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BEREZIN NUMBER OF OPERATORS AND RELATED QUESTIONS

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Abstract. We prove some estimates for the Berezin number of operators on the reproducing kernel Hilbert spaces. We also give in terms of Berezin number necessary and sufficient conditions providing unitarity of invertible operator on the reproducing kernel Hilbert space. Moreover, we give a lower estimate for any operator on the Hardy space $H^2(\mathbb{D})$ over the unit disc $\mathbb{D}$.

1. Introduction and Notations

A Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ consisting of functions defined on some set Ω in any topological space is called reproducing kernel Hilbert space if all point evaluations $f \to f(\lambda)$ $(\lambda \in \Omega)$ are continuous. Equivalently, there exists a function $k : \Omega \times \Omega \to \mathbb{C}$ such that all functions of the form $k(.,\lambda) : \Omega \to \mathbb{C}$ belong to $\mathcal{H}$ and, moreover, satisfy the equality

$$\langle f, k(.,\lambda)\rangle = f(\lambda) \quad (f \in \mathcal{H},\ \lambda \in \Omega).$$

The function k is easily seen to be unique with these properties and is usually called the reproducing kernel of $\mathcal{H}$.

Let $A : \mathcal{H} \to \mathcal{H}$ be a linear bounded operator (*i.e.*, $A \in \mathcal{B}(\mathcal{H})$). Its Berezin symbol is defined by

$$\widetilde{A}(\lambda) := \left\langle A\frac{k(.,\lambda)}{\|k(.,\lambda)\|}, \frac{k(.,\lambda)}{\|k(.,\lambda)\|}\right\rangle \quad (\lambda \in \Omega).$$

The function $\widehat{k_\lambda}(z) := \frac{k(z,\lambda)}{\|k(z,\lambda)\|}$ is called the normalized reproducing kernel of $\mathcal{H}$. A detailed presentation of the theory of functional Hilbert spaces, reproducing kernels and Berezin symbols is given, for instance, in Aronzajn [1], Saitoh [6, 7] and Zhu [10].

Let us denote

$$\mathrm{Ber}(A) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\} \quad \text{and} \quad \mathrm{ber}(A) := \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|,$$

which is called Berezin set and Berezin number of the operator A, respectively. We recall that

$$W(A) := \{\langle Af, f\rangle : \|f\|_{\mathcal{H}} = 1\}$$

is the numerical range and

$$w(A) := \sup\{|\langle Af, f\rangle| : \|f\|_{\mathcal{H}} = 1\}$$

is the numerical radius of A. It is obvious that $\mathrm{ber}(A) \le w(A)$ and $\mathrm{Ber}(A) \subset W(A)$. The investigation of these new numerical characteristics of the linear bounded operators apparently has a great interest in the spectral theory of operators. For more informations about Berezin set and Berezin number, see Karaev [3].

It is well known that unitary operators can be characterized as invertible contractions with contractive inverses, i.e., as operators A with $\|A\| \le 1$ and $\left\|A^{-1}\right\| \le 1$.

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Recently Sano and Uchiyama [8] proved that if A is an invertible operator on the abstract Hilbert space H such that $w(A) \leq 1$ and $w\left(A^{-1}\right) \leq 1$, then A is unitary (see also Stampfli [9, Corollary 1]).

Here we obtain in terms of Berezin number necessary and sufficient conditions which guarantee unitarity of an invertible operator on the reproducing kernel Hilbert space.We also estimate the Berezin number of an arbitrary operator on the Hardy space $H^2(\mathbb{D})$.

2. A characterization of unitary operators

In this section we characterize unitary operators in terms of Berezin number.

Definition 1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space of complex-valued functions defined on some set Ω. We say that $\mathcal{H}$ possesses "(Ber) property", if for any two operators $A_1, A_2 \in \mathcal{B}(\mathcal{H})$ $\widetilde{A_1}(\lambda) = \widetilde{A_2}(\lambda)$ for all $\lambda \in \Omega$ implies $A_1 = A_2$.*

It is well known, for example, that any reproducing kernel Hilbert space of analytic functions in the unit disc $\mathbb{D}$ (including Hardy and Bergman spaces) has the (Ber) property (see Zhu [10], Proposition 6.2 and Karaev [3], Lemma 2).

Note that A is said to be unitary, if $A^*A = AA^* = I$, that is, $A^* = A^{-1}$.

The main result of this note is the following.

Theorem 1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space with (Ber) property and $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then A is unitary if and only if* $\operatorname{ber}\left(A^{-1*}A^{-1}\right) \leq 1$, $\operatorname{ber}\left(A^{-1}A^{-1*}\right) \leq 1$, $\operatorname{ber}(AA^*) \leq 1$ *and* $\operatorname{ber}(A^*A) \leq 1$.

