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ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH GENERAL POINT INTERACTIONS

J. F. BRASCHE AND L. P. NIZHNIK

To Myroslav Lvovych Gorbachuk on the occasion of his 75th birthday

Abstract. We consider various forms of boundary-value conditions for general one-dimensional Schrödinger operators with point interactions that include $\delta-$ and $\delta'-$ interactions, $\delta'-$ potential, and $\delta-$ magnetic potential. We give most simple spectral properties of such operators, and consider a possibility of finding their norm resolvent approximations.

1. Introduction

One important problem in the theory of singular perturbations of a Schrödinger operator is to construct non–trivial self–adjoint operators that describe interactions on a set Γ of Lebesgue measure zero [3, 4, 27]. The most studied case is the one where Γ consists of isolated points. In this case the corresponding interaction is called point interaction and leads to solvable models in quantum mechanics [3, 4].

For an arbitrary closed set Γ of Lebesgue measure zero, the Schrödinger operator with interaction on Γ is defined as a self-adjoint extension of the minimal operator $-\dfrac{d^2}{dx^2}$ defined on functions in the space $C_0^\infty(\mathbb{R}\setminus\Gamma)$ [3, 4, 9, 31]. In some cases, other definitions of the Schrödinger operator with interaction on Γ are possible. Such definitions are given in terms of certain boundary conditions [3, 4], singular perturbations [4, 5, 14, 18], quadratic forms [1, 16], construction of BVS [15, 26, 27], and other methods [2, 10, 11, 12, 13, 30, 32, 34]. If Γ is endowed with a Radon measure, then Schrödinger operators with interactions on Γ can be defined using analogues of the usual boundary conditions on Γ [9, 17, 31].

In this paper, we give various forms of boundary-value conditions for general one-dimensional Schrödinger operators with point interactions. The classification of point interactions for a one–dimensional Schrödinger operator is briefly given in Section 2.

In Section 3, we consider simplest spectral properties of Schrödinger operators with $\delta-$ and $\delta'-$ interactions, $\delta'-$ potential, and $\delta-$ magnetic potential.

In Section 4, we discuss a possibility to find a norm resolvent approximation for a Schrödinger operator with general point interactions.

2. Point interactions

The one-dimensional Schrödinger operator that describes a one-point interaction in a point x_0 is a self-adjoint operator in the space $L_2(\mathbb{R})$ and, for $x \neq x_0$, is given by the differential expression $-\frac{d^2}{dx^2}$. The maximal domain of the operator $-\frac{d^2}{dx^2}$ for $x \neq x_0$ is the

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Sobolev space $W_2^2(\mathbb{R}\setminus\{x_0\})$. For functions φ, $\psi \in W_2^2(\mathbb{R}\setminus\{x_0\})$, we have the Lagrange formula

$$(-\psi'',\varphi)_{L_2} - (\psi,-\varphi'')_{L_2} = \omega(\Gamma\psi,\Gamma\varphi), \tag{2.1}$$

where the boundary form ω is defined on the space E^4 of boundary values of the functions ψ and φ,

$$\Gamma\psi = \mathrm{col}\,(\psi(x_0+0),\psi(x_0-0),\psi'(x_0+0),\psi'(x_0-0)) \in E^4,$$

by the formula

$$\begin{aligned}\omega(\Gamma\psi,\Gamma\varphi) = \psi'(x_0+0)\bar{\varphi}(x_0+0) - \psi(x_0+0)\bar{\varphi}'(x_0+0)\\ -\psi'(x_0-0)\bar{\varphi}(x_0-0) + \psi(x_0-0)\bar{\varphi}'(x_0-0).\end{aligned} \tag{2.2}$$

Self-adjoint restrictions of the maximal operator are defined by domains in terms of the corresponding boundary data that make a Lagrangian plane in the space E^4; it is a maximal subspace on which the boundary form satisfies $\omega(\Gamma\psi,\Gamma\psi) = 0$. Since the boundary form (2.2) can be represented as

$$\omega(\Gamma\psi,\Gamma\varphi) = (\Gamma_1\psi,\Gamma_2\varphi)_{E^2} - (\Gamma_2\psi,\Gamma_1\varphi)_{E^2}, \tag{2.3}$$

where $\Gamma_1\psi = \mathrm{col}\,(\psi'(x_0+0),-\psi'(x_0-0))$, $\Gamma_2\psi = \mathrm{col}\,(\psi(x_0+0),\psi(x_0-0))$, the general self-adjoint boundary conditions are given by a unitary matrix U operating on the space E^2,

$$\Gamma_1\psi + i\Gamma_2\psi = U(\Gamma_1\psi - i\Gamma_2\psi). \tag{2.4}$$

The matrix U uniquely parametrizes the Lagrangian planes. This gives rise to a Schrödinger operator A_U in the space $L_2(\mathbb{R})$ with domain consisting of all functions in the space $W_2^2(\mathbb{R}\setminus\{x_0\})$ satisfying boundary condition (2.4) and $A_U\psi = -\psi''(x)$, $x \neq x_0$. The Schrödinger operator A_U that describes a point interaction in the point x_0 is characterized with the matrix U. The description of all self-adjoint restrictions of the maximal operator in terms of boundary conditions (2.4) may also be obtained from abstract BVS-theory [26]. If the matrix $U = \begin{pmatrix} 0, & -1, \\ -1, & 0 \end{pmatrix}$, that is, the boundary conditions have the form

$$\begin{aligned}\psi(x_0+0) &= \psi(x_0-0),\\ \psi'(x_0+0) &= \psi'(x_0-0),\end{aligned} \tag{2.5}$$

then there is no interaction in x_0. The boundary conditions (2.5) are called trivial.

The conditions (2.4) contain split boundary conditions of the form

$$\begin{aligned}\psi(x_0+0)\cos\alpha_+ - \psi'(x_0+0)\sin\alpha_+ = 0,\\ \psi(x_0-0)\cos\alpha_- - \psi'(x_0-0)\sin\alpha_- = 0,\end{aligned} \tag{2.6}$$

where $\alpha_\pm \in (-\frac{\pi}{2},\frac{\pi}{2}]$. These boundary conditions define a non-transparent interaction in the point x_0. The conditions (2.6) correspond to a self-adjoint Schrödinger operator A in the space $L_2(\mathbb{R}) = L_2(-\infty,x_0)\oplus L_2(x_0,+\infty)$. This operator can be decomposed into the direct sum $A = A_1\oplus A_2$ of self-adjoint operators A_1 and A_2 acting in the spaces $L_2(-\infty,x_0)$ and $L_2(x_0,+\infty)$ that correspond to the boundary conditions (2.6) in the points $x = x_0 - 0$ and $x = x_0 + 0$, respectively.

A converse statement also holds true. If a self-adjoint Schrödinger operator A describes a one point interaction and admits a representation as a direct sum, $A = A_1\oplus A_2$, then the functions in its domain satisfy the boundary conditions (2.6) with some real numbers $\alpha_\pm$.

The boundary conditions (2.4) split if and only if the unitary matrix U is diagonal, $U = \mathrm{diag}\,(e^{2i\alpha_+}, e^{-2i\alpha_-})$. In this case, the boundary conditions (2.4) are equivalent to the conditions (2.6).

