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SCHRÖDINGER OPERATORS WITH COMPLEX SINGULAR POTENTIALS

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To Myroslav Lvovych Gorbachuk on the occasion of his 75th birthday

Abstract. We study one-dimensional Schrödinger operators $\mathrm{S}(q)$ on the space $L^2(\mathbb{R})$ with potentials q being complex-valued generalized functions from the negative space $H^{-1}_{\mathrm{unif}}(\mathbb{R})$. Particularly the class $H^{-1}_{\mathrm{unif}}(\mathbb{R})$ contains periodic and almost periodic $H^{-1}_{\mathrm{loc}}(\mathbb{R})$-functions. We establish an equivalence of the various definitions of the operators $\mathrm{S}(q)$, investigate their approximation by operators with smooth potentials from the space $L^1_{\mathrm{unif}}(\mathbb{R})$ and prove that the spectrum of each operator $\mathrm{S}(q)$ lies within a certain parabola.

1. Introduction and main results

In the complex Hilbert space $L^2(\mathbb{R})$ we consider a Schrödinger operator

$$\mathrm{S}(q) = -\frac{d^2}{dx^2} + q(x)$$

with potential q that is a complex-valued distribution from the space $H^{-1}_{\mathrm{unif}}(\mathbb{R}) \subset H^{-1}_{\mathrm{loc}}(\mathbb{R})$. Recall that $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ is a dual to the space $H^1_{\mathrm{comp}}(\mathbb{R})$ of functions in $H^1(\mathbb{R})$ with compact support and that every $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$ can be represented as Q' for $Q \in L^2_{\mathrm{loc}}(\mathbb{R})$. Then the operator $\mathrm{S}(q)$ can be rigorously defined, e.g., by the so-called regularization method that was used in [1] in the particular case $q(x) = 1/x$ and then developed for generic distributional potential functions in $H^{-1}_{\mathrm{loc}}(\mathbb{R})$ in [21, 22]; see also recent extensions to more general differential expressions in [7, 8]. Namely, the regularization method suggests to define $\mathrm{S}(q)$ via

$$\mathrm{S}(q)y = l[y] = -(y' - Qy)' - Qy' \tag{1}$$

on the natural maximal domain

$$\mathrm{Dom}(\mathrm{S}(q)) = \left\{ y \in L^2(\mathbb{R}) \;\middle|\; y, y' - Qy \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}),\ l[y] \in L^2(\mathbb{R}) \right\}, \tag{2}$$

where $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$ is the space of functions that are locally absolutely continuous. It is easy to see that $\mathrm{S}(q)y = -y'' + qy$ in the sense of distributions and the above definition does not depend on the particular choice of the primitive $Q \in L^2_{\mathrm{loc}}(\mathbb{R})$.

One can also introduce the minimal operator $\mathrm{S}_0(q)$, which is the closure of the restriction $\mathrm{S}_{00}(q)$ of $\mathrm{S}(q)$ onto the set of functions with compact support, i.e., onto

$$\mathrm{Dom}(\mathrm{S}_{00}(q)) := \left\{ y \in L^2_{\mathrm{comp}}(\mathbb{R}) \;\middle|\; y, y' - Qy \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}),\ l[y] \in L^2(\mathbb{R}) \right\}.$$

In the case where the potential function q is real-valued, the operator $\mathrm{S}_{00}(q)$ (and hence $\mathrm{S}_0(q)$) is symmetric; moreover, in a standard manner [18] one can prove that $\mathrm{S}(q)$ is an adjoint of $\mathrm{S}_0(q)$. An important question preceding any further analysis of the operator

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$\mathrm{S}(q)$ is whether it is self-adjoint, i.e., $\mathrm{S}(q) = \mathrm{S}_0(q)$. The case where the potential belongs to the space $H^{-1}_{\mathrm{unif}}(\mathbb{R})$ was investigated in [10]. We recall [10] that any $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$ can be represented (not uniquely) in the form

$$q = Q' + \tau, \tag{3}$$

where the derivative is understood in the sense of distributions and Q and τ belong to the Stepanov spaces $L^2_{\mathrm{unif}}(\mathbb{R})$ and $L^1_{\mathrm{unif}}(\mathbb{R})$ respectively, i.e.,

$$\|Q\|^2_{L^2_{\mathrm{unif}}(\mathbb{R})} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |Q(s)|^2 ds < \infty,$$

$$\|\tau\|_{L^1_{\mathrm{unif}}(\mathbb{R})} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |\tau(s)| ds < \infty.$$

Given such a representation, the operator S is defined as

$$\mathrm{S}(q)y = -(y' - Qy)' - Qy' + \tau y \tag{4}$$

on the domain (2). This definition also does not depend on a particular choice of Q and τ above. Theorem 3.5 of the paper [10] claims that for real-valued $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$ the operator $\mathrm{S}(q)$ as defined by (4) and (2) is self-adjoint and coincides with the operator $\mathrm{S}_{fs}(q)$ constructed by the form-sum method. However the proof given in [10] is incomplete.

The fact that $\mathrm{S}(q)$ is indeed self-adjoint is rigorously justified in the paper [18] for the particular case where $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$ is periodic. The authors prove therein that $\mathrm{S}_0(q)$, $\mathrm{S}(q)$, $\mathrm{S}_{fs}(q)$ and the Friedrichs extension $\mathrm{S}_F(q)$ of $\mathrm{S}_0(q)$ all coincide. However the arguments heavily use periodicity of q and can not be applied to generic real-valued $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$. This gap in the proof of Theorem 3.5 of [10] is filled in by the authors in their recent paper [11], see also [14].

This paper deals with the case when the potential $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$ is complex-valued. One can easily see that in this case all the operators $\mathrm{S}_0(q)$, $\mathrm{S}(q)$, $\mathrm{S}_{fs}(q)$ and $\mathrm{S}_F(q)$ are well-defined and are related by

$$\mathrm{S}_0(q) \subset \mathrm{S}_F(q) = \mathrm{S}_{fs}(q) \subset \mathrm{S}(q), \quad \mathrm{Dom}(\mathrm{S}_F(q)) \subset H^1(\mathbb{R}), \quad \mathrm{Dom}(\mathrm{S}(q)) \subset H^1_{\mathrm{loc}}(\mathbb{R}) \cap L^2(\mathbb{R}).$$

The main purpose of this paper is to prove that these operators coincide and to investigate their approximation and spectral properties. Let us state the main results.

Theorem A. *For every function $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$, the operators $\mathrm{S}_0(q)$, $\mathrm{S}(q)$, $\mathrm{S}_{fs}(q)$ and $\mathrm{S}_F(q)$ are m-sectorial and coincide.*

Theorem A allows to link the known results for the Schrödinger operators in the space $L^2(\mathbb{R})$ which are defined in different ways, see e. g. [2, 5, 9, 13, 24].

