Methods of Functional Analysis and Topology
Vol. 19 (2013), no. 1, pp. 55–67

ON THE RANGE AND KERNEL OF TOEPLITZ AND LITTLE HANKEL OPERATORS

NAMITA DAS AND PABITRA KUMAR JENA

Abstract. In this paper we study the interplay between the range and kernel of Toeplitz and little Hankel operators on the Bergman space. Let T_ϕ denote the Toeplitz operator on $L^2_a(\mathbb{D})$ with symbol $\phi \in L^\infty(\mathbb{D})$ and S_ψ denote the little Hankel operator with symbol $\psi \in L^\infty(\mathbb{D})$. We have shown that if Ran $(T_\phi) \subseteq$ Ran (S_ψ) then $\phi \equiv 0$ and find necessary and sufficient conditions for the existence of a positive bounded linear operator X defined on the Bergman space such that $T_\phi X = S_\psi$ and Ran $(S_\psi) \subseteq$ Ran (T_ϕ). We also obtain necessary and sufficient conditions on $\psi \in L^\infty(\mathbb{D})$ such that Ran (T_ψ) is closed.

1. Introduction

Let $dA(z)$ be the Lebesgue area measure on the open unit disk $\mathbb{D}$ normalized so that the measure of the disk $\mathbb{D}$ equals 1. The Bergman space $L^2_a(\mathbb{D})$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dA)$. For $z \in \mathbb{D}$, the Bergman reproducing kernel is the function $K_z \in L^2_a(\mathbb{D})$ such that $f(z) = \langle f, K_z\rangle$ for every $f \in L^2_a(\mathbb{D})$. The normalized reproducing kernel k_z is the function $\frac{K_z}{\|K_z\|_2}$. Here the norm $\|\cdot\|_2$ and the inner product $\langle,\rangle$ are taken in the space $L^2(\mathbb{D}, dA)$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. The sequence $\{e_n\}_{n=0}^\infty$ forms an orthonormal basis for $L^2_a(\mathbb{D})$. Let

$$K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1 - z\overline{w})^2} = \sum_{n=0}^\infty e_n(z)\overline{e_n(w)}.$$

For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $L^2_a(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. For $\phi \in L^\infty(\mathbb{D})$, the big Hankel operator H_ϕ is a mapping from $L^2_a(\mathbb{D})$ into $\left(L^2_a(\mathbb{D})\right)^\perp$ defined by $H_\phi f = (I - P)(\phi f)$ for $f \in L^2_a(\mathbb{D})$. Let $\overline{L^2_a(\mathbb{D})}$ be the space of conjugate analytic functions in $L^2(\mathbb{D}, dA)$. Clearly, $\overline{L^2_a(\mathbb{D})} = \{\overline{f} : f \in L^2_a(\mathbb{D})\}$ is closed in $L^2(\mathbb{D}, dA)$. For $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator h_ϕ is a mapping from $L^2_a(\mathbb{D})$ into $\overline{L^2_a(\mathbb{D})}$ defined by $h_\phi f = \overline{P}(\phi f), f \in L^2_a(\mathbb{D})$ where $\overline{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $\overline{L^2_a(\mathbb{D})}$.

There are also many equivalent ways of defining little Hankel operators. For example, define the map S_ϕ from $L^2_a(\mathbb{D})$ into $L^2_a(\mathbb{D})$ by $S_\phi f = PJ(\phi f)$, where J is the self-adjoint, unitary mapping from $L^2(\mathbb{D}, dA)$ into itself given by $Jh(z) = h(\overline{z})$. Notice that $JS_\phi = h_\phi$. Thus S_ϕ is unitarily equivalent to h_ϕ.

Let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let Aut$(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in Aut$(\mathbb{D})$ such that

(i) $(\phi_a \; o \; \phi_a)(z) \equiv z$;

2000 *Mathematics Subject Classification.* 47B38, 47B35.

Key words and phrases. Toeplitz operators, little Hankel operators, Bergman space, inner functions, range and kernel of operators.

(ii) $\phi_a(0) = a, \phi_a(a) = 0$;

(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\overline{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a(z)} = |k_a(z)|^2 = \frac{\left(1-|a|^2\right)^2}{|1-\overline{a}z|^4}$. Let $h^\infty(\mathbb{D})$ be the space of bounded harmonic functions on $\mathbb{D}$. Then $h^\infty(\mathbb{D}) \subset L^\infty(\mathbb{D})$. For H a nonzero complex Hilbert space, let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators from the Hilbert space H into itself. Hence $\mathcal{L}(L^2_a(\mathbb{D}))$ is the space of all bounded linear operators from $L^2_a(\mathbb{D})$ into itself. Let $C(\overline{\mathbb{D}})$ be the algebra of complex-valued, continuous functions on $\overline{\mathbb{D}}$, the Euclidean closure of $\mathbb{D}$ and $C_0(\mathbb{D})$ be the subalgebra of $C(\overline{\mathbb{D}})$ consisting of functions f with $f(z) \to 0$ as $|z| \to 1^-$.

In this paper we study the interplay between the range and kernel of little Hankel and Toeplitz operators defined on the Bergman space. In Section 2, we show that Ran $(S_\phi) = L^2_a(\mathbb{D})$ if and only if S_ϕ is bounded below and $\ker(S_\phi) = \{0\}$ if and only if $\ker(S^2_\phi) = \{0\}$. Further if S_ϕ is normal then $\ker(S_\phi) = \ker(S_{\phi^+}) = \ker(S_\phi S_{\phi^+})$ where $\phi^+(z) = \overline{\phi(\overline{z})}$. We then establish that if $T_\phi S_\psi, T_\phi S^2_\psi, T^2_\phi S_\psi$ are positive then $\ker(S_\psi) = \{0\}$ implies $T_\phi \geq 0$ and $\ker(T_\phi) = \{0\}$ implies $S_\psi \geq 0$ and if further $T_\phi S_\psi$ is invertible then $T_\phi \geq 0$ and $S_\psi \geq 0$. Thus if we know a priori that the products $T_\phi S_\psi, T_\phi S^2_\psi, T^2_\phi S_\psi$ are positive and if the kernel of the operators T_ϕ, S_ψ are trivial then the operators T_ϕ, S_ψ are positive. This gives us the motivation to investigate when the products $T_\phi S_\psi$ and $S_\psi T_\phi$ are positive.

We then proceed to show that there is no nonzero Toeplitz and Hankel operators T such that T^k is compact for some $k \in \mathbb{Z}_+$ and $\ker(T) = \ker(T^2)$ and Ran $(T) =$ Ran (T^2). On the other hand, if there is a little Hankel operator S_ψ such that S^k_ψ is compact for some $k \in \mathbb{Z}_+$ and $\ker(S_\psi) = \ker(S^2_\psi)$ and Ran $(S_\psi) =$ Ran (S^2_ψ) then the symbol ψ admits a particular form. More precisely, in this case $\psi = \phi + \overline{\chi}$ where $\chi \in (\overline{L^2_a})^\perp \cap L^\infty(\mathbb{D})$ and $\overline{\phi}$ is a linear combination of the Bergman kernels and some of their derivatives.

We further establish that there is no nonzero little Hankel operator whose range contains the range of a Toeplitz operator and we obtain sufficient conditions on ϕ and ψ such that $\ker(T_\phi) \subset \ker(S_\psi)$. We obtain conditions on $\phi, \psi \in L^\infty(\mathbb{D})$ such that T_ϕ commutes with S_ψ and showed that if S_ψ intertwines T_ϕ and $T_{\overline{\phi}}$ and $\ker(S_\psi)$ is trivial then T_ϕ and $T_{\overline{\phi}}$ are unitarily equivalent.

In Section 3, we show that if $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$ then $\ker(T_{1-\overline{\phi}}) = \{0\}$ if and only if $T^n_{\frac{1+\phi}{2}}$ converges to 0 weakly. We further prove that if $\|\phi\|_\infty \leq 1$ then $T^n_{\frac{1+\phi}{2}}$ converges to 0 in norm if and only if $\ker(T_{1-\overline{\phi}}) = \{0\}$ and Ran $(T_{1-\phi})$ is closed. We find necessary and sufficient conditions for the existence of a positive bounded linear operator X defined on the Bergman space such that $T_\phi X = S_\psi$ and Ran $(S_\psi) \subseteq$ Ran (T_ϕ). We also obtain necessary and sufficient conditions on $\psi \in L^\infty(\mathbb{D})$ such that Ran (T_ψ) is closed.

2. Kernel of little Hankel operators

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. It is well known in case of Hardy space [20] that the Hankel operator S has a nontrivial kernel if and only if the range of S is not dense and in this case the symbol of the Hankel operator S is of the form $z\overline{u}h$ where u is an inner function in $H^\infty(\mathbb{T})$ and $h \in H^\infty(\mathbb{T})$. These results play important roles in deriving the algebraic and asymptotic properties of Toeplitz and Hankel operators. In this paper we investigate whether similar results are possible for operators on the Bergman space.

In the following lemma, we show that Ran $(S_\phi) = L^2_a(\mathbb{D})$ if and only if S_ϕ is bounded below and $\ker(S_\phi)$ is trivial if and only if $\ker(S^2_\phi)$ is trivial.