Proof. First note the following:

1) $\operatorname{ber}\left(A^{-1*}A^{-1}\right) \leq 1$ if and only if $\left\|A^{-1*}\widehat{k}_\lambda\right\| \leq 1 \ (\forall\lambda \in \Omega)$;

2) $\operatorname{ber}\left(A^{-1}A^{-1*}\right) \leq 1$ if and only if $\left\|A^{-1*}\widehat{k}_\lambda\right\| \leq 1 \ (\forall\lambda \in \Omega)$;

3) $\operatorname{ber}(AA^*) \leq 1$ if and only if $\left\|A^*\widehat{k}_\lambda\right\| \leq 1 \ (\forall\lambda \in \Omega)$;

4) $\operatorname{ber}(A^*A) \leq 1$ if and only if $\left\|A\widehat{k}_\lambda\right\| \leq 1 \ (\forall\lambda \in \Omega)$.

Indeed, for all $\lambda \in \Omega$ we have

$$\left\|A^{-1}\widehat{k}_\lambda\right\|^2 = \left\langle A^{-1}\widehat{k}_\lambda, A^{-1}\widehat{k}_\lambda\right\rangle = \left\langle A^{-1*}A^{-1}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle = \widetilde{A^{-1*}A^{-1}}(\lambda)$$

and

$$\left\|A^{-1*}\widehat{k}_\lambda\right\|^2 = \left\langle A^{-1*}\widehat{k}_\lambda, A^{-1*}\widehat{k}_\lambda\right\rangle = \left\langle A^{-1}A^{-1*}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle = \widetilde{A^{-1}A^{-1*}}(\lambda).$$

Thus

$$\sup_{\lambda\in\Omega}\left\|A^{-1}\widehat{k}_\lambda\right\| \leq 1 \iff \operatorname{ber}\left(A^{-1*}A^{-1}\right) \leq 1 \tag{1}$$

and

$$\sup_{\lambda\in\Omega}\left\|A^{-1*}\widehat{k}_\lambda\right\| \leq 1 \iff \operatorname{ber}\left(A^{-1}A^{-1*}\right) \leq 1. \tag{2}$$

Analogously, it can be proved that

$$\sup_{\lambda\in\Omega}\left\|A^{*}\widehat{k}_\lambda\right\| \leq 1 \iff \operatorname{ber}(AA^*) \leq 1, \tag{3}$$

and

$$\sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\| \leq 1 \iff \operatorname{ber}(A^*A) \leq 1, \tag{4}$$

which prove statements 1)–4), respectively.

It is easy to show that A is unitary if and only if

$$\left\|A\widehat{k}_\lambda\right\| = \left\|A^*\widehat{k}_\lambda\right\| = 1 \quad (\forall\lambda \in \Omega)\,. \tag{5}$$

Indeed, by using (Ber) property of the space $\mathcal{H}$ we assert that

$$A \text{ is unitary} \iff AA^* = A^*A = I \iff \widetilde{AA^*}(\lambda) = \widetilde{A^*A}(\lambda) = 1 \quad (\forall\lambda \in \Omega)\,.$$

Since $\widetilde{AA^*}(\lambda) = \left\|A^*\widehat{k}_\lambda\right\|^2$ and $\widetilde{A^*A}(\lambda) = \left\|A\widehat{k}_\lambda\right\|^2$ for all $\lambda \in \Omega$, we assert that A is unitary if and only if $\left\|A\widehat{k}_\lambda\right\| = \left\|A^*\widehat{k}_\lambda\right\| = 1$ for all $\lambda \in \Omega$.

Now by using these and the equalities

$$AA^{-1} = A^{-1}A = I,$$

we have

$$\left\langle AA^{-1}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle = \left\langle A^{-1}A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle = 1$$

for all $\lambda \in \Omega$. Then by considering (1)–(4) we obtain that

$$1 \leq \left\|A^{-1}\widehat{k}_\lambda\right\| \left\|A^*\widehat{k}_\lambda\right\| \leq \left\|A^*\widehat{k}_\lambda\right\| \leq 1 \quad (\forall\lambda \in \Omega)\,,$$

and hence

$$\left\|A^*\widehat{k}_\lambda\right\| = 1 \quad (\forall\lambda \in \Omega)\,. \tag{6}$$

Also

$$1 \leq \left\|A\widehat{k}_\lambda\right\| \left\|A^{-1*}\widehat{k}_\lambda\right\| \leq \left\|A\widehat{k}_\lambda\right\| \leq 1 \quad (\forall\lambda \in \Omega)\,,$$

and hence

$$\left\|A\widehat{k}_\lambda\right\| = 1 \quad (\forall\lambda \in \Omega)\,. \tag{7}$$

Now by considering (5) we deduce from (6) and (7) that an operator A is unitary, as desired.

