondence between all spectral functions of  $A$  and all solutions of the moment problem is bijective.

By (11) and (12) we conclude that the formula

$$(13) \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} dm_{k,j}(\lambda) = (\mathbf{R}_z x_k, x_j)_H, \quad 0 \leq k, j \leq N - 1, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

establishes a one-to-one correspondence between all generalized resolvents of  $A$  and all solutions of the moment problem (1).

Let  $B$  be a closed symmetric operator in the Hilbert space  $H$ , with the domain  $D(B)$ ,  $\overline{D(B)} = H$ . Set  $\Delta_B(\lambda) = (B - \lambda E_H)D(B)$ , and  $N_\lambda = N_\lambda(B) = H \ominus \Delta_B(\lambda)$ ,  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Consider an arbitrary bounded linear operator  $C$ , which maps  $N_i$  into  $N_{-i}$ . For

$$(14) \quad g = f + C\psi - \psi, \quad f \in D(B), \quad \psi \in N_i,$$

we set

$$(15) \quad B_C g = Bf + iC\psi + i\psi.$$

The operator  $B_C$  is said to be a *quasiself-adjoint extension of the operator  $B$ , defined by the operator  $C$* . By Theorem 4 in [14], the following relation:

$$(16) \quad \int_{\mathbb{R}} \frac{1}{x - \lambda} dm_{k,j}(x) = ((A_{F(\lambda)} - \lambda E_H)^{-1} x_k, x_j)_H, \quad \lambda \in \mathbb{C}_+,$$

establishes a bijective correspondence between all solutions of the moment problem (1) and all analytic in  $\mathbb{C}_+$  operator-valued functions  $F(\lambda)$ , which values are contractions which map  $N_i(\overline{A})$  into  $N_{-i}(\overline{A})$ . Here  $A_{F(\lambda)}$  is the quasiself-adjoint extension of  $\overline{A}$  defined by  $F(\lambda)$ .

Set

$$\begin{aligned}
 & y_k^- := (A - iE_H)x_k = x_{k+N} - ix_k, \\
 & y_k^+ := (A + iE_H)x_k = x_{k+N} + ix_k, \quad k \in \mathbb{Z}_+, \\
 (17) \quad & L^- := \text{Lin}\{y_k^-\}_{k=0}^\infty = (A - iE_H)D(A), \quad L^+ := \text{Lin}\{y_k^+\}_{k=0}^\infty = (A + iE_H)D(A), \\
 & H^- := \overline{L^-} = (\overline{A} - iE_H)D(\overline{A}), \quad H^+ := \overline{L^+} = (\overline{A} + iE_H)D(\overline{A}).
 \end{aligned}$$

Let us apply the Gram-Schmidt orthogonalization procedure to the sequence  $\{y_k^-\}_{k=0}^\infty$ , removing the linear dependent elements if they appear. We shall get a sequence  $\mathfrak{A}^- = \{u_k^-\}_{k=0}^{\tau^- - 1}$ ,  $0 \leq \tau^- \leq +\infty$ . The case  $\tau^- = 0$  means that  $y_k^- = 0, \forall k \in \mathbb{Z}_+$ , and  $\mathfrak{A}^-$  is an empty set.

In a similar manner, we apply the Gram-Schmidt orthogonalization procedure to the sequence  $\{y_k^+\}_{k=0}^\infty$ , and obtain a sequence  $\mathfrak{A}^+ = \{u_k^+\}_{k=0}^{\tau^+ - 1}$ ,  $0 \leq \tau^+ \leq +\infty$ . The case  $\tau^+ = 0$  means that  $y_k^+ = 0, \forall k \in \mathbb{Z}_+$ , and  $\mathfrak{A}^+ = \emptyset$ .

If not empty, the set  $\mathfrak{A}^\pm$  forms an orthonormal basis in  $H^\pm$ , respectively. Denote by  $\nu^\pm(k)$  the index of the last used  $y_j^\pm$  for the construction of  $u_k^\pm$ . In other words,  $\nu^\pm(k)$  is the number of a step in the Gram-Schmidt orthogonalization procedure, counting from zero and including steps which produced linearly dependent elements, in which  $u_k^\pm$  was built. Notice that, by the construction, each element  $u_k^\pm, k \in \overline{0, \tau^\pm - 1}$ , is a linear combination of  $y_j^\pm, 0 \leq j \leq \nu^\pm(k)$ , respectively. Let

$$(18) \quad u_k^\pm = \sum_{j=0}^{\nu^\pm(k)} \xi_{k;j}^\pm y_j^\pm, \quad \xi_{k;j}^\pm \in \mathbb{C}, \quad k \in \overline{0, \tau^\pm - 1}.$$

Observe that by (9) we may write

$$\begin{aligned}
 (19) \quad (x_n, u_k^\pm)_H &= \sum_{j=0}^{\nu^\pm(k)} \overline{\xi_{k;j}^\pm} (x_n, y_j^\pm)_H = \sum_{j=0}^{\nu^\pm(k)} \overline{\xi_{k;j}^\pm} (x_n, x_{j+N} \pm ix_j)_H \\
 &= \sum_{j=0}^{\nu^\pm(k)} \overline{\xi_{k;j}^\pm} (\Gamma_{n,j+N} \pm i\Gamma_{n,j}), \quad n \in \mathbb{Z}_+, \quad k \in \overline{0, \tau^\pm - 1}.
 \end{aligned}$$

By representation (17), the condition  $\tau^- = 0$  ( $\tau^+ = 0$ ) is equivalent to the condition  $D(A) = \{0\}$ , and therefore to the condition  $H = \{0\}$ . By (8), (9), the condition  $H = \{0\}$  is equivalent to the condition  $S_n = 0, \forall n \in \mathbb{Z}_+$ .