Lemma 2.1. *Let S_ϕ be a little Hankel operator on $L^2_a(\mathbb{D})$ with symbol $\phi \in L^\infty(\mathbb{D})$. Then the following holds:*

- **(a):** $\ker(S_\phi) = \{0\}$ *if and only if* $\ker(S^2_\phi) = \{0\}$.
- **(b):** Ran $(S_\phi) = L^2_a(\mathbb{D})$ *if and only if S_ϕ is bounded below (i.e., there exists an $\epsilon > 0$ such that $\|S_\phi f\| \geq \epsilon \|f\|$ for all $f \in L^2_a(\mathbb{D})$).*

Proof. **(a)** To prove **(a)** the points to note are the following:
(i) $S^*_\phi = S_{\phi^+}$ where $\phi^+(z) = \overline{\phi(\bar{z})}$.
(ii) $\ker(S_\phi) = \{0\}$ if and only if $\ker(S_{\phi^+}) = \{0\}$.
(iii) $\ker(S^2_\phi) = \{0\}$ if and only if $\ker(S^2_{\phi^+}) = \{0\}$.
These can be verified as follows :
(i) For $f, g \in L^2_a(\mathbb{D})$,

$$\begin{aligned}\langle S^*_\phi f, g\rangle &= \langle f, S_\phi g\rangle = \langle f, PJ(\phi g)\rangle = \langle f, (J\phi)Jg\rangle = \langle \overline{J\phi} f, Jg\rangle = \langle \phi^+ f, Jg\rangle \\ &= \langle J(\phi^+ f), g\rangle = \langle PJ(\phi^+ f), g\rangle = \langle S_{\phi^+} f, g\rangle.\end{aligned}$$

Thus $S^*_\phi = S_{\phi^+}$ where $\phi^+(z) = \overline{\phi(\bar{z})}$.
(ii) Let $f \in \ker S_\phi$. Then $S_\phi f = PJ(\phi f) = 0$. This implies $\phi f \in (\overline{L^2_a(\mathbb{D})})^\perp$. That is, $\int_{\mathbb{D}} \phi(z) f(z) g(z) dA(z) = 0$ for all $g \in L^2_a(\mathbb{D})$. Thus $\int_{\mathbb{D}} \phi^+(z) f^+(z) g^+(z) dA(z) = 0$ for all $g^+ \in L^2_a(\mathbb{D})$ and therefore $\int_{\mathbb{D}} \phi^+(z) f^+(z) g(z) dA(z) = 0$ for all $g \in L^2_a(\mathbb{D})$. Hence $\langle \phi^+ f^+, \bar{g}\rangle = 0$ for all $g \in L^2_a(\mathbb{D})$. That is, $f^+ \in \ker(S_{\phi^+})$. Similarly one can verify that if $f^+ \in \ker(S_{\phi^+})$ then $f \in \ker(S_\phi)$. Thus $f \in \ker(S_\phi)$ if and only if $f^+ \in \ker(S_{\phi^+})$. It hence follows that $\ker(S_\phi) = \{0\}$ if and only if $\ker(S_{\phi^+}) = \{0\}$. This proves (ii).
(iii) Now let $f \in \ker(S^2_\phi)$. Then either $f \in \ker(S_\phi)$ or $f \notin \ker(S_\phi)$.

Case 1: If $f \in \ker(S_\phi)$ then $f^+ \in \ker(S_{\phi^+})$.

Case 2: If $f \notin \ker(S_\phi)$ then since $S^2_\phi f = 0$ hence $(S_\phi f)^+ \in \ker(S_{\phi^+})$. Thus $\ker(S_{\phi^+}) = \{0\}$ implies $\ker(S^2_\phi) = \{0\}$.

Conversely, if $g \in \ker(S_{\phi^+})$, then $g^+ \in \ker(S_\phi)$. Hence $S^2_\phi g^+ = 0$. Thus $\ker(S^2_\phi) = \{0\}$ implies $\ker(S_{\phi^+}) = \{0\}$.

To prove (iii) suppose $\ker(S^2_\phi) = \{0\}$. This happens if and only if $\ker(S_{\phi^+}) = \{0\}$. But from (ii) this is true if and only if $\ker(S_\phi) = \{0\}$. Proceeding similarly one can show that $\ker(S_\phi) = \{0\}$ if and only if $\ker(S^2_{\phi^+}) = \{0\}$.

Thus **(a)** follows.

(b) Suppose Ran $(S_\phi) = L^2_a(\mathbb{D})$. Then $\ker(S^*_\phi) = \{0\}$ and hence from **(a)** it follows that $\ker(S_\phi) = \{0\}$. Thus from bounded inverse theorem [21] and [5], it follows that S_ϕ is bounded below. Conversely, if S_ϕ is bounded below then $\ker(S_\phi) = \{0\}$ and hence from **(a)** it follows that $\ker(S^*_\phi) = \{0\}$. This implies $\overline{\text{Ran } (S_\phi)} = L^2_a(\mathbb{D})$. Since S_ϕ is bounded below and it has dense range, hence from [5], it follows that S_ϕ is invertible and Ran $(S_\phi) = L^2_a(\mathbb{D})$.

This proves **(b)**. □

In Lemma 2.2 we establish that if S_ϕ is normal then $\ker(S_\phi)$ coincides with $\ker(S^*_\phi) = \ker(S_{\phi^+}) = \ker(S_\phi S_{\phi^+})$.

Lemma 2.2. *If S_ϕ is normal then* $\ker(S_\phi) = \ker(S_{\phi^+}) = \ker(S_\phi S_{\phi^+})$ *where* $\phi^+(z) = \overline{\phi(\overline{z})}$.

Proof. Suppose S_ϕ is normal. Let $f \in \ker(S_\phi)$. Then $S_\phi S_{\phi^+} f = S_{\phi^+} S_\phi f = 0$. Thus $S_{\phi^+} f \in \ker(S_\phi) = (\overline{\text{Ran } (S_{\phi^+})})^\perp$. But $S_{\phi^+} f \in \text{Ran } (S_{\phi^+}) \subseteq \overline{\text{Ran } (S_{\phi^+})}$. Hence $S_{\phi^+} f = 0$. That is, $f \in \ker(S_{\phi^+})$. Therefore

$$\ker(S_\phi) \subset \ker(S_{\phi^+}). \tag{2.1}$$

Conversely, let $f \in \ker(S_{\phi^+})$. This implies, $f^+ \in \ker(S_\phi)$. From (2.1) it follows that, $f^+ \in \ker(S_{\phi^+})$. Therefore $f \in \ker(S_\phi)$. Hence

$$\ker(S_{\phi^+}) \subset \ker(S_\phi). \tag{2.2}$$

From (2.1) and (2.2), we obtain $\ker(S_\phi) = \ker(S_{\phi^+})$.

We shall now show that $\ker(S_{\phi^+}) = \ker(S_\phi S_{\phi^+})$. Let $f \in \ker(S_{\phi^+})$. This implies, $S_{\phi^+} f = 0$. That is, $S_\phi S_{\phi^+} f = 0$. So $f \in \ker(S_\phi S_{\phi^+})$. Therefore, $\ker(S_{\phi^+}) \subseteq \ker(S_\phi S_{\phi^+})$.

Now let $f \in \ker(S_\phi S_{\phi^+})$. This implies, $S_\phi S_{\phi^+} f = 0$. Then $S_{\phi^+} f \in \ker(S_\phi) = (\overline{\text{Ran }(S_{\phi^+})})^\perp$. But $S_{\phi^+} f \in \text{Ran }(S_{\phi^+}) \subseteq \overline{\text{Ran }(S_{\phi^+})}$. Hence $S_{\phi^+} f = 0$. That is, $f \in \ker(S_{\phi^+})$. Therefore, $\ker(S_\phi S_{\phi^+}) \subseteq \ker(S_{\phi^+})$. Hence $\ker(S_{\phi^+}) = \ker(S_\phi S_{\phi^+})$. □

Using Lemma 2.1 and Lemma 2.2, we now prove the following proposition which gives us sufficient conditions for T_ϕ and S_ψ to be positive.

Proposition 2.3. *Let $\phi, \psi \in L^\infty(\mathbb{D})$. Suppose $T_\phi S_\psi \geq 0, T_\phi^2 S_\psi \geq 0, T_\phi S_\psi^2 \geq 0$.*

(i) *If $\ker(S_\psi) = \{0\}$ then $T_\phi \geq 0$. Similarly if $\ker(T_\phi) = \{0\}$ then $S_\psi \geq 0$.*

(ii) *If $T_\phi S_\psi$ is invertible then S_ψ is similar to S_{ψ^+}, $T_\phi \geq 0$ and $S_\psi \geq 0$. Here $\psi^+(z) = \overline{\psi(\overline{z})}$.*

Proof. (i) From Lemma 2.1 it follows that $\ker(S_\psi) = \{0\}$ if and only if $\ker(S_\psi^*) = \{0\}$. Since $T_\phi S_\psi \geq 0$, hence $T_\phi S_\psi^2 = (T_\phi S_\psi^2)^* = S_\psi^*(T_\phi S_\psi)^* = S_\psi^* T_\phi S_\psi$. Thus we have, for all $f \in L_a^2(\mathbb{D})$,

$$\langle T_\phi S_\psi f, S_\psi f\rangle = \langle S_\psi^* T_\phi S_\psi f, f\rangle = \langle T_\phi S_\psi^2 f, f\rangle \geq 0.$$

Now since $\ker(S_\psi) = \{0\}$, we obtain $\ker(S_\psi^*) = \{0\}$. Hence $\overline{\text{Ran }(S_\psi)} = (\ker(S_\psi^*))^\perp = \{0\}^\perp = L_a^2(\mathbb{D})$. It therefore follows that $T_\phi \geq 0$.

Further $T_\phi^2 S_\psi = (T_\phi^2 S_\psi)^* = (T_\phi T_\phi S_\psi)^* = (T_\phi S_\psi)^* T_\phi^* = T_\phi S_\psi T_\phi^*$. Hence

$$\langle S_\psi T_\phi^* f, T_\phi^* f\rangle = \langle T_\phi S_\psi T_\phi^* f, f\rangle = \langle T_\phi^2 S_\psi f, f\rangle \geq 0.$$

Now if $\ker(T_\phi) = \{0\}$ then $\overline{\text{Ran }(T_\phi^*)} = L_a^2(\mathbb{D})$. Thus it follows that, $S_\psi \geq 0$.