It is obvious that if A is unitary, then $\operatorname{ber}\left(A^{-1*}A^{-1}\right) = \operatorname{ber}\left(A^{-1}A^{-1*}\right) = \operatorname{ber}(AA^*) = \operatorname{ber}(A^*A) = 1$. The theorem is proved. □

3. An inequality for the Berezin number of operators

Note that always $\operatorname{ber}(A) \leq w(A)$ and $\frac{\|A\|}{2} \leq w(A) \leq \|A\|$. However, it is known (see, for example, Englis [2]) that

$$\operatorname{ber}(T_f) \geq C\,\|T_f\| \quad \forall f \in L^\infty(\mathbb{D}, dm_2)$$

can not hold for any constant $C > 0$; here T_f is the Toeplitz operator on the Bergman space $L_a^2 := L_a^2(\mathbb{D}, dm_2)$ (which is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D}, dm_2)$) defined by

$$T_f g(z) := P(fg)(z) = \int_{\mathbb{D}} \frac{f(w)\,g(w)}{(1 - z\overline{w})^2} dm_2(w)\,,$$

where dm_2 denotes the Lebesgue measure on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, normalized so that the measure of $\mathbb{D}$ equals 1, and $P : L^2(\mathbb{D}, dm_2) \to L_a^2$ is the orthogonal projection defined by

$$P(h)(z) := \int_{\mathbb{D}} \frac{h(w)}{(1 - z\overline{w})^2} dm_2(w)$$

for each $z \in \mathbb{D}$ and $h \in L^2(\mathbb{D}, dm_2)$. Thus, it is interesting and natural to find a lower estimate for $\operatorname{ber}(A)$ in terms of the operator A.

In this connection we will prove the following theorem, which estimates $\operatorname{ber}(A)$ in terms of some quantity associated with the operator A.

Before giving our result, let us introduce some necessary notations. The Hardy space $H^2 = H^2(\mathbb{D})$ of the disk is defined as the set of all functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and

$$\|f\|_{H^2} = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^2 dt \right)^{1/2} < \infty.$$

H^2 has a reproducing kernel $k_\lambda(z) := \frac{1}{1-\overline{\lambda} z}$ $(\lambda, z \in \mathbb{D})$. Let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle. For any $\varphi \in L^\infty(\mathbb{T})$, the corresponding Toeplitz operator on H^2 is defined by

$$T_\varphi f = P_+ \varphi f, \quad f \in H^2,$$

where $P_+ : L^2(\mathbb{T}) \to H^2$ is the orthogonal projector (the Riesz projector). The Banach algebra of bounded analytic functions on $\mathbb{D}$ is denoted by $H^\infty = H^\infty(\mathbb{D})$. Every $\psi \in H^\infty$ acts as a multiplication operator on H^2, $T_\psi f = \psi f$, $f \in H^2$. It is easy to see that $T_{\overline{\psi}} k_\lambda = \overline{\psi(\lambda)} k_\lambda$ for each $\lambda \in \mathbb{D}$. Also it is well known that the Berezin symbol $\widetilde{T_\varphi}$ is an "asymptotic eigenvalue" of the Teoplitz operator T_φ, i.e.,

$$\lim_{r \to 1^-} \left\| \left(T_\varphi - \widetilde{T_\varphi}\left(re^{it}\right) I \right) \widehat{k}_{re^{it}} \right\|_{H^2} = 0 \tag{8}$$

for almost all $t \in [0, 2\pi)$. Moreover,

$$\widetilde{T_\varphi}(\lambda) = \widetilde{\varphi}(\lambda) \quad (\lambda \in \mathbb{D}), \tag{9}$$

where $\widetilde{\varphi}$ denotes the harmonic extension of the function $\varphi \in L^\infty(\mathbb{T})$ into the unit disk $\mathbb{D}$. For more details, see Englis [2] and Karaev [4, 5].

Theorem 2. *Let $\varphi \in L^\infty(\partial\mathbb{D})$ and $\psi \in H^\infty(\mathbb{D})$ be any two nonzero functions such that $\|\varphi\|_{L^\infty} \le 1$ and $\|\psi\|_{H^\infty} \le 1$. For any operator $A : H^2 \to H^2$, we define $N_{\varphi,\psi,A} := T_\varphi\left(I - T_\psi A T_{\overline{\psi}}\right)$, where T_φ, T_ψ and $T_{\overline{\psi}}$ are Toeplitz operators on the Hardy space $H^2 = H^2(\mathbb{D})$. Then $\widetilde{N}^{\mathrm{rad}}_{\varphi,\psi,A}(e^{it}) := \lim_{r\to1^-} \widetilde{N}_{\varphi,\psi,A}\left(re^{it}\right)$ exists for almost all $t \in [0, 2\pi)$ and*

$$\mathrm{ber}(A) \ge \left\| \varphi - \widetilde{N}^{\mathrm{rad}}_{\varphi,\psi,A} \right\|_{L^\infty}.$$