In the paper [18] the authors proved that for every real-valued 1-periodic function $q \in H^{-1}_{\mathrm{loc}}(\mathbb{R})$ a sequence of *smooth* 1-periodic functions q_n exists such that the sequence of operators $\mathrm{S}(q_n)$ converges to the operator $\mathrm{S}(q)$ in the sense of norm resolvent convergence. It is sufficient to establish

$$\|q - q_n\|_{H^{-1}(0,1)} \to 0, \quad n \to \infty.$$

The following theorem generalizes this result in two directions. The potential q may be complex-valued and non-periodic.

Theorem B. *Let q, q_n, $n \geq 1$, belong to the space $H^{-1}_{\mathrm{unif}}(\mathbb{R})$. Then the sequence of operators $\mathrm{S}(q_n)$, $n \geq 1$, converges to the operator $\mathrm{S}(q)$ in the sense of norm resolvent convergence, $\mathrm{R}(\lambda, \mathrm{S}) := (\mathrm{S} - \lambda \mathrm{Id})^{-1}$:*

$$\|\mathrm{R}(\lambda, \mathrm{S}(q)) - \mathrm{R}(\lambda, \mathrm{S}(q_n))\| \to 0, \quad n \to \infty, \quad \lambda \in \mathrm{Resolv}(\mathrm{S}(q)) \neq \emptyset, \tag{5}$$

if

$$q_n \overset{H^{-1}_{\mathrm{unif}}(\mathbb{R})}{\longrightarrow} q, \quad n \to \infty \tag{6}$$

or, equivalently,

$$Q_n \overset{L^2_{\mathrm{unif}}(\mathbb{R})}{\longrightarrow} Q, \quad \tau_n \overset{L^1_{\mathrm{unif}}(\mathbb{R})}{\longrightarrow} \tau, \quad n \to \infty. \tag{7}$$

Since the set $C^\infty(\mathbb{R}) \cap L^1_{\mathrm{unif}}(\mathbb{R})$ is dense in the space $H^{-1}_{\mathrm{unif}}(\mathbb{R})$ (see Section 3.2 below), the following corollary holds.

Corollary B.1. *For every function $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$ there is a sequence of functions $q_n \in C^\infty(\mathbb{R}) \cap L^1_{\mathrm{unif}}(\mathbb{R})$ such that the limit relation* (5) *is true. If the function q is real-valued, then the functions q_n can be chosen to be real-valued as well.*

In particular, if Q and τ are almost periodic Stepanov functions then Q_n and τ_n can be chosen to be trigonometrical polynomials [15, Theorem I.5.7.2]. If Q and τ are are bounded and uniformly continuous on the whole real axis $\mathbb{R}$, then Q_n and τ_n can be chosen to be entire analytic functions [15, Theorem I.1.10.1, Remark].

The following theorem allows to describe the localization of the spectrum of the operators $\mathrm{S}(q)$.

Theorem C. *The numerical ranges of the operators* $\mathrm{S}(q)$ *(and therefore their spectra) lie within the parabola*

$$\begin{aligned} &|\mathrm{Im}\,\lambda| \le 5K \left(\mathrm{Re}\,\lambda + 4(2K+1)^4\right)^{3/4}, \\ &K = 2\left(\|Q\|_{L^2_{\mathrm{unif}}(\mathbb{R})} + \|\tau\|_{L^1_{\mathrm{unif}}(\mathbb{R})}\right). \end{aligned} \tag{8}$$

If the potential q is real-valued, then the self-adjoint operator $\mathrm{S}(q)$ *is bounded below by the number*

$$m(K) = \begin{cases} -4K, & \text{if} \quad K \in [0, 1/2), \\ -32K^4, & \text{if} \quad K \ge 1/2. \end{cases}$$

Note that if a complex-valued potential $q \in H^{-1}_{\mathrm{unif}}(\mathbb{R})$ is a periodic generalized function, then the spectrum of the operator $\mathrm{S}(q)$ lies within a quadratic parabola [16, Theorem 6]. A similar result holds for certain complex-valued measures, see [24] and Section 3.3, formula (38).

Similar problems are considered in the papers [3, 4, 6, 17, 20, 23].

2. Preliminaries

This section contains several statements that are used in the proof of Theorem A.

We begin with introduction of the dual operators $\mathrm{S}^+_{00}(q)$ $\mathrm{S}^+(q)$.

The formally adjoint quasi-differential expression l^+ for l is defined by [25]

$$v^{\{0\}} := v, \quad v^{\{1\}} := v' - \overline{Q}v, \quad v^{\{2\}} := (v^{\{1\}})' + \overline{Q}v^{\{1\}} + (\overline{Q^2} - \overline{\tau})v,$$

$$\mathrm{l}^+[v] := -v^{\{2\}}, \quad \mathrm{Dom}(\mathrm{l}^+) := \left\{ v : \mathbb{R} \to \mathbb{C} \;\middle|\; v, v^{\{1\}} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) \right\}.$$

By $\overline{\cdot}$ we denote the complex conjugation.

Then

$$\mathrm{S}^+ v \equiv \mathrm{S}^+(q)v := \mathrm{l}^+[v], \;\; \mathrm{Dom}(\mathrm{S}^+) := \left\{ v \in L^2(\mathbb{R}) \;\middle|\; v,\, v^{\{1\}} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}), \mathrm{l}^+[v] \in L^2(\mathbb{R}) \right\},$$

$$\mathrm{S}^+_{00} v \equiv \mathrm{S}^+_{00}(q)v := \mathrm{l}^+[v], \;\; \mathrm{Dom}(\mathrm{S}^+_{00}) := \left\{ v \in \mathrm{Dom}(\mathrm{S}^+) \;\mid\; \mathrm{supp}\, v \Subset \mathbb{R} \right\}.$$

One can easily see that if $\mathrm{Im}\, q \equiv 0$ then the operators $\mathrm{S}_{00}(q)$ and $\mathrm{S}^+_{00}(q)$, $\mathrm{S}(q)$ and $\mathrm{S}^+(q)$ coincide.

Lemma 1. [Theorem 1, Corollary 1 [25]]. *For arbitrary functions* $u \in \mathrm{Dom}(\mathrm{S})$*,* $v \in \mathrm{Dom}(\mathrm{S}^+)$ *and finite interval* $[a,b]$ *the following equality holds:*

$$\int_a^b l[u]\overline{v}\,dx - \int_a^b u\overline{l^+[v]}\,dx = [u,v]_a^b, \tag{9}$$

where

$$[u,v](t) := u(t)\overline{v^{\{1\}}(t)} - u^{[1]}(t)\overline{v(t)},$$
$$[u,v]_a^b := [u,v](b) - [u,v](a).$$

Lemma 2. *For arbitrary functions* $u \in \mathrm{Dom}(\mathrm{S})$ *and* $v \in \mathrm{Dom}(\mathrm{S}^+)$ *the following limits exist and are finite:*

$$[u,v](-\infty) := \lim_{t\to-\infty}[u,v](t), \quad [u,v](\infty) := \lim_{t\to\infty}[u,v](t).$$

Proof. Let us fix the number b in the equality (9) and then pass to the limit as $a \to -\infty$. Due to the assumptions of the lemma $u, v, l[u], l^+[v] \in L^2(\mathbb{R})$, the limit $[u,v](-\infty)$ exists and is finite. Similarly one can prove that the limit $[u,v](\infty)$ exists and is finite.