We emphasize that the numbers  $\xi_{k;j}$  in (18) can be computed explicitly by using relations (8), (9). Moreover, the processes of orthogonalization which appear in this paper are based on the use of relations (8), (9). In fact, any norm or any scalar product which appear during orthogonalization is expressed in terms of the prescribed moments.

**Theorem 1.** *Let the matrix Hamburger moment problem (1) be given and condition (3), with  $\Gamma_n$  from (2), be satisfied. Let the operator  $A$  in the Hilbert space  $H$  be constructed as in (10). The following conditions are equivalent:*

- (A) *The moment problem (1) is determinate;*
- (B) *One of the defect numbers of  $A$  is equal to zero (or the both of them are zero);*
- (C)  *$S_r = 0, \forall r \in \mathbb{Z}_+$ , or,  $\exists S_l \neq 0, l \in \mathbb{Z}_+$ , and one of the following conditions holds (or the both of them hold):*
  - (a) *For each  $n, 0 \leq n \leq N - 1$ , the following equality holds:*

$$(20) \quad \Gamma_{n,n} = \sum_{k=0}^{\tau^- - 1} \left| \sum_{j=0}^{\nu^-(k)} \overline{\xi_{k;j}^-} (\Gamma_{n,j+N} - i\Gamma_{n,j}) \right|^2;$$

(b) For each  $n, 0 \leq n \leq N - 1$ , the following equality holds:

$$(21) \quad \Gamma_{n,n} = \sum_{k=0}^{\tau^+-1} \left| \sum_{j=0}^{\nu^+(k)} \overline{\xi_{k;j}^+} (\Gamma_{n,j+N} + i\Gamma_{n,j}) \right|^2.$$

Here  $\Gamma_{\cdot,\cdot}$  are from (7), and  $\xi_{\cdot,\cdot}^\pm$  are from (18).

If the above conditions are satisfied then the unique solution of the moment problem (1) is given by the following relation:

$$(22) \quad M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}, \quad m_{k,j}(t) = (E_t x_k, x_j)_H,$$

where  $E_t$  is the unique left-continuous spectral function of the operator  $A$ .

*Proof.* (A) $\Rightarrow$ (B). If the both defect numbers are greater than zero, then we can choose unit vectors  $u_1 \in N_i(\bar{A})$  and  $u_2 \in N_{-i}(\bar{A})$ . We set

$$F(\lambda)(cu_1 + u) = cu_2, \quad c \in \mathbb{C}, \quad u \in N_i(\bar{A}) \ominus \text{Lin}\{u_1\}.$$

On the other hand, we set  $\tilde{F}(\lambda) \equiv 0$ . Functions  $F(\lambda)$  and  $\tilde{F}(\lambda)$  produce different solutions of the moment problem (1) by relation (16).

(B) $\Rightarrow$ (A). If one of the defect numbers is zero, then the only admissible function  $F(\lambda)$  in relation (16) is  $F(\lambda) \equiv 0$ .

(B) $\Rightarrow$ (C). If  $H = \{0\}$  then condition (C) holds. Let  $H \neq \{0\}$ .

Notice that by (9) and (19), condition (C),(a) may be written as

$$\|x_n\|^2 = \sum_{k=0}^{\tau^--1} |(x_n, u_k^-)_H|^2, \quad n = 0, 1, \dots, N - 1,$$

while condition (C),(b) is equivalent to

$$\|x_n\|^2 = \sum_{k=0}^{\tau^+-1} |(x_n, u_k^+)_H|^2, \quad n = 0, 1, \dots, N - 1.$$

Therefore condition (C),(a) is equivalent to relations

$$(23) \quad x_n \in H^-, \quad n = 0, 1, \dots, N - 1,$$

and condition (C),(b) is equivalent to condition

$$(24) \quad x_n \in H^+, \quad n = 0, 1, \dots, N - 1.$$

By the formula (37) in [14, p. 278], each element of  $L$  belongs to the linear span of elements  $\{x_n\}_{n=0}^{N-1}, \{y_k^-\}_{k=0}^\infty$ , as well as to the linear span of elements  $\{x_n\}_{n=0}^{N-1}, \{y_k^+\}_{k=0}^\infty$ . Consequently, condition (23) is equivalent to the condition

$$(25) \quad H = H^-,$$

and condition (24) is equivalent to the condition

$$(26) \quad H = H^+.$$

Since one of the defect numbers is equal to zero then either (25), or (26) holds.

(C) $\Rightarrow$ (B). If  $H = \{0\}$  then condition (B) holds. Let  $H \neq \{0\}$ . If condition (C),(a) (condition (C),(b)) holds, then by the above considerations before (25) we obtain  $H = H^-$  (respectively  $H = H^+$ ). Therefore one of the defect numbers of  $A$  is equal to zero.

The last assertion of the theorem follows from formula (12). □

We shall continue our considerations started before the statement of Theorem 1. In what follows we assume that *the moment problem (1) is indeterminate*. Let the defect numbers of  $A$  are equal to  $\delta = \delta(A) = \dim H \ominus H^-$ , and  $\omega = \omega(A) = \dim H \ominus H^+$ ,  $\delta, \omega \geq 1$ .