Proof. By considering (9), let us calculate the Berezin symbol of the operator $N_{\varphi,\psi,A}$,

$$\begin{aligned}
\widetilde{N}_{\varphi,\psi,A}(\lambda) &= \left\langle N_{\varphi,\psi,A} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&= \left\langle \left(I - T_\psi A T_{\overline{\psi}} \right) \widehat{k}_\lambda, T_{\overline{\varphi}} \widehat{k}_\lambda - \widetilde{T_{\overline{\varphi}}}(\lambda) \widehat{k}_\lambda \right\rangle \\
&\quad + \left\langle \widehat{k}_\lambda - \overline{\psi(\lambda)} T_\psi A \widehat{k}_\lambda, \widetilde{T_{\overline{\varphi}}}(\lambda) \widehat{k}_\lambda \right\rangle \\
&= \left\langle \left(I - T_\psi A T_{\overline{\psi}} \right) \widehat{k}_\lambda, \left(T_{\overline{\varphi}} - \widetilde{T_{\overline{\varphi}}}(\lambda) I \right) \widehat{k}_\lambda \right\rangle \\
&\quad + \widetilde{\varphi}(\lambda) - \widetilde{\varphi}(\lambda) \overline{\psi(\lambda)} \left\langle T_\psi A \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&= \left\langle \left(I - T_\psi A T_{\overline{\psi}} \right) \widehat{k}_\lambda, \left(T_{\overline{\varphi}} - \widetilde{T_{\overline{\varphi}}}(\lambda) I \right) \widehat{k}_\lambda \right\rangle \\
&\quad + \widetilde{\varphi}(\lambda) - \widetilde{\varphi}(\lambda) |\psi(\lambda)|^2 \widetilde{A}(\lambda)
\end{aligned}$$

for all $\lambda \in \mathbb{D}$. Then, by considering that $\sup_{\lambda\in\mathbb{D}} |\widetilde{\varphi}(\lambda)| \le \|\varphi\|_{L^\infty} \le 1$ and $\|\psi\|_{H^\infty} \le 1$, we have

$$\begin{aligned}
\left| \widetilde{\varphi}(\lambda) - \widetilde{N}_{\varphi,\psi,A}(\lambda) \right| &\le \mathrm{ber}(A) + \left\| I - T_\psi A T_{\overline{\psi}} \right\| \left\| \left(T_{\overline{\varphi}} - \widetilde{T_{\overline{\varphi}}}(\lambda) I \right) \widehat{k}_\lambda \right\| \\
&\le \mathrm{ber}(A) + (1 + \|A\|) \left\| \left(T_{\overline{\varphi}} - \widetilde{T_{\overline{\varphi}}}(\lambda) I \right) \widehat{k}_\lambda \right\|
\end{aligned}$$

for all $\lambda \in \mathbb{D}$. Thus, by using (8), it follows from the latter that

$$\lim_{r\to 1^-} \left| \widetilde{\varphi}\left(re^{it}\right) - \widetilde{N}_{\varphi,\psi,A}\left(re^{it}\right) \right| \leq \operatorname{ber}(A)$$

for almost all $t \in [0, 2\pi)$. Since $\lim_{r\to 1^-} \widetilde{\varphi}\left(re^{it}\right) = \varphi\left(e^{it}\right)$, the last inequality implies that

$$\operatorname{ess}\sup_{t\in[0,2\pi)} \left| \varphi\left(e^{it}\right) - \lim_{r\to 1^-} \widetilde{N}_{\varphi,\psi,A}\left(re^{it}\right) \right| \leq \operatorname{ber}(A),$$

that is, $\widetilde{N}^{\mathrm{rad}}_{\varphi,\psi,A}\left(e^{it}\right) := \lim_{r\to 1^-} \widetilde{N}_{\varphi,\psi,A}\left(re^{it}\right)$ exists almost everywhere on the unit circle $\mathbb{T}$ and

$$\operatorname{ess}\sup_{t\in[0,2\pi)} \left| \varphi\left(e^{it}\right) - \widetilde{N}^{\mathrm{rad}}_{\varphi,\psi,A}\left(e^{it}\right) \right| \leq \operatorname{ber}(A),$$

or

$$\left\| \varphi - \widetilde{N}^{\mathrm{rad}}_{\varphi,\psi,A} \right\|_{L^\infty} \leq \operatorname{ber}(A),$$

as desired. The proof is completed. □

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