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PARAMETER–ELLIPTIC OPERATORS ON THE EXTENDED SOBOLEV SCALE

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Dedicated to Professor Gorbachuk on the occasion of his 75th birthday

Abstract. Parameter–elliptic pseudodifferential operators given on a closed smooth manifold are investigated on the extended Sobolev scale. This scale consists of all Hilbert spaces that are interpolation spaces with respect to a Hilbert–Sobolev scale. We prove that these operators set isomorphisms between appropriate spaces of the scale provided the absolute value of the parameter is large enough. For solutions to the corresponding parameter–elliptic equations, we establish two-sided a priori estimates, in which the constants are independent of the parameter.

1. Introduction

Parameter–elliptic operators occupy a special position in the theory of elliptic differential equations. These operators are distinguished by the following fundamental property: if the absolute value of the complex parameter is large enough, then the elliptic operator defines an isomorphism between appropriate Sobolev spaces, and moreover the solution of the elliptic equation admits an a priory estimate in which the constant does not depend on the parameter. Elliptic operators with spectral parameter are simple and important examples of the operators discussed. Various classes of parameter–elliptic equations and boundary–value problems were introduced and investigated in the papers by S. Agmon [1], S. Agmon and L. Nirenberg [2], M. S. Agranovich and M. I. Vishik [3], M. S. Agranovich [4, 5], G. Grubb [6, Ch. 2], A. N. Kozhevnikov [7–10], R. Denk, R. Mennicken, and L. R. Volevich [11, 12], R. Denk and M. Fairman [13] and other papers (also see the surveys [14, 15] and the references therein). Such classes have important applications to the spectral theory of elliptic operators, to parabolic differential equations and others; note that the most significant results are obtained in the case of Hilbert spaces.

In this connection, of interest is an investigation of parameter–elliptic operators in classes of Hilbert spaces that are calibrated much finer than the Sobolev scale. For such classes, a sufficiently general function, not a number parameter, serves as the smoothness index. Among them, we consider the class of all Hilbert spaces that are interpolation spaces for the Hilbert Sobolev scale. This class consists of the Hörmander spaces $B_{2,k}$ [16, Sec. 2.2] for which the smoothness index k is an arbitrary radial function RO-varying at $+\infty$. Such a class is naturally called an extended Sobolev scale (by means of interpolation spaces); this scale is distinguished and investigated in [17] and [18, Sec. 2.4]. Since the isomorphism and Fredholm properties of linear operators are preserved under the

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interpolation of spaces, the extended Sobolev scale proved to be convenient and efficient in the theory of general elliptic operators (see [17, 19, 20] and [18, Sec. 2.4.3]).

In this paper we investigate parameter–elliptic pseudodifferential operators given on a closed smooth manifold and acting on the extended Sobolev scale. Our purpose is to show that these operators possess the above–mentioned property on this scale. Namely, we will prove a theorem on isomorphisms realized by a parameter–elliptic pseudodifferential operator and on a priory estimates of a solution to the corresponding elliptic equation.

Note that the theory of general elliptic equations and elliptic boundary–value problems is built for a narrower class of Hörmander spaces (called the refined Sobolev scale) by V. A. Mikhailets and the second author in a series of papers, among them we mention the articles [21–28], survey [29], and monograph [18]. Specifically, parameter–elliptic equations are investigated therein.

Nowadays Hörmander spaces and their various analogs, called the spaces of generalized smoothness, are of considerable interest both by themselves and in applications [30–33].

2. Statement of the problem

Let Γ be a closed (i.e. compact and without boundary) infinitely smooth manifold of dimension $n \geq 1$. A certain C^∞-density dx is supposed to be given on Γ. The linear topological spaces $C^\infty(\Gamma)$ of test functions and $\mathcal{D}'(\Gamma)$ of distributions defined on Γ are considered as antidual spaces with respect to the inner product in $L_2(\Gamma, dx)$. We suppose that functions and distributions are complex-valued, and interpret distributions as antilinear functionals.

Following [14, Sec. 4.1], we recall the definition of a parameter–elliptic pseudodifferential operator on Γ.

Let $\Psi^r_{\mathrm{ph}}(\Gamma)$ denote the class of polyhomogeneous (i.e. classical) pseudodifferential operators (PsDOs) of order $r \in \mathbb{R}$ defined on the manifold Γ. The principal symbol of a PsDO belonging to $\Psi^r_{\mathrm{ph}}(\Gamma)$ is an infinitely smooth and complex-valued function defined on the cotangent bundle $T^*\Gamma \backslash 0$ (here 0 is the zero-section) and being positively homogeneous of the degree r with respect to ξ in every section $T^*_x\Gamma \setminus \{0\}$, where $x \in \Gamma$. We admit that the principal symbol can be equal to zero identically, then $\Psi^r_{\mathrm{ph}}(\Gamma) \subset \Psi^k_{\mathrm{ph}}(\Gamma)$ whenever $r < k$. A linear differential operator of order $r \geq 1$ given on Γ and having infinitely smooth coefficients is an important special case of a PsDO belonging to $\Psi^r_{\mathrm{ph}}(\Gamma)$. Note that the PsDOs under consideration are linear and continuous on both topological spaces $C^\infty(\Gamma)$ and $\mathcal{D}'(\Gamma)$.

Let numbers $m > 0$ and $q \in \mathbb{N}$ be chosen arbitrarily. We consider a PsDO $A(\lambda)$ that belongs to $\Psi^{mq}_{\mathrm{ph}}(\Gamma)$ and depends on the complex-valued parameter λ in the following way:

$$A(\lambda) = \sum_{j=0}^{q} \lambda^{q-j} A_j. \tag{1}$$

Here $A_j \in \Psi^{mj}_{\mathrm{ph}}(\Gamma)$ for each $j \in \{0, \ldots, q\}$, and moreover A_0 is the operator of multiplication by a function $a_0 \in C^\infty(\Gamma)$. Note that since $m(q-j) + \operatorname{ord} A_j = \operatorname{ord} A(\lambda)$, the weight m is ascribed to λ in (1).