For simplicity of notations we set  $\tau := \tau^-$ , and

$$u_k := u_{\bar{k}}, \quad k \in \overline{0, \tau - 1}.$$

Let us apply the Gram-Schmidt orthogonalization procedure to the vectors

$$\{u_k\}_{k=0}^{\tau-1}, \quad \{x_n\}_{n=0}^{N-1}.$$

Notice that the elements  $\{u_k\}_{k=0}^{\tau-1}$  are already orthonormal. Then we get an orthonormal set in  $H$

$$\mathfrak{A}_u := \{u_k\}_{k=0}^{\tau-1} \cup \{u'_l\}_{l=0}^{\delta-1}.$$

Notice that  $\mathfrak{A}' := \{u'_l\}_{l=0}^{\delta-1}$  is an orthonormal basis in  $H \ominus H^-$ .

Set

$$(27) \quad V = V_{\bar{A}} = (\bar{A} + iE_H)(\bar{A} - iE_H)^{-1} = E_H + 2i(\bar{A} - iE_H)^{-1}.$$

The operator  $V$  is a closed isometric operator with the domain  $H^-$  and the range  $H^+$ . Set

$$v_k := Vu_k, \quad k \in \overline{0, \tau - 1}.$$

Observe that by (18) we may write

$$v_k = \sum_{j=0}^k \xi_{k;j}^- Vy_j^- = \sum_{j=0}^k \xi_{k;j}^- y_j^+, \quad k \in \overline{0, \tau - 1}.$$

Notice that

$$\mathfrak{A}_v^- := \{v_k\}_{k=0}^{\tau-1}$$

is an orthonormal basis in  $H^+$ .

Let us apply the Gram-Schmidt orthogonalization procedure to the vectors

$$\{v_k\}_{k=0}^{\tau-1}, \{x_n\}_{n=0}^{N-1}.$$

The elements  $\{v_k\}_{k=0}^{\tau-1}$  are already orthonormal. Then we get another orthonormal basis in  $H$

$$\mathfrak{A}_v := \{v_k\}_{k=0}^{\tau-1} \cup \{v'_l\}_{l=0}^{\omega-1}.$$

Observe that  $\mathfrak{A}'_v := \{v'_l\}_{l=0}^{\omega-1}$  is an orthonormal basis in  $H \ominus H^+$ .

Let  $\mathbf{R}_\lambda$  be an arbitrary generalized resolvent of the operator  $A$ . Let us check that

$$(28) \quad (\mathbf{R}_z x_k, x_j)_H = \frac{1}{z^2 + 1} (\mathbf{R}_z y_k^-, y_j^-)_H - \frac{1}{z^2 + 1} (x_{k+N}, x_j)_H - \frac{z}{z^2 + 1} (x_k, x_j)_H, \\ z \in \mathbb{C}_+ \setminus \{i\}, \quad 0 \leq k, j \leq N - 1.$$

In fact, let  $\tilde{A} \supseteq A$  be a self-adjoint operator in a Hilbert space  $\tilde{H} \supseteq H$ , such that  $P_{\tilde{H}}^{\tilde{H}} R_z(\tilde{A}) = \mathbf{R}_z$ ,  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then

$$(29) \quad (\mathbf{R}_z x_k, x_j)_H = (R_z(\tilde{A})(A - iE_H)^{-1}(A - iE_H)x_k, x_j)_{\tilde{H}} \\ = (R_z(\tilde{A})R_i(\tilde{A})y_k^-, x_j)_{\tilde{H}} = \frac{1}{z - i} ((R_z(\tilde{A}) - R_i(\tilde{A}))y_k^-, x_j)_{\tilde{H}} \\ = \frac{1}{z - i} (R_z(\tilde{A})y_k^-, x_j)_{\tilde{H}} - \frac{1}{z - i} (x_k, x_j)_{\tilde{H}};$$

$$\begin{aligned}
 (R_z(\tilde{A})y_k^-, x_j)_{\tilde{H}} &= (R_z(\tilde{A})y_k^-, R_i(\tilde{A})y_j^-)_{\tilde{H}} \\
 (30) \qquad &= (R_{-i}(\tilde{A})R_z(\tilde{A})y_k^-, y_j^-)_{\tilde{H}} = -\frac{1}{i+z}((R_{-i}(\tilde{A}) - R_z(\tilde{A}))y_k^-, y_j^-)_{\tilde{H}} \\
 &= -\frac{1}{i+z}(y_k^-, x_j)_{\tilde{H}} + \frac{1}{i+z}(\mathbf{R}_z y_k^-, y_j^-)_{\tilde{H}}.
 \end{aligned}$$

By substitution of the last expression for  $(R_z(\tilde{A})y_k^-, x_j)_{\tilde{H}}$  into the right-hand side of (29), we get relation (28).

Let  $\hat{U} \supseteq V$  be an arbitrary unitary extension of  $V$  in a Hilbert space  $\hat{H} \supseteq H$ . Recall [18] that the following function:

$$(31) \qquad \mathbf{R}_\zeta(V) = P_{\hat{H}}^{\hat{H}}(E_{\hat{H}} - \zeta\hat{U})^{-1}, \quad \zeta \in \mathbb{C} \setminus \mathbb{T},$$

is said to be a *generalized resolvent* of  $V$ .