The one-dimensional Schrödinger operator corresponding to point interactions on a finite set $X = \{x_1, \dots, x_n\}$ is a self-adjoint operator in the space $L_2(\mathbb{R})$ and an extension of the minimal operator $L_{\min,X}$ defined on the space $C_0^\infty(\mathbb{R} \setminus X)$ by $L_{\min,X}\varphi(x) = -\varphi''(x)$ [3, 4]. All such self-adjoint extensions are described by Lagrangian planes in the Euclidean space E^{4n} of boundary data for the functions $\psi \in W_2^2(\mathbb{R} \setminus X)$. This leads to self-adjoint boundary conditions given by unitary matrices acting on E^{2n}. Localized self-adjoint boundary conditions have the form of (2.4) in every point $x_k \in X$, whereas localized indecomposable boundary conditions have the form [3]

$$\operatorname{col}(\psi(x_k+0), \psi'(x_k+0)) = \Lambda_k \operatorname{col}(\psi(x_k-0), \psi'(x_k-0)), \tag{2.7}$$

where the transmission matrices Λ_k can be written as $\Lambda_k = e^{i\eta_k} R_k$, where R_k is a real matrix, and $\det R_k = 1$, η_k is a real constant.

The boundary form (2.2) can be represented equivalently as

$$\omega(\Gamma\psi, \Gamma\varphi) = (\hat{\Gamma}_1\psi, \hat{\Gamma}_2\varphi)_{E_2} - (\hat{\Gamma}_2\psi, \hat{\Gamma}_1\varphi)_{E_2}, \tag{2.8}$$

where

$$\hat{\Gamma}_1\psi = \operatorname{col}(\psi_s', \psi_s), \quad \hat{\Gamma}_2\psi = \operatorname{col}(\psi_r, -\psi_r'), \tag{2.9}$$

$$\begin{aligned} \psi_s &= \psi(x_0+0) - \psi(x_0-0); & \psi_s' &= \psi'(x_0+0) - \psi'(x_0-0); \\ \psi_r &= \tfrac{1}{2}[\psi(x_0+0) + \psi(x_0-0)]; & \psi_r' &= \tfrac{1}{2}[\psi'(x_0+0) + \psi'(x_0-0)]. \end{aligned} \tag{2.10}$$

By (2.8), general self-adjoint boundary conditions in the point x_0 are defined with a unitary matrix $\hat{U}$ acting on the space E^2 and have the form

$$\hat{\Gamma}_1\psi + i\Gamma_2\psi = \hat{U}(\hat{\Gamma}_1\psi - i\Gamma_2\psi). \tag{2.11}$$

The matrices $\hat{U}$ and U in the boundary conditions (2.4) and (2.11) are connected with each other via the relations

$$\begin{aligned} \hat{U} &= (3C^{tr}UC + 1)(3 + C^{tr}UC)^{-1}, \\ U &= \overline{C}(3 - \hat{U})^{-1}(3\hat{U} - 1)C^*, \end{aligned}$$

where $C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ is a unitary matrix.

Among one-point interactions, the following four cases are important.

1) The $\delta-$ interaction, or $\delta-$ potential, with intensity α is defined by the boundary conditions

$$\psi_s(x_0) = 0, \quad \psi_s'(x_0) = \alpha\psi_r(x_0), \tag{2.12}$$

where x_0 is the interaction point. In this case, the Λ-matrix in the boundary conditions (2.7) has the form $\Lambda = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$.

2) The $\delta'-$ interaction with intensity β is defined by the boundary conditions

$$\psi_s'(x_0) = 0, \quad \psi_s(x_0) = \beta\psi_r'(x_0). \tag{2.13}$$

In this case, the Λ-matrix in the boundary conditions (2.7) has the form $\Lambda = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$.

3) The $\delta'-$ potential with intensity γ is defined by the boundary conditions

$$\psi_s(x_0) = \gamma\psi_r(x_0), \quad \psi_s'(x_0) = -\gamma\psi_r'(x_0). \tag{2.14}$$

An equivalent form of the boundary conditions (2.14) is $\psi(x_0+0) = \theta\psi(x_0-0)$, $\psi'(x_0+0) = \theta^{-1}\psi'(x_0-0)$, where $\theta = \dfrac{2+\gamma}{2-\gamma}$. In this case, the matrix Λ in the boundary conditions (2.7) is $\Lambda = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$.

4) The $\delta-$ magnetic potential with intensity μ is defined in terms of the boundary conditions

$$\psi_s(x_0) = i\mu\psi_r(x_0), \quad \psi_s'(x_0) = i\mu\psi_r'(x_0), \tag{2.15}$$

where i is the imaginary unit. An equivalent form of the boundary conditions (2.15) is $\psi(x_0+0) = e^{i\eta}\psi(x_0-0)$, $\psi'(x_0+0) = e^{i\eta}\psi'(x_0-0)$, where $\frac{\mu}{2} = \tan\frac{\eta}{2}$. In this case, Λ in the boundary conditions (2.7) is a multiple of the identity matrix, $\Lambda = e^{i\eta}I$.

To explain the names and the physical meaning of the four types of interactions listed above, consider at first the formal Schrödinger operators L

$$L = -\frac{d^2}{dx^2} + \varepsilon\delta^{(j)}(x-x_0), \quad j=0,1; \quad \varepsilon=\alpha, \quad j=0; \quad \varepsilon=\gamma, \quad j=1, \tag{2.16}$$

the expression $L\psi$ can be defined in the sense of distribution theory for functions $\psi \in W_2^2(\mathbb{R}\setminus\{x_0\})$.

Indeed, the expression $-\frac{d^2}{dx^2}$ on such functions ψ, in the sense of distribution theory, is given by the expression

$$-\frac{d^2}{dx^2}\psi(x) = -\psi''(x) - \delta'(x-x_0)\psi_s(x_0) - \delta(x-x_0)\psi_s'(x_0). \tag{2.17}$$

The product $\delta^{(j)}(x-x_0)\psi(x)$ is well defined if $\psi \in C^\infty(\mathbb{R})$, that is, the function ψ is a multiplicator for the Schwartz space $C_0^\infty(\mathbb{R})$ of test functions. In this case

$$\begin{gathered} \delta(x-x_0)\psi(x) = \psi(x_0)\delta(x-x_0), \\ \delta'(x-x_0)\psi(x) = \psi_r(x_0)\delta'(x-x_0) - \psi_r'(x_0)\delta(x-x_0). \end{gathered} \tag{2.18}$$

The identity (2.18) can be extended as to also encompass discontinuous functions $\psi \in C^\infty(\mathbb{R}\setminus\{x_0\})$ by defining the functionals $\delta^{(j)}(x-x_0)$ by $(\delta^{(j)}(x-x_0), \varphi(x)) = (-1)^j\varphi_r^{(j)}(x_0)$ [4]. Hence, with such a definition, formulas (2.18) hold if all $\psi^{(j)}(x_0)$ in the right-hand sides of formulas (2.18) are replaced with $\psi_r^{(j)}(x_0)$.

If (2.17) and (2.18) are used in (2.16), then the condition $L\psi \in L_2(\mathbb{R})$ leads to (2.12) if $j=0$ and to (2.14) if $j=1$.

Consider now a one-dimensional Schrödinger operator with magnetic field potential a and potential V, that is, $L = \left(i\frac{d}{dx} + a\right)^2 + V$, in the particular case where $L = -\frac{d^2}{dx^2} + 2ia\frac{d}{dx} + ia'$ and $a(x) = \mu\delta(x)$, so that

$$L_\mu\psi = -\frac{d^2\psi}{dx^2} + 2ia\delta(x)\frac{d\psi}{dx} + i\mu\delta'(x)\psi(x). \tag{2.19}$$

If we use expressions (2.17), (2.18) in (2.19), then imposing the condition on $\psi(x) \in W_2^2(\mathbb{R}\setminus\{x_0\})$ that the distribution $L_\mu\psi$ is a usual function in $L_2(\mathbb{R})$ leads to (2.15). Hence, the boundary conditions (2.15) describe a magnetic field with the potential $a(x) = \mu\delta(x)$.

Particular forms of boundary conditions (2.12)–(2.15) can be represented as

$$\begin{pmatrix} \psi_s'(x_0) \\ \psi_s(x_0) \end{pmatrix} = B \begin{pmatrix} \psi_r(x_0) \\ -\psi_r'(x_0) \end{pmatrix}, \tag{2.20}$$

where ψ_s, ψ_s', ψ_r, and ψ_r' are defined in (2.10). The matrix $B = \begin{pmatrix} \alpha & \gamma - i\mu \\ \gamma + i\mu & -\beta \end{pmatrix}$ is self-adjoint and each condition in (2.11)–(2.15) follows from (2.20) by setting three of the four parameters α, β, γ, μ to zero. For an arbitrary self-adjoint matrix B, conditions (2.20) make a particular case of self-adjoint boundary conditions of form (2.11) with the unitary matrix $\hat{U} = (B-i)^{-1}(B+i)$.

Note that the boundary conditions (2.20) do not contain all non-splitting self-adjoint boundary conditions of the form (2.7). In particular, they do not include boundary conditions of the form

$$\psi'(x_0+0) = i\lambda_0\psi(x_0-0), \quad \psi'(x_0-0) = i\lambda_0\psi(x_0+0) \tag{2.21}$$

with a real constant λ_0. The boundary conditions (2.21) describe a point interaction, in the point $x = x_0$, transparent for the waves $e^{i\lambda x}$ with $\lambda = \lambda_0$. In this case, the function $\psi = e^{i\lambda_0 x}$ satisfies the boundary conditions (2.21) and the Schrödinger equation. Boundary conditions (2.21) have the form (2.7) with the matrix $\Lambda = i\begin{pmatrix} 0 & -\lambda_0^{-1} \\ \lambda_0 & 0 \end{pmatrix}$.

Let us also give a relation between the matrix Λ from the boundary condition (2.7) and the matrix B from the conditions (2.20)

$$\Lambda = \frac{1}{D}\left\| \begin{matrix} \theta_+ & \beta \\ \alpha & \theta_- \end{matrix} \right\|,$$

where $D = \left(1 - \frac{i}{2}\mu\right)^2 - \frac{1}{4}\alpha\beta - \frac{1}{4}\gamma^2$, $\theta_\pm = \left(1 \pm \frac{\gamma}{2}\right)^2 + \frac{1}{4}\alpha\beta + \frac{1}{4}\mu^2$.

The Schrödinger operator L_B corresponding to the boundary conditions (2.20) for a point interaction in the point $x_0 = 0$ can formally be represented with the following expression containing the Dirac δ-function and its derivative $\delta'(x)$:

$$L_B = -\frac{d^2}{dx^2} + \alpha\delta(x)(\cdot,\delta) - \beta\delta'(x)(\cdot,\delta') + (\gamma + i\mu)\delta'(x)(\cdot,\delta) + (\gamma - i\mu)\delta(x)(\cdot,\delta'). \tag{2.22}$$

Here the differentiation $\frac{d^2}{dx^2}$ is understood in the distribution sense, and the functionals $(\cdot,\delta)$ and $(\cdot,\delta')$ are defined by $(\psi,\delta) = \psi_r(0) = \frac{1}{2}\left[\psi(+0) + \psi(-0)\right]$, $(\psi,\delta') = -\psi_r'(0) = -\frac{1}{2}\left[\psi'(+0) + \psi'(-0)\right]$. The domain of the operator L_B is defined by the condition $L_B\psi \in L_2(\mathbb{R})$ imposed on the functions ψ [4].

3. Spectral properties of Schrödinger operator with point interactions

Let L_X be a Schrödinger operator with point interactions in a finite number of points $\{x_k\}_{k=1}^n = X$. The operator L_X is a self-adjoint extension, in the space $L_2(\mathbb{R})$, of the minimal symmetric operator $L_{\min,X}$ with finite deficiency indices. Here, $L_{\min,X}\psi(x) = -\psi''(x)$, if $x \neq x_k \in X$, and its domain is $D(L_{\min,X}) = \left\{\psi : \psi \in W_2^2(\mathbb{R});\ \psi(x_k) = \psi'(x_k) = 0,\ x_k \in X\right\}$. Since the free Schrödinger operator $-\Delta$, $D(-\Delta) = W_2^2(\mathbb{R})$, is also a self-adjoint extension of the operator $L_{\min,X}$ and the spectrum $\sigma(-\Delta) = [0,+\infty)$, we see that the continuous spectum of the operator L_X also consists of the positive half-axis.

Proposition 3.1. *Let the Schrödinger operator L_X describe point interactions on a finite set $X = \{x_k\}_{k=1}^n$ with locally indecomposable boundary-value conditions (2.7) in every point $x_k \in X$. Then the operator L_X does not have any nonnegative eigenvalues.*

Proof. Let $\psi_\lambda \neq 0$ be an eigenfunction of the operator L_X corresponding to an eigenvalue $\lambda^2 \geq 0$. If $x \notin X$, then the function ψ_λ satisfies the equation $-\psi_\lambda''(x) = \lambda^2\psi_\lambda(x)$. Since $\psi_\lambda \in L_2(\mathbb{R})$, we see that $\psi_\lambda(x) = 0$ to the left and to the right of the set X. If a function equals zero to the left of a point $x_k \in X$, then $\psi_\lambda(x_k - 0) = \psi_\lambda'(x_k - 0) = 0$. Using the boundary-value condition (2.7) we see that $\psi_\lambda(x_k + 0) = \psi_\lambda'(x_k + 0) = 0$. But then $\psi_\lambda(x) = 0$ for $x_k \leq x \leq x_{k+1}$, since $\psi_\lambda'' + \lambda^2\psi_\lambda(x) = 0$. Continuing this construction by induction starting with $k = 1$ and ending with $k = n$ we see that $\psi_\lambda \equiv 0$. Hence, $\lambda^2 \geq 0$ can not be an eigenvalue of the operator L_X. ∎

It is well known that a one-dimensional Schrödinger operator with one-point $\delta-$ interaction in a point $x = x_0$ with intensity α can have a negative eigenvalue if and only if $\alpha < 0$. In such a case, the eigenfunction is $\psi = C \exp\left(\frac{\alpha}{2}|x - x_0|\right)$ and it corresponds to the eigenvalue $\lambda^2 = -\frac{\alpha^2}{4}$.

For Schrödinger operators $L_{X,\alpha}$ with $\delta-$ interactions in points $x_k \in X = \{x_k\}_{k=1}^n$ having intensities $\alpha_k \in \alpha = \{\alpha_k\}_{k=1}^n$, if the number of point interactions is finite or countable, there exist effective algorithms for finding the number $n_-(L_{X,\alpha})$ of negative eigenvalues of the operator $L_{X,\alpha}$ [7, 8, 23, 28, 29, 33]. It is always the case thought that the number $n_-(L_{X,\alpha})$ of negative eigenvalues does not exceed the number $n_-(\alpha)$ of point interactions having negative intensities α_k of the point $\delta-$ interactions. There are also necessary and sufficient conditions for $n_-(L_{X,\alpha}) = n_-(\alpha)$. In particular, this is always true if the distances between the point $\delta-$ interactions are sufficiently large. If the $\delta-$ interactions occur in two points $x_l < x_r$ with intensities α_l and α_r, the equality $n_-(L_{X,\alpha}) = n_-(\alpha)$ is true if and only if the distance $d = x_r - x_l$ between the $\delta-$ interactions satisfies the condition

$$d + \frac{1}{\alpha_l} + \frac{1}{\alpha_r} > 0. \tag{3.1}$$

Hence, if $\alpha_l < 0$ and $\alpha_r < 0$, then, for d satisfying inequality (3.1), the operator $L_{X,\alpha}$ will have 2 negative eigenvalues. If condition (3.1) does not hold, then $n_-(L_{X,\alpha}) = 1$.

This property of two-point $\delta-$ interaction is generalized to the case where the number of $\delta-$ interactions is finite in [7, 8, 33]. We will formulate this result using the following notations. Let $X_l = \{x_{l,k}\}_{k=1}^{n_1}$ and $X_r = \{x_{r,k}\}_{k=1}^{n_2}$ be two finite sets of points of the real line. All points of the set X_l are located to the left of points of the set X_r. We also assume that the points of the sets X_l and X_r are indexed so that the distances between the points is increasing. The distance d between the sets X_l and X_r is defined by $d = x_{r,1} - x_{l,1}$. Let $\alpha_l = \{\alpha_{l,k}\}_{k=1}^{n_1}$ and $\alpha_r = \{\alpha_{r,k}\}_{k=1}^{n_2}$ be real nonzero numbers, and L_{X_l,α_l} and L_{X_r,α_r} be two Schrödinger operators with $\delta-$ interactions on the sets X_l and X_r having intensities α_l and α_r, correspondingly, and let $L_{X,\alpha}$ be a Schrödinger operator with $\delta-$ interaction in points of the set $X = X_l \cup X_r$ with intensities $\alpha = \alpha_l \cup \alpha_r$.

Proposition 3.2. *For*

$$n_-\left(L_{X,\alpha}\right) = n_-\left(L_{X_l,\alpha_l}\right) + n_-\left(L_{X_r,\alpha_r}\right) \tag{3.2}$$

to hold, it is necessary and sufficient that the distance between the sets X_l and X_r satisfy the condition

$$d(X_r, X_l) + \frac{1}{\widetilde{\alpha}_l} + \frac{1}{\widetilde{\alpha}_r} > 0, \tag{3.3}$$

where $\widetilde{\alpha}_l$ and $\widetilde{\alpha}_r$ are equivalent intensities for (X_l, α_l) and (X_r, α_r) defined by the following identities $(g = r, l)$:

$$\widetilde{\alpha}_g = \alpha_{g,1} + \cfrac{1}{|x_{g,1} - x_{g,2}| + \cfrac{1}{\alpha_{g,2} + \cfrac{1}{|x_{g,2} - x_{g,3}| + \cfrac{1}{\alpha_{g,3} + \dots}}}} \tag{3.4}$$

where the intensities and the distances between subsequent points of the $\delta-$ interactions enter in turns. If inequality (3.3) *ceases to hold, then the left-hand side in* (3.2) *is less by* 1 *than the nonzero right-hand of* (3.2)*.*

For a one-point $\delta'-$ interaction with intensity β, the Schrödinger operator has a negative eigenvalue only if $\beta < 0$ and $\lambda^2 = -\frac{4}{\beta^2}$. However, as opposed to the case of $\delta-$ interactions, for the Schrödinger operator $L_{X,\beta}$ with $\delta'-$ interactions in points of a set $X = \{x_k\}_{k=1}^n$ with intensities $\beta = \{\beta_k\}_{k=1}^n$, the number of negative eigenvalues,

$n_-(L_{X,\beta})$, is always equal to the number $n_-(\beta)$ of negative values of the intensities of the $\delta'-$ interactions,

$$n_-(L_{X,\beta}) = n_-(\beta). \tag{3.5}$$

In this connection, let us consider an example of two-point $\delta'-$ interaction with negative intensities β_l and β_r of the $\delta'-$ interactions in points $x = x_l$ and $x = x_r$ as $x_r \to x_l$. Such an example of two-point $\delta-$ interaction, due to (3.1), shows that the Schrödinger operator $L_{X,\alpha}$ will have two negative eigenvalues if $d = x_r - x_l$ is large. As $x_r \to x_l$, inequality (3.1) becomes invalid, and the Schrödinger operator will have a single eigenvalue that, if $x_r = x_l$, will correspond to the total intensity $\alpha_l + \alpha_r$. As far as $\delta'-$ interaction is concerned, it follows from (3.5) that, as $x_r \to x_l$, the Schrödinger operator will always have two eigenvalues. A simple analysis of the characteristic equation for negative eigenvalues shows that, as $x_r \to x_l$, one of the negative eigenvalues approach $-\infty$, and the other one approaches the eigenvalue that corresponds to a one-point $\delta'-$ interaction with total intensity $\beta_l + \beta_r$.

Consider now interactions with $\delta'-$ potentials in points of the set $X = \{x_k\}_{k=1}^n$ having intensities $\gamma = \{\gamma_j\}_{j=1}^n$.

Proposition 3.3. *The Schrödinger operator $L_{X,\gamma}$ with point interactions of $\delta'-$ potential type occuring in a finite number of points of the set $X = \{x_j\}_{j=1}^n$ and having intensities $\gamma = \{\gamma_j\}_{j=1}^n$ is a positive operator on the space $L_2(\mathbb{R})$ and does not have any eigenvalues.*

Proof. Functions $\psi, \varphi \in W_2^2(\mathbb{R} \setminus X)$ satisfy the Green formula

$$(-\psi'', \varphi) = (\psi', \varphi') + \sum_{x_k \in X} \left[\psi_s'(x_k)\, \overline{\varphi_r(x_k)} + \psi_r'(x_k)\, \overline{\varphi_s(x_k)} \right]. \tag{3.6}$$

If $\psi, \varphi \in D(L_{X,\gamma})$, then the following boundary-value conditions hold: $\psi_s'(x_k) = -\gamma_k \psi_r(x_k)$, $\psi_s(x_k) = \gamma_k \psi_r'(x_k)$. Hence, (3.6) imply that $(L_{X,\gamma}\psi, \varphi) = (\psi', \varphi')$, i.e., the operator $L_{X,\gamma}$ is positive. Positive operators can only have nonnegative eigenvalues. Then, by Proposition 3.1, the operator $L_{X,\gamma}$ has no eigenvalues. □

Finally, consider the case of point interactions of $\delta-$ magnetic potential type.