To prove (ii) assume $T_\phi S_\psi$ is invertible. This implies $\ker(T_\phi S_\psi) = \{0\}$. Notice that $\ker(S_\psi) \subset \ker(T_\phi S_\psi)$. Hence $\ker(S_\psi) = \{0\}$. Therefore $\ker(S_\psi^*) = \{0\}$. This implies $\overline{\text{Ran }(S_\psi)} = L_a^2(\mathbb{D})$. Now $S_\psi = (T_\phi S_\psi)^{-1} T_\phi S_\psi^2 = (T_\phi S_\psi)^{-1} S_\psi^* T_\phi S_\psi$. Thus S_ψ is similar to $S_\psi^* = S_{\psi^+}$ and $\overline{\text{Ran }(S_\psi)} = (\ker(S_\psi^*))^\perp = (\ker(S_\psi))^\perp = L_a^2(\mathbb{D})$. From (i) it follows that $T_\phi \geq 0$. Since $\ker(T_\phi^*) \subset \ker(S_\psi^* T_\phi^*) = \{0\}$, we obtained that T_ϕ^* is injective. Now $T_\phi^* = (T_\phi S_\psi)^{-1}(T_\phi S_\psi) T_\phi^* = (T_\phi S_\psi)^{-1} T_\phi S_\psi T_\phi^* = (T_\phi S_\psi)^{-1} T_\phi^2 S_\psi = (T_\phi S_\psi)^{-1} T_\phi (T_\phi S_\psi)$. Hence T_ϕ^* is similar to T_ϕ. Since $\ker(T_\phi^*) = \{0\}$, hence $\ker(T_\phi) = \{0\}$. This implies $\overline{\text{Ran }(T_\phi^*)} = L_a^2(\mathbb{D})$. From (i) it follows that $S_\psi \geq 0$. □

In the following theorem we show that there is no nonzero Toeplitz and Hankel operators T such that T^k is compact for some $k \in \mathbb{Z}_+$ and $\ker(T) = \ker(T^2)$ and $\text{Ran }(T) = \text{Ran }(T^2)$. On the other hand, if there is a little Hankel operator S_ψ such that S_ψ^k is compact for some $k \in \mathbb{Z}_+$ and $\ker(S_\psi) = \ker(S_\psi^2)$ and $\text{Ran }(S_\psi) = \text{Ran }(S_\psi^2)$ then the symbol ψ admits a particular form. More precisely, in this case $\psi = \phi + \overline{\chi}$ where $\chi \in (\overline{L_a^2})^\perp \cap L^\infty(\mathbb{D})$ and $\overline{\phi}$ is a linear combination of the Bergman kernels and some of their derivatives.

Theorem 2.4. *If $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ is such that $\ker(T) = \ker(T^2)$, $\text{Ran }(T) = \text{Ran }(T^2)$ and T^k is compact for some $k \in \mathbb{Z}_+$, then the following holds:*

- **(i):** *If $T = T_\phi$ for some $\phi \in L^\infty(\mathbb{D})$ then $\phi \equiv 0$.*
- **(ii):** *If $T = H_\phi$ for some $\phi \in L^\infty(\mathbb{D})$ then $\phi \in H^\infty(\mathbb{D})$.*
- **(iii):** *If $T = S_\psi$ for some $\psi \in L^\infty(\mathbb{D})$ then $\psi = \phi + \chi$ where $\chi \in (\overline{L_a^2})^\perp \cap L^\infty(\mathbb{D})$ and $\overline{\phi}$ is a linear combination of the Bergman kernels and some of their derivatives.*

Proof. **(i)** Suppose $T = T_\phi$ for some $\phi \in L^\infty(\mathbb{D})$. By our hypothesis it follows from [21] that $L^2_a(\mathbb{D}) = \ker(T_\phi) \oplus \text{Ran } (T_\phi)$. Since $\ker(T_\phi)$ is always a closed subspace of $L^2_a(\mathbb{D})$, we obtain from [9] that Ran (T_ϕ) is closed. But Ran $(T_\phi) =$ Ran (T^m_ϕ) for all integers $m \geq 1$. This can be verified by Mathematical induction.

We shall only verify for $m = 3$. Let $g \in$ Ran (T^3_ϕ). That implies, $g = T^3_\phi f = T_\phi(T^2_\phi f)$ for some $f \in L^2_a(\mathbb{D})$. That is, $g \in$ Ran $(T_\phi) =$ Ran (T^2_ϕ). Therefore, Ran $(T^3_\phi) \subset$ Ran (T^2_ϕ). Conversely, let $g \in$ Ran (T^2_ϕ). That implies, $g = T^2_\phi h = T_\phi(T_\phi h)$. Since $T_\phi h \in$ Ran $(T_\phi) =$ Ran (T^2_ϕ). This implies, $T_\phi h = T^2_\phi k$ for some $k \in L^2_a(\mathbb{D})$. That is, $g = T_\phi(T_\phi h) = T_\phi(T^2_\phi k) = T^3_\phi k$. So $g \in$ Ran (T^3_ϕ). Therefore, Ran $(T^2_\phi) \subset$ Ran (T^3_ϕ). Hence Ran $(T^3_\phi) =$ Ran (T^2_ϕ).

Using induction one can show that Ran $(T^m_\phi) =$ Ran (T_ϕ) for all $m \geq 1$. But T^k_ϕ is compact for some $k \in \mathbb{Z}_+$. Hence [5] Ran (T^k_ϕ) shall not contain any closed infinite dimensional subspace of $L^2_a(\mathbb{D})$. Thus Ran (T_ϕ) is a finite dimensional subspace of $L^2_a(\mathbb{D})$ and T_ϕ is a finite rank operator. It then follows from [17] that $\phi \equiv 0$.

(ii) Let $T = H_\phi$, the big Hankel operator with symbol $\phi \in L^\infty(\mathbb{D})$. Proceeding similarly as in **(i)**, one can show that H_ϕ is a finite rank operator. Thus $\ker(H_\phi) \neq \{0\}$ and hence it is clear from [7] that $\ker(H_\phi) = L^2_a(\mathbb{D})$ and $\phi \in H^\infty(\mathbb{D})$.

(iii) Let $T = S_\psi$, the little Hankel operator with symbol $\psi \in L^\infty(\mathbb{D})$. Proceeding similarly as in **(i)**, one can show that S_ψ is a finite rank operator. Therefore [7] $\psi = \phi + \chi$ where $\chi \in (\overline{L^2_a})^\perp \cap L^\infty(\mathbb{D})$ and $\overline{\phi}$ is a linear combination of the Bergman kernels and some of their derivatives. □

In Theorem 2.5, we show that the range of a nonzero little Hankel operator can never contain the range of a Toeplitz operator and if Ran $(S_\phi) \subseteq$ Ran (T_ϕ) then $\|P(\overline{\phi}Jf)\|^2 \leq c\|P(\overline{\phi}f)\|^2$ for some constant $c > 0$ and for all $f \in L^2_a(\mathbb{D})$.

Theorem 2.5. *Let $\phi, \psi \in L^\infty(\mathbb{D})$. Then the following holds:*

- **(i):** Ran $(T_\phi) \subseteq$ Ran (S_ψ) *if and only if* $\phi \equiv 0$.
- **(ii):** *If* Ran $(S_\phi) \subseteq$ Ran (T_ϕ) *then* $\|P(\overline{\phi}Jf)\|^2 \leq c\|P(\overline{\phi}f)\|^2$ *for all* $f \in L^2_a(\mathbb{D})$ *and for some constant* $c > 0$.

Proof. **(i)** If $\phi = 0$, then $T_\phi = 0$. Hence Ran $(T_\phi) = \{0\} \subseteq$ Ran (S_ψ).

Suppose Ran $(T_\phi) \subseteq$ Ran (S_ψ). By [6] there is a constant $c > 0$ such that $T_\phi T^*_\phi \leq cS_\psi S^*_\psi$. Hence $\langle T_\phi T^*_\phi f, f\rangle \leq c\langle S_\psi S^*_\psi f, f\rangle$ for all $f \in L^2_a(\mathbb{D})$. That is,

$$\begin{aligned}\|T_{\overline{\phi}}f\|^2 &\leq c\|S^*_\psi f\| = c\|S_{\psi^+}f\|^2 = c\|Jh_{\psi^+}f\|^2 = c\|h_{\psi^+}f\|^2\\ &\leq c\|H_{\psi^+}f\|^2 \quad \text{for all} \quad f \in L^2_a(\mathbb{D}).\end{aligned}$$

Thus

$$\begin{aligned}c^{-1}\|P(\overline{\phi}f)\|^2 &\leq \|H_{\psi^+}f\|^2 = \|(I-P)(\psi^+f)\|^2\\ &= \big\langle \psi^+f - P(\psi^+f), \psi^+f - P(\psi^+f)\big\rangle = \|\psi^+f\|^2 - \|P(\psi^+f)\|^2.\end{aligned}$$

Hence $c^{-1}\|P(\overline{\phi}f)\|^2 + \|P(\psi^+f)\|^2 \leq \|\psi^+f\|^2 \leq \|\psi^+\|^2_\infty\|f\|^2$. This implies

$$c^{-1}\frac{\|P(\overline{\phi}f)\|^2}{\|f\|^2} + \frac{\|P(\psi^+f)\|^2}{\|f\|^2} \leq \|\psi^+\|^2_\infty \quad \text{for all} \quad f \in L^2_a(\mathbb{D}).$$

Thus $c^{-1}\|\overline{\phi}\|^2_\infty + \|\psi^+\|^2_\infty \leq \|\psi^+\|^2_\infty$. Hence $\|\overline{\phi}\|^2_\infty = 0$ and $\phi \equiv 0$.