The Lemma is proved. □

Lemma 3. (Generalized Lagrange identity). *For all functions* $u \in \mathrm{Dom}(\mathrm{S})$*,* $v \in \mathrm{Dom}(\mathrm{S}^+)$ *the equality*

$$\int_{-\infty}^{\infty} l[u]\overline{v}\,dx - \int_{-\infty}^{\infty} u\overline{l^+[v]}\,dx = [u,v]_{-\infty}^{\infty}, \tag{10}$$
$$[u,v]_{-\infty}^{\infty} := [u,v](\infty) - [u,v](-\infty).$$

holds.

Proof. The identity (10) is true due to Lemma 1 and Lemma 2. □

In the following proposition we describe properties of minimal and maximal operators and their adjoints.

Proposition 4. *For the operators* S*,* S_{00} *and* S^+*,* S_{00}^+ *the following statements are fulfilled.*

1^0. *The operators* S_{00} *and* S_{00}^+ *are densely defined in the Hilbert space* $L^2(\mathbb{R})$*.*

2^0. *The following relations hold:*

$$(\mathrm{S}_{00})^* = \mathrm{S}^+, \quad \left(\mathrm{S}_{00}^+\right)^* = \mathrm{S}.$$

3^0. *The operators* S*,* S^+ *are closed and the operators* S_{00}*,* S_{00}^+ *are closable,*

$$\mathrm{S}_0 := (\mathrm{S}_{00})^\sim, \quad \mathrm{S}_0^+ := \left(\mathrm{S}_{00}^+\right)^\sim.$$

4^0. *Domains of the operators* S_0*,* S_0^+ *can be described in the following way:*

$$\mathrm{Dom}(\mathrm{S}_0) = \left\{u \in \mathrm{Dom}(\mathrm{S}) \,\middle|\, [u,v]_{-\infty}^{\infty} = 0 \quad \forall v \in \mathrm{Dom}(\mathrm{S}^+)\right\},$$
$$\mathrm{Dom}(\mathrm{S}_0^+) = \left\{v \in \mathrm{Dom}(\mathrm{S}^+) \,\middle|\, [u,v]_{-\infty}^{\infty} = 0 \quad \forall u \in \mathrm{Dom}(\mathrm{S})\right\}.$$

5^0. *Domains of the operators* S*,* S_0*,* S_{00} *and* S^+*,* S_0^+*,* S_{00}^+ *satisfy the following relations:*

$$u \in \mathrm{Dom}(\mathrm{S}) \Leftrightarrow \overline{u} \in \mathrm{Dom}(\mathrm{S}^+),$$
$$u \in \mathrm{Dom}(\mathrm{S}_0) \Leftrightarrow \overline{u} \in \mathrm{Dom}(\mathrm{S}_0^+),$$
$$u \in \mathrm{Dom}(\mathrm{S}_{00}) \Leftrightarrow \overline{u} \in \mathrm{Dom}(\mathrm{S}_{00}^+).$$

The proof of properties 1^0–4^0 in Proposition 4 is similar to the proof of similar statements for symmetric operators on semi-axis [25], see also [19]. The property 5^0 is proved by a direct calculation.

We use the following estimates obtained in [10, Lemma 3.2] to prove the main theorems.

Lemma 5. *Let the functions $Q \in L^2_{\text{unif}}(\mathbb{R})$, $\tau \in L^1_{\text{unif}}(\mathbb{R})$ and $u \in H^1(\mathbb{R})$. Then $\forall \varepsilon \in (0,1]$ and $\forall \eta \in (0,1]$ we have the estimates*

$$\left|(Q, \overline{u}'u)_{L^2(\mathbb{R})}\right| \leq \|Q\|_{L^2_{\text{unif}}(\mathbb{R})} \left(\varepsilon \|u'\|^2_{L^2(\mathbb{R})} + 4\varepsilon^{-3}\|u\|^2_{L^2(\mathbb{R})}\right),$$
$$\left|(\tau, |u|^2)_{L^2(\mathbb{R})}\right| \leq \|\tau\|_{L^1_{\text{unif}}(\mathbb{R})} \left(\eta \|u'\|^2_{L^2(\mathbb{R})} + 8\eta^{-1}\|u\|^2_{L^2(\mathbb{R})}\right).$$

3. Proofs

3.1. **Proof of Theorem A.** Consider the sesquilinear forms generated by preminimal operators $\mathrm{S}_{00}(q)$

$$\dot{t}_{\mathrm{S}_{00}}[u,v] := (\mathrm{S}_{00}(q)u, v)_{L^2(\mathbb{R})} = (u', v')_{L^2(\mathbb{R})} - (Q, \overline{u}'v + \overline{u}v')_{L^2(\mathbb{R})} + (\tau, \overline{u}v)_{L^2(\mathbb{R})},$$
$$\mathrm{Dom}(\dot{t}_{\mathrm{S}_{00}}) := \mathrm{Dom}(\mathrm{S}_{00}(q)).$$

The corresponding quadratic forms are

$$\dot{t}_{\mathrm{S}_{00}}[u] = (u', u')_{L^2(\mathbb{R})} - (Q, \overline{u}'u + \overline{u}u')_{L^2(\mathbb{R})} + (\tau, |u|^2)_{L^2(\mathbb{R})}.$$

Introduce the notation

$$t_{Q,\tau}[u,v] := -(Q, \overline{u}'v + \overline{u}v')_{L^2(\mathbb{R})} + (\tau, \overline{u}v)_{L^2(\mathbb{R})}, \qquad \mathrm{Dom}(t_{Q,\tau}) := \mathrm{Dom}(\mathrm{S}_{00}(q)),$$
$$\dot{t}_0[u,v] := (u', v')_{L^2(\mathbb{R})}, \qquad \mathrm{Dom}(\dot{t}_0) := \mathrm{Dom}(\mathrm{S}_{00}(q)).$$

Then, due to Lemma 5, the forms $t_{Q,\tau}$ are 0-bounded with respect to the densely defined positive form $\dot{t}_0$,

$$|t_{Q,\tau}[u]| \leq K\varepsilon \dot{t}_0[u] + 4K\varepsilon^{-3}\|u\|^2_{L^2(\mathbb{R})} \quad \forall \varepsilon \in (0,1], \quad u \in \mathrm{Dom}(\dot{t}_0), \tag{11}$$
$$K := 2\left(\|Q\|_{L^2_{\text{unif}}(\mathbb{R})} + \|\tau\|_{L^1_{\text{unif}}(\mathbb{R})}\right).$$