Proposition 3.4. *The Schrödinger operator $L_{X,\mu}$ of $\delta-$ magnetic potential type with point interactions in a finite number of points of the set $X = \{x_j\}_{j=1}^n$ and having intensities $\mu = \{\mu_j\}_{j=1}^n$ is unitary equivalent to a free Schrödinger operator.*

Proof. The domain of the operator $L_{X,\mu}$ consists of all the functions $\psi \in W_2^2(\mathbb{R} \setminus X)$ that satisfy the boundary-value conditions

$$\psi(x_k + 0) = e^{i\eta_k}\psi(x_k - 0), \quad \psi'(x_k + 0) = e^{i\eta_k}\psi'(x_k - 0), \quad x_k \in X, \tag{3.7}$$

where $\tan \frac{\eta_k}{2} = \frac{\mu_k}{2}$.

Let U be a unitary operator on the space $L_2(\mathbb{R})$, which an operator of multiplication by the function $e^{i\omega(x)}$, where $\omega(x) = 0$ for $x < x_1$, and $\omega(x) = \sum_{j=1}^{k} \eta_j$ for $x_k < x < x_{k+1}$. If $\psi \in W_2^2(\mathbb{R})$, then $U\psi = \widehat{\psi} \in D(L_{X,\mu})$ and $L_{X,\mu}\widehat{\psi} = U(-\psi'')$. Hence, $L_{X,\mu} = U(-\Delta)U^*$. □

4. Norm resolvent approximation

It is well known [3, 4] that a model for point interactions is exactly solvable and can serve as a good approximation of real Schrödinger operators if the potential v has small support in a neighborhood of the point x_0, that is, $v(x) = 0$ for $|x - x_0| > \varepsilon$, and the processes under the study have the energy λ^2 much less than ε^{-2}. Here it is assumed that, for the energies under consideration, the matrix Λ_ε that connects values of solutions ψ of the Schrödinger equation $\Big[-\dfrac{d^2}{dx^2}+v\Big]\psi=\lambda^2\psi$ and their derivatives $\psi'(x)$ for $x=x_0-\varepsilon$ and $x_0+\varepsilon$, that is, $\mathrm{col}\,(\psi(x_0+\varepsilon),\psi'(x_0+\varepsilon))=\Lambda_\varepsilon \mathrm{col}\,(\psi(x_0-\varepsilon),\psi'(x_0-\varepsilon))$, is close to the matrix Λ that defines the boundary conditions (2.7) for the point interaction. Thus the Schrödinger operator with point interaction can be considered as a limit (in a certain sense, e.g., in the sense of uniform resolvent convergence), as $\varepsilon\to 0$, of Schrödinger operators with the potentials $v_\varepsilon(x)$ with $\Lambda_\varepsilon\to\Lambda$ for $\varepsilon\to 0$. Here, the potentials $v_\varepsilon(x)$ themselves may or may not have a limit as $\varepsilon\to 0$ even in the sense of distributions. It can happen that their limit values, even if they exist, do not determine the character and the intensity of the point interaction.

Let us look at this phenomenon in greater details for the case of $\delta'-$ potentials; this case was considered in a number of papers [2, 6, 19, 20, 21, 22, 24, 25, 35, 36, 37, 38, 39, 40, 41]. For a model of $\delta-$ potentials with intensity α, one can take a sequence of regular potentials $v_\varepsilon(x)\to\alpha\delta(x)$ with $\varepsilon\to 0$, for example, $v_\varepsilon(x)=\alpha\varepsilon^{-1}\varphi(\frac{x}{\varepsilon})$, where the compactly supported function φ is such that $\int\varphi(x)\,dx=1$. More complex potentials can be well modeled on small intervals by a sum of several δ-functions

$$v_\varepsilon(x)=\sum_{j=1}^{n}\alpha_j(\varepsilon)\delta(x-x_j(\varepsilon)), \tag{4.1}$$

where all $x_j(\varepsilon)\to x_0$ for $\varepsilon\to 0$. It is shown in [7] that the $\delta'-$ interaction is well modeled with three approaching δ-functions that have special opposite sign increasing intensities $\alpha_j(\varepsilon)$

$$v_\varepsilon(x)=\beta\varepsilon^{-2}\delta(x)-\varepsilon^{-1}(1+2\varepsilon\beta^{-1})^{-1}[\delta(x+\varepsilon)+\delta(x-\varepsilon)].$$

When modeling a $\delta'-$ potential of intensity γ, the number of terms in representation (4.1) depends on the conditions to be satisfied. Since the matrix Λ in the boundary conditions (2.7) is diagonal for the $\delta'-$ potential of intensity γ, there are two necessary conditions on the elements of the matrix Λ_ε

1) $\lim\limits_{\varepsilon\to 0}(\Lambda_\varepsilon)_{2,1}=0,$
2) $\lim\limits_{\varepsilon\to 0}(\Lambda_\varepsilon)_{1,1}=\Big(1+\frac{\gamma}{2}\Big)\Big(1-\frac{\gamma}{2}\Big)^{-1}.$

These two conditions can be satisfied with two terms in approximation (4.1)

$$v_\varepsilon(x)=\alpha_1\varepsilon^{-1}\delta(x)+\alpha_2\varepsilon^{-1}\delta(x-\varepsilon), \tag{4.2}$$

where $\alpha_1+\alpha_2+\alpha_1\alpha_2=0$, $\alpha_1=\gamma\Big(1-\frac{\gamma}{2}\Big)^{-1}$, $\alpha_2=-\gamma\Big(1+\frac{\gamma}{2}\Big)^{-1}$.

Here, the potentials v_ε do not have a limit as $\varepsilon\to 0$ in the sense of distributions. In this case, the matrix Λ_ε can be written as a product of three matrices $\Lambda_\varepsilon=\Lambda_2\Lambda_\varepsilon^0\Lambda_1$, where $\Lambda_j=\begin{pmatrix}1 & 0\\ \alpha_j\varepsilon^{-1} & 1\end{pmatrix}$, $j=1,2$, $\Lambda_\varepsilon^0=\begin{pmatrix}\cos\lambda\varepsilon & \frac{\sin\lambda\varepsilon}{\lambda}\\ -\lambda\sin\lambda\varepsilon & \cos\lambda\varepsilon\end{pmatrix}$. These matrices give a relation between the solutions $\psi(x)$ of the Schrödinger equation

$$-\frac{d^2}{dx^2}\psi+v_\varepsilon\psi=\lambda^2\psi$$

and its derivatives $\psi'(x)$ in different points x

$$\begin{aligned}
&\mathrm{col}\,(\psi(+0),\psi'(+0))=\Lambda_1\mathrm{col}\,(\psi(-0),\psi'(-0)),\\
&\mathrm{col}\,(\psi(\varepsilon-0),\psi'(\varepsilon-0))=\Lambda_\varepsilon^0\mathrm{col}\,(\psi(+0),\psi'(+0)),\\
&\mathrm{col}\,(\psi(\varepsilon+0),\psi'(\varepsilon+0))=\Lambda_2\mathrm{col}\,(\psi(\varepsilon-0),\psi'(\varepsilon-0)).
\end{aligned}$$

Using the explicit form of α_j we get

$$\Lambda_\varepsilon = \begin{pmatrix} 1+\alpha_1 & 0 \\ 0 & 1+\alpha_2 \end{pmatrix} + O(\varepsilon), \quad \lim_{\varepsilon\to 0} \Lambda_\varepsilon = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix},$$

where $\theta = \dfrac{2+\gamma}{2-\gamma}$. Hence, the limit Schrödinger operator corresponds to a point interaction having δ'– potential of intensity γ.