(ii) If Ran $(S_\phi) \subseteq$ Ran (T_ϕ) then by [6] we have $S_\phi S^*_\phi \leq cT_\phi T^*_\phi$ for some constant $c > 0$. That is,

$$\|S_{\phi^+}f\|^2 = \langle S_\phi S^*_\phi f, f\rangle \leq c\langle T_\phi T^*_\phi f, f\rangle = c\|T_{\overline{\phi}}f\|^2 \quad \text{for all} \quad f \in L^2_a(\mathbb{D}).$$

Hence $\|PJ(\phi^+f)\|^2 \leq c\|P(\overline{\phi}f)\|^2$ for all $f \in L^2_a(\mathbb{D})$. Thus $\|P(\overline{\phi}Jf)\|^2 \leq c\|P(\overline{\phi}f)\|^2$ for all $f \in L^2_a(\mathbb{D})$. □

Definition 1. A function $G \in L^2_a(\mathbb{D})$ is called an inner function in $L^2_a(\mathbb{D})$ if $|G|^2 - 1$ is orthogonal to $H^\infty(\mathbb{D})$.

For more details about Bergman space inner divisors, see [14]. In the following theorem, we find conditions on $\phi, \psi \in L^\infty(\mathbb{D})$ such that $\ker(T_\phi) \subseteq \ker(S_\psi)$.

Theorem 2.6. *Suppose $\phi \in L^\infty(\mathbb{D})$ is such that $\ker(T_\phi) = \{0\}$. Let*

$$\overline{\psi}(z) = \sum_{j=1}^{N} \sum_{v=0}^{m_j-1} c_{jv} \frac{\partial^v}{\partial \overline{b_j^v}} K_{b_j}(z),$$

where $\boldsymbol{b} = \{b_j\}_{j=1}^N$ is a finite set of points in $\mathbb{D}$, $c_{jv} \neq 0$ for all j, v and m_j is the number of times b_j appears in $\boldsymbol{b}$. Let $S_\psi T_\phi = T_\phi S_\psi$. Then there exists an inner function $G \in H^\infty(\mathbb{D})$ such that $\ker(T^*_\phi) \subseteq \ker(S^*_\psi) = GL^2_a(\mathbb{D})$.

Proof. Since $\overline{\psi}(z) = \sum_{j=1}^N \sum_{v=0}^{m_j-1} c_{jv} \frac{\partial^v}{\partial \overline{b_j^v}} K_{b_j}(z)$ where $\mathbf{b} = \{b_j\}_{j=1}^N$ is a finite set of points in $\mathbb{D}$, $c_{jv} \neq 0$ for all j, v and m_j is the number of times b_j appears in $\mathbf{b}$, hence the operator S_ψ is a [7] finite rank operator on $L^2_a(\mathbb{D})$ and there exists an inner function $G \in H^\infty(\mathbb{D})$ such that $\ker(S^*_\psi) = GL^2_a(\mathbb{D})$. Thus there exists a system of linearly independent vectors $\zeta_i, i = 1, 2, \dots, n$ and a system of nonzero bounded linear functionals ϕ_i for $i = 1, 2, \dots, n$ on $L^2_a(\mathbb{D})$ such that

$$S_\psi f = \sum_{i=1}^{n} \phi_i(f)\zeta_i, f \in L^2_a(\mathbb{D}).$$

Moreover,

$$\sum_{i=1}^{n} \phi_i(f) T_\phi \zeta_i = T_\phi S_\psi f = S_\psi T_\phi f = \sum_{i=1}^{n} \phi_i(T_\phi f)\zeta_i, f \in L^2_a(\mathbb{D}).$$

On the other hand, since T_ϕ is injective, it is clear that the vectors $T_\phi \zeta_i, i = 1, 2, \dots, n$ are linearly independent. Hence $S_\psi f \in \operatorname{span}\{\zeta_1, \zeta_2, \dots, \zeta_n\} = \operatorname{span}\{T_\phi\zeta_1, \dots, T_\phi\zeta_n\}$ for all $f \in L^2_a(\mathbb{D})$. Thus $\overline{\text{Ran } (S_\psi)} = \text{Ran } (S_\psi) \subseteq \text{Ran } (T_\phi) \subseteq \overline{\text{Ran } (T_\phi)}$ and therefore $\ker(T^*_\phi) = (\overline{\text{Ran } (T_\phi)})^\perp \subseteq (\overline{\text{Ran } (S_\psi)})^\perp = \ker(S^*_\psi) = GL^2_a(\mathbb{D})$ for some inner [7] function $G \in H^\infty(\mathbb{D})$. $\square$

Definition 2. An operator A defined on a Hilbert space H is said to be hyponormal if and only if $A^*A - AA^* \geq 0$.

In the following theorem we find conditions on $\phi \in L^\infty(\mathbb{D})$ such that T_ϕ commutes with S_ψ where S_ψ is a positive little Hankel operator on $L^2_a(\mathbb{D})$ with trivial kernel.

Theorem 2.7. *Let $\phi \in C_0(\mathbb{D}), \|\phi\|_\infty \leq 1$. Suppose $\psi \in L^\infty(\mathbb{D})$ and S_ψ is a positive little Hankel operator on $L^2_a(\mathbb{D})$ such that $\ker(S_\psi) = \{0\}$ and $S_\psi \leq T_{\overline{\phi}} S_\psi T_\phi$. Then T_ϕ is unitary and $T_\phi S_\psi = S_\psi T_\phi$.*

Proof. The operator $S_\psi^{\frac{1}{2}} T_\phi$ is compact [23] since $\phi \in C_0(\mathbb{D})$. By Lemma 2.1, $\ker(S^*_\psi) = \{0\}$. Let $S_\psi^{\frac{1}{2}} T_\phi = A$. Then

$$AA^* = S_\psi^{\frac{1}{2}} T_\phi T_{\overline{\phi}} S_\psi^{\frac{1}{2}} \leq S_\psi.$$

Now

$$0 \leq T_{\overline{\phi}} S_\psi T_\phi - S_\psi \leq T_{\overline{\phi}} S_\psi T_\phi - S_\psi^{\frac{1}{2}} T_\phi T_{\overline{\phi}} S_\psi^{\frac{1}{2}} = A^*A - AA^*.$$

Thus the operator A is hyponormal and as A is compact, the [10] operator A is normal. Therefore,

$$S_\psi = T_{\overline{\phi}} S_\psi T_\phi = S_\psi^{\frac{1}{2}} T_\phi T_{\overline{\phi}} S_\psi^{\frac{1}{2}}$$

and $T_{\overline{\phi}}$ is an isometry in $\overline{\text{Ran }(S_\psi)} = (\ker(S_\psi^*))^\perp = \{0\}^\perp = L_a^2(\mathbb{D})$. Further, S_ψ commutes with T_ϕ and $T_{\overline{\phi}}$. It follows therefore that

$$T_{\overline{\phi}}T_\phi S_\psi = T_{\overline{\phi}}S_\psi T_\phi = S_\psi = S_\psi T_\phi T_{\overline{\phi}}.$$

Hence T_ϕ is unitary. □

In Theorem 2.8, we show that if S_ψ is a little Hankel operator on $L_a^2(\mathbb{D})$ with trivial kernel and S_ψ intertwines T_ϕ and $T_{\overline{\phi}}$ then T_ϕ and $T_{\overline{\phi}}$ are unitarily equivalent.

Theorem 2.8. *Suppose $T_\phi S_\psi = S_\psi T_\phi^*$ and $T_\phi^* S_\psi = S_\psi T_\phi$ and $\ker(S_\psi) = \{0\}, \phi, \psi \in L^\infty(\mathbb{D})$. Then T_ϕ and $T_{\overline{\phi}}$ are unitarily equivalent. Further, $T_\phi^* T_\phi$ and $T_\phi T_\phi^*$ are unitarily equivalent.*

Proof. $T_\phi S_\psi S_\psi^* = S_\psi T_\phi^* S_\psi^* = S_\psi S_\psi^* T_\phi$. Thus T_ϕ commutes with $S_\psi S_\psi^*$. Proceeding similarly as in Lemma 2.2, one can verify that

$$\overline{\text{Ran }(S_\psi)} = (\ker(S_\psi^*))^\perp = (\ker(S_\psi S_\psi^*))^\perp = \overline{\text{Ran }(S_\psi S_\psi^*)}.$$

This can also be verified as follows: Let $g \in \ker(S_{\psi^+})$. Then $S_\psi S_{\psi^+} g = 0$ and $g \in \ker(S_\psi S_{\psi^+})$. Further, let $h \in \ker(S_\psi S_{\psi^+})$. Then $S_{\psi^+} h \in \ker(S_\psi)$. But $S_{\psi^+} h \in \text{Ran }(S_{\psi^+})$ $= \text{Ran }(S_\psi^*) \subset \overline{\text{Ran }(S_\psi^*)} = (\ker(S_\psi))^\perp$. Hence $S_{\psi^+} h = 0$ and $h \in \ker(S_{\psi^+})$. Thus we verify that $\ker(S_{\psi^+}) = \ker(S_\psi S_{\psi^+})$.

We now show that $T_\phi(\overline{\text{Ran }(S_\psi S_\psi^*)}) \subset \overline{\text{Ran }(S_\psi S_\psi^*)}$. Let $g \in \text{Ran }(S_\psi S_\psi^*)$. Then $g = S_\psi S_\psi^* f$ for some $f \in L_a^2(\mathbb{D})$. Hence

$$T_\phi g = T_\phi S_\psi S_\psi^* f = S_\psi S_\psi^* T_\phi f \in \text{Ran }(S_\psi S_\psi^*).$$

Thus $T_\phi(\text{Ran }(S_\psi S_\psi^*)) \subset \text{Ran }(S_\psi S_\psi^*)$. Now let $g \in \overline{\text{Ran }(S_\psi S_\psi^*)}$ and $g = \lim_{n\to\infty} g_n, g_n \in \text{Ran }(S_\psi S_\psi^*)$. Then $T_\phi g = \lim_{n\to\infty} T_\phi g_n$ and $T_\phi g_n \in \text{Ran }(S_\psi S_\psi^*)$. Therefore, $T_\phi g \in \overline{\text{Ran }(S_\psi S_\psi^*)}$. Thus $T_\phi(\overline{\text{Ran }(S_\psi S_\psi^*)}) \subset \overline{\text{Ran }(S_\psi S_\psi^*)}$.

We now proceed to verify that $T_\phi(\ker(S_\psi S_\psi^*)) \subset \ker(S_\psi S_\psi^*)$. Let $g \in \ker(S_\psi S_\psi^*)$. Then $S_\psi S_\psi^* g = 0$. Hence $T_\phi S_\psi S_\psi^* g = 0$. This implies, $S_\psi S_\psi^* T_\phi g = 0$. That is, $T_\phi g \in \ker(S_\psi S_\psi^*)$. Thus $\overline{\text{Ran }(S_\psi)}$ is a reducing subspace of T_ϕ. Proceeding similarly one can show that T_ϕ commutes $S_\psi^* S_\psi$ and $\ker(S_\psi) = \ker(S_\psi^* S_\psi)$ reduces T_ϕ.

Further, let $S_\psi = VQ$ be the polar decomposition of S_ψ such that $\ker(V) = \ker(Q)$. Here V is the partial isometry and Q is the positive operator. Let $f \in \ker(Q)$. Then $Qf = 0$ and therefore $S_\psi f = VQf = 0$. Hence $f \in \ker(S_\psi) = \{0\}$. Thus $f = 0$ and $\ker(V) = \ker(Q) = \{0\}$. Since $V^*Vf = f, f \in (\ker(V))^\perp = \{0\}^\perp = L_a^2(\mathbb{D})$, hence V is an isometry.

Since $T_\phi S_\psi^* S_\psi = S_\psi^* S_\psi T_\phi$, we obtain $T_\phi QV^*VQ = QV^*VQT_\phi$. Thus $T_\phi Q^2 = Q^2 T_\phi$ and therefore [5], we have $T_\phi Q = QT_\phi$ and $QT_\phi^* = T_\phi^* Q$. Now $T_\phi S_\psi = S_\psi T_\phi^*$ implies $T_\phi VQ = VQT_\phi^* = VT_\phi^* Q$. Thus

$$(T_\phi V - VT_\phi^*)Qf = 0 \quad \text{for all} \quad f \in L_a^2(\mathbb{D}). \tag{2.3}$$

Notice that $(\overline{\text{Ran }(S_\psi)})^\perp = \ker(S_\psi^*) = \{0\}$, hence $\overline{\text{Ran }(S_\psi)} = L_a^2(\mathbb{D})$ and $\overline{\text{Ran }(Q)} = (\ker(Q))^\perp = (\ker(S_\psi))^\perp = \{0\}^\perp = L_a^2(\mathbb{D})$. From equation (2.3) it follows that Ran $(Q) \subset \ker(T_\phi V - VT_\phi^*)$.

Let $h \in \overline{\text{Ran }(Q)}$ and $h = \lim_{n\to\infty} h_n$ where $h_n \in \text{Ran }(Q)$. Then

$$(T_\phi V - VT_\phi^*)h = (T_\phi V - VT_\phi^*)(\lim_{n\to\infty} h_n) = \lim_{n\to\infty}(T_\phi V - VT_\phi^*)h_n = 0.$$

Thus $L_a^2(\mathbb{D}) = \overline{\text{Ran }(Q)} \subset \ker(T_\phi V - VT_\phi^*)$ and $T_\phi V = VT_\phi^*$. Similarly, since $T_\phi^* S_\psi = S_\psi T_\phi$ we obtain $T_\phi^* V = VT_\phi$. Thus $V^* T_\phi V = T_{\overline{\phi}}$.

Since $S_\psi = VQ$, we have $S_\psi^* = S_{\psi^+} = QV^*$. Let $f \in \ker V^*$. Then $V^* f = 0$ and hence $S_\psi^* f = QV^* f = 0$. Thus by Lemma 2.1, $f \in \ker(S_\psi^*) = \{0\}$ and $f = 0$. Thus

$\ker(V^*) = \{0\}$. Therefore $\{0\} = \ker(V^*) = (\text{Ran }(V))^{\perp}$ and $\overline{\text{Ran }(V)} = L^2_a(\mathbb{D})$. Since $\|Vf\| = \|f\|$, hence V is bounded below with dense range. By [5], V is invertible. Therefore T_ϕ and T^*_ϕ are unitarily equivalent. Further, $T^*_\phi T_\phi = V^* T_\phi T^*_\phi V$ and $T^*_\phi T_\phi$ and $T_\phi T^*_\phi$ are unitarily equivalent. □

3. Range and kernel of Toeplitz operators

A well known lemma attributed to Coburn [5] states that a bounded Toeplitz operator with nontrivial kernel acting on the Hardy space must have dense range. That is, if ϕ is a function in $L^\infty(\mathbb{T})$ not almost everywhere zero, then either $\ker(T_\phi) = \{0\}$ or $\ker(T^*_\phi) = \{0\}$. Vukotic [22] showed that the range of a nonzero Toeplitz operator with $\ker(T_\phi) \neq \{0\}$ must contain all polynomials. Further, if $\ker(T_\phi) \neq \{0\}$ then [11] $\ker(T_\phi) = g(H^2(\mathbb{T}) \ominus z\theta H^2(\mathbb{T}))$ where g is an outer function and θ is an inner function in $H^\infty(\mathbb{T})$.

In this section we show that if $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$ then $\ker(T_{1-\overline{\phi}}) = \{0\}$ if and only if $T^n_{\frac{1+\phi}{2}}$ converges to 0 weakly. We further prove that if $\|\phi\|_\infty \leq 1$ then $T^n_{\frac{1+\phi}{2}}$ converges to 0 in norm if and only if $\ker(T_{1-\overline{\phi}}) = \{0\}$ and Ran $(T_{1-\phi})$ is closed. We find necessary and sufficient conditions for the existence of a positive bounded linear operator X defined on the Bergman space such that $T_\phi X = S_\psi$ and Ran $(S_\psi) \subseteq$ Ran (T_ϕ). We also obtain necessary and sufficient conditions on $\psi \in L^\infty(\mathbb{D})$ such that Ran (T_ψ) is closed.

Theorem 3.1. *Let $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$ and* Ran $(T_{1-\phi})$ *be closed. If* $\ker(T_{1-\phi}) \oplus$ Ran $(T_{1-\phi})$ *is closed then there exists a closed complementary subspace M of* $\ker(T_{1-\phi})$ *containing* Ran $(T_{1-\phi})$.

Proof. First we shall show that $\ker(T_{1-\phi}) \cap \text{Ran }(T_{1-\phi}) = \{0\}$. Let $V = T_{\frac{1+\phi}{2}}$. Then the range and the kernel of the operator $I - V$ coincide with those of $I - T_\phi$. Let $g \in \ker(I - V) \cap \text{Ran }(I - V)$. Since $(I - V)g = 0$, that is, $Vg = g$, we have $V^n g = g$ for every n. Further $(I - V)f = g$ for some $f \in L^2_a(\mathbb{D})$, that is, $g = f - Vf$. Hence $g = V^n f - V^{n+1} f$. By [12], $\|V^n f - V^{n+1} f\| \longrightarrow 0$ as $n \to \infty$, which implies that $g = 0$. So $\ker(I - V) \cap \text{Ran }(I - V) = \{0\}$. Thus $\ker(I - T_\phi) \cap \text{Ran }(I - T_\phi) = \{0\}$.

Let $L = (\ker(T_{1-\phi}) \oplus \text{Ran }(T_{1-\phi}))^\perp$ be the orthogonal complement of the closed subspace $\ker(T_{1-\phi}) \oplus \text{Ran }(T_{1-\phi})$. Then $L^2_a(\mathbb{D}) = (\ker(T_{1-\phi}) \oplus \text{Ran }(T_{1-\phi})) \oplus L$. Hence Ran $(T_{1-\phi}) \oplus L$ is closed and $M = \text{Ran }(T_{1-\phi}) \oplus L$ is the desired complementary subspace of $\ker(T_{1-\phi})$. □

Let $L^2(\mathbb{R})$ be the usual Lebesgue space considered with the Lebesgue measure. Since both the infinite dimensional Hilbert spaces $L^2_a(\mathbb{D})$ and $L^2(\mathbb{R})$ are separable, they are isomorphic. Therefore, there exists a unitary map U from $L^2_a(\mathbb{D})$ onto $L^2(\mathbb{R})$.

For each $n \in \mathbb{N}$, define the operator $\breve{L}_n$ on $L^2(\mathbb{R})$ by $(\breve{L}_n f)(s) := e^{\frac{iq(s)}{n}} f(s), s \in \mathbb{R}, f \in L^2(\mathbb{R})$ where $q : \mathbb{R} \longrightarrow [0, 1]$ is strictly monotone. It is not difficult to see that

$$\begin{aligned} \|\breve{L}_n - I_{\mathcal{L}(L^2(\mathbb{R}))}\| &= \sup_{s \in \mathbb{R}} |e^{\frac{iq(s)}{n}} - 1| \\ &\leq |e^{\frac{i}{n}} - 1| \longrightarrow 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Let $L_n = U^* \breve{L}_n U$. Then $L_n \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $\|L_n - I_{\mathcal{L}(L^2_a(\mathbb{D}))}\| \longrightarrow 0$ as $n \to \infty$. That is, $L_n \longrightarrow I_{\mathcal{L}(L^2_a(\mathbb{D}))}$ in norm operator topology.