Observe that the generalized resolvents of  $V$  and  $\bar{A}$  are connected by the following relation [21, pp. 370–371]:

$$(32) \qquad (1 - \zeta)\mathbf{R}_\zeta(V) = E_H + (z - i)\mathbf{R}_z(\bar{A}), \quad z \in \mathbb{C}_+, \quad \zeta = \frac{z - i}{z + i} \in \mathbb{D}.$$

(The latter relation follows from the fact that the usual resolvents of  $V$  and  $\bar{A}$  are related by a similar relation, and then one applies the projection operator  $P_{\hat{H}}^{\hat{H}}$  to the both sides of that relation.) Correspondence (32) between all generalized resolvents of  $V$  and all generalized resolvents of  $\bar{A}$  is bijective. Then

$$(33) \qquad \mathbf{R}_z(\bar{A}) = \frac{2i}{z^2 + 1}\mathbf{R}_{\frac{z-i}{z+i}}(V) - \frac{1}{z - i}E_H, \quad z \in \mathbb{C}_+ \setminus \{i\}.$$

By (33),(28) and (9) we get

$$\begin{aligned}
 (34) \qquad (\mathbf{R}_z x_k, x_j)_H &= \frac{2i}{(z^2 + 1)^2}(\mathbf{R}_{\frac{z-i}{z+i}}(V_A)y_k^-, y_j^-)_H - \frac{1}{(z^2 + 1)(z - i)}\varphi_{j,k}(z), \\
 &z \in \mathbb{C}_+ \setminus \{i\}, \quad 0 \leq k, j \leq N - 1,
 \end{aligned}$$

where

$$\begin{aligned}
 (35) \qquad \varphi_{j,k}(z) &:= \Gamma_{k+N,j+N} - i\Gamma_{k+N,j} - i\Gamma_{k,j+N} + \Gamma_{k,j} + (z - i)\Gamma_{k+N,j} + z(z - i)\Gamma_{k,j} \\
 &= \Gamma_{k+N,j+N} - i\Gamma_{k,j+N} + (z - 2i)\Gamma_{k+N,j} + (z^2 - iz + 1)\Gamma_{k,j}, \quad z \in \mathbb{C}_+.
 \end{aligned}$$

Observe that an arbitrary generalized resolvent  $\mathbf{R}_\zeta$  of the closed isometric operator  $V_{\bar{A}}$  has the following representation [18, Theorem 3]:

$$(36) \qquad \mathbf{R}_\zeta = [E - \zeta(V \oplus \Phi_\zeta)]^{-1}, \quad \zeta \in \mathbb{D}.$$

Here  $\Phi_\zeta$  is an analytic in  $\mathbb{D}$  operator-valued function which values are linear contractions from  $H \ominus H^-$  into  $H \ominus H^+$ . The correspondence between all such functions  $\Phi_\zeta$  and all generalized resolvents of  $V$  is bijective.

By (34), (36) we get

$$\begin{aligned}
 (37) \qquad (\mathbf{R}_z x_k, x_j)_H &= \frac{2i}{(z^2 + 1)^2} \left( \left[ E - \frac{z - i}{z + i}(V \oplus \Phi_{\frac{z-i}{z+i}}) \right]^{-1} y_k^-, y_j^- \right)_H \\
 &- \frac{1}{(z^2 + 1)(z - i)}\varphi_{j,k}(z), \quad z \in \mathbb{C}_+ \setminus \{i\}, \quad 0 \leq k, j \leq N - 1,
 \end{aligned}$$

where  $\Phi$  is an analytic in  $\mathbb{D}$  operator-valued function which values are linear contractions from  $H \ominus H^-$  into  $H \ominus H^+$ .



By (13) and (37) we conclude that the formula

$$(38) \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} dm_{k,j}(\lambda) = \frac{2i}{(z^2 + 1)^2} \left( \left[ E - \frac{z-i}{z+i} (V \oplus \Phi_{\frac{z-i}{z+i}}) \right]^{-1} y_k^-, y_j^- \right)_H - \frac{1}{(z^2 + 1)(z - i)} \varphi_{j,k}(z), \quad 0 \leq k, j \leq N - 1, \quad z \in \mathbb{C}_+ \setminus \{i\},$$

establishes a one-to-one correspondence between all analytic in  $\mathbb{D}$  operator-valued functions  $\Phi$ , which values are linear contractions from  $H \ominus H^-$  into  $H \ominus H^+$ , and all solutions  $M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}$  of the moment problem (1).

It turns out that formula (38) is more convenient than formula (16), in order to obtain a Nevanlinna-type formula for the moment problem (1).

Denote by  $\mathcal{M}_{1,\zeta}(\Phi)$  the matrix of the operator  $E_H - \zeta(V \oplus \Phi_\zeta)$  in the basis  $\mathfrak{A}_u$ ,  $\zeta \in \mathbb{D}$ . Here  $\Phi_\zeta$  is an analytic in  $\mathbb{D}$  operator-valued function, which values are linear contractions from  $H \ominus H^-$  into  $H \ominus H^+$ . Then

$$\mathcal{M}_{1,\zeta}(\Phi) = \begin{pmatrix} A_{0,\zeta} & B_{0,\zeta}(\Phi) \\ C_{0,\zeta} & D_{0,\zeta}(\Phi) \end{pmatrix},$$

where

$$(39) \quad \begin{aligned} A_{0,\zeta} &= \left( ([E_H - \zeta(V \oplus \Phi_\zeta)] u_k, u_j)_H \right)_{j,k=0}^{\tau-1} \\ &= \left( (u_k - \zeta V u_k, u_j)_H \right)_{j,k=0}^{\tau-1} = I_\tau - \zeta \left( (v_k, u_j)_H \right)_{j,k=0}^{\tau-1}, \\ B_{0,\zeta}(\Phi) &= \left( ([E_H - \zeta(V \oplus \Phi_\zeta)] u'_k, u_j)_H \right)_{0 \leq j \leq \tau-1, 0 \leq k \leq \delta-1} \\ &= \left( (u'_k - \zeta \Phi_\zeta u'_k, u_j)_H \right)_{0 \leq j \leq \tau-1, 0 \leq k \leq \delta-1} \\ &= -\zeta \left( (\Phi_\zeta u'_k, u_j)_H \right)_{0 \leq j \leq \tau-1, 0 \leq k \leq \delta-1}, \end{aligned}$$

$$(40) \quad \begin{aligned} C_{0,\zeta} &= \left( ([E_H - \zeta(V \oplus \Phi_\zeta)] u_k, u'_j)_H \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \tau-1} \\ &= \left( (u_k - \zeta V u_k, u'_j)_H \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \tau-1} \\ &= -\zeta \left( (v_k, u'_j)_H \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \tau-1}, \\ D_{0,\zeta}(\Phi) &= \left( ([E_H - \zeta(V \oplus \Phi_\zeta)] u'_k, u'_j)_H \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \delta-1} \\ &= \left( (u'_k - \zeta \Phi_\zeta u'_k, u'_j)_H \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \delta-1} \\ &= I_\delta - \zeta \left( (\Phi_\zeta u'_k, u'_j)_H \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \delta-1}, \quad \zeta \in \mathbb{D}. \end{aligned}$$

Notice that matrices  $A_{0,\zeta}, C_{0,\zeta}, \zeta \in \mathbb{D}$ , can be calculated explicitly using relations (9) and (8).

Denote by  $F_\zeta, \zeta \in \mathbb{D}$ , the matrix of the operator  $\Phi_\zeta$ , acting from  $H \ominus H^-$  into  $H \ominus H^+$ , with respect to the bases  $\mathfrak{A}'$  and  $\mathfrak{A}'_v$

$$\begin{aligned} F_\zeta &= (f_\zeta(j, k))_{0 \leq j \leq \omega-1, 0 \leq k \leq \delta-1}, \\ f_\zeta(j, k) &:= (\Phi_\zeta u'_k, v'_j)_H. \end{aligned}$$

Then

$$\Phi_\zeta u'_k = \sum_{l=0}^{\omega-1} f_\zeta(l, k) v'_l, \quad 0 \leq k \leq \delta - 1,$$

and

$$\begin{aligned} B_{0,\zeta}(\Phi) &= -\zeta \left( \left( \sum_{l=0}^{\omega-1} f_\zeta(l, k) v'_l, u_j \right)_H \right)_{0 \leq j \leq \tau-1, 0 \leq k \leq \delta-1} \\ &= -\zeta \left( \sum_{l=0}^{\omega-1} \left( v'_l, u_j \right)_H f_\zeta(l, k) \right)_{0 \leq j \leq \tau-1, 0 \leq k \leq \delta-1}, \quad \zeta \in \mathbb{D}. \end{aligned}$$

Set

$$(41) \quad W := \left( (v'_l, u_j)_H \right)_{0 \leq j \leq \tau-1, 0 \leq l \leq \omega-1}.$$

Then

$$B_{0,\zeta}(\Phi) = -\zeta W F_\zeta, \quad \zeta \in \mathbb{D}.$$

We may write

$$\begin{aligned} D_{0,\zeta}(\Phi) &= I_\delta - \zeta \left( \left( \sum_{l=0}^{\omega-1} f_\zeta(l, k) v'_l, u'_j \right)_H \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \delta-1} \\ &= I_\delta - \zeta \left( \sum_{l=0}^{\omega-1} \left( v'_l, u'_j \right)_H f_\zeta(l, k) \right)_{0 \leq j \leq \delta-1, 0 \leq k \leq \delta-1}, \quad \zeta \in \mathbb{D}. \end{aligned}$$

Set

$$(42) \quad T := \left( (v'_l, u'_j)_H \right)_{0 \leq j \leq \delta-1, 0 \leq l \leq \omega-1}.$$

Then

$$D_{0,\zeta}(\Phi) = I_\delta - \zeta T F_\zeta, \quad \zeta \in \mathbb{D}.$$

Thus, we may write

$$\mathcal{M}_{1,\zeta}(\Phi) = \begin{pmatrix} A_{0,\zeta} & -\zeta W F_\zeta \\ C_{0,\zeta} & I_\delta - \zeta T F_\zeta \end{pmatrix}, \quad \zeta \in \mathbb{D},$$

where  $A_{0,\zeta}, C_{0,\zeta}$  are given by (39), (40), and  $W, T$  are given by (41), (42).

Consider the block representation of the operator  $E_H - \zeta(V \oplus \Phi_\zeta)$  with respect to the decomposition  $H^- \oplus (H \ominus H^-)$

$$(43) \quad E_H - \zeta(V \oplus \Phi_\zeta) = \begin{pmatrix} \mathcal{A}_{0,\zeta} & \mathcal{B}_{0,\zeta}(\Phi) \\ \mathcal{C}_{0,\zeta} & \mathcal{D}_{0,\zeta}(\Phi) \end{pmatrix}, \quad \zeta \in \mathbb{D}.$$

Of course, the matrices of operators  $\mathcal{A}_{0,\zeta}, \mathcal{B}_{0,\zeta}, \mathcal{C}_{0,\zeta}, \mathcal{D}_{0,\zeta}$  are matrices  $A_{0,\zeta}, B_{0,\zeta}, C_{0,\zeta}, D_{0,\zeta}$ , respectively. Observe that the matrix  $A_{0,\zeta}$  is invertible, since  $A_{0,\zeta} = P_{H^-}(E_H - \zeta V)P_{H^-} = E_{H^-} - \zeta P_{H^-} V P_{H^-}$ , is invertible,  $\zeta \in \mathbb{D}$ . Set

$$(44) \quad V_0 := P_{H^-} V P_{H^-}.$$

The matrix of  $V_0$  in the basis  $\mathfrak{A}^-$  we denote by  $\mathfrak{V}$

$$(45) \quad \mathfrak{V} = \left( (v_k, u_j)_H \right)_{j,k=0}^{\tau-1}.$$

Observe that using definitions of  $v_k, u_j$ , the elements of the matrix  $\mathfrak{V}$  can be calculated explicitly by the prescribed moments.

We may write for the resolvent function of  $V_0$

$$(46) \quad \mathcal{R}_\zeta(V_0) = \mathcal{A}_{0,\zeta}^{-1} = E_{H^-} + \sum_{k=1}^{\infty} V^k \zeta^k, \quad \zeta \in \mathbb{D}.$$

Then for the corresponding matrices we may write

$$(47) \quad A_{0,\zeta}^{-1} = I_\infty + \sum_{k=1}^{\infty} \mathfrak{V}_k \zeta^k, \quad \zeta \in \mathbb{D},$$

where

$$(48) \quad \mathfrak{A}_k := \mathfrak{A}^k, \quad k \in \mathbb{Z}^+.$$

By the convergence in (47) we mean the convergence of the corresponding entries of matrices.

Observe that the Frobenius formula for the inverse of the block matrix ([22, p. 59]) is still valid for the block representations of bounded operators as in (43), if the following operator

$$\mathcal{H}_\zeta := \mathcal{D}_{0,\zeta} - \mathcal{C}_{0,\zeta} \mathcal{A}_{0,\zeta}^{-1} \mathcal{B}_{0,\zeta}$$

has a bounded inverse. This can be verified by the direct multiplication of the corresponding block representations. Notice that in our case  $\mathcal{H}_\zeta$  has a bounded inverse. In fact, we may write

$$\begin{pmatrix} E_{H^-} & 0 \\ -\mathcal{C}_{0,\zeta} \mathcal{A}_{0,\zeta}^{-1} & E_{H \ominus H^-} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{0,\zeta} & \mathcal{B}_{0,\zeta}(\Phi) \\ \mathcal{C}_{0,\zeta} & \mathcal{D}_{0,\zeta}(\Phi) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{0,\zeta} & \mathcal{B}_{0,\zeta}(\Phi) \\ 0 & \mathcal{H}_\zeta \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} E_{H^-} & 0 \\ -\mathcal{C}_{0,\zeta} \mathcal{A}_{0,\zeta}^{-1} & E_{H \ominus H^-} \end{pmatrix}^{-1} = \begin{pmatrix} E_{H^-} & 0 \\ \mathcal{C}_{0,\zeta} \mathcal{A}_{0,\zeta}^{-1} & E_{H \ominus H^-} \end{pmatrix}.$$

Therefore the operator  $\mathcal{Q} := \begin{pmatrix} \mathcal{A}_{0,\zeta} & \mathcal{B}_{0,\zeta}(\Phi) \\ 0 & \mathcal{H}_\zeta \end{pmatrix}$  is invertible.

Suppose that there exists  $y \in H \ominus H^-, y \neq 0$ , such that  $\mathcal{H}_\zeta y = 0$ . Set  $u := -\mathcal{A}_{0,\zeta}^{-1} \mathcal{B}_{0,\zeta} y$ . Then

$$\begin{pmatrix} \mathcal{A}_{0,\zeta} & \mathcal{B}_{0,\zeta}(\Phi) \\ 0 & \mathcal{H}_\zeta \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = 0.$$

This contradicts to the invertibility of  $\mathcal{Q}$ . Since  $\mathcal{H}_\zeta^{-1}$  acts in the finite-dimensional space  $H \ominus H^-$ , it is bounded.

Applying the Frobenius formula we get

$$(49) \quad (E_H - \zeta(V \oplus \Phi_\zeta))^{-1} = \begin{pmatrix} \mathcal{A}_{0,\zeta}^{-1} + \mathcal{A}_{0,\zeta}^{-1} \mathcal{B}_{0,\zeta} \mathcal{H}_\zeta^{-1} \mathcal{C}_{0,\zeta} \mathcal{A}_{0,\zeta}^{-1} & * \\ * & * \end{pmatrix}, \quad \zeta \in \mathbb{D},$$

where by stars  $*$  we denote the blocks which are not of interest for us.