Proposition 4.1. *The Schrödinger operator $-\frac{d^2}{dx^2} + v_\varepsilon(x)$, where the potential $v_\varepsilon(x)$ is given by (4.2), converges, as $\varepsilon \to 0$, with respect to the norm of the resolvent to the Schrödinger operator $L_{\{0\},\gamma}$ with point interaction of the δ'– type potential of intensity γ.*

Proof. Let us show that if $z, \operatorname{Im} z \neq 0$, is fixed and $\varepsilon \to 0$, then

$$\Big|\Big|\Big[-\frac{d^2}{dx^2} + v_\varepsilon(x) - z\Big]^{-1} - [L_{\{0\},\gamma} - z]^{-1}\Big|\Big| \to 0. \tag{4.3}$$

The functions $\varphi \in \mathcal{D}(L_{\min}) = \{\varphi : \varphi \in W_2^2(\mathbb{R}), \varphi(0) = \varphi'(0) = \varphi(\varepsilon) = \varphi'(\varepsilon) = 0\}$ belong to domains of both the operators $L_1 = L_{\{0\},\gamma}$ and $L_2 = -\frac{d^2}{dx^2} + v_\varepsilon(x)$. Hence, the resolvents of the operators L_1 and L_2 coincide on $\mathcal{R} = (L_1 - z)\mathcal{D}(L_{\min})$. Since the dimension of the orthogonal complements $\mathfrak{N} = L_2 \ominus \mathcal{R}$ is 4, to prove (4.3) is is sufficient to show that $||(L_1 - z)^{-1}h - (L_2 - z)^{-1}h|| \to 0$ as $\varepsilon \to 0$ for any $h \in \mathfrak{N}$. The subspace $\mathfrak{N}$ consists of the functions h that are solutions of the equation $-h''(x) - \overline{z}h(x) = 0$ for $x \neq 0, \varepsilon$. For the sake of definiteness, assume that $z = 2i$ and $\sqrt{z} = 1 + i$. Then one of the functions $h \in \mathfrak{N}$ is of the form $h(x) = \theta(-x)e^{(1+i)x}$. Define $\psi_j = (L_j - z)^{-1}h \;\; j = 1, 2$. The functions ψ_j are solutions of the equations $(L_j - z)\psi_j = h$ and, hence, they admit the following representations:

$$\begin{aligned} \psi_1(x) &= \frac{i}{4}h(x) + \theta(-x)C_-^{(1)}e^{(1-i)x} + \theta(x)C_+^{(1)}e^{-(1-i)x}, \\ \psi_2(x) &= \frac{i}{4}h(x) + \theta(-x)C_-^{(2)}e^{(1-i)x} + \theta(x-\varepsilon)C_+^{(2)}e^{-(1-i)(x-\varepsilon)} \\ &\quad + \chi_{[0,\varepsilon]}(x)\Big[\Big(C_-^{(2)} + \frac{i}{4}\Big)\frac{\sin(1-i)(\varepsilon - x)}{\sin(1-i)\varepsilon} + C_+^{(2)}\frac{\sin(1-i)x}{\sin(1-i)\varepsilon}\Big], \end{aligned} \tag{4.4}$$

where $\chi_{[0,\varepsilon]}(x)$ is the indicator function for the line segment $[0, \varepsilon]$, that is, $\chi_{[0,\varepsilon]}(x) = 1$ if $x \in [0, \varepsilon]$ and $\chi_{[0,\varepsilon]}(x) = 0$ if $x \notin [0, \varepsilon]$. The constants $C_\pm^{(j)}$, $j = 1, 2$, that enter (4.4) can be found from the condition that ψ_j belong to domains of the operators L_j. The condition $\psi_1 \in \mathcal{D}(L_{\{0\},\gamma})$ leads to the boundary-value conditions (2.14) for the δ'– potential, which make the following system:

$$\operatorname{col}\big(C_+^{(1)}, -(1-i)C_+^{(1)}\big) = \Lambda \operatorname{col}\Big(\frac{i}{4} + C_-^{(1)}, \frac{-1+i}{4} + (1-i)C_-^{(1)}\Big). \tag{4.5}$$

The condition $\psi_2 \in \mathcal{D}\Big(-\frac{d^2}{dx^2} + v_\varepsilon\Big)$ leads to a system for $C_-^{(2)}$ and $C_+^{(2)}$ similar to (4.5) with the matrix Λ replaced with $\Lambda_\varepsilon = \Lambda + O(\varepsilon)$ and $C_\pm^{(1)}$ with $C_\pm^{(2)}$. Hence, $|C_+^{(2)} - C_+^{(1)}| + |C_-^{(2)} - C_-^{(1)}| \leq k \cdot \varepsilon$. Using an explicit form of ψ_j in (4.4) we see that $||\psi_2 - \psi_1||_{L_2} \to 0$ as $\varepsilon \to 0$. The cases where $h(x) = \theta(x - \varepsilon)e^{-(1+i)x}$ and $h(x) = \chi_{[0,\varepsilon]}(x)[Ae^{(1+i)x} + Be^{-(1+i)x}]$ are treated similarly. $\square$

One can additionally require that $v_\varepsilon(x) \to \kappa\delta'(x)$ in (4.1) as $\varepsilon \to 0$. This can be achieved if we take

$$v_\varepsilon(x) = \alpha_1\varepsilon^{-1}\delta(x+\varepsilon) + \alpha_2\varepsilon^{-1}\delta(x) + \alpha_3\varepsilon^{-1}\delta(x-\varepsilon) \tag{4.6}$$

in (4.1), where $\alpha_2 = \pm 2\gamma[\gamma^2 - 4]^{-\frac{1}{2}}$, $\alpha_1 = \frac{\gamma}{2} - \frac{\alpha_2}{2}\Big(1 + \frac{\gamma}{2}\Big)$, $\alpha_3 = -\frac{\gamma}{2} - \frac{\alpha_2}{2}\Big(1 - \frac{\gamma}{2}\Big)$.