In Theorem 3.2, we show that if $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$ then $T^n_{\frac{1+\phi}{2}} \xrightarrow{w} 0$ if and only if $T_{1-\overline{\phi}}$ has trivial kernel. We also present necessary and sufficient conditions such that $T^n_{\frac{1+\phi}{2}} \longrightarrow 0$ in norm.

Theorem 3.2. *Let $\phi \in L^\infty(\mathbb{D})$ be such that $\|\phi\|_\infty \leq 1$. Then the following holds:*

(i): $\ker(T_{1-\overline{\phi}}) = \{0\}$ *if and only if $T^n_{\frac{1+\phi}{2}}$ converges to zero weakly as $n \to \infty$.*

(ii): *If* $\ker(T_{1-\overline{\phi}}) = \{0\}$ *then* $T^n_{\frac{1+\phi}{2}} L_n \xrightarrow{w} 0$.

(iii): $\ker(T_{1-\overline{\phi}}) = \{0\}$ *and* $\text{Ran }(T_{1-\phi})$ *is closed if and only if* $T^n_{\frac{1+\phi}{2}}$ *converges to zero in norm as* $n \to \infty$.

Proof. Since $\|\phi\|_\infty \leq 1$ we have $\|T_{\frac{1+\phi}{2}}\| \leq 1$. Hence the sequence $\{T^{*n}_{\frac{1+\phi}{2}}\}_{n=0}^\infty$ is bounded. So by [1] the sequence $\{T^{*^n}_{\frac{1+\phi}{2}}\}_{n=0}^\infty$ has a subsequence which converges to an operator $K \in \mathcal{L}(L^2_a(\mathbb{D}))$ in the weak operator topology. Without loss of generality, we shall assume the original sequence $\{T^{*n}_{\frac{1+\phi}{2}}\}_{n=0}^\infty$ converges to an operator $K \in \mathcal{L}(L^2_a(\mathbb{D}))$ in the weak operator topology. Hence $\left\langle \left(T^{*n+1}_{\frac{1+\phi}{2}} f - T^{*n}_{\frac{1+\phi}{2}} f\right), g\right\rangle \longrightarrow 0$ for every $f, g \in L^2_a(\mathbb{D})$ and $\left\{\langle T^{*n+1}_{\frac{1+\phi}{2}} f, g\rangle\right\}_{n=0}^\infty$ converges to $\langle Kf, g\rangle$ as n tends to ∞ for all $f, g \in L^2_a(\mathbb{D})$. This implies

$$\langle T^{*^n}_{\frac{1+\phi}{2}} f, T_{\frac{1+\phi}{2}} g\rangle \longrightarrow \langle Kf, g\rangle \quad \text{for all} \quad f, g \in L^2_a(\mathbb{D}).$$

Thus $\langle Kf, T_{\frac{1+\phi}{2}} g\rangle = \langle Kf, g\rangle$ for all $f, g \in L^2_a(\mathbb{D})$ and therefore $T^*_{\frac{1+\phi}{2}} K = K$. Further since $\left\{\langle T^{*^n}_{\frac{1+\phi}{2}} T^*_{\frac{1+\phi}{2}} f, g\rangle\right\}_{n=0}^\infty$ converges to $\langle Kf, g\rangle$ for all $f, g \in L^2_a(\mathbb{D})$, hence

$$\left\langle KT^*_{\frac{1+\phi}{2}} f, g\right\rangle = \langle Kf, g\rangle \quad \text{for all} \quad f, g \in L^2_a(\mathbb{D}).$$

Thus $KT^*_{\frac{1+\phi}{2}} = K$ and $T^{*^n}_{\frac{1+\phi}{2}} K = K$ for all $n \in \mathbb{Z}_+$. That is,

$$\langle T^{*^n}_{\frac{1+\phi}{2}} Kf, g\rangle = \langle Kf, g\rangle \quad \text{for all} \quad f, g \in L^2_a(\mathbb{D}).$$

Taking limit both the sides, we obtain $K^2 = K$. This proves that the operator K is an idempotent. Moreover, $T^*_{\frac{1+\phi}{2}} K = K$ implies $T^*_\phi K = K$ and $KT^*_{\frac{1+\phi}{2}} = K$ implies $KT^*_\phi = K$. So $\text{Ran }(K) \subseteq \ker(T_{1-\overline{\phi}})$.

On the other hand, for $f \in \ker(T_{1-\overline{\phi}})$, we have $T^*_\phi f = f$, so $T^*_{\frac{1+\phi}{2}} f = f$. Hence $T^{*^n}_{\frac{1+\phi}{2}} f = f$ for all $n \in \mathbb{Z}_+$ and this implies $Kf = f$. Hence $\text{Ran }(K) = \ker(T_{1-\overline{\phi}})$.

To prove the inclusion $\text{Ran }(T_{1-\overline{\phi}}) \subseteq \ker(K)$, let $f \in L^2_a(\mathbb{D})$ be an arbitrary element and $g = f - T^*_\phi f$. Then we have $Kg = Kf - KT^*_\phi f = Kf - Kf = 0$. Hence $g \in \ker(K)$.

Thus we have shown that, if $\|\phi\|_\infty \leq 1$ then there exists an idempotent $K \in \mathcal{L}(L^2_a(\mathbb{D}))$ whose range is $\ker(T_{1-\overline{\phi}})$ and kernel contains $\text{Ran }(T_{1-\overline{\phi}})$ and there exists a subsequence of $\{T^{*^n}_{\frac{1+\phi}{2}}\}$ which converges to K weakly.

To prove **(i)**, assume that $\langle T^n_{\frac{1+\phi}{2}} f, g\rangle \longrightarrow 0$ for every $f, g \in L^2_a(\mathbb{D})$. Then

$$\langle f, T^{*^n}_{\frac{1+\phi}{2}} g\rangle \longrightarrow 0 \quad \text{for all} \quad f, g \in L^2_a(\mathbb{D}). \tag{3.1}$$

That is, the sequence $\{\langle f, T^{*^n}_{\frac{1+\phi}{2}} g\rangle\}_{n=0}^\infty$ is a Cauchy sequence. Thus if any subsequence of $\{\langle f, T^{*^n}_{\frac{1+\phi}{2}} g\rangle\}_{n=0}^\infty$ converges to some $r \in \mathbb{C}$, then the sequence $\{\langle f, T^{*^n}_{\frac{1+\phi}{2}} g\rangle\}_{n=0}^\infty$ itself converges to r. We have already seen in the first part that there exists a subsequence of $\{\langle f, T^{*^n}_{\frac{1+\phi}{2}} g\rangle\}_{n=0}^\infty$ which converges to $\langle f, Kg\rangle$ for all $f, g \in L^2_a(\mathbb{D})$. Thus

$$T^{*^n}_{\frac{1+\phi}{2}} \xrightarrow{w} K$$

in $\mathcal{L}(L^2_a(\mathbb{D}))$ and the operator K is an idempotent. Thus it follows from (3.1) that $K = 0$, $T^{*^n}_{\frac{1+\phi}{2}} \xrightarrow{w} 0$ and $\ker(T_{1-\overline{\phi}}) = \text{Ran }(K) = \{0\}$.

Conversely, assume $\ker(T_{1-\overline{\phi}}) = \{0\}$. Then since $T^{*^n}_{\frac{1+\phi}{2}} \xrightarrow{w} K$ and the operator $K \in \mathcal{L}(L^2_a(\mathbb{D}))$ is an idempotent and Ran $K = \ker T_{1-\overline{\phi}} = \{0\}$ we obtain $K = 0$ and $T^{*^n}_{\frac{1+\phi}{2}} \xrightarrow{w} 0$.

To prove **(ii)** suppose $\ker(T_{1-\overline{\phi}}) = \{0\}$. Then by **(i)**, $T^n_{\frac{1+\phi}{2}}$ converges to zero weakly as $n \to \infty$. Since $L_n \longrightarrow I_{\mathcal{L}(L^2_a(\mathbb{D}))}$ in norm operator topology, hence it follows from [8] that $T^n_{\frac{1+\phi}{2}} L_n \xrightarrow{w} 0$.

To prove **(iii)**, suppose $\ker(T^*_{1-\phi}) = \{0\}$ and Ran $(T_{1-\phi})$ is closed. Then Ran $(T_{1-\phi}) = L^2_a(\mathbb{D})$ and from Theorem 3.1, it follows that, $\ker(T_{1-\phi}) = \{0\}$. Thus $T_{1-\phi}$ is invertible and $1 \notin \sigma(T_\phi)$. But

$$I - T_{\frac{1+\phi}{2}} = I - \frac{I}{2} - \frac{T_\phi}{2} = \frac{I - T_\phi}{2}.$$

Hence $1 \notin \sigma(T_{\frac{1+\phi}{2}})$. It therefore follows from [13] that if $\mathrm{Ran}(T_{1-\phi}) = L^2_a(\mathbb{D})$ then $\sigma(T_{\frac{1+\phi}{2}}) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset$.