Formula (11) implies that the sesquilinear forms $\dot{t}_{\mathrm{S}_{00}} = \dot{t}_0 + t_{Q,\tau}$ are closable, $t_{\mathrm{S}_{00}} := (\dot{t}_{\mathrm{S}_{00}})^{\sim}$:

$$t_{\mathrm{S}_{00}}[u,v] = (u', v')_{L^2(\mathbb{R})} - (Q, \overline{u}'v + \overline{u}v')_{L^2(\mathbb{R})} + (\tau, \overline{u}v)_{L^2(\mathbb{R})}, \quad \mathrm{Dom}(t_{\mathrm{S}_{00}}) = H^1(\mathbb{R}).$$

Forms $t_{\mathrm{S}_{00}}$ are densely defined, closed and sectorial. Then due to the First Representation Theorem [12], to the sesqulinear forms $t_{\mathrm{S}_{00}}$ we associate m-sectorial operators $\mathrm{S}_F(q)$ that are the Friedrichs extensions of the operators $\mathrm{S}_{00}(q)$.

Proposition 6. *The m-sectorial operators $\mathrm{S}_F(q)$ are described in the following way:*

$$\mathrm{S}_F u \equiv \mathrm{S}_F(q)u = \mathrm{l}[u], \quad \mathrm{Dom}(\mathrm{S}_F) = \left\{u \in H^1(\mathbb{R}) \;\middle|\; u,\, u^{[1]} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}), \mathrm{l}[u] \in L^2(\mathbb{R})\right\}.$$

The proof of Proposition 6 is similar to the proof of [10, Theorem 3.5] for real-valued distributions $q \in H^{-1}_{\text{unif}}(\mathbb{R})$.

Thus we have established that the following relations hold:

$$\mathrm{S}_{00} \subset \mathrm{S}_0 \subset \mathrm{S}_F \subset \mathrm{S}. \tag{12}$$

Passing to the adjoint operators (12) and using property 2^0 of Proposition 4, we obtain

$$\mathrm{S}^+_{00} \subset \mathrm{S}^+_0 \subset \mathrm{S}^*_F \subset \mathrm{S}^+. \tag{13}$$

One can easily prove that the operators S^*_F coincide with Friedrichs extensions S^+_F of the operators S^+_{00}.

Let us now define the operators (1) as form-sums.

Consider the sesquilinear forms generated by the distributions $q \in H^{-1}_{\text{unif}}(\mathbb{R})$

$$\dot{t}_q[u,v] := \langle q(x)u, v\rangle, \quad \operatorname{Dom}(\dot{t}_q) := C_0^\infty(\mathbb{R}),$$

where $\langle \cdot,\cdot\rangle$ is a sesquilinear form pairing the spaces of generalized functions $\mathfrak{D}'(\mathbb{R})$ and test functions $C_0^\infty(\mathbb{R})$ with respect to the space $L^2(\mathbb{R})$.

Due to Lemma 5, the forms

$$\dot{t}_q[u] = \langle q(x)u, v\rangle = -(Q, \overline{u}'u + \overline{u}u')_{L^2(\mathbb{R})} + (\tau, |u|^2)_{L^2(\mathbb{R})}, \quad u \in C_0^\infty(\mathbb{R}),$$

satisfy the following estimates:

$$\left|\dot{t}_q[u]\right| \leq 2\left(\|Q\|_{L^2_{\text{unif}}(\mathbb{R})} + \|\tau\|_{L^1_{\text{unif}}(\mathbb{R})}\right)\left(\|u'\|^2_{L^2(\mathbb{R})} + 4\|u\|^2_{L^2(\mathbb{R})}\right), \quad u \in C_0^\infty(\mathbb{R}).$$

Therefore, the forms $\dot{t}_q$ allow for a continuous extension onto the space $H^1(\mathbb{R})$ [20]. The sesquilinear forms $\dot{t}_q[u,v]$ on the space $H^1(\mathbb{R})$ are represented as

$$t_q[u,v] = -(Q, \overline{u}'v + \overline{u}v')_{L^2(\mathbb{R})} + (\tau, \overline{u}v)_{L^2(\mathbb{R})}, \quad \operatorname{Dom}(t_q) = H^1(\mathbb{R}). \tag{14}$$

One can easily see that the following Lemma is true applying the estimates of Lemma 5.

Lemma 7. *The sesquilinear forms t_q are 0-bounded with respect to the sesquilinear form*

$$t_0[u,v] := (u', v')_{L^2(\mathbb{R})}, \quad \operatorname{Dom}(t_0) := H^1(\mathbb{R}).$$

Thus, the sesquilinear forms

$$t[u,v] := t_0[u,v] + t_q[u,v], \quad \operatorname{Dom}(t) := H^1(\mathbb{R}), \tag{15}$$

are densely defined, closed and sectorial. According to the First Representation Theorem [12], the forms t can be associated with m-sectorial operators $\mathrm{S}_{fs}(q)$, which are called the *form-sums* and denoted by

$$\begin{aligned}
&\mathrm{S}_{fs} \equiv \mathrm{S}_{fs}(q) := -\frac{d^2}{dx^2} \dotplus q(x),\\
&\operatorname{Dom}(\mathrm{S}_{fs}(q)) := \left\{u \in H^1(\mathbb{R}) \,\middle|\, -u'' + q(x)u \in L^2(\mathbb{R})\right\}.
\end{aligned}$$

Since the forms t coincide with the forms $t_{\mathrm{S}_{00}}$, the form-sum operators $\mathrm{S}_{fs}(q)$ and the Friedrichs extensions $\mathrm{S}_F(q)$ of operators $\mathrm{S}_{00}(q)$ coincide, $\mathrm{S}_F(q) = \mathrm{S}_{fs}(q)$.