Let K be a fixed closed angle on the complex plain with the vertex at the origin (we do not exclude the case where K degenerates into a ray).

The PsDO $A(\lambda)$ is said to be parameter–elliptic in the angle K on the manifold Γ if

$$\sum_{j=0}^{q} \lambda^{q-j} a_{j,0}(x, \xi) \neq 0 \tag{2}$$

for each point $x \in \Gamma$, covector $\xi \in T^*_x\Gamma$ and the parameter $\lambda \in K$ such that $(\xi, \lambda) \neq 0$. Here $a_{j,0}(x, \xi)$ is the principle symbol of A_j, so $a_{0,0}(x, \xi) \equiv a_0(x)$. We also admit that the functions $a_{1,0}(x,\xi)$, $a_{2,0}(x,\xi),\ldots$ are equal to zero at $\xi=0$ (this assumption is connected with the fact that the principal symbols are not initially defined at $\xi=0$).

For instance, let a PsDO be of the form $A-\lambda I$, where $A\in\Psi^{m}_{\mathrm{ph}}(\Gamma)$ (as usual I denotes the identity operator). Then, for $A-\lambda I$, the parameter–ellipticity condition in K means that $a_0(x,\xi)\notin K$ whenever $\xi\neq 0$; here $a_0(x,\xi)$ is the principal symbol of A. This example is important in the spectral theory of elliptic operators.

We investigate properties of the parameter–elliptic PsDO $A(\lambda)$ on an extended Sobolev scale.

3. The extended Sobolev scale

Following [18, Sec. 2.4], we will introduce the spaces that form an extended Sobolev scale. They are parametrized with a function $\varphi\in\mathrm{RO}$ that characterizes regularity properties of the distributions belonging to the space. Here RO is the set of all Borel measurable functions $\varphi:[1,\infty)\to(0,\infty)$ for which there exist numbers $a>1$ and $c\geq 1$ such that

$$c^{-1}\leq\frac{\varphi(\lambda t)}{\varphi(t)}\leq c\quad\text{for each}\quad t\geq 1\quad\text{and}\quad\lambda\in[1,a] \tag{3}$$

(generally, the constants a and c depend on $\varphi\in\mathrm{RO}$). These functions are said to be RO-varying at $+\infty$. The class of RO-varying functions was introduced by V. G. Avakumović [34] in 1936 and has been sufficiently investigated [35, 36].

The class RO admits the following description:

$$\varphi\in\mathrm{RO}\quad\Longleftrightarrow\quad\varphi(t)=\exp\biggl(\beta(t)+\int_1^t\frac{\gamma(\tau)}{\tau}\,d\tau\biggr),\quad t\geq 1,$$

where the real-valued functions β and γ are Borel measurable and bounded on $[1,\infty)$. Note also that condition (3) is equivalent to the bilateral inequality

$$c^{-1}\lambda^{s_0}\leq\frac{\varphi(\lambda t)}{\varphi(t)}\leq c\lambda^{s_1}\quad\text{for each}\quad t\geq 1\quad\text{and}\quad\lambda\geq 1, \tag{4}$$

in which (another) constant $c\geq 1$ is independent of t and λ. Hence, for every function $\varphi\in\mathrm{RO}$, we can define the lower and the upper Matuszewska indices [37] as follows:

$$\sigma_0(\varphi):=\sup\{s_0\in\mathbb{R}:\text{ the left-hand inequality in (4) holds}\}, \tag{5}$$

$$\sigma_1(\varphi):=\inf\{s_1\in\mathbb{R}:\text{ the right-hand inequality in (4) holds}\} \tag{6}$$

(see [36, Theorem 2.2.2]); here $-\infty<\sigma_0(\varphi)\leq\sigma_1(\varphi)<\infty$.

Now let $\varphi\in\mathrm{RO}$ and introduce the necessary function spaces over $\mathbb{R}^n$ and then over Γ.

The linear space $H^{\varphi}(\mathbb{R}^n)$ is defined to consists of all distributions $w\in\mathcal{S}'(\mathbb{R}^n)$ such that their Fourier transforms $\widehat{w}:=\mathcal{F}w$ is locally Lebesgue integrable over $\mathbb{R}^n$ and satisfy the condition

$$\int_{\mathbb{R}^n}\varphi^2(\langle\xi\rangle)\,|\widehat{w}(\xi)|^2\,d\xi<\infty.$$

Here, as usual, $\mathcal{S}'(\mathbb{R}^n)$ is the linear topological space of tempered distributions given in $\mathbb{R}^n$, and $\langle\xi\rangle:=(1+|\xi|^2)^{1/2}$ is the smoothed modulus of $\xi\in\mathbb{R}^n$. The inner product in $H^{\varphi}(\mathbb{R}^n)$ is defined by the formula

$$(w_1,w_2)_{H^{\varphi}(\mathbb{R}^n)}:=\int_{\mathbb{R}^n}\varphi^2(\langle\xi\rangle)\,\widehat{w_1}(\xi)\,\overline{\widehat{w_2}(\xi)}\,d\xi.$$

It endows $H^{\varphi}(\mathbb{R}^n)$ with the Hilbert space structure and induces the norm $\|w\|_{H^{\varphi}(\mathbb{R}^n)}:=(w,w)^{1/2}_{H^{\varphi}(\mathbb{R}^n)}$.

The space $H^{\varphi}(\mathbb{R}^n)$ is a Hilbert and isotropic case of the spaces $B_{p,k}$ introduced and systematically investigated by L. Hörmander [16, Sec. 2.2] (also see [38, Sec. 10.1]). Namely, $H^{\varphi}(\mathbb{R}^n)=B_{p,k}$ provided $p=2$ and $k(\xi)=\varphi(\langle\xi\rangle)$ for all $\xi\in\mathbb{R}^n$. Not that, if

$p = 2$, then the Hörmander spaces coincide with the spaces introduced and investigated by L. R. Volevich and B. P. Paneah [39, Sec. 2].

To define an analog of $H^{\varphi}(\mathbb{R}^n)$ for the manifold Γ, choose a finite atlas belonging to the C^{∞}-structure on Γ. Let this atlas consist of local charts $\alpha_j : \mathbb{R}^n \leftrightarrow \Gamma_j$, $j = 1, \ldots, p$, where the open sets Γ_j form a finite covering of Γ. Also choose functions $\chi_j \in C^{\infty}(\Gamma)$, $j = 1, \ldots, p$, that satisfy the condition $\operatorname{supp} \chi_j \subset \Gamma_j$ and that form a partition of unity on Γ.

The linear space $H^{\varphi}(\Gamma)$ is defined to consist of all distributions $u \in \mathcal{D}'(\Gamma)$ such that $(\chi_j u) \circ \alpha_j \in H^{\varphi}(\mathbb{R}^n)$ for every $j \in \{1, \ldots, p\}$. Here $(\chi_j u) \circ \alpha_j$ is the representation of the distribution $\chi_j u$ in the local chart α_j. The inner product in $H^{\varphi}(\Gamma)$ is defined by the formula

$$(u_1, u_2)_{\varphi} := \sum_{j=1}^{p} ((\chi_j u_1) \circ \alpha_j, (\chi_j u_2) \circ \alpha_j)_{H^{\varphi}(\mathbb{R}^n)},$$

where $u_1, u_2 \in H^{\varphi}(\mathbb{R}^n)$. This inner product endows $H^{\varphi}(\mathbb{R}^n)$ with the Hilbert space structure and induces the norm $\|u\|_{\varphi} := (u, u)_{\varphi}^{1/2}$.

The Hilbert space $H^{\varphi}(\Gamma)$ does not depend (up to equivalence of norms) on our choice of local charts and partition of unity on Γ [18, Theorem 2.21]. This space is separable, and the continuous and dense embeddings $C^{\infty}(\Gamma) \hookrightarrow H^{\varphi}(\Gamma) \hookrightarrow \mathcal{D}'(\Gamma)$ hold.

If $\varphi(t) = t^s$ for each $t \geq 1$ with some $s \in \mathbb{R}$, then $H^{\varphi}(\mathbb{R}^n) =: H^{(s)}(\mathbb{R}^n)$ and $H^{\varphi}(\Gamma) =: H^{(s)}(\Gamma)$ are the inner product Sobolev spaces (of the differentiation order s) given over $\mathbb{R}^n$ and Γ respectively.

The class of Hilbert function spaces

$$\{H^{\varphi}(\mathbb{R}^n \text{ or } \Gamma) : \varphi \in \mathrm{RO}\}$$

is naturally said to be the extended Sobolev scale over $\mathbb{R}^n$ or Γ.

We mention some properties of the extended Sobolev scale on Γ connected with embedding of spaces. Let $\varphi, \varphi_1 \in \mathrm{RO}$; the function $\varphi(t)/\varphi_1(t)$ is bounded on a neighborhood of $+\infty$ if and only if $H^{\varphi_1}(\Gamma) \hookrightarrow H^{\varphi}(\Gamma)$. This embedding is continuous and dense; moreover, it is compact if and only if $\varphi(t)/\varphi_1(t) \to 0$ as $t \to +\infty$. Specifically, the following compact and dense embeddings hold:

$$H^{(s_1)}(\Gamma) \hookrightarrow H^{\varphi}(\Gamma) \hookrightarrow H^{(s_0)}(\Gamma) \quad \text{for each} \quad s_1 > \sigma_1(\varphi) \quad \text{and} \quad s_0 < \sigma_0(\varphi). \tag{7}$$

This properties result from the corresponding properties of the Hörmander spaces $B_{2,k}$ [16, Sec. 2.2].

4. The main result

Put $\varrho(t) := t$ for $t \geq 1$. The PsDO $A(\lambda)$, which order is mq, defines the bounded operator

$$A(\lambda) : \; H^{\varphi\varrho^{mq}}(\Gamma) \;\to\; H^{\varphi}(\Gamma) \quad \text{for each} \quad \lambda \in \mathbb{C} \quad \text{and} \quad \varphi \in \mathrm{RO}. \tag{8}$$

This fact will be proved in Section 6. Note here that $\varphi\rho^{mq} \in \mathrm{RO}$, and therefore operator (8) acts on the extended Sobolev scale.

The main result of the paper is the following.

Theorem. *Suppose that the PsDO $A(\lambda)$ is parameter–elliptic in the corner $K \subset \mathbb{C}$ on the manifold Γ. Then there exists a number $\lambda_0 > 0$ such that for every $\lambda \in K$ and $\varphi \in \mathrm{RO}$ we have the isomorphism*

$$A(\lambda) : \; H^{\varphi\varrho^{mq}}(\Gamma) \;\leftrightarrow\; H^{\varphi}(\Gamma) \quad \textit{whenever} \quad |\lambda| \geq \lambda_0. \tag{9}$$

Moreover, for each fixed $\varphi \in \mathrm{RO}$ there exists a number $c = c(\varphi) \geq 1$ such that

$$c^{-1} \|A(\lambda)u\|_{\varphi} \leq \big(\|u\|_{\varphi\varrho^{mq}} + |\lambda|^{q} \, \| u\|_{\varphi} \big) \leq c \, \|A(\lambda)u\|_{\varphi} \tag{10}$$

for every $\lambda \in K$, *with* $|\lambda| \geq \lambda_0$, *and all* $u \in H^{s+mq,\varphi}(\Gamma)$. *Here the number* c *does not depend on* λ *and* u.

This theorem is known in the Sobolev case, where $\varphi(t) \equiv t^s$ and $s \in \mathbb{R}$ (see [14, Theorem 4.1.2]). We will prove Theorem for arbitrary $\varphi \in \mathrm{RO}$ in Section 7 by applying interpolation with function parameter.

Note that the left-hand side of inequality in (10) remains true without the parameter–ellipticity assumption (see Lemma 2 in Section 6).