Notice that $\sigma(T_{\frac{1+\phi}{2}})$ is a compact subset [5] of $\mathbb{C}$ and since $T_{\frac{1+\phi}{2}}$ is a contraction, the spectral radius $r(T_{\frac{1+\phi}{2}}) \leq \|T_{\frac{1+\phi}{2}}\| \leq 1$. Further, as $\sigma(T_{\frac{1+\phi}{2}}) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset$, hence $\sigma(T_{\frac{1+\phi}{2}})$ is a compact subset of $\mathbb{D}$ and therefore $r(T_{\frac{1+\phi}{2}}) < 1$. By [15],

$$\|T^n_{\frac{1+\phi}{2}}\| \longrightarrow 0.$$

Conversely, assume that $\|T^n_{\frac{1+\phi}{2}}\| \longrightarrow 0$. Then by [15], $r(T_{\frac{1+\phi}{2}}) < 1$. This implies that $\sigma(T_{\frac{1+\phi}{2}}) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset$. Therefore $1 \notin \sigma(T_{\frac{1+\phi}{2}})$. Since

$$I - T_{\frac{1+\phi}{2}} = \frac{I - T_\phi}{2}.$$

So $1 \notin \sigma(T_\phi)$. Hence $T_{1-\phi}$ is invertible. Hence $\mathrm{Ran}(T_{1-\phi}) = L^2_a(\mathbb{D})$. □

It is not difficult to find examples of operators $T \in \mathcal{L}(H)$ such that Ran (T) is closed but Ran (T^2) is not closed.

Let $\{u_j, f_j, h_j, j = 1, 2, \ldots\}$ be an orthonormal basis for H. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers converging to zero such that $a_n \leq 1$ for all n. For each n, let $b_n = (1 - a_n^2)^{\frac{1}{2}}$ and set $v_n = a_n u_n + b_n h_n, w_n = a_n f_n + b_n h_n$.

Let $M^\perp$ be the closed span of $\{v_j\}$ and define N to be the closed span of $\{w_j\}$. Then $M^\perp \cap N = \{0\}$ and the angle between $M^\perp$ and N is zero since

$$\langle v_j, w_j \rangle = b_n^2 = 1 - a_n^2.$$

Let T be a partially isometry with initial space M and the final space N. Then Ran (T) is closed but Ran (T^2) is not closed. For more details see [4].

In the following lemma we shall show that if $\|\phi\|_\infty \leq 1$ and Ran $(T_{1-\overline{\phi}})$ is closed then Ran $(T^2_{1-\overline{\phi}})$ is closed.

Lemma 3.3. *Let $\phi \in L^\infty(\mathbb{D})$ be such that $\|\phi\|_\infty \leq 1$ and $\ker(T_{1-\overline{\phi}}) = \{0\}$. Then the following conditions are equivalent:*

- **(i):** Ran $(T_{1-\overline{\phi}})$ *is closed.*
- **(ii):** Ran $(T_{1-\overline{\phi}} T_{1-\phi})$ *is closed.*
- **(iii):** Ran $(T^2_{1-\overline{\phi}})$ *is closed.*

Proof. To show (**i**) $\Longleftrightarrow$ (**ii**), suppose $T_{1-\overline{\phi}}$ has closed range. From [18], it follows that Ran $(T_{1-\overline{\phi}} T_{1-\phi})$ is closed and Ran $(T_{1-\overline{\phi}})$ = Ran $(T_{1-\overline{\phi}} T_{1-\phi})$. Conversely, if $T_{1-\overline{\phi}} T_{1-\phi}$

has closed range then

$$\begin{aligned} F &= \text{Ran } (T_{1-\overline{\phi}}T_{1-\phi}) \oplus \ker(T_{1-\overline{\phi}}T_{1-\phi}) = \text{Ran } (T_{1-\overline{\phi}}T_{1-\phi}) \oplus \ker(T_{1-\phi}) \\ &\subset \text{Ran } (T_{1-\overline{\phi}}) \oplus \ker(T_{1-\phi}) \subset F, \end{aligned}$$

which implies $T_{1-\overline{\phi}}$ has closed range.

To prove (**i**) $\Longrightarrow$ (**iii**) suppose Ran $(T_{1-\overline{\phi}})$ is closed. It is enough [16] to prove that the space Ran $(T_{1-\overline{\phi}})+\ker(T_{1-\overline{\phi}})$ is closed. Let $\{T_{1-\overline{\phi}}f_n+g_n\}$ be a sequence in Ran $(T_{1-\overline{\phi}})+\ker(T_{1-\overline{\phi}})$ that converges to $f \in L^2_a(\mathbb{D})$. Since $\|\phi\|_\infty \leq 1$, from the first part of the proof of Theorem 3.2, it follows that there exists an idempotent operator $K \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that

$$T^*_\phi K = K = KT^*_\phi, \text{Ran } (T_{1-\overline{\phi}}) \subset \ker(K)$$

and

$$\ker(T_{1-\overline{\phi}}) = \text{Ran } (K).$$

Thus we obtain

$$K(T_{1-\overline{\phi}}f_n + g_n) = Kg_n \longrightarrow Kf.$$

As $g_n \in \ker(T_{1-\overline{\phi}}) = \text{Ran } (K)$, we obtain $g_n = Kx_n$ for some $x_n \in L^2_a(\mathbb{D})$ and $Kg_n = KKx_n = Kx_n = g_n$. Since $Kg_n \longrightarrow Kf$, we obtain $g_n \longrightarrow Kf$. Thus the sequence $\{T_{1-\overline{\phi}}f_n\}$ converges to $f - Kf$ which must be in Ran $(T_{1-\overline{\phi}})$, as the space Ran $(T_{1-\overline{\phi}})$ is closed.

To prove (**iii**) $\Longrightarrow$ (**i**) suppose Ran $(T^2_{1-\overline{\phi}})$ is closed. We have to show Ran $(T_{1-\overline{\phi}})$ is closed. Let $\{h_n\} \in \text{Ran } (T_{1-\overline{\phi}})$, suppose $h_n \to h$ and $h_n = T_{1-\overline{\phi}}f_n, f_n \in L^2_a(\mathbb{D})$.

Now $T_{1-\overline{\phi}}f_n \longrightarrow h$ implies $T^2_{1-\overline{\phi}}f_n \longrightarrow T_{1-\overline{\phi}}h$. That is, $T_{1-\overline{\phi}}h \in \text{Ran } (T^2_{1-\overline{\phi}})$. Thus $T_{1-\overline{\phi}}h = T^2_{1-\overline{\phi}}k$ for some $k \in L^2_a(\mathbb{D})$. Therefore, $T^2_{1-\overline{\phi}}k - T_{1-\overline{\phi}}h = 0$. That is, $T_{1-\overline{\phi}}(T_{1-\overline{\phi}}k - h) = 0$. So $h = T_{1-\overline{\phi}}k$ as $\ker(T_{1-\overline{\phi}}) = \{0\}$. Hence $h \in \text{Ran } (T_{1-\overline{\phi}})$. Therefore Ran $(T_{1-\overline{\phi}})$ is closed. □

Remark 1. It follows from Lemma 3.3 that if $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$ then Ran $(T_{1-\overline{\phi}})$ is closed implies Ran $(T^2_{1-\overline{\phi}})$ is closed. We do not need the condition $\ker(T_{1-\overline{\phi}}) = \{0\}$ in this case.

Theorem 3.4. *Let $\psi = 1-\phi$ where $\phi \in L^\infty(\mathbb{D})$ and $\|\phi\|_\infty \leq 1$. Then* Ran (T_ψ) *is closed if and only if there exists an invertible operator $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ which commutes with T_ψ and ST_ψ is a projection operator.*

Proof. If Ran (T_ψ) is closed, then from [4] it follows that Ran $(T_{\overline{\psi}})$ is closed and hence by Lemma 3.3 and Remark 1, the space Ran $(T^2_{\overline{\psi}})$ is also closed. Again from [4], it follows that the space Ran (T^2_ψ) is closed. Since $\|\phi\|_\infty \leq 1$, we obtain (see the proof of Theorem 3.1) that $\ker(T_\psi) \cap$ Ran $(T_\psi) = \{0\}$.

Let $f \in \ker(T^2_{\overline{\psi}})$. Then the element $g = T_{\overline{\psi}}f \in \ker(T_{\overline{\psi}}) \cap$ Ran $(T_{\overline{\psi}}) = \{0\}$. Hence $f \in \ker(T_{\overline{\psi}})$. Thus $\ker(T^2_{\overline{\psi}}) \subseteq \ker(T_{\overline{\psi}})$. The inclusion relation $\ker(T_{\overline{\psi}}) \subseteq \ker(T^2_{\overline{\psi}})$ is always true. Hence $\ker(T_{\overline{\psi}}) = \ker(T^2_{\overline{\psi}})$.

Now,

$$(\text{Ran } (T_\psi))^\perp = \ker(T_{\overline{\psi}}) = \ker(T^2_{\overline{\psi}}) = (\text{Ran } (T^2_\psi))^\perp$$

and Ran (T_ψ) is closed. Thus we obtain

$$\text{Ran } (T^2_\psi) = \overline{\text{Ran } (T^2_\psi)} = \text{Ran } (T_\psi).$$

So for every $f \in L^2_a(\mathbb{D})$, there exists $g \in L^2_a(\mathbb{D})$ such that $T_\psi f = T^2_\psi g$. Thus $f - T_\psi g$ is in $\ker(T_\psi)$ and from Theorem 3.1, we obtain

$$\text{Ran } (T_\psi) \oplus \ker(T_\psi) = L^2_a(\mathbb{D}).$$

Now, define $R : L^2_a(\mathbb{D}) \longrightarrow L^2_a(\mathbb{D})$ such that $R(h+g) = T_\psi h + g$ where $h \in \text{Ran }(T_\psi)$ and $g \in \ker(T_\psi)$. The mapping R is well-defined, linear. Now suppose $h_n + g_n \to h + g$ and $R(h_n + g_n) \to l$. Then $l = R(h+g)$. This can be verified as follows:

$$\begin{aligned} l &= \lim_{n\to\infty} R(h_n + g_n) = \lim_{n\to\infty}(T_\psi h_n + g_n) \\ &= T_\psi(\lim_{n\to\infty} h_n) + (\lim_{n\to\infty} g_n) = T_\psi h + g = R(h+g). \end{aligned}$$

Thus R is a closed operator. By the closed graph theorem, R is bounded. We claim that R is invertible. That is, R is onto and one-one.

Let $f = h + g \in L^2_a(\mathbb{D})$ where $h \in \text{Ran }(T_\psi) = \text{Ran }(T^2_\psi)$ and $g \in \ker(T_\psi)$. Thus $h = T_\psi p$ for some $p \in \text{Ran }(T_\psi)$ and $R(p+g) = T_\psi p + g = h + g = f$. Hence R is onto.

Again let $f = h+g \in \ker(R)$ where $h \in \text{Ran }(T_\psi)$ and $g \in \ker(T_\psi)$. Then $R(h+g) = 0$. That is, $T_\psi h = -g$. From Theorem 3.1, it follows that $T_\psi h = g = 0$. This implies h is in the intersection of the spaces $\ker(T_\psi)$ and $\text{Ran }(T_\psi)$ and so it is 0 as well. Hence $f = h + g = 0$ and R is one-one.

Let Θ be the projection with range $\text{Ran}(T_\psi)$ and kernel $\ker(T_\psi)$. Now let $f = h+g, h \in \text{Ran }(T_\psi), g \in \ker(T_\psi)$. Then

$$R\Theta f = Rh = T_\psi h = T_\psi(h+g) = T_\psi f$$

and

$$\Theta R f = \Theta(T_\psi h + g) = T_\psi h = T_\psi(h+g) = T_\psi f.$$

Thus $T_\psi = R\Theta = \Theta R$. Hence $R^{-1}T_\psi = \Theta = T_\psi R^{-1}$. Let $S = R^{-1}$. Then $ST_\psi = T_\psi S$ and $ST_\psi = \Theta$ is a projection operator.

To prove the converse, suppose there exists an invertible operator $S \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $ST_\psi = T_\psi S$ and $ST_\psi = \Theta$, a projection operator. Let $S^{-1} = R$. Then $T_\psi = R\Theta = \Theta R, \Theta$ is a projection and R is invertible. We shall show that $\text{Ran }(T_\psi)$ is closed. Let $M = \text{Ran}(\Theta)$. Then M is a closed subspace of $L^2_a(\mathbb{D})$. The map R is linear, one-one, onto, bounded and R^{-1} is also bounded. Hence R is a homeomorphism. Thus $\text{Ran }(T_\psi) = R(M)$ is a closed subspace of $L^2_a(\mathbb{D})$ as R is a closed map. □

Recall that, if H, K are two Hilbert spaces and $C \in \mathcal{L}(H, K)$ has closed range, then there exists a unique $C^\dagger \in \mathcal{L}(K, H)$ such that $CC^\dagger C = C, C^\dagger CC^\dagger = C^\dagger$ and $CC^\dagger, C^\dagger C$ are Hermitian, $C^\dagger$ is called the Moore-Penrose inverse of C (For more details see [3]). If C is positive then $C^\dagger \geq 0$.

Theorem 3.5. *Let $\phi \in L^\infty(\mathbb{D})$ and $\text{Ran }(T_\phi)$ is closed. Suppose $\psi \in L^\infty(\mathbb{D})$ and $\text{Ran }(S_\psi)$ is a closed subspace of $L^2_a(\mathbb{D})$ of finite codimension. Then $\text{Ran }(S_\psi) \subset \text{Ran }(T_\phi)$ and there exists a positive operator $X \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $T_\phi X = S_\psi$ if and only if $S_\psi T_{\overline{\phi}} \geq 0$ and $S_\psi = S_\psi T_{\overline{\phi}} C$ for some $C \in \mathcal{L}(L^2_a(\mathbb{D}))$. The operator X is unique if $\ker(X) = \ker(T_\phi) = \ker(S_\psi)$. If X is invertible then $\phi \equiv 0$.*

Proof. Suppose $\psi \in L^\infty(\mathbb{D})$ and $\text{Ran }(S_\psi)$ is a closed subspace of $L^2_a(\mathbb{D})$ of finite codimension. Then $\ker(S^*_\psi)$ is finite dimensional. Hence by Lemma 2.1, $\ker(S_\psi)$ is of finite dimension. Since $\text{Ran }(T_\phi)$ is closed and $\ker(S_\psi)$ is finite dimensional hence from [4] it follows that $\text{Ran }(T^*_\phi)$ is closed and $\text{Ran }(S_\psi T_{\overline{\phi}})$ is closed. Now suppose $S_\psi T_{\overline{\phi}} \geq 0$ and $S_\psi = S_\psi T_{\overline{\phi}} C$ for some $C \in \mathcal{L}(L^2_a(\mathbb{D}))$. Then $\text{Ran }(S_\psi) \subset \text{Ran }(S_\psi T_{\overline{\phi}})$ and $(S_\psi T_{\overline{\phi}})(S_\psi T_{\overline{\phi}})^\dagger S_\psi = S_\psi$. Set $X = S_{\psi^+}(S_\psi T_{\overline{\phi}})^\dagger S_\psi$. Then $X \geq 0$ and

$$T_\phi X = (T_\phi S_{\psi^+})(S_\psi T_{\overline{\phi}})^\dagger S_\psi = (S_\psi T_{\overline{\phi}})(S_\psi T_{\overline{\phi}})^\dagger S_\psi = S_\psi.$$

From [5] and [2], it follows that $\text{Ran }(S_\psi) \subseteq \text{Ran }(T_\phi)$.

We now prove the converse. Since $T_\phi X = S_\psi$ and $X \geq 0$, hence $S_\psi T_{\overline{\phi}} = T_\phi X T_{\overline{\phi}} \geq 0$. We shall show that $\ker(S_\psi T_{\overline{\phi}}) \subset \ker(S_{\psi^+})$. Suppose $f \in L^2_a(\mathbb{D})$ be such that $S_\psi T_{\overline{\phi}} f =$

$T_\phi X T_{\overline{\phi}} f = 0$. Then by Reid's inequality [19], we have

$$\begin{aligned}\|S_{\psi^+} f\|^2 &= \|X T_{\overline{\phi}} f\|^2 \\ &\leq \|X\| \langle X T_{\overline{\phi}} f, T_{\overline{\phi}} f\rangle = \|X\| \langle T_\phi X T_{\overline{\phi}} f, f\rangle = 0.\end{aligned}$$

Hence $\ker(S_\psi T_{\overline{\phi}}) \subset \ker(S_{\psi^+})$ and therefore Ran $(S_\psi) = \overline{\text{Ran } S_\psi} \subset$ Ran $(S_\psi T_{\overline{\phi}})$. Thus by [6], we have $S_\psi = S_\psi T_{\overline{\phi}} C$ for some $C \in \mathcal{L}(L^2_a(\mathbb{D}))$.

From [6] it follows that the operator X is unique if $\ker(X) = \ker(T_\phi) = \ker(S_\psi)$. Further if the operator X is invertible and $T_\phi X = S_\psi$ then it follows from [6], [2] that Ran $(T_\phi) =$ Ran (S_ψ). From Theorem 2.5, it follows that $\phi \equiv 0$. $\square$

References

1. N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Monographs and Studies in Mathematics, vol. 9, Pitman, Boston—London, 1981.
2. B. A. Barnes, *Majorization, range inclusion and factorization for bounded linear operators*, Proc. Amer. Math. Soc. **133** (2004), no. 1, 155–162.
3. A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Robert E. Krieger publishing Co., Inc., Huntington, New York, 1980.
4. R. Bouldin, *The product of operators with closed range*, Tohoku Math. J. **25** (1973), 359–363.
5. R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.
6. R. G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.
7. N. Das, *The kernel of a Hankel operator on the Bergman space*, Bull. London Math. Soc. **31** (1999), 75–80.
8. D. Deckard, R. G. Douglas, C. Pearcy, *On invariant subspaces of quasitriangular operators*, Amer. J. Math. **91** (1969), no. 3, 637–647.
9. S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966.
10. P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer-Verlag, New York—Heidelberg—Berlin, 1982.
11. E. Hayashi, *The kernel of a Toeplitz operator*, Integr. Equ. Oper. Theory **9** (1986), 588–591.
12. S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71.
13. Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. **68** (1986), 313–328.
14. B. Korenblum and M. Stessin, *On Toeplitz-invariant subspaces of the Bergman space*, J. Funct. Anal. **111** (1983), 76–96.
15. C. S. Kubrusly, *Similarity to contractions and weak stability*, Adv. Math. Sci. Appl. **2** (1993), 335–343.
16. T. Kato, *Pertubation theory for nullity, deficiency, and other quantities of linear operators*, J. Analyse Math. **6** (1958), 261–322.
17. D. Luecking, *Finite rank Toelitz operators on the Bergman space*, Proc. Amer. Math. Soc. **136** (2008), no. 5, 1717–1723.
18. E. C. Lance, *Hilbert C^*-Modules*, LMS Lecture Note Series 210, Cambridge University Press, 1995.
19. C. S. Lin, *Inequalities of Reid type and Furuta*, Proc. Amer. Math. Soc. **129** (2001), no. 3, 855–859.
20. S. C. Power, *Hankel operators on Hilbert space*, Bull. Lond. Math. Soc. **12** (1980), 422–442.
21. A. E. Taylor, *Introduction to Functional Analysis*, Wiley, New York, 1958.
22. D. Vukotic, *A note on the range of Toeplitz operators*, Integr. Equ. Oper. Theory **50** (2004), 565–567.
23. K. H. Zhu, *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 139, Marcel Dekker Inc., New York, 1990.

P. G. Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar, 751004, Orissa, India

E-mail address: namitadas440@yahoo.co.in

P. G. Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar, 751004, Orissa, India

E-mail address: pabitramath@gmail.com

Received 23/02/2012