Thus, relations (12) and (13) take the following form:

$$\begin{gathered}
\mathrm{S}_{00} \subset \mathrm{S}_0 \subset \mathrm{S}_F = \mathrm{S}_{fs} \subset \mathrm{S},\\
\operatorname{Dom}(\mathrm{S}_F) \subset H^1(\mathbb{R}), \quad \operatorname{Dom}(\mathrm{S}) \subset H^1_{\text{loc}}(\mathbb{R}),
\end{gathered} \tag{16}$$

$$\begin{gathered}
\mathrm{S}^+_{00} \subset \mathrm{S}^+_0 \subset \mathrm{S}^+_F = \mathrm{S}^+_{fs} = \mathrm{S}^*_F = \mathrm{S}^*_{fs} \subset \mathrm{S}^+,\\
\operatorname{Dom}(\mathrm{S}^+_F) \subset H^1(\mathbb{R}), \quad \operatorname{Dom}(\mathrm{S}^+) \subset H^1_{\text{loc}}(\mathbb{R}).
\end{gathered} \tag{17}$$

Proposition 8. *Suppose* $\operatorname{Dom}(\mathrm{S}) \subset H^1(\mathbb{R})$. *Then the operators* $\mathrm{S}_0(q)$ *and* $\mathrm{S}^+_0(q)$ *are m-sectorial and*

$$\begin{aligned}
&\mathrm{S}_0 = \mathrm{S}_F = \mathrm{S}_{fs} = \mathrm{S},\\
&\mathrm{S}^+_0 = \mathrm{S}^+_F = \mathrm{S}^+_{fs} = \mathrm{S}^*_F = \mathrm{S}^*_{fs} = \mathrm{S}^+.
\end{aligned}$$

Proof. Let the assumptions of Proposition 8 be fulfilled. Then due to property 5^0 of Proposition 4 we also have $\operatorname{Dom}(\mathrm{S}^+) \subset H^1(\mathbb{R})$.

For $Q \in L^2_{\text{unif}}(\mathbb{R})$ and $u \in H^1(\mathbb{R})$ we have $Qu \in L^2(\mathbb{R})$ [10, Theorem 3.5] and therefore

$$u^{[1]} = u' - Qu \in L^2(\mathbb{R}), \quad v^{\{1\}} = v' - \overline{Q}v \in L^2(\mathbb{R}).$$

So, $\forall u \in \mathrm{Dom}(\mathrm{S})$ and $\forall v \in \mathrm{Dom}(\mathrm{S}^{+})$ we obtain

$$[u,v](-\infty) = \lim_{t\to-\infty}[u,v](t) = \lim_{t\to-\infty}\left(u(t)\overline{v^{\{1\}}(t)} - u^{[1]}(t)\overline{v(t)}\right) = 0, \tag{18}$$

$$[u,v](\infty) = \lim_{t\to\infty}[u,v](t) = \lim_{t\to\infty}\left(u(t)\overline{v^{\{1\}}(t)} - u^{[1]}(t)\overline{v(t)}\right) = 0. \tag{19}$$

Taking into account (18) and (19), property 4^0 of Proposition 4 implies the equalities

$$\mathrm{S}_0 = \mathrm{S}_F = \mathrm{S}, \quad \mathrm{S}_0^{+} = \mathrm{S}_F^{+} = \mathrm{S}^{+}.$$

Proposition is proved. □

Due to Proposition 14 (see Section 3.3) the operators $\mathrm{S}_0(q)$ are quasiaccretive,

$$\mathrm{Re}\,(\mathrm{S}_0 u, u)_{L^2(\mathbb{R})} \ge -4\,(2K+1)^4\,\|u\|^2_{L^2(\mathbb{R})}, \quad u \in \mathrm{Dom}(\mathrm{S}_0).$$

In what follows w.l.a.g. we assume that

$$\mathrm{Re}\,(\mathrm{S}_0 u, u)_{L^2(\mathbb{R})} \ge \|u\|^2_{L^2(\mathbb{R})}, \quad u \in \mathrm{Dom}(\mathrm{S}_0). \tag{20}$$

Obviously, together with (20) the following is also true:

$$\mathrm{Re}\,(\mathrm{S}_0^{+} v, v)_{L^2(\mathbb{R})} \ge \|v\|^2_{L^2(\mathbb{R})}, \quad v \in \mathrm{Dom}(\mathrm{S}_0^{+}). \tag{21}$$

Indeed, taking into account the property 5^0 of Proposition 4 we get

$$\mathrm{Re}\,(\mathrm{S}_0^{+} v, v)_{L^2(\mathbb{R})} = \mathrm{Re}\,\overline{(\mathrm{S}_0^{+} v, v)}_{L^2(\mathbb{R})} = \mathrm{Re}\,(\mathrm{S}_0 \overline{v}, \overline{v})_{L^2(\mathbb{R})} \ge \|v\|^2_{L^2(\mathbb{R})} \quad \forall v \in \mathrm{Dom}(\mathrm{S}_0^{+}).$$

The following lemma is used in the proof of Theorem A. It is proved by direct calculation.

Lemma 9. *Suppose* $u \in \mathrm{Dom}(\mathrm{S})$. *Then* $\forall \varphi \in C_0^{\infty}(\mathbb{R})$

$$i)\ \mathrm{l}[\varphi u] = \varphi \mathrm{l}[u] - \varphi'' u - 2\varphi' u'; \quad ii)\ \varphi u \in \mathrm{Dom}(\mathrm{S}_{00}).$$

Now let us prove Theorem A.

Let us prove that the operators $\mathrm{S}_0(q)$ are quasi-m-accretive. It is sufficient to show that

$$\mathrm{def}\,\mathrm{S}_0(q) := \dim\,(\mathrm{ran}\,\mathrm{S}_0(q))^{\perp} \equiv \dim\,(\ker\,\mathrm{S}^{+}(q)) = 0.$$

Let $v(x)$ be a solution of the equation

$$\mathrm{S}^{+}(q)v = 0. \tag{22}$$

Let us show that $v(x) \equiv 0$.

For any real function $\varphi \in C_0^{\infty}(\mathbb{R})$ due to Lemma 9 and property 5^0 of Proposition 4 we have $\varphi v \in \mathrm{Dom}(\mathrm{S}_{00}^{+})$. Therefore, taking into consideration that $\mathrm{l}^{+}[v] = 0$ due to (22), one calculates

$$(\mathrm{S}^{+}\varphi v, \varphi v)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (\varphi')^2 |v|^2 dx + \int_{\mathbb{R}} \varphi\varphi'(v\overline{v}' - v'\overline{v})\,dx. \tag{23}$$

Considering (20) and that

$$\mathrm{Re}\int_{\mathbb{R}} \varphi\varphi'(v\overline{v}' - v'\overline{v})\,dx = 0,$$

from (23) we obtain

$$\int_{\mathbb{R}} (\varphi')^2 |v|^2 d\,x \ge \int_{\mathbb{R}} \varphi^2 |v|^2 d\,x \quad \forall \varphi \in C_0^{\infty}(\mathbb{R}), \quad \mathrm{Im}\,\varphi = 0. \tag{24}$$

Let us then take a sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ such that the following is true:

i) $\varphi_n \in C_0^{\infty}(\mathbb{R})$, $\mathrm{Im}\,\varphi_n \equiv 0$;
ii) $\mathrm{supp}\,\varphi_n \subset [-n-1, n+1]$;
iii) $\varphi_n(x) = 1$, $x \in [-n, n]$;

iv) $|\varphi_n'(x)| \le C$.