Denote by  $\mathcal{M}_{2,\zeta}(\Phi)$  the matrix of the operator  $(E_H - \zeta(V \oplus \Phi_\zeta))^{-1}$  in the basis  $\mathfrak{A}_u$ ,  $\zeta \in \mathbb{D}$ . Then

$$(50) \quad \begin{aligned} \mathcal{M}_{2,\zeta}(\Phi) &= \begin{pmatrix} A_{0,\zeta}^{-1} + A_{0,\zeta}^{-1} B_{0,\zeta} (D_{0,\zeta} - C_{0,\zeta} A_{0,\zeta}^{-1} B_{0,\zeta})^{-1} C_{0,\zeta} A_{0,\zeta}^{-1} & * \\ * & * \end{pmatrix} \\ &= \begin{pmatrix} A_{0,\zeta}^{-1} - \zeta A_{0,\zeta}^{-1} W F_\zeta (I_\delta - \zeta T F_\zeta + \zeta C_{0,\zeta} A_{0,\zeta}^{-1} W F_\zeta)^{-1} C_{0,\zeta} A_{0,\zeta}^{-1} & * \\ * & * \end{pmatrix}, \quad \zeta \in \mathbb{D}. \end{aligned}$$

Let  $\{u_j\}_{j=0}^{\rho-1}$  be a set of elements which were obtained by the Gram-Schmidt orthogonalization of  $\{y_k^-\}_{k=0}^{N-1}$ . Observe that  $\rho \geq 1$ . In the opposite case we have  $y_k^- = 0$ ,  $0 \leq k \leq N-1$ . By (38) we obtain that the moment problem (1) is determinate, what contradicts to our assumptions. Set

$$H_\rho^- := \text{Lin}\{y_k^-\}_{k=0}^{N-1} = \text{Lin}\{u_j\}_{j=0}^{\rho-1}.$$

Consider the following operator:

$$\mathcal{J}_\zeta := P_{H_\rho^-}^H (E_H - \zeta(V \oplus \Phi_\zeta))^{-1} P_{H_\rho^-}^H, \quad \zeta \in \mathbb{D},$$

as an operator in the (finite-dimensional) Hilbert space  $H_\rho^-$ . Its matrix in the basis  $\{u_j\}_{j=0}^{\rho-1}$  we denote by  $J_\zeta$ . It is given by

$$J_\zeta = A_{1,\zeta} - \zeta A_{2,\zeta} W F_\zeta (I_\delta - \zeta T F_\zeta + \zeta C_{0,\zeta} A_{0,\zeta}^{-1} W F_\zeta)^{-1} C_{0,\zeta} A_{3,\zeta}, \quad \zeta \in \mathbb{D}.$$

Here  $A_{1,\zeta}$  is a matrix standing in the first  $\rho$  rows and the first  $\rho$  columns of the matrix  $A_{0,\zeta}^{-1}$ ;  $A_{2,\zeta}$  is a matrix standing in the first  $\rho$  rows of the matrix  $A_{0,\zeta}^{-1}$ ;  $A_{3,\zeta}$  is a matrix standing in the first  $\rho$  columns of the matrix  $A_{0,\zeta}^{-1}$ .

Consider the following operator from  $\mathbb{C}^N$  to  $H_\rho^-$ :

$$\mathcal{K} \sum_{n=0}^{N-1} c_n \vec{e}_n = \sum_{n=0}^{N-1} c_n y_n^-, \quad c_n \in \mathbb{C},$$

where  $\vec{e}_n = (\delta_{n,0}, \delta_{n,1}, \dots, \delta_{n,N-1}) \in \mathbb{C}^N$ . Let  $K$  be the matrix of  $\mathcal{K}$  with respect to the orthonormal bases  $\{\vec{e}_n\}_{n=0}^{N-1}$  and  $\{u_j\}_{j=0}^{\rho-1}$

$$(51) \quad K = ((\mathcal{K} \vec{e}_k, u_j)_H)_{0 \leq j \leq \rho-1, 0 \leq k \leq N-1} = ((y_k^-, u_j)_H)_{0 \leq j \leq \rho-1, 0 \leq k \leq N-1}.$$

By (38) we may write that

$$(52) \quad \begin{aligned} & \int_{\mathbb{R}} \frac{1}{\lambda - z} dm_{k,j}(\lambda) \\ &= \frac{2i}{(z^2 + 1)^2} \left( P_{H_\rho^-}^H [E - \zeta(V \oplus \Phi_\zeta)]^{-1} P_{H_\rho^-}^H \mathcal{K} \vec{e}_k, \mathcal{K} \vec{e}_j \right)_H - \frac{1}{(z^2 + 1)(z - i)} \varphi_{j,k}(z) \\ &= \frac{2i}{(z^2 + 1)^2} (\mathcal{K}^* \mathcal{J}_\zeta \mathcal{K} \vec{e}_k, \vec{e}_j)_{\mathbb{C}^N} - \frac{1}{(z^2 + 1)(z - i)} \varphi_{j,k}(z), \\ & \quad 0 \leq k, j \leq N - 1, \quad z \in \mathbb{C}_+ \setminus \{i\}, \quad \zeta = \frac{z - i}{z + i}, \end{aligned}$$

establishes a one-to-one correspondence between all analytic in  $\mathbb{D}$  operator-valued functions  $\Phi_\cdot$ , which values are linear contractions from  $H \ominus H^-$  into  $H \ominus H^+$ , and all solutions  $M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}$  of the moment problem (1).