In the limit as $\varepsilon \to 0$, the Schrödinger operators with the potentials $v_\varepsilon(x)$ of the form (4.6) define point interaction of δ'– potential type with intensity γ, and the limit $v_\varepsilon(x) \to \kappa\delta'(x)$ exists in the distribution sense, where the constant $\kappa = \alpha_1 - \alpha_3 = \gamma\Big(1 - \frac{\alpha_2}{2}\Big)$ depends on the choice of the sign of α_2 and, consequently, it does not determine the intensity γ. Moreover, considering an expression of the form (4.1) for the potentials $v_\varepsilon(x)$ with four terms

$$v_\varepsilon(x) = \alpha_1\varepsilon^{-1}\delta(x) + \alpha_2\varepsilon^{-1}\delta(x-\varepsilon) + \alpha_3\varepsilon^{-1}\delta(x-2\varepsilon) + \alpha_4\varepsilon^{-1}\delta(x-3\varepsilon), \tag{4.7}$$

where $\alpha_1 = -1$, $\alpha_2 = 6$, $\alpha_3 = -3$, $\alpha_4 = -2$ we obtain $\lim\limits_{\varepsilon\to 0} v_\varepsilon(x) = 6\delta'(x)$ in the sense of distributions. On the other hand, it is easy to see that $\lim\limits_{\varepsilon\to 0} \Lambda_{3\varepsilon} = I$, that is, if $\varepsilon \to 0$, the Schrödinger operators with potentials (4.7) converge to a free Schrödinger operator. By taking $\alpha_1 = \alpha_4 = 3$, $\alpha_2 = \alpha_3 = -3$ in (4.7), we have $v_\varepsilon(x) \to 0$ and the Schrödinger operators converge to a direct sum of operators on the spaces $L_2(-\infty, 0)$ and $L_2(0, +\infty)$ corresponding to the Dirichlet conditions $\psi(\pm 0) = 0$.

Let us remark that if the Schrödinger operators have potentials in the form of (4.1), then the kernels of the resolvents for these operators can be written explicitly similarly to the case of the limit Schrödinger operator. This yields that these operators converge, as $\varepsilon \to 0$, in the sense of uniform resolvent convergence. The above conclusions about Schrödinger operators with potentials (4.1) remain also true if v_ε are piecewise constant or even $v_\varepsilon \in C_0^\infty(\mathbb{R})$ if they can well approximate each term in (4.1).

Let us also make a remark on one more feature of point interactions. If the support of the potential $v_\varepsilon(x)$ belongs to the interval $(-\varepsilon, \varepsilon)$ and its components $v_\varepsilon^-(x) = \theta(-x)v_\varepsilon(x)$, $v_\varepsilon^+(x) = \theta(x)v_\varepsilon(x)$, where θ is the unit Heaviside function, determine point interactions with the corresponding matrices Λ^- and Λ^+, as $\varepsilon \to 0$, then the potential $v_\varepsilon(x)$ also gives rise to a point interaction, as $\varepsilon \to 0$, with the matrix $\Lambda = \Lambda^+\Lambda^-$. This leads to additivity of intensities α and β for δ- and δ'– interactions, since they correspond to triangular matrices Λ^-, Λ^+, Λ. For point interactions with δ'– type potentials and δ– magnetic potentials, the intensities γ and μ do not have such an additivity property. Here, if γ_- and γ_+ are intensities of δ'– potentials corresponding to v_ε^- and v_ε^+, then the total intensity γ is found as $\gamma = (\gamma_- + \gamma_+)\Big(1 + \frac{1}{4}\gamma_-\gamma_+\Big)^{-1}$. Thus, for point interactions with δ'– type potential and δ– magnetic potential, the "additive" characteristics of the intensities are useful. The additive characteristic ξ for δ'– potential with intensity γ are defined by the identities $\frac{2+\gamma}{2-\gamma} = \pm e^{\xi_\pm}$, where the sign "+" is taken if $|\gamma| < 2$ and we take the sign "−" if $|\gamma| > 2$. A more exact definition of additive characteristic for point interactions with δ'– potential is the following. Additive characteristic is a pair (ξ, s) consisting of the number ξ and the sign $s = \pm 1$. As two-point interactions with δ'– potentials having characteristics (ξ_1, s_1) and (ξ_2, s_2) approach, the total characteristic (ξ, s) is found as $(\xi, s) = (\xi_1 + \xi_2, s_1 \cdot s_2)$, which corresponds to the above "adding" rule for the intensities γ_- and γ_+.

For a point interaction with δ– magnetic potential of intensity μ, the Λ-matrix in the boundary condition (2.7) is a multiple of the identity matrix, $\Lambda = e^{i\eta}I$. Hence, it is convenient to take the number η to be an "additive" characteristic of the δ– magnetic potential. There is a relation between μ and η, $\mu = 2\tan\frac{\eta}{2}$. For two approaching point interactions with δ– magnetic potentials having characteristics η_1 and η_2, the corresponding total characteristic is $\eta = \eta_1 + \eta_2$.

Remark 4.1. *A general one-point interaction in a point x_0 with indecomposable boundary condition (2.7) can be obtained by the limiting process of contracting the interaction points to the point x_0, of not more than 4 simple point interactions of δ– , δ'–interactions, δ'– potential, and δ– magnetic potential type.*

This follows from the possibility of representing an arbitrary matrix Λ in (2.7) as a product of 4 simplest matrices Λ in (2.12)–(2.15). In particular, the matrix Λ in (2.21) can be represented as

$$i\begin{pmatrix} 0 & -\lambda_0^{-1} \\ \lambda_0 & 0 \end{pmatrix} = iI\cdot\begin{pmatrix} 1 & 0 \\ \lambda_0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & -\lambda_0^{-1} \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ \lambda_0 & 1 \end{pmatrix}.$$

It is not true that if the Schrödinger operators $-\frac{d^2}{dx^2}+v_\varepsilon(x)$ converge, as $\varepsilon\to 0$, to a Schrödinger operator with point interaction of a certain type then the operators $-\frac{d^2}{dx^2}+kv_\varepsilon(x)$, where $k\neq 1$ is an arbitrary real constant, also converge to a Schrödinger operator with point interaction of the same type. In the general case, this is true only for δ– potential. It is shown in [24, 25] that, for special approximations of $k\delta'$-functions where $v_\varepsilon = k\varepsilon^{-2}\psi\left(\frac{x}{\varepsilon}\right)$, $\int\psi(x)\,dx=0$, $\int x\psi(x)\,dx=-1$, the Schrödinger operators have a limit that defines a point interaction of δ'– potential only for special "resonance" values of k.

For potentials in (4.1), which give norm resolvent convergence, as $\varepsilon\to 0$, to a one point δ'–interaction or δ'– potential there can exist only a finite number of resonance values $k\le n-1$, and this number depends on the choice of $\alpha_j(\varepsilon)$ in (4.1).

References

1. S. Albeverio, J. F. Brasche, M. Röckner, *Dirichlet forms and generalized Schrödinger operators*, Schrödinger Operators (H. Holden, A. Jensen, eds.). Lecture Notes in Physics, vol. 345, Springer, Berlin, 1989, pp. 1–42.
2. S. Albeverio, C. Cacciapuoti, and D. Finco, *Coupling in the singular limit of thin quantum waveguides*, J. Math. Phys. **48** (2007), no. 3, 032103, 21 p.
3. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer Verlag, Berlin, 1988; 2nd edition with an Appendix by P. Exner, AMS Chelsea Publishing, Providence, RI, 2005.
4. S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators. Solvable Schrödinger Type Operators*, Cambridge University Press, Cambridge, 2000.
5. S. Albeverio, S. Kuzhel, and L. Nizhnik, *On the perturbation theory of self-adjoint operators*, Tokyo Journal of Mathematics **31** (2008), no. 2, 273–292.
6. S. Albeverio and L. Nizhnik, *Approximation of general zero-range potentials*, Ukrain. Mat. Zh. **52** (2000), no. 5, 582–589; English transl. Ukrainian Math. J. **52** (2001), no. 5, 664–672.
7. S. Albeverio, L. Nizhnik, *On the number of negative eigenvalues of a one-dimensional Schrödinger operator with point interaction*, Letters Math. Phys. **65** (2003), 27–35.
8. S. Albeverio and L. Nizhnik, *Schrödinger operators with a number of negative eigenvalues equal to the number of point interactions*, Methods Funct. Anal. Topology **9** (2003), no. 4, 273–286.
9. S. Albeverio, L. Nizhnik, *A Schrödinger operator with δ'-interaction on a Cantor set and Krein-Feller operators*, Math. Nachr. **279** (2006), no. 5–6, 467–476.
10. J. F. Brasche, *On extension theory in L^2-spaces*, Potential Anal. **4** (1995), no. 3, 297–307.
11. J. F. Brasche, *On eigenvalues and eigensolutions of the Schrödinger equation on the complement of a set with classical capacity zero*, Methods Funct. Anal. Topology **9** (2003), no. 3, 189–206.
12. J. F. Brasche, *Interactions along Brownian paths in R^d, $d\le 5$*, J. Phys. A **38** (2005), no. 22, 4755–4767.
13. J. F. Brasche, *Interactions along Brownian paths: completeness and eigenvalues*, Rep. Math. Phys. **59** (2007), no. 3, 331–335.
14. J. Brasche, R. Figari, and A. Teta, *Singular Schrödinger operators as limits of point interaction Hamiltonian*, Potential Anal. **8** (1998), no. 2, 163–178.
15. J. F. Brasche, M. M. Malamud, H. Neidhardt, *Selfadjoint extensions with several gaps: finite deficiency indices*, Oper. Theory Adv. Appl. **162** (2006), 85–101.
16. J. Brasche, L. Nizhnik, *A generalized sum of quadratic forms*, Methods Funct. Anal. Topology **8** (2002), no. 3, 13–19.
17. J. Brasche, L. Nizhnik, *One-dimensional Schrödinger operators with δ'-interactions on a set of Lebesgue measure zero, 2011, arXiv: 1112.2545v1[math.FA].*
18. J. F. Brasche, K. Ožanová, *Convergence of Schrödinger operators*, SIAM J. Math. Anal. **39** (2007), no. 1, 281–297.

19. C. Cacciapuoti and P. Exner, *Nontrivial edge coupling from a Dirichlet network squeezing: the case of a bent waveguide*, J. Phys. A: Math. Theor. **40** (2007), no. 26, F511–F523.
20. P. L. Christiansen, H. C. Arnbak, A. V. Zolotaryuk, V. N. Ermakov, Y. B. Gaididei, *On the existence of resonances in the transmission probability for interactions arising from derivatives of Dirac's delta function*, J. Phys. A: Math. Gen. **36** (2003), 7589–7600.
21. P. Exner, *The absence of the absolutely continuous spectrum for δ' Wannier-Stark ladders*, J. Math. Phys. **36** (1995), 4561–4570.
22. P. Exner, H. Neidhardt, and V. A. Zagrebnov, *Potential approximation to δ': An inverse Klauder phenomenon with norm-resolvent convergence*, Comm. Math. Phys. **224** (2001), 593–612.
23. N. Goloschapova and L. L. Oridoroga, *On the number of negative squares of one-dimensional Schrödinger operators with point interactions*, Integr. Equ. Oper. Theory **6** (2010), no. 1, 1–14.
24. Yu. D. Golovaty, *Schrödinger operators with $(\alpha\delta' + \beta\delta)$–like potentials: norm resolvent convergence and solvable models*, Methods Funct. Anal. Topology **18** (2012), no. 3, 243–255.
25. Yu. D. Golovaty and R. O. Hryniv, *On norm resolvent convergence of Schrödinger operators with δ'-like potentials*, J. Phys. A: Math. Theor. **43** (2010), no. 15, 155204, 15 p.
26. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984)
27. A. N. Kochubei, *Elliptic operators with boundary conditions on a subset of measure zero*, Funct. Anal. Appl. **16** (1982), 137–139.
28. A. Kostenko, M. Malamud, *1–D Schrödinger operators with local point interactions on a discrete set*, J. Diff. Equat. **249** (2010), 253–304.
29. A. Kostenko, M. Malamud, *Schrödinger operator with δ'-interactions and the Krein-Stieltjes string*, Dokl. Math. **81** (2010), no. 3, 342–347.
30. Vladimir Mikhailets, Volodymyr Molyboga, *Schrödinger operators with complex singular potentials*, Methods Funct. Anal. Topology **19** (2013), no. 1.
31. L. P. Nizhnik, *Schrödinger operator with δ'-interaction*, J. Funct. Anal. Appl. **37** (2003), no. 1, 85–88.
32. L. P. Nizhnik, *One–dimensional Schrödinger operators with point interactions on Sobolev spaces*, J. Funct. Anal. Appl. **40** (2006), no. 2, 74–79.
33. O. Ogurisu, *On the number of negative eigenvalues of a Schrödinger operator with δ-interactions*, Methods Funct. Anal. Topology **16** (2010), no. 1, 42–50.
34. P. Šeba, *Schrödinger particle on a half line*, Letters Math. Phys. **10** (1985), no. 1, 21–27.
35. P. Šeba, *Some remarks on the δ'–interaction in one dimension*, Rep. Math. Phys. **24** (1986), 111–120.
36. F. M. Toyama, Y. Nogami, *Transmission-reflection problem with a potential of the form of the derivative of the delta function*, J. Phys. A: Math. Theor. **40** (2007), F685–F690.
37. A. V. Zolotaryuk, P. L. Christiansen, and S. V. Iermakova, *Scattering problems of point dipole interactions*, J. Phys. A: Math. Gen. **39** (2006), 9329–9338.
38. A. V. Zolotaryuk, P. L. Christiansen, and S. V. Iermakova, *Resonant tunneling through short-range singular potentials*, J. Phys. A: Math. Gen. **40** (2007), 5443–5457.
39. A. V. Zolotaryuk, *Two-parametric resonant tunneling across the $\delta'(x)$ potential*, Advanced Science Letters **1** (2008), no. 2, 187–191.
40. A. V. Zolotaryuk, *Point interactions of the dipole type defined through a three-parametric power regularization*, J. Phys. A: Math. Theor. **43** (2010), no. 10, 105302, 21 p.
41. A. V. Zolotaryuk, *Boundary conditions for the states with resonant tunnelling across the δ'-potential*, Phys. Lett. A **374** (2010), no. 15-16, 1636–1641.

Institute of Mathematics, TU Clausthal, Clausthal, Germany
E-mail address: `johannes.brasche@tu-clausthal.de`

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: `nizhnik@imath.kiev.ua`

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