5. Interpolation with function parameter

The extended Sobolev scale possesses an important interpolation property, which we will use. Namely, every space $H^{\varphi}(\Gamma)$, with $\varphi \in \mathrm{RO}$, is the result of the interpolation (with an appropriate function parameter) between the Sobolev spaces $H^{(s_0)}(\Gamma)$ and $H^{(s_1)}(\Gamma)$ appearing in (7). (An analogous result holds for the spaces defined over $\mathbb{R}^n$.) In this connection we recall the definition of interpolation with function parameter in the case of general Hilbert spaces and then state some properties of the interpolation (see [18, Sec. 1.1]). It is sufficient to restrict ourselves to separable complex Hilbert spaces.

Let $X := [X_0, X_1]$ be an ordered couple of separable complex Hilbert spaces such that the continuous and dense embedding $X_1 \hookrightarrow X_0$ holds. We say that this couple is admissible. For X there exists an isometric isomorphism $J : X_1 \leftrightarrow X_0$ such that J is a self-adjoint positive operator on X_0 with the domain X_1. The operator J is called a generating operator for the couple X. This operator is uniquely determined by X.

Let $\psi \in \mathcal{B}$, where $\mathcal{B}$ is the set of all Borel measurable functions $\psi : (0, \infty) \to (0, \infty)$ such that ψ is bounded on each compact interval $[a, b]$, with $0 < a < b < \infty$, and that $1/\psi$ is bounded on every set $[r, \infty)$, with $r > 0$.

Consider the operator $\psi(J)$, which is defined (and positive) in X_0 as the Borel function ψ of J. Denote by $[X_0, X_1]_\psi$ or simply by X_ψ the domain of the operator $\psi(J)$ endowed with the inner product $(u_1, u_2)_{X_\psi} := (\psi(J)u_1, \psi(J)u_2)_{X_0}$ and the corresponding norm $\|u\|_{X_\psi} = \|\psi(J)u\|_{X_0}$. The space X_ψ is Hilbert and separable.

A function $\psi \in \mathcal{B}$ is called an interpolation parameter if the following condition is fulfilled for all admissible couples $X = [X_0, X_1]$ and $Y = [Y_0, Y_1]$ of Hilbert spaces and for an arbitrary linear mapping T given on X_0: if the restriction of T to X_j is a bounded operator $T : X_j \to Y_j$ for each $j \in \{0, 1\}$, then the restriction of T to X_ψ is also a bounded operator $T : X_\psi \to Y_\psi$.

If ψ is an interpolation parameter, then we say that the Hilbert space X_ψ is obtained by the interpolation of X with the function parameter ψ. In this case, the dense and continuous embeddings $X_1 \hookrightarrow X_\psi \hookrightarrow X_0$ hold.

Note that a function $\psi \in \mathcal{B}$ is an interpolation parameter if and only if ψ is pseudoconcave on a neighborhood of $+\infty$ (see [18, Theorem 1.9]). The latter condition means that there exists a concave function $\psi_1 : (b, \infty) \to (0, \infty)$, with $b \gg 1$, such that both functions ψ/ψ_1 and ψ_1/ψ are bounded on (b, ∞).

The above-mentioned interpolation property of the extended Sobolev is stated in the following way [18, Theorems 2.18 and 2.22].

Proposition 1. *Let a function* $\varphi \in \mathrm{RO}$ *and numbers* $s_0, s_1 \in \mathbb{R}$ *be such that* $s_0 < \sigma_0(\varphi)$ *and* $s_1 > \sigma_1(\varphi)$. *Set*

$$\psi(t) := \begin{cases} t^{-s_0/(s_1-s_0)}\, \varphi\bigl(t^{1/(s_1-s_0)}\bigr) & \text{for} \quad t \geq 1, \\ \varphi(1) & \text{for} \quad 0 < t < 1. \end{cases} \tag{11}$$

Then $\psi\in\mathcal{B}$ is an interpolation parameter, and

$$[H^{(s_0)}(\mathbb{R}^n),H^{(s_1)}(\mathbb{R}^n)]_\psi=H^{\varphi}(\mathbb{R}^n)\quad\textit{with equality of norms},$$
$$[H^{(s_0)}(\Gamma),H^{(s_1)}(\Gamma)]_\psi=H^{\varphi}(\Gamma)\quad\textit{with equivalence of norms.}$$

We will also use two properties of interpolation between abstract Hilbert spaces. The first of them is the following estimate of the operator norm in interpolation spaces [18, Theorem 1.8].

Proposition 2. *For every interpolation parameter $\psi\in\mathcal{B}$ there exists a number $\widetilde{c}=\widetilde{c}(\psi)>0$ such that*

$$\|T\|_{X_\psi\to Y_\psi}\leq\widetilde{c}\,\max\bigl\{\,\|T\|_{X_j\to Y_j}:\,j=0,\,1\,\bigr\}.$$

Here $X=[X_0,X_1]$ and $Y=[Y_0,Y_1]$ are arbitrary normal admissible couples of Hilbert spaces, and T is an arbitrary linear mapping given on X_0 and defining the bounded operators $T:X_j\to Y_j$, with $j=0,\,1$. The number $c_\psi>0$ does not depend on X, Y, and T.

Recall here that an admissible couple of Hilbert spaces $X=[X_0,X_1]$ is said to be normal if $\|u\|_{X_0}\leq\|u\|_{X_1}$ for each $u\in X_1$. Note that each admissible couple $[X_0,X_1]$ can be transformed into a normal couple by replacing the norm $\|\,u\,\|_{X_0}$ with the proportional norm $k\,\|u\|_{X_0}$, where k is the norm of the embedding operator $X_1\hookrightarrow X_0$.

The second property is useful when we interpolate between direct sums of Hilbert spaces.

Proposition 3. *Let $\bigl[X_0^{(j)},X_1^{(j)}\bigr]$, with $j=1,\ldots,p$, be a finite collection of admissible couples of Hilbert spaces. Then for every function $\psi\in\mathcal{B}$ we have*

$$\Bigl[\bigoplus_{j=1}^{p}X_0^{(j)},\,\bigoplus_{j=1}^{p}X_1^{(j)}\Bigr]_\psi=\bigoplus_{j=1}^{p}\bigl[X_0^{(j)},\,X_1^{(j)}\bigr]_\psi\quad\textit{with equality of norms.}$$

6. Some auxiliary results

Here we will prove some auxiliary results regarding the boundedness of the PsDO $A(\lambda)$ on the extended Sobolev scale.

Lemma 1. *Let $T\in\Psi^{r}_{\mathrm{ph}}(\Gamma)$ for some $r\in\mathbb{R}$. Then the PsDO T defines the bounded operator*

$$T:H^{\varphi\varrho^r}(\Gamma)\to H^{\varphi}(\Gamma)\quad\textit{for each}\quad\varphi\in\mathrm{RO}.$$

Proof. This lemma is known in the Sobolev case [14, Theorem 2.1.2]. We prove the lemma for arbitrary $\varphi\in\mathrm{RO}$ by applying Proposition 1. Choose numbers s_0 and s_1 so that $s_0<\sigma_0(\varphi)$ and $s_1>\sigma_1(\varphi)$. Let ψ be the interpolation parameter appearing in Proposition 1. Consider the bounded operators

$$T:\,H^{(s_j+r)}(\Gamma)\to H^{(s_j)}(\Gamma),\quad j=0,1,\tag{12}$$

which map between Sobolev spaces. Applying the interpolation with the function parameter ψ to (12), we obtain, by Proposition 1, the required bounded operator

$$T:\,H^{\varphi\varrho^r}(\Gamma)=\bigl[H^{(s_0+r)}(\Gamma),H^{(s_1+r)}(\Gamma)\bigr]_\psi\to\bigl[H^{(s_0)}(\Gamma),H^{(s_1)}(\Gamma)\bigr]_\psi=H^{\varphi}(\Gamma).$$

Note that the first equality is true here because $s_0+r<\sigma_0(\varphi\rho^r)$, $s_1+r>\sigma_1(\varphi\rho^r)$, and ψ satisfies formula (11), in which s_0, s_1, and φ should be replaced with s_0+r, s_1+r, and $\varphi\rho^r$ respectively. □

According to Lemma 1, the operator (8) is well-defined and bounded for each $\lambda\in\mathbb{C}$ and $\varphi\in\mathrm{RO}$. The next lemma refines this result.

Lemma 2. *For an arbitrary $\varphi \in \mathrm{RO}$ there exists a number $c' = c'(\varphi) > 0$ such that*

$$\|A(\lambda)u\|_{\varphi} \leq c'\big(\,\|u\|_{\varphi\varrho^{mq}} + |\lambda|^{q}\,\|\,u\|_{\varphi}\,\big) \tag{13}$$

for every $\lambda \in \mathbb{C}$ and each $u \in H^{\varphi\varrho^{mq}}(\Gamma)$. Here c' does not depend on λ and u.

Proof. We will use the following interpolation inequality:

$$r^{\varepsilon}\,\|u\|_{\eta} \leq \sqrt{2}\,\big(\,\|u\|_{\eta\varrho^{\varepsilon}} + r^{\varepsilon+\delta}\,\|u\|_{\eta\varrho^{-\delta}}\big), \tag{14}$$

where the number parameters $r, \varepsilon, \delta \geq 0$, the function parameter $\eta \in \mathrm{RO}$ and the distribution $u \in H^{\eta\varrho^{\varepsilon}}(\Gamma)$ are all arbitrary. (Similar inequalities are known for Sobolev spaces; see, e.g., [3, § 1, Sec. 6].)

Formula (14) follows from the evident inequality $1 \leq (k/r)^{\varepsilon} + (r/k)^{\delta}$ for all positive numbers r and k. Indeed, if we put $k := \langle\xi\rangle$ in this inequality and multiply both sides by $r^{\varepsilon}\eta(\langle\xi\rangle)\,|\widehat{w}(\xi)|$, where $\xi \in \mathbb{R}^n$ and $w \in H^{\eta\varrho^{\varepsilon}}(\mathbb{R}^n)$ are arbitrary, then we obtain an analog of (14) for spaces over $\mathbb{R}^n$. Namely, we can write the following:

$$\begin{aligned} r^{\varepsilon}\,\|w\|_{H^{\eta}(\mathbb{R}^n)} &= \|\,r^{\varepsilon}\eta(\langle\xi\rangle)\,|\widehat{w}(\xi)|\,\|_{L_2(\mathbb{R}^n,d\xi)} \\ \leq \|\,\eta(\langle\xi\rangle)\langle\xi\rangle^{\varepsilon}|\widehat{w}(\xi)|\,\|_{L_2(\mathbb{R}^n,d\xi)} &+ \|\,r^{\varepsilon+\delta}\eta(\langle\xi\rangle)\langle\xi\rangle^{-\delta}|\widehat{w}(\xi)|\,\|_{L_2(\mathbb{R}^n,d\xi)} \\ &= \|w\|_{H^{\eta\varrho^{\varepsilon}}(\mathbb{R}^n)} + r^{\varepsilon+\delta}\,\|w\|_{H^{\eta\varrho^{-\delta}}(\mathbb{R}^n)}. \end{aligned}$$

Here, as usual, $L_2(\mathbb{R}^n, d\xi)$ denotes the Hilbert space of functions square integrable over $\mathbb{R}^n$ with respect to the Lebesgue measure $d\xi$, where ξ is their argument. Whence we directly obtain (14) according to the definition of the spaces over Γ. Certainly, we should use the same collection of local charts and partition of unity for these spaces.

Now let $\varphi \in \mathrm{RO}$ be chosen arbitrarily. Then for each $\lambda \in \mathbb{C}$ and $u \in H^{\varphi\varrho^{mq}}(\Gamma)$, we can write

$$\begin{aligned} \|A(\lambda)u\|_{\varphi} &\leq \sum_{j=0}^{q} |\lambda|^{q-j}\,\|A_j u\|_{\varphi} \leq c_1 \sum_{j=0}^{q} |\lambda|^{q-j}\,\|u\|_{\varphi\varrho^{mj}} \\ &\leq c_1\,\sqrt{2}\,\big(\,\|u\|_{\varphi\varrho^{mq}} + |\lambda|^{q}\,\|\,u\|_{\varphi}\,\big). \end{aligned}$$

Here we apply (1), Lemma 1, and (14) in succession. According to Lemma 1, the number $c_1 > 0$ is independent of both λ and u in these inequalities. Note that we use (14) for $\eta := \varphi\varrho^{mj}$, $\varepsilon := m(q-j)$, $\delta := mj$ and $r := |\lambda|^{1/m}$, with $j = 0, \ldots, q$. Thus, we have the required inequality (13) with $c' := c_1\,\sqrt{2}$. ☐

7. Proof of the main result

Our proof of Theorem is based on an interpolation property of some parameter–dependent spaces. Therefore we will first introduce these spaces, establish this property, and then prove Theorem.

Let a function $\eta \in \mathrm{RO}$ and numbers $r, \theta \geq 0$ be given. We let $H^{\eta}(\Gamma, r, \theta)$ denote the space $H^{\eta}(\Gamma)$ which is endowed with the norm depending on the parameters r and θ in the following way:

$$\|u\|_{\eta,r,\theta} := \big(\,\|u\|_{\eta}^{2} + r^{2}\,\|u\|_{\eta\varrho^{-\theta}}^{2}\big)^{1/2}, \quad u \in H^{\eta}(\Gamma).$$

The space $H^{\eta}(\Gamma, r, \theta)$ is well-defined, and the norms in $H^{\eta}(\Gamma, r, \theta)$ and $H^{\eta}(\Gamma)$ are equivalent. This directly follows from the continuous embedding $H^{\eta}(\Gamma) \hookrightarrow H^{\eta\varrho^{-\theta}}(\Gamma)$. Note that the norm in the space $H^{\eta}(\Gamma, r, \theta)$ is induced by the inner product

$$(u_1, u_2)_{\eta,r,\theta} := (u_1, u_2)_{\eta} + r^{2}\,(u_1, u_2)_{\eta\varrho^{-\theta}}, \quad u_1, u_2 \in H^{\eta}(\Gamma);$$

therefore this space is Hilbert. If we consider the Sobolev case where $\eta(t) \equiv t^{s}$ for some $s \in \mathbb{R}$, then the space $H^{\eta}(\Gamma, r, \theta)$ is denoted by $H^{(s)}(\Gamma, r, \theta)$.

Returning to Theorem note that

$$\|u\|_{\varphi\varrho^{mq},|\lambda|^{q},mq} \leq \big(\,\|u\|_{\varphi\varrho^{mq}} + |\lambda|^{q}\,\|\,u\|_{\varphi}\,\big) \leq \sqrt{2}\,\|u\|_{\varphi\varrho^{mq},|\lambda|^{q},mq} \tag{15}$$

for each $u\in H^{\varphi\varrho^{mq}}(\Gamma)$.

According to Proposition 1, the spaces

$$\bigl[\,H^{(l_0)}(\Gamma,r,\theta),\,H^{(l_1)}(\Gamma,r,\theta)\,\bigr]_{\psi}\quad\text{and}\quad H^{\eta}(\Gamma,r,\theta)$$

are equal up to equivalence of norms. Here both the numbers $l_0<\sigma_0(\eta)$ and $l_1>\sigma_1(\eta)$ are arbitrary, whereas the interpolation parameter ψ is defined by the formula

$$\psi(t):=\begin{cases} t^{-l_0/(l_1-l_0)}\,\eta\bigl(t^{1/(l_1-l_0)}\bigr) & \text{for}\quad t\geq 1,\\ \eta(1) & \text{for}\quad 0<t<1.\end{cases}\tag{16}$$

We now refine this result in the following way.

Lemma 3. *Let a function $\eta\in\mathrm{RO}$ and numbers $l_0<\sigma_0(\eta)$, $l_1>\sigma_1(\eta)$, $\theta\geq 0$ be all chosen arbitrarily. Then there exists a number $c_0\geq 1$ such that*

$$c_0^{-1}\,\|u\|_{\eta,r,\theta}\leq\|u\|_{[H^{(l_0)}(\Gamma,r,\theta),H^{(l_1)}(\Gamma,r,\theta)]_{\psi}}\leq c_0\,\|u\|_{\eta,r,\theta}\tag{17}$$

for every number $r\geq 0$ and each distribution $u\in H^{\eta}(\Gamma)$. Here ψ is the interpolation parameter defined by (16)*, and the number c_0 does not depend on r and u.*

Proof. Let a number $r\geq 0$ be arbitrary. We will first prove the following: if we replace Γ with $\mathbb{R}^n$ in formula (17), then it holds for $c_0=1$.

Let $H^{\eta}(\mathbb{R}^n,r,\theta)$ denote the space $H^{\eta}(\mathbb{R}^n)$ endowed with the Hilbert norm

$$\begin{aligned}\|w\|_{H^{\eta}(\mathbb{R}^n,r,\theta)}:=&\Bigl(\,\|w\|^2_{H^{\eta}(\mathbb{R}^n)}+r^2\,\|w\|^2_{H^{\eta\varrho^{-\theta}}(\mathbb{R}^n)}\Bigr)^{1/2}\\ =&\Bigl(\,\int_{\mathbb{R}^n}\bigl(1+r^2\,\langle\xi\rangle^{-2\theta}\bigr)\,\eta^2(\langle\xi\rangle)\,|\widehat{w}(\xi)|^2\,d\xi\Bigr)^{1/2};\end{aligned}\tag{18}$$

here $w\in H^{\eta}(\mathbb{R}^n)$. This norm is equivalent to the norm in $H^{\eta}(\mathbb{R}^n)$ for every fixed $r\geq 0$. Hence, the space $H^{\eta}(\mathbb{R}^n,r,\theta)$ is Hilbert. If $\eta(t)\equiv t^s$ for some $s\in\mathbb{R}$ (the Sobolev case), then the space $H^{\eta}(\mathbb{R}^n,r,\theta)$ is denoted by $H^{(s)}(\mathbb{R}^n,r,\theta)$.

Calculate the norm in the interpolation space

$$\bigl[\,H^{(l_0)}(\mathbb{R}^n,r,\theta),\,H^{(l_1)}(\mathbb{R}^n,r,\theta)\,\bigr]_{\psi}.\tag{19}$$

Let J denote the PsDO in $\mathbb{R}^n$ with the symbol $\langle\xi\rangle^{l_1-l_0}$, where $\xi\in\mathbb{R}^n$ is an argument. We may verify directly that J is the generating operator for the couple of spaces appearing in (19). Applying the isometric isomorphism

$$\mathcal{F}:H^{(l_0)}(\mathbb{R}^n,r,\theta)\leftrightarrow L_2\bigl(\mathbb{R}^n,(1+r^2\,\langle\xi\rangle^{-2\theta})\,\langle\xi\rangle^{2l_0}\,d\xi\bigr),$$

we reduce the operator J to the form of multiplication by the function $\langle\xi\rangle^{l_1-l_0}$; here $\mathcal{F}$ is the Fourier transform. Therefore the operator $\psi(J)$ is reduced to the form of multiplication by the function $\psi(\langle\xi\rangle^{l_1-l_0})=\langle\xi\rangle^{-l_0}\eta(\langle\xi\rangle)$ in view of (16). Hence, given any $w\in H^{\eta}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|w\|^2_{[H^{(l_0)}(\mathbb{R}^n,r,\theta),H^{(l_1)}(\mathbb{R}^n,r,\theta)]_{\psi}}&=\|\psi(J)\,w\|^2_{H^{(l_0)}(\mathbb{R}^n,r,\theta)}\\ =\int_{\mathbb{R}^n}\bigl|\langle\xi\rangle^{-l_0}\,\eta(\langle\xi\rangle)\,\widehat{w}(\xi)\bigr|^2\,\bigl(1+r^2\,\langle\xi\rangle^{-2\theta}\bigr)\,\langle\xi\rangle^{2l_0}\,d\xi&=\|w\|^2_{H^{\eta}(\mathbb{R}^n,r,\theta)}<\infty;\end{aligned}$$

here (18) is used. Thus

$$\bigl[\,H^{(l_0)}(\mathbb{R}^n,r,\theta),\,H^{(l_1)}(\mathbb{R}^n,r,\theta)\,\bigr]_{\psi}=H^{\eta}(\mathbb{R}^n,r,\theta)\quad\text{with equality of norms.}\tag{20}$$

We will now prove (17) by applying property (20) and the definition of spaces over Γ. Fix a finite atlas $\{\alpha_j\}$ and partition of unity $\{\chi_j\}$ on Γ used in this definition (see Section 3); here $j=1,\ldots,p$.

Consider the linear mapping of the "rectification" of Γ, namely,

$$T:\,u\mapsto\bigl((\chi_1u)\circ\alpha_1,\ldots,(\chi_pu)\circ\alpha_p\bigr),\quad u\in\mathcal{D}'(\Gamma).$$

We may directly verify that this mapping defines the isometric operators

$$T:\, H^{\eta}(\Gamma,r,\theta)\to\bigl(H^{\eta}(\mathbb{R}^n,r,\theta)\bigr)^{p}, \tag{21}$$

$$T:\, H^{(l_j)}(\Gamma,r,\theta)\to\bigl(H^{(l_j)}(\mathbb{R}^n,r,\theta)\bigr)^{p},\quad j\in\{0,\,1\}. \tag{22}$$

Applying the interpolation with the parameter ψ to (22), we obtain the bounded operator

$$T:\,\bigl[H^{(l_0)}(\Gamma,r,\theta),\,H^{(l_1)}(\Gamma,r,\theta)\bigr]_{\psi}\to\bigl[\bigl(H^{(l_0)}(\mathbb{R}^n,r,\theta)\bigr)^{p},\,\bigl(H^{(l_1)}(\mathbb{R}^n,r,\theta)\bigr)^{p}\bigr]_{\psi}. \tag{23}$$

Here the couples of spaces are normal. Therefore, according to Proposition 2, the norm of the operator (23) does not exceed a certain number $\widetilde{c}=\widetilde{c}(\psi)>0$, which is independent of the parameter r, specifically. Whence, by Proposition 3 and property (20), we obtain the bounded operator

$$T:\,\bigl[H^{(l_0)}(\Gamma,r,\theta),\,H^{(l_1)}(\Gamma,r,\theta)\bigr]_{\psi}\to\bigl(H^{\eta}(\mathbb{R}^n,r,\theta)\bigr)^{p},\quad\text{whose norm }\leq\widetilde{c}. \tag{24}$$

Along with T, consider the linear mapping of “sewing”

$$K:\,(w_1,\dots,w_p)\mapsto\sum_{j=1}^{p}\Theta_j\bigl((\eta_j w_j)\circ\alpha_j^{-1}\bigr),$$

where $w_1,\dots,w_p$ are distributions defined in $\mathbb{R}^n$. Here the function $\eta_j\in C^{\infty}(\mathbb{R}^n)$ is equal to 1 on the set $\alpha_j^{-1}(\operatorname{supp}\chi_j)$ and is compactly supported, whereas Θ_j denotes the operator of extension by zero from Γ_j onto Γ. We have the bounded operators

$$K:\,\bigl(H^{(s)}(\mathbb{R}^n)\bigr)^{p}\to H^{(s)}(\Gamma)\quad\text{for each}\quad s\in\mathbb{R}, \tag{25}$$

$$K:\,\bigl(H^{\varphi}(\mathbb{R}^n)\bigr)^{p}\to H^{\varphi}(\Gamma)\quad\text{for each}\quad \varphi\in\mathrm{RO}. \tag{26}$$

Note that the boundedness of the operator (25) is a known property of Sobolev spaces (see, e.g., [16, Sec. 2.6] or [18, p. 86]). The boundedness of the operator (26) follows from this property with the help of interpolation. Namely, let φ, s_0, s_1, and ψ be the same as that in Proposition 1. Then applying the interpolation with the function parameter ψ to (25) with $s\in\{s_0,s_1\}$, we get the boundedness of the operator (26) by virtue of Propositions 1 and 3.

Let c_1 be the maximum of the norms of the operators (25) and (26), where $s\in\{l_0,l_0-\theta,l_1,l_1-\theta\}$ and $\varphi\in\{\eta,\eta\varrho^{-\theta}\}$. The number $c_1>0$ does not depend on the parameter r. We may directly verify that the norms of the operators

$$K:\,\bigl(H^{\eta}(\mathbb{R}^n,r,\theta)\bigr)^{p}\to H^{\eta}(\Gamma,r,\theta), \tag{27}$$

$$K:\,\bigl(H^{(l_j)}(\mathbb{R}^n,r,\theta)\bigr)^{p}\to H^{(l_j)}(\Gamma,r,\theta),\quad j=0,\,1, \tag{28}$$

does not exceed the number c_1. Applying the interpolation with the parameter ψ to (28), we obtain the bounded operator

$$K:\,\bigl[\,\bigl(H^{(l_0)}(\mathbb{R}^n,r,\theta)\bigr)^{p},\,\bigl(H^{(l_1)}(\mathbb{R}^n,r,\theta)\bigr)^{p}\bigr]_{\psi}\to\bigl[\,H^{(l_0)}(\Gamma,r,\theta),\,H^{(l_1)}(\Gamma,r,\theta)\bigr]_{\psi}. \tag{29}$$

Its norm does not exceed $\widetilde{c}\,c_1$ in view of Proposition 2 (note that both couples of spaces are normal in (29)). Whence, by (20) and Proposition 3, we obtain the bounded operator

$$K\,:\,\bigl(H^{\eta}(\mathbb{R}^n,r,\theta)\bigr)^{p}\,\to\,\bigl[\,H^{(l_0)}(\Gamma,r,\theta),\,H^{(l_1)}(\Gamma,r,\theta)\bigr]_{\psi},\quad\text{whose norm}\quad\leq\widetilde{c}\,c_1. \tag{30}$$

By the choice of the functions χ_j and η_j, we may write

$$KTu=\sum_{j=1}^{p}\,\Theta_j\bigl((\eta_j\,((\chi_j u)\circ\alpha_j))\circ\alpha_j^{-1}\bigr)=\sum_{j=1}^{p}\,\Theta_j\bigl((\chi_j u)\circ\alpha_j\circ\alpha_j^{-1}\bigr)=\sum_{j=1}^{p}\,\chi_j u=u,$$

that is, $KTu=u$ for each $u\in\mathcal{D}'(\Gamma)$. Therefore, multiplying (30) by the isometric operator (21), we obtain the bounded identity operator

$$I=KT:\,H^{\eta}(\Gamma,r,\theta)\to\bigl[\,H^{(l_0)}(\Gamma,r,\theta),\,H^{(l_1)}(\Gamma,r,\theta)\bigr]_{\psi},$$

whose norm $\leq \widetilde{c}\,c_1$. Besides, taking the product of the operators (27) and (24) (the norm of (27) does not exceed c_1), we get another bounded identity operator

$$I = KT:\ \bigl[\,H^{(l_0)}(\Gamma,r,\theta),\,H^{(l_1)}(\Gamma,r,\theta)\,\bigr]_{\psi} \to H^{\eta}(\Gamma,r,\theta),$$

whose norm $\leq \widetilde{c}\,c_1$. These identity operators yield the required estimate (17), where the number $c_0 := \widetilde{c}\,c_1 \geq 1$ does not depend on $r \geq 0$ and $u \in H^{\eta}(\Gamma)$. □

Now, applying Lemma 3, we can give a proof of the theorem.

Proof of Theorem. As has been mentioned in Section 4, Theorem is known in the case of Sobolev inner–product spaces. By using parameter-depended spaces introduced above, we can reformulate Theorem for the Sobolev scale in the following way. There exists a number $\lambda_0 > 0$ such that the isomorphism

$$A(\lambda):\ H^{s+mq}(\Gamma,|\lambda|^q,mq) \leftrightarrow H^{s}(\Gamma) \tag{31}$$

holds for each $s \in \mathbb{R}$ and $\lambda \in K$ with $|\lambda| \geq \lambda_0$. Moreover, the norms of the operator (31) and its inverse are uniformly bounded with respect to λ.

Let $\varphi \in \mathrm{RO}$, choose numbers $s_0 < \sigma_0(\varphi)$ and $s_1 > \sigma_1(\varphi)$, and define the interpolation parameter ψ by (11). Applying the interpolation with this parameter to (31) for $s \in \{s_0, s_1\}$, we obtain the isomorphism

$$A(\lambda):\ \bigl[\,H^{s_0+mq}(\Gamma,|\lambda|^q,mq),\,H^{s_1+mq}(\Gamma,|\lambda|^q,mq)\,\bigr]_{\psi} \leftrightarrow \bigl[\,H^{s_0}(\Gamma),\,H^{s_1}(\Gamma)\,\bigr]_{\psi}. \tag{32}$$

According to Proposition 2, the norms of the operator (32) and its inverse are uniformly bounded with respect to λ. Now, by Lemma 3 and Proposition 1, we draw a conclusion that (32) yields the isomorphism

$$A(\lambda):\ H^{\varphi\varrho^{mq}}(\Gamma,|\lambda|^q,mq) \leftrightarrow H^{\varphi}(\Gamma) \tag{33}$$

such that the norms of the operator (33) and its inverse are uniformly bounded with respect to λ. Here we apply Lemma 3 for $\eta := \varphi\varrho^{mq}$, $l_0 := s_0 + mq < \sigma_0(\eta)$, $l_1 := s_1 + mq > \sigma_1(\eta)$, $\theta := mq$, and $r := |\lambda|^q$, and we also note that ψ satisfies (16). The isomorphism (33) and the norms property just proved mean in view of (15) that Theorem is true. □

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