Substituting functions φ_n into (24) we obtain

$$\int_{-n}^{n} |v|^2 dx \le \int_{\mathbb{R}} \varphi_n^2 |v|^2 dx \le \int_{\mathbb{R}} (\varphi_n')^2 |v|^2 dx \le C^2 \int\limits_{n \le |x| \le n+1} |v|^2 dx,$$

that is,

$$\int_{-n}^{n} |v|^2 dx \le C^2 \int\limits_{n \le |x| \le n+1} |v|^2 dx. \tag{25}$$

Taking into account that $v(x) \in L^2(\mathbb{R})$, passing in (25) to the limit as $n \to \infty$ we obtain $v(x) \equiv 0$.

Thus, the operators $\mathrm{S}_0(q)$ are proved to be quasi-m-accretive. Due to Proposition 14 they are m-sectorial.

Therefore, by the properties of the Friedrichs extensions [12] we have

$$\mathrm{S}_0(q) = \mathrm{S}_F(q). \tag{26}$$

Then taking into account property 2^0 of Proposition 4 from (26) we derive

$$\mathrm{S}^+(q) = \mathrm{S}_F^+(q), \quad \mathrm{Dom}(\mathrm{S}^+(q)) \subset H^1(\mathbb{R}).$$

Due to the property 5^0 of Proposition 4 from Proposition 8 we finally get the necessary result,

$$\mathrm{S}_0 = \mathrm{S}_F = \mathrm{S}_{fs} = \mathrm{S}.$$

Theorem A is proved completely. □

3.2. **Proof of Theorem B.** Let us suppose that assumptions of the theorem, that is, the formula (6) (or equivalently (7)), hold. Consider the sesquilinear forms

$$\dot{t}_0[u,v] := (\mathrm{S}(q)u, v)_{L^2(\mathbb{R})}, \quad \mathrm{Dom}(\dot{t}_0) := \mathrm{Dom}(\mathrm{S}(q)),$$
$$\dot{t}_n[u,v] := (\mathrm{S}(q_n)u, v)_{L^2(\mathbb{R})}, \quad \mathrm{Dom}(\dot{t}_n) := \mathrm{Dom}(\mathrm{S}(q_n)), \quad n \in \mathbb{N}.$$

The forms $\dot{t}_0$ and $\dot{t}_n$, $n \in \mathbb{N}$, are densely defined, closable and sectorial. Their closures may be represented in the following way:

$$t_0[u,v] = (u',v')_{L^2(\mathbb{R})} - (Q, \overline{u}'v + \overline{u}v')_{L^2(\mathbb{R})} + (\tau, \overline{u}v)_{L^2(\mathbb{R})}, \qquad \mathrm{Dom}(t_0) = H^1(\mathbb{R}),$$
$$t_n[u,v] = (u',v')_{L^2(\mathbb{R})} - (Q_n, \overline{u}'v + \overline{u}v')_{L^2(\mathbb{R})} + (\tau_n, \overline{u}v)_{L^2(\mathbb{R})}, \quad \mathrm{Dom}(t_n) = H^1(\mathbb{R}).$$

Further, applying the estimates of Lemma 5 we get

$$|t_n[u] - t_0[u]| \le a_n \|u'\|^2_{L^2(\mathbb{R})} + 4a_n \|u\|^2_{L^2(\mathbb{R})}, \tag{27}$$

where

$$a_n := 2\left(\|Q - Q_n\|_{L^2_{\mathrm{unif}}(\mathbb{R})} + \|\tau - \tau_n\|_{L^1_{\mathrm{unif}}(\mathbb{R})}\right),$$

and similarly to the proof of Lemma 11 (see below) we obtain

$$2\mathrm{Re}\, t_0[u] + 4\|u\|^2_{L^2(\mathbb{R})} \ge \|u'\|^2_{L^2(\mathbb{R})}. \tag{28}$$

Formulas (27) and (28) together with (6), (7) imply

$$|t_n[u] - t_0[u]| \le 2a_n \mathrm{Re}\, t_0[u] + 8a_n \|u\|^2_{L^2(\mathbb{R})}, \quad a_n \to 0, \quad n \to \infty.$$

To complete the proof we only need to apply [12, Theorem VI.3.6].

Theorem B is proved completely. □

To prove Corollary B.1 we need an auxiliary result. It is also of an independent interest.

Theorem 10. *The set*

$$C^{\infty}(\mathbb{R})\cap L^p_{\mathrm{unif}}(\mathbb{R}) \tag{29}$$

is everywhere dense in the Stepanov space $L^p_{\mathrm{unif}}(\mathbb{R})$, $1\le p<\infty$.

Proof. Set for $f\in L^p_{\mathrm{loc}}(\mathbb{R})$, $f_n:=\chi_{[n,n+1)}f$, $n\in\mathbb{Z}$. Let $\varepsilon>0$ be given. Since the set $C_0^{\infty}(a,b)$ is dense in the space $L^p(a,b)$, there is a function sequence $g_n\in C_0^{\infty}(\mathbb{R})$, $\operatorname{supp} g_n\subset(n,n+1)$, such that $\|f_n-g_n\|_{L^p(\mathbb{R})}<\varepsilon 2^{-|n|-2}$. Set $g_\varepsilon:=\sum_{n\in\mathbb{Z}}g_n$. Then $g_\varepsilon\in C^{\infty}(\mathbb{R})$ and $\|f-g_\varepsilon\|_{L^p(\mathbb{R})}<\varepsilon$. If $f\in L^p_{\mathrm{unif}}(\mathbb{R})$, then $g_\varepsilon\in L^p_{\mathrm{unif}}(\mathbb{R})$, since $\|f-g_\varepsilon\|_{L^p_{\mathrm{unif}}(\mathbb{R})}<\varepsilon$. If the function f is real-valued, then such are the functions f_n as well. Therefore, the functions g_ε may be chosen to be real-valued. □

Theorem 10 and [10, Theorem 2.1] imply the following important statement.

Corollary 10.1. *The set*

$$C^{\infty}(\mathbb{R})\cap L^1_{\mathrm{unif}}(\mathbb{R})$$

is everywhere dense in the space $H^{-1}_{\mathrm{unif}}(\mathbb{R})$.

Then Corollary B.1 follows from Theorem B and Corollary 10.1.

3.3. **Proof of Theorem C.** Theorem C follows from Theorem 13 below regarding perturbations of a positive quadratic form. It is abstract and can be of independent interest.

Let in an abstract Hilbert space H a densely defined closed positive sesquilinear form $\alpha_0[u,v]$ with domain $\mathrm{Dom}(\alpha_0)\subset H$ be given. Let $\beta[u,v]$ be a sesquilinear form defined on H with a domain $\mathrm{Dom}(\beta)\supset\mathrm{Dom}(\alpha_0)$.

