

## EXTENDED WEYL THEOREMS AND PERTURBATIONS

M. H. M. RASHID

**ABSTRACT.** In this paper we study the properties (gaw), (aw), (gab) and (ab), a variant of Weyl's type theorems introduced by Berkani. We established for a bounded linear operator defined on a Banach space several sufficient and necessary conditions for which the properties (gaw), (aw), (gab) and (ab) hold. Among other things, we study the stability of the properties (gaw), (aw), (gab) and (ab) for a polaroid operator  $T$  acting on a Banach space, under perturbations by finite rank operators, by nilpotent operators and, more generally, by algebraic operators commuting with  $T$ .

### 1. INTRODUCTION

Throughout this paper,  $\mathbf{L}(\mathbb{X})$  denotes the Banach algebra of all bounded linear operators acting on a Banach space  $\mathbb{X}$ . For  $T \in \mathbf{L}(\mathbb{X})$ , let  $T^*$ ,  $\ker(T)$ ,  $\mathfrak{R}(T)$ ,  $\sigma(T)$  and  $\sigma_a(T)$  denote the adjoint, the null space, the range, the spectrum and the approximate point spectrum of  $T$  respectively. Let  $\alpha(T)$  and  $\beta(T)$  be the nullity and the deficiency of  $T$  defined by  $\alpha(T) = \dim \ker(T)$  and  $\beta(T) = \text{codim}(T)$ . Recall that an operator  $T \in \mathbf{L}(\mathbb{X})$  is called upper semi-Fredholm if  $\alpha(T) < \infty$  and  $\mathfrak{R}(T)$  is closed, and is called lower semi-Fredholm if  $\beta(T) < \infty$ . Let  $SF_+(\mathbb{X})$  and  $SF_-(\mathbb{X})$  denote the class of all upper (resp. lower) semi-Fredholm operators. If  $T \in \mathbf{L}(\mathbb{X})$  is either upper or lower semi-Fredholm operator, then  $T$  is called a semi-Fredholm operator, and the index of  $T$  is defined by

$$\text{ind}(T) = \alpha(T) - \beta(T).$$

If both  $\alpha(T)$  and  $\beta(T)$  are finite, then  $T$  is called a *Fredholm operator*. An operator  $T \in \mathbf{L}(\mathbb{X})$  is called a *Weyl operator* if it is a Fredholm operator of index 0. Define

$$SF_+^-(\mathbb{X}) = \{T \in SF_+(\mathbb{X}) : \text{ind}(T) \leq 0\}$$

and

$$SF_-^+(\mathbb{X}) = \{T \in SF_-(\mathbb{X}) : \text{ind}(T) \geq 0\}.$$

The classes of operators defined above generate the following spectra : The *Weyl spectrum* is defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl operator}\},$$

the *Weyl essential approximate spectrum* is defined by

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(\mathbb{X})\},$$

while the *Weyl essential surjective spectrum* is defined by

$$\sigma_{SF_-^+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_-^+(\mathbb{X})\}.$$

Let  $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$  and  $\Delta_a(T) = \sigma_a(T) \setminus \sigma_+^-(T)$ . Following Coburn [21], we say that *Weyl's theorem* holds for  $T \in \mathbf{L}(\mathbb{X})$  if  $\Delta(T) = E_0(T)$ , where  $E_0(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$ . Here and elsewhere in this paper, for  $K \subset \mathbb{C}$ ,  $\text{iso}K$  is the set of isolated points of  $K$ . According to Rakočević [30], an operator

---

2000 *Mathematics Subject Classification.* Primary 47A53, 47A55; Secondary 47A10, 47A11, 47A20.

*Key words and phrases.* Generalized Weyl's theorem, generalized  $a$ -Weyl's theorem, property (gb), property (gw), polaroid operators, perturbation theory.

$T \in \mathbf{L}(\mathbb{X})$  is said to satisfy *a-Weyl's theorem* if  $\Delta_a(T) = E_0^a(T)$ , where  $E_0^a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$ .