Observe that  $(\mathcal{K}^* \mathcal{J}_\zeta \mathcal{K} \vec{e}_k, \vec{e}_j)_{\mathbb{C}^N}$  is the element in the  $j$ -th row and  $k$ -th column of the matrix  $\mathcal{M}_{3,\zeta}$  of the operator  $\mathcal{J}_{1,\zeta} := \mathcal{K}^* \mathcal{J}_\zeta \mathcal{K}$  in the basis  $\{e_n\}_{n=0}^{N-1}$ . We may write

$$\begin{aligned} \mathcal{M}_{3,\zeta} &= K^* J_\zeta K \\ &= K^* A_{1,\zeta} K - \zeta K^* A_{2,\zeta} W F_\zeta (I_\delta + \zeta(C_{0,\zeta} A_{0,\zeta}^{-1} W - T) F_\zeta)^{-1} C_{0,\zeta} A_{3,\zeta} K, \quad \zeta \in \mathbb{D}. \end{aligned}$$

Set

$$\Delta(z) := (\varphi_{j,k}(z))_{j,k=0}^{N-1}, \quad z \in \mathbb{C}_+.$$

Then the following relation

$$(53) \quad \begin{aligned} \int_{\mathbb{R}} \frac{1}{\lambda - z} dM^T(\lambda) &= \frac{2i}{(z^2 + 1)^2} K^* A_{1,\zeta} K - \frac{1}{(z^2 + 1)(z - i)} \Delta(z) \\ & - \frac{2i}{(z^2 + 1)^2} \zeta K^* A_{2,\zeta} W F_\zeta (I_\delta + \zeta(C_{0,\zeta} A_{0,\zeta}^{-1} W - T) F_\zeta)^{-1} C_{0,\zeta} A_{3,\zeta} K, \\ & \quad z \in \mathbb{C}_+ \setminus \{i\}, \quad \zeta = \frac{z - i}{z + i}, \end{aligned}$$

establishes a one-to-one correspondence between all analytic in  $\mathbb{D}$ ,  $\mathbb{C}_{\omega \times \delta}$ -valued functions  $F_\zeta$ , which values are such that  $F_\zeta^* F_\zeta \leq I_\delta$ , and all solutions  $M(\lambda)$  of the moment problem (1). Set

$$(54) \quad \mathbf{A}(z) = 2iK^* A_{1,\zeta} K - (z + i)\Delta(z), \quad \mathbf{B}(z) = -2i\zeta K^* A_{2,\zeta} W,$$

$$(55) \quad \mathbf{C}(z) = \zeta(C_{0,\zeta} A_{0,\zeta}^{-1} W - T), \quad \mathbf{D}(z) = C_{0,\zeta} A_{3,\zeta} K, \quad z \in \mathbb{C}_+ \setminus \{i\}, \quad \zeta = \frac{z - i}{z + i}.$$

Then the right-hand side of (53) becomes

$$\frac{1}{(z^2+1)^2}\mathbf{A}(z) + \frac{1}{(z^2+1)^2}\mathbf{B}(z)F_\zeta(I_\delta + \mathbf{C}(z)F_\zeta)^{-1}\mathbf{D}(z).$$

**Theorem 2.** *Let the matrix Hamburger moment problem (1) be given and condition (3), with  $\Gamma_n$  from (2), be satisfied. Suppose that the moment problem is indeterminate. All solutions of the moment problem (1) can be obtained from the following relation:*

$$(56) \quad \int_{\mathbb{R}} \frac{1}{\lambda - z} dM^T(\lambda) = \frac{1}{(z^2+1)^2}\mathbf{A}(z) + \frac{1}{(z^2+1)^2}\mathbf{B}(z)\mathbf{F}(z)(I_\delta + \mathbf{C}(z)\mathbf{F}(z))^{-1}\mathbf{D}(z), \quad z \in \mathbb{C}_+ \setminus \{i\},$$

where  $\mathbf{A}(z)$ ,  $\mathbf{B}(z)$ ,  $\mathbf{C}(z)$ ,  $\mathbf{D}(z)$  are analytic in  $\mathbb{C}_+$ , matrix-valued functions defined by (54), (55), with values in  $\mathbb{C}_{N \times N}$ ,  $\mathbb{C}_{N \times \omega}$ ,  $\mathbb{C}_{\delta \times \omega}$ ,  $\mathbb{C}_{\delta \times N}$ , respectively. Here  $\mathbf{F}(z)$  is an analytic in  $\mathbb{C}_+$ ,  $\mathbb{C}_{\omega \times \delta}$ -valued function which values are such that  $\mathbf{F}(z)^*\mathbf{F}(z) \leq I_\delta$ ,  $\forall z \in \mathbb{C}_+$ . Conversely, each analytic in  $\mathbb{C}_+$ ,  $\mathbb{C}_{\omega \times \delta}$ -valued function such that  $\mathbf{F}(z)^*\mathbf{F}(z) \leq I_\delta$ ,  $\forall z \in \mathbb{C}_+$ , generates by relation (56) a solution of the moment problem (1). The correspondence between all analytic in  $\mathbb{C}_+$ ,  $\mathbb{C}_{\omega \times \delta}$ -valued functions such that  $\mathbf{F}(z)^*\mathbf{F}(z) \leq I_\delta$ ,  $\forall z \in \mathbb{C}_+$ , and all solutions of the moment problem (1) is bijective.

*Proof.* The proof follows from the preceding considerations.  $\square$

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