Suppose the form β satisfies the following estimate:

$$\exists\, a,b,s>0:\quad |\beta[u]|\le a\varepsilon\alpha_0[u]+b\varepsilon^{-s}\|u\|_H^2\quad \forall\varepsilon>0,\quad u\in\mathrm{Dom}(\alpha_0). \tag{30}$$

Consider on the Hilbert space H the sum of forms α_0 and β,

$$\alpha[u,v]:=\alpha_0[u,v]+\beta[u,v],\quad \mathrm{Dom}(\alpha):=\mathrm{Dom}(\alpha_0).$$

A sesquilinear form α is a densely defined closed and sectorial form on the Hilbert space H. Let $\Theta(\alpha)$ be a numerical range of α,

$$\Theta(\alpha):=\alpha[u],\quad u\in\mathrm{Dom}(\alpha),\quad \|u\|_H=1.$$

According to our assumptions, $\Theta(\alpha_0)\subset[0,\infty)$. Let us establish properties of the set $\Theta(\alpha)$. To do that we require the following two lemmas.

Lemma 11. *The following estimates hold:*

$$|\operatorname{Im}\alpha[u]|\le 2a\varepsilon\operatorname{Re}\alpha[u]+2b\varepsilon^{-s}\|u\|_H^2,\quad 0<\varepsilon\le(2a+1)^{-1}. \tag{31}$$

Proof. According to our assumptions we have

$$\operatorname{Re}\alpha[u]=\alpha_0[u]+\operatorname{Re}\beta[u],\quad \operatorname{Im}\alpha[u]=\operatorname{Im}\beta[u],$$

and, due to (30),

$$|\operatorname{Im}\alpha[u]|\le a\varepsilon\alpha_0[u]+b\varepsilon^{-s}\|u\|_H^2. \tag{32}$$

Furthermore given that $0<\varepsilon\le(2a+1)^{-1}$ and therefore $1-a\varepsilon\ge\frac12$, for $\operatorname{Re}\alpha[u]$, we have

$$\operatorname{Re}\alpha[u]\ge\alpha_0[u]-|\operatorname{Re}\beta[u]|\ge(1-a\varepsilon)\alpha_0[u]-b\varepsilon^{-s}\|u\|_H^2\ge\frac12\alpha_0[u]-b\varepsilon^{-s}\|u\|_H^2,$$

and

$$\begin{gathered}2a\varepsilon\operatorname{Re}\alpha[u]\ge a\varepsilon\alpha_0[u]-2a\varepsilon\cdot b\varepsilon^{-s}\|u\|_H^2\ge a\varepsilon\alpha_0[u]-b\varepsilon^{-s}\|u\|_H^2,\\ 2a\varepsilon\operatorname{Re}\alpha[u]+b\varepsilon^{-s}\|u\|_H^2\ge a\varepsilon\alpha_0[u].\end{gathered} \tag{33}$$

From (32) and (33) we receive the required estimates,

$$|\operatorname{Im}\alpha[u]| \le 2a\varepsilon \operatorname{Re}\alpha[u] + 2b\varepsilon^{-s}\|u\|_H^2.$$

Lemma is proved. □

We introduce the following notation:

$$\mathcal{S}_{a,b,s,\varepsilon} := \left\{\lambda \in \mathbb{C} \,\big|\, |\operatorname{Im}\lambda| \le 2a\varepsilon \operatorname{Re}\lambda + 2b\varepsilon^{-s}\right\},$$
$$\mathcal{M}_{a,b,s} := \bigcap_{0<\varepsilon\le(2a+1)^{-1}} \mathcal{S}_{a,b,s,\varepsilon}.$$

Then due to Lemma 11 we have $\Theta(\alpha) \subset \mathcal{M}_{a,b,s}$.

Lemma 12. *The set $\mathcal{M}_{a,b,s}$ can be written as*

$$\mathcal{M}_{a,b,s} = \begin{cases} \left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \le \dfrac{2a}{2a+1}\operatorname{Re}\lambda + 2b(2a+1)^s\right\}, & \lambda_0 \le \operatorname{Re}\lambda \le \lambda_1, \\ \left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \le 2(s+1)b^{1/(s+1)}\left(\dfrac{a}{s}\right)^{s/(s+1)}(\operatorname{Re}\lambda)^{s/(s+1)}\right\}, & \lambda_1 < \operatorname{Re}\lambda, \end{cases}$$

where $\lambda_0 := -\dfrac{b}{a}(2a+1)^{s+1}$ is the vertex of the sector

$$\left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \le \frac{2a}{2a+1}\operatorname{Re}\lambda + 2b(2a+1)^s\right\},$$

and $\lambda_1 := \dfrac{bs}{a}(2a+1)^{s+1}$.

Proof. For convenience we will describe the set $\mathcal{M}_{a,b,s}$ in $\mathbb{R}^2$.

Let

$$y = 2a\varepsilon x + 2b\varepsilon^{-s} \tag{34}$$

be the line which bounds the corresponding sector from above. Let us find the locus of points of intersection of these lines when $0 < \varepsilon < (2a+1)^{-1}$,

$$2a\varepsilon_1 x + 2b\varepsilon_1^{-s} = 2a\varepsilon_2 x + 2b\varepsilon_2^{-s},$$
$$2a(\varepsilon_1 - \varepsilon_2)x = 2b(\varepsilon_2^{-s} - \varepsilon_1^{-s}),$$
$$x = -\frac{b}{a}\cdot\frac{\varepsilon_1^{-s} - \varepsilon_2^{-s}}{\varepsilon_1 - \varepsilon_2},$$
$$x \underset{\varepsilon_2\to\varepsilon_1}{\longrightarrow} \frac{sb}{a}\varepsilon_1^{-s-1},$$

and

$$y = 2a\varepsilon_1\cdot\frac{sb}{a}\varepsilon_1^{-s-1} + 2b\varepsilon_1^{-s} = 2(s+1)b\varepsilon_1^{-s}.$$

So, for $\varepsilon = (2a+1)^{-1}$ the set $\mathcal{M}_{a,b,s}$ is bounded from above by the line

$$y = \frac{2a}{2a+1}x + 2b(2a+1)^s, \tag{35}$$

and for $0 < \varepsilon < (2a+1)^{-1}$ by the curves

$$\begin{cases} x = \frac{sb}{a}\varepsilon^{-s-1}, \\ y = 2(s+1)b\varepsilon^{-s}. \end{cases} \tag{36}$$

If we express ε in terms of x in the first equality of (36) and substitute it in the equality for y, we obtain an explicit equation for curves (36),

$$y = 2(s+1)b^{1/(s+1)}\left(\frac{a}{s}\right)^{s/(s+1)} x^{s/(s+1)}, \quad x > x_1, \quad x_1 := \frac{bs}{a}(2a+1)^{s+1}. \tag{37}$$

The set $\mathcal{M}_{a,b,s}$ is bounded from below by curves of the form (35) and (37) with $-y$ instead of y.

Thus the set $\mathcal{M}_{a,b,s}$ in $\mathbb{R}^2$ can be represented in the following way:

$$\mathcal{M}_{a,b,s} = \begin{cases} \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, |y| \leq \dfrac{2a}{2a+1}x + 2b(2a+1)^s \right\}, & x_0 \leq x \leq x_1, \\ \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, |y| \leq 2(s+1)b^{1/(s+1)} \left(\dfrac{a}{s}\right)^{s/(s+1)} x^{s/(s+1)} \right\}, & x_1 < x, \end{cases}$$

where $x_0 := -\dfrac{b}{a}(2a+1)^{s+1}$ is the vertex of the sector

$$\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, |y| \leq \frac{2a}{2a+1}x + 2b(2a+1)^s \right\},$$

and $x_1 = \dfrac{bs}{a}(2a+1)^{s+1}$.

Lemma is proved. □

Lemmas 11 and 12 imply the following theorem.

Theorem 13. *The numerical range $\Theta(\alpha)$ of the sesquilinear form α is a subset of the set $\mathcal{M}_{a,b,s}$,*

$$\mathcal{M}_{a,b,s} = \begin{cases} \left\{ \lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq \dfrac{2a}{2a+1}\operatorname{Re}\lambda + 2b(2a+1)^s \right\}, & \lambda_0 \leq \operatorname{Re}\lambda \leq \lambda_1, \\ \left\{ \lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq 2(s+1)b^{1/(s+1)} \left(\dfrac{a}{s}\right)^{s/(s+1)} (\operatorname{Re}\lambda)^{s/(s+1)} \right\}, & \lambda_1 < \operatorname{Re}\lambda, \end{cases}$$

where $\lambda_0 = -\dfrac{b}{a}(2a+1)^{s+1}$ is the vertex of the sector

$$\left\{ \lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq \frac{2a}{2a+1}\operatorname{Re}\lambda + 2b(2a+1)^s \right\},$$

and $\lambda_1 = \dfrac{bs}{a}(2a+1)^{s+1}$.

Remark 13.1. Direct calculations show that the following inclusion is valid:

$$\mathcal{M}_{a,b,s} \subset \left\{ \lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq 2(s+1)b^{1/(s+1)} \left(\frac{a}{s}\right)^{s/(s+1)} \left(\operatorname{Re}\lambda + \frac{b}{a}(2a+1)^{s+1}\right)^{s/(s+1)} \right\}.$$

Theorem 13 is useful for preliminary localization of spectra of various operators.

For instance, if the potential $q \in H^{-1}_{\text{unif}}(\mathbb{R})$ is a complex-valued regular Borel measure such that

$$q = Q', \quad Q \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}): \quad |q(I)| \equiv \left| \int_I dQ \right| \leq K_0, \quad K_0 > 0,$$

for any interval $I \subset \mathbb{R}$ of unit length, then the forms satisfy the estimates (30) with $a = b = 4K_0$, $s = 1$ [24],

$$|t_q[u]| \equiv \left| \int_I |u|^2 dQ \right| \leq 4K_0\varepsilon \|u'\|^2_{L^2(\mathbb{R})} + 4K_0\varepsilon^{-1}\|u\|^2_{L^2(\mathbb{R})} \quad \forall \varepsilon \in (0,1], \quad u \in H^1(\mathbb{R}).$$

Then due to Theorem 13 the spectra $\operatorname{spec}(\mathrm{S}(q))$ of the operators $\mathrm{S}(q)$ belong to a quadratic parabola

$$\operatorname{spec}(S(q)) \subset \left\{ \lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq 16K_0 \left(\operatorname{Re}\lambda + (8K_0+1)^2\right)^{1/2} \right\}, \tag{38}$$

compare with [24, Proposition 2.3].

Applying Theorem 13 and estimates (11) we obtain a description of the numerical ranges of the preminimal operators $\mathrm{S}_{00}(q)$ and $\mathrm{S}_{00}^{+}(q)$.

Proposition 14. *The operators* $\mathrm{S}_{00}(q)$ *and* $\mathrm{S}_{00}^{+}(q)$ *are sectorial: for arbitrary* $\varepsilon > 0$ *the numerical ranges* $\Theta(\mathrm{S}_{00}(q))$ *and* $\Theta(\mathrm{S}_{00}^{+}(q))$ *are located within the sector*

$$\mathcal{S}_{K,\varepsilon} := \left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq 2K\varepsilon \operatorname{Re}\lambda + 8K\varepsilon^{-3}\right\}, \quad 0 < \varepsilon \leq (2K+1)^{-1}.$$

Furthermore,

$$\Theta(\mathrm{S}_{00}(q)) \subset \mathcal{M}_K, \quad \Theta(\mathrm{S}_{00}^{+}(q)) \subset \mathcal{M}_K,$$

where

$$\mathcal{M}_K := \begin{cases} \left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq \dfrac{2K}{2K+1}\operatorname{Re}\lambda + 8K(2K+1)^3\right\}, & \lambda_0 \leq \operatorname{Re}\lambda \leq \lambda_1 \\ \left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Im}\lambda| \leq \dfrac{32}{12^{3/4}}K\,(\operatorname{Re}\lambda)^{3/4}\right\}, & \lambda_1 < \operatorname{Re}\lambda, \end{cases}$$

with $\lambda_0 := -4(2K+1)^4$ *and* $\lambda_1 := 12(2K+1)^4$.

Estimates (8) result from Proposition 14 and Remark 13.1.

Now let $\operatorname{Im} q \equiv 0$. We estimate the lower bound of the operator $\mathrm{S}(q)$. From (11) for $K\varepsilon \leq 1/2$ we get

$$\begin{aligned}(39)\quad (\mathrm{S}(q)u,u)_{L^2(\mathbb{R})} &= \|u'\|_{L^2(\mathbb{R})} + t_{Q,\tau}[u] \geq \|u'\|_{L^2(\mathbb{R})} - K\varepsilon\|u'\|_{L^2(\mathbb{R})} - 4K\varepsilon^{-3}\|u\|^2_{L^2(\mathbb{R})} \\ &= (1-K\varepsilon)\|u'\|_{L^2(\mathbb{R})} - 4K\varepsilon^{-3}\|u\|^2_{L^2(\mathbb{R})} \geq -4K\varepsilon^{-3}\|u\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

The estimates (39) with $\varepsilon := \min\{1,(2K)^{-1}\}$ give us the required result

$$(\mathrm{S}(q)u,u)_{L^2(\mathbb{R})} \geq \begin{cases} -4K\|u\|^2_{L^2(\mathbb{R})}, & \text{if} \quad K < 1/2, \\ -32K^4\|u\|^2_{L^2(\mathbb{R})}, & \text{if} \quad K \geq 1/2. \end{cases}$$

Thus Theorem C is proved completely. □

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