For  $T \in \mathbf{L}(\mathbb{X})$  and a nonnegative integer  $n$  define  $T_{[n]}$  to be the restriction of  $T$  to  $\mathfrak{R}(T^n)$  viewed as a map from  $\mathfrak{R}(T^n)$  into  $\mathfrak{R}(T^n)$  (in particular  $T_{[0]} = T$ ). If for some integer  $n$  the range space  $\mathfrak{R}(T^n)$  is closed and  $T_{[n]}$  is an upper (resp. a lower) semi-Fredholm operator, then  $T$  is called an *upper (resp. a lower) semi-B-Fredholm*. In this case the index of  $T$  is defined as the index of the semi-Fredholm operator  $T_{[n]}$ , see [11]. Moreover, if  $T_{[n]}$  is a Fredholm operator, then  $T$  is called a *B-Fredholm operator*. An operator  $T$  is called a *B-Weyl* [13, Definition 1.1] if it is a *B-Fredholm operator* of index zero. The *B-Weyl spectrum* of  $T$  is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a } B\text{-Weyl operator}\}.$$

Recall that the *ascent*,  $a(T)$ , of an operator  $T$  is the smallest non-negative integer  $p$  such that  $\ker(T^p) = \ker(T^{p+1})$ . If such an integer does not exist we put  $a(T) = \infty$ . Analogously, the *descent*,  $d(T)$ , of an operator  $T$  is the smallest non-negative integer  $q$  such that  $\mathfrak{R}(T^q) = \mathfrak{R}(T^{q+1})$ , and if such an integer does not exist we put  $d(T) = \infty$ . It is well known that if  $a(T)$  and  $d(T)$  are both finite then  $a(T) = d(T)$  [22, Proposition 1.49]. Moreover,  $0 < a(T - \lambda I) = d(T - \lambda I) < \infty$  precisely when  $\lambda$  is a pole of the resolvent of  $T$ , see Dowson [22, Theorem 1.54].

An operator  $T \in \mathbf{L}(\mathbb{X})$  is called *Drazin invertible* if it has a finite ascent and descent. The *Drazin spectrum* of  $T$  is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

Define also the set

$$LD(\mathbb{X}) := \left\{ T \in \mathbf{L}(\mathbb{X}) : a(T) < \infty \text{ and } \mathfrak{R}(T^{a(T)+1}) \text{ is closed} \right\}$$

and

$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(\mathbb{X})\}.$$

Following [15], an operator  $T \in \mathbf{L}(\mathbb{X})$  is said to be *left Drazin invertible* if  $T \in LD(\mathbb{X})$ . We say that  $\lambda \in \sigma_a(T)$  is a *left pole* of  $T$  if  $T - \lambda I \in LD(\mathbb{X})$ , and that  $\lambda \in \sigma_a(T)$  is a *left pole of finite rank* if  $\lambda$  is a left pole of  $T$  and  $\alpha(T - \lambda I) < \infty$ . Let  $\pi^a(T)$  denotes the set of all left poles of  $T$  and let  $\pi_0^a(T)$  denotes the set of all left poles of  $T$  of finite rank. From [15, Theorem 2.8], it follows that if  $T \in \mathbf{L}(\mathbb{X})$  is left Drazin invertible, then  $T$  is upper semi-*B-Fredholm* of index less or equal than 0.

We say that *Browder's theorem* holds for  $T \in \mathbf{L}(\mathbb{X})$  if  $\pi_0(T)$ , where  $\pi_0(T)$  is the set of all poles of  $T$  of finite rank and that *a-Browder's theorem* holds for  $T$  if  $\Delta_a(T) = \pi_0^a(T)$ . Let  $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$ . Following [13], we say that *generalized Weyl's theorem* holds for  $T \in \mathbf{L}(\mathbb{X})$  if  $\Delta^g(T) = E(T)$ , where  $E(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I)\}$ , and that *generalized Browder's theorem* holds for  $T$  if  $\Delta^g(T) = \pi(T)$ , where  $\pi(T)$  is the set of all poles of  $T$ . It is proved in [10, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem. In [15, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but not conversely, and under the assumption  $E(T) = \pi(T)$ , it is proved in [14, Theorem 2.9] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let  $SBF_+(\mathbb{X})$  be the class of all *upper semi-B-Fredholm operators*,  $SBF_+(\mathbb{X}) = \{T \in SBF_+(X) : \text{ind}(T) \leq 0\}$  and let  $SBF_-(\mathbb{X})$  be the class of all *lower semi-B-Fredholm operators*,  $SBF_-(\mathbb{X}) = \{T \in SBF_-(\mathbb{X}) : \text{ind}(T) \geq 0\}$ . The *upper B-Weyl spectrum* of  $T$  is defined by

$$\sigma_{SBF_+}^-(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_+(\mathbb{X})\},$$

while the *lower B-Weyl spectrum* of  $T$  is defined by

$$\sigma_{SBF_-}^+(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SBF_-(\mathbb{X})\}.$$

Let  $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ . We say that *generalized  $a$ -Weyl's theorem* holds for  $T \in \mathbf{L}(\mathbb{X})$  if  $\Delta_a^g(T) = E^a(T)$ , where  $E^a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I)\}$  and that  $T \in \mathbf{L}(\mathbb{X})$  obeys *generalized  $a$ -Browder's theorem* if  $\Delta_a^g(T) = \pi^a(T)$ . It is proved in [10, Theorem 2.2] that generalized  $a$ -Browder's theorem is equivalent to  $a$ -Browder's theorem, and it is known [15, Theorem 3.11] that an operator satisfying generalized  $a$ -Weyl's theorem satisfies  $a$ -Weyl's theorem, but not conversely, and under the assumption  $E^a(T) = \pi^a(T)$ , it is proved in [14, Theorem 2.10] that generalized  $a$ -Weyl's theorem is equivalent to  $a$ -Weyl's theorem.

Following [29], we say that  $T \in \mathbf{L}(\mathbb{X})$  possesses *property (w)* if  $\Delta_a(T) = E_0(T)$ . The property (w) has been studied in [4, 29]. Following [9], we say that  $T \in \mathbf{L}(\mathbb{X})$  possesses *property (gw)* if  $\Delta_a^g(T) = E(T)$ . The property (gw) has been introduced and studied in [9], which is an extension to the context of  $B$ -Fredholm theory of the property (w). It is shown [9, Theorem 2.3] that an operator possessing property (gw) possesses property (w), but not conversely, and it is shown [9, Theorem 2.4] that an operator possessing property (gw) satisfies generalized Weyl's theorem, but not conversely. For more details about these properties the reader should refer to [32, 34].

Following [17], we say that *property (b)* holds for  $T \in \mathbf{L}(\mathbb{X})$  if  $\Delta_a(T) = \pi_0(T)$  and that property (gb) holds for  $T$  if  $\Delta_a^g(T) = \pi(T)$ . Property (gb) extends property (b) to the context of  $B$ -Fredholm theory. It is known [17, Theorem 2.5] that an operator possessing property (b) satisfies  $a$ -Browder's theorem, but the converse does not hold in general and it is proved in [17, Theorem 2.3] that an operator possessing property (gb) possesses property (b), but not conversely and under the assumption  $\pi(T) = \pi_0(T)$ , it is proved in [17, Theorem 2.10] that the two properties are equivalent. Recently in [31], property (gb) and perturbations were extensively studied by Rashid.

Following [18], we say that *property (ab)* holds for  $T \in \mathbf{L}(\mathbb{X})$  if  $\Delta(T) = \pi_0^a(T)$ , and is said to possess property (gab) if  $\Delta^g(T) = \pi^a(T)$ . It is known that [18, Theorem 2.2] that an operator possessing property (gab) possesses property (ab), but not conversely and under the assumption  $\pi(T) = \pi^a(T)$ , it is proved in [18, Theorem 2.8] that the two properties are equivalent.

Following [18], we say that *property (aw)* holds for  $T \in \mathbf{L}(\mathbb{X})$  if  $\Delta(T) = E_0^a(T)$ , and is said to possess property (gaw) if  $\Delta^g(T) = E^a(T)$ . It is known that [18, Theorem 3.3] that an operator possessing property (gaw) possesses property (gab), but not conversely and under the assumption  $E^a(T) = \pi^a(T)$ , it is proved in [18, Theorem 3.5] that the two properties are equivalent.

The *quasinilpotent part*  $H_0(T - \lambda I)$  and the *analytic core*  $K(T - \lambda I)$  of  $T - \lambda I$  are defined by

$$H_0(T - \lambda I) := \{x \in \mathbb{X} : \lim_{n \rightarrow \infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(T - \lambda I) = & \{x \in \mathbb{X} : \text{there exists a sequence } \{x_n\} \subset \mathbb{X} \text{ and } \delta > 0 \\ & \text{for which } x = x_0, (T - \lambda I)x_{n+1} = x_n \text{ and} \\ & \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \dots\}. \end{aligned}$$

We note that  $H_0(T - \lambda I)$  and  $K(T - \lambda I)$  are generally non-closed hyper-invariant subspaces of  $T - \lambda I$  such that  $(T - \lambda I)^{-p}(0) \subseteq H_0(T - \lambda I)$  for all  $p = 0, 1, \dots$  and  $(T - \lambda I)K(T - \lambda I) = K(T - \lambda I)$ . Recall that if  $\lambda \in \text{iso}(\sigma(T))$ , then  $H_0(T - \lambda I) = \chi_T(\{\lambda\})$ , where  $\chi_T(\{\lambda\})$  is the local spectral subspace consisting of all  $x \in \mathcal{H}$  for which there exists an analytic function  $f : \mathbb{C} \setminus \{\lambda\} \rightarrow \mathbb{X}$  that satisfies  $(T - \mu I)f(\mu) = x$  for all  $\mu \in \mathbb{C} \setminus \{\lambda\}$  (see [24]).

## 2. NEW EXTENDED WEYL'S TYPE THEOREMS

Let  $\text{Hol}(\sigma(T))$  be the space of all functions that analytic in an open neighborhoods of  $\sigma(T)$ . Following [25] we say that  $T \in \mathbf{L}(\mathbb{X})$  has the single-valued extension property (SVEP) at point  $\lambda \in \mathbb{C}$  if for every open neighborhood  $U_\lambda$  of  $\lambda$ , the only analytic function  $f : U_\lambda \rightarrow \mathcal{H}$  which satisfies the equation  $(T - \mu)f(\mu) = 0$  is the constant function  $f \equiv 0$ . It is well-known that  $T \in \mathbf{L}(\mathbb{X})$  has SVEP at every point of the resolvent  $\rho(T) := \mathbb{C} \setminus \sigma(T)$ . Moreover, from the identity Theorem for analytic function it easily follows that  $T \in \mathbf{L}(\mathbb{X})$  has SVEP at every point of the boundary  $\partial\sigma(T)$  of the spectrum. In particular,  $T$  has SVEP at every isolated point of  $\sigma(T)$ . In [26, Proposition 1.8], Laursen proved that if  $T$  is of finite ascent, then  $T$  has SVEP, see also [33].

**Lemma 2.1.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . Then the following statements hold:*

- (i)  $T^*$  has the SVEP at every  $\lambda \in \sigma_{SF_+^-}(T)$  if and only if  $T$  possesses property (ab) and  $\sigma_W(T) = \sigma_{SF_+^-}(T)$ .
- (ii)  $T$  has the SVEP at every  $\lambda \in \sigma_{SF_+^+}(T)$  if and only if  $T$  possesses property (ab) and  $\sigma_W(T) = \sigma_{SF_+^+}(T)$ .
- (iii) If  $T$  possesses property (ab), then  $T$  has the SVEP at every  $\lambda \in \sigma_W(T)$ .

*Proof.* (i) Assume that  $T^*$  has the SVEP at every  $\lambda \notin \sigma_{SF_+^-}(T)$ . From [4, Theorem 2.2],  $T$  satisfies  $a$ -Browder's theorem, that is  $\Delta_a(T) = \pi_0^a(T)$ . By [19, Lemma 2.1], we have  $\Delta(T) = \Delta_a(T)$  and  $\sigma_{SF_+^-}(T) = \sigma_W(T)$ . Therefore  $\Delta(T) = \pi_0^a(T)$ , i.e.  $T$  possesses property (ab) and  $\sigma_{SF_+^-}(T) = \sigma_W(T)$ . Conversely, assume that  $T$  possesses property (ab) and  $\sigma_{SF_+^-}(T) = \sigma_W(T)$ . Let  $\lambda \in \sigma_{SF_+^-}(T) = \sigma_W(T)$ . If  $\lambda \in \sigma(T)$ , then  $\lambda \in \Delta(T) = \pi_0^a(T)$ . Hence  $\lambda \in \text{iso}\sigma_a(T)$  and  $T^*$  has the SVEP at  $\lambda$ . If  $\lambda \notin \sigma(T)$ , then  $\lambda \notin \sigma_W(T)$ . Therefore  $T - \lambda I$  is surjective. Hence  $T^*$  has the SVEP at  $\lambda$ .

(ii) Suppose that  $T$  has the SVEP at every  $\lambda \in \sigma_{SF_+^+}(T)$ . From [4, Theorem 2.2],  $T^*$  satisfies  $a$ -Browder's theorem which implies that  $T^*$  satisfies Browder's theorem, that is  $\Delta_a(T^*) = \pi_0^a(T^*)$ . From [19, Lemma 2.1] we have  $\Delta(T^*) = \Delta_a(T^*)$ . Hence property (ab) holds for  $T^*$ . Conversely, assume that  $T^*$  possesses property (ab) and  $\sigma_W(T) = \sigma_{SF_+^+}(T)$ . Let  $\lambda \notin \sigma_{SF_+^+}(T)$ . If  $\lambda \in \sigma(T^*)$ , then  $\lambda \in \Delta(T^*) = \pi_0^a(T^*)$ . Hence  $T$  has the SVEP at  $\lambda$ . If  $\lambda \notin \sigma(T^*)$ , then  $\lambda \notin \sigma_W(T)$ . Hence  $T - \lambda I$  is injective and  $T$  has the SVEP at  $\lambda$ .

(iii) Suppose that  $T$  possesses property (ab) and let  $\lambda \notin \sigma_W(T)$ . If  $\lambda \in \sigma(T)$ , then  $\lambda \in \pi_0^a(T)$  and hence  $T$  has the SVEP at  $\lambda$ . If  $\lambda \notin \sigma(T)$ , then  $T - \lambda I$  is invertible and hence  $T$  has the SVEP at  $\lambda$ .  $\square$

**Example 2.2.** 1) The following example shows that property (ab) does not imply the SVEP for  $T^*$  at every  $\lambda \in \sigma_W(T)$ . Consider the operator  $T = R \oplus S$  defined on the Banach space  $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ , where  $R$  is the right shift operator defined on  $\ell^2(\mathbb{N})$  and  $S$  is defined on  $\ell^2(\mathbb{N})$  by

$$S(x_1, x_2, x_3, \dots) = \left( \frac{1}{2}x_2, \frac{1}{3}x_3, \dots \right).$$

Then  $\sigma(T) = \sigma_W(T) = \mathbf{D}(0, 1)$  the closed unit disc in  $\mathbb{C}$ . This implies that  $\pi_0(T) = \emptyset$ . On the other hand,  $\sigma_a(T) = \sigma_{SF_+^-}(T) = \mathbf{C}(0, 1) \cup \{0\}$ ; where  $\mathbf{C}(0, 1)$  is the unit circle of  $\mathbb{C}$ . This implies that  $\pi_0^a(T) = \emptyset$ . Suppose that  $T^*$  has the SVEP at every  $\lambda \notin \sigma_{SF_+^-}(T)$ . Then  $\sigma_W(T) = \sigma_{SF_+^-}(T)$ . But this is impossible. Therefore there exists at least one scalar  $\mu_0 \notin \sigma_{SF_+^-}(T)$  such that  $T^*$  does not have the SVEP at  $\mu_0$ . Hence  $T$  possesses property (ab), but  $T^*$  does not have the SVEP at  $\mu_0 \notin \sigma_{SF_+^-}(T)$ . Now, we consider  $U = T^*$ . Then  $U^* = T$  possesses property (ab). But  $U$  does not have the SVEP at

$$\mu_0 \notin \sigma_{SF_+^-}(T) = \sigma_{SF_-^+}(U).$$

2) The converse of the statement (iii) of Lemma 2.1 is not true in general as the following example [4, Example 2.14] shows : Let  $R \in (\ell^2(\mathbb{N}))$  the unilateral right shift and  $S \in (\ell^2(\mathbb{N}))$  the operator defined by

$$S(x_1, x_2, \dots) = (0, x_2, x_3, \dots).$$

Consider the operator  $T = R \oplus S$ , then  $\sigma(T) = \sigma_W(T) = \mathbf{D}(0, 1)$  the closed unit disc in  $\mathbb{C}$ ,  $\text{iso}\sigma(T) = \emptyset$  and  $\sigma_a(T) = \mathbf{C}(0, 1) \cup \{0\}$ , where  $\mathbf{C}(0, 1)$  is the unit circle of  $\mathbb{C}$ . Therefore  $\sigma_a(T)$  has empty interior and  $T$  has the SVEP. On the other hand,  $\sigma_{SF_+^-}(T) = \mathbf{C}(0, 1)$  and  $\pi_0^\alpha(T) = \{0\}$ . Hence  $T$  does not possess property (ab).

**Lemma 2.3.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . Then the following statements hold:*

- (i)  $T^*$  has the SVEP at every  $\lambda \notin \sigma_{SBF_+^-}(T)$  if and only if  $T$  possesses property (gab) and  $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$ .
- (ii)  $T$  has the SVEP at every  $\lambda \notin \sigma_{SBF_-^+}(T)$  if and only if  $T$  possesses property (gab) and  $\sigma_{SBF_-^+}(T) = \sigma_{BW}(T)$ .
- (iii) If  $T$  possesses property (gab), then  $T$  has the SVEP at every  $\lambda \notin \sigma_{BW}(T)$ .

*Proof.* (i) Suppose that  $T^*$  has the SVEP at every  $\lambda \notin \sigma_{SBF_+^-}(T)$ . From [19, Lemma 2.4], we have  $\sigma_{BW}(T) = \sigma_D(T)$ , so  $T$  satisfies generalized  $\alpha$ -Browder's theorem  $\Delta_a^g(T) = \pi^\alpha(T)$ . Again by [19, Lemma 2.4], we have  $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$ ,  $\Delta_a^g(T) = \Delta^g(T)$  and  $T$  possesses property (gab). Conversely, suppose that property (gab) holds for  $T$  and  $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$ . Let  $\lambda \notin \sigma_{BW}(T)$ . If  $\lambda \in \sigma(T)$ , then  $\lambda \in \Delta^g(T) = \pi^\alpha(T)$ . Hence  $T^*$  has the SVEP at  $\lambda$ . The second possibility is that  $\lambda \notin \sigma(T)$ , in this case we have  $\lambda \notin \sigma_{BW}(T)$ , it follows that  $T - \lambda I$  is surjective. Hence  $T^*$  has the SVEP at  $\lambda$ .

(ii) Suppose that  $T$  has the SVEP at every  $\lambda \notin \sigma_{SBF_-^+}(T)$ . Then  $T^*$  satisfies generalized  $\alpha$ -Browder's theorem  $\Delta_a^g(T^*) = \pi^\alpha(T^*)$ , and again by [19, Lemma 2.4] we have  $\sigma_{BW}(T) = \sigma_{SBF_-^+}(T)$ ,  $\Delta_a^g(T^*) = \Delta^g(T^*)$  and  $T^*$  possesses property (gab). Conversely, suppose that  $T^*$  possesses property (gab) and  $\sigma_{BW}(T) = \sigma_{SBF_-^+}(T)$ . Let  $\lambda \notin \sigma_{SBF_-^+}(T)$ . If  $\lambda \in \sigma(T^*)$ , then  $\lambda \in \Delta^g(T^*) = \pi^\alpha(T^*)$ . Hence  $T$  has the SVEP at  $\lambda$ . If  $\lambda \notin \sigma(T^*)$ , then  $\lambda \notin \sigma_{BW}(T^*)$ . Hence  $T - \lambda I$  is injective and  $T$  has the SVEP at  $\lambda$ .

(iii) Suppose that  $T$  possesses property (gab) and let  $\lambda \notin \sigma_{BW}(T)$ . If  $\lambda \in \sigma(T)$ , then  $\lambda \in \pi^\alpha(T)$  and hence  $T$  has the SVEP at  $\lambda$ . If  $\lambda \notin \sigma(T)$ , then  $T - \lambda$  is injective and hence  $T$  has the SVEP at  $\lambda$ .  $\square$

**Example 2.4.** 1) The following example shows that property (gab) does not imply the SVEP for  $T^*$  at every  $\lambda \in \sigma_{BW}(T)$ . Consider the operator  $T = R \oplus S$  defined on the Banach space  $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ , where  $R$  is the right shift operator defined on  $\ell^2(\mathbb{N})$  and  $S$  is defined on  $\ell^2(\mathbb{N})$  by

$$S(x_1, x_2, x_3, \dots) = \left( \frac{1}{2}x_2, \frac{1}{3}x_3, \dots \right).$$

Then  $\sigma(T) = \sigma_{BW}(T) = \mathbf{D}(0, 1)$  the closed unit disc in  $\mathbb{C}$ . This implies that  $\pi(T) = \emptyset$ . On the other hand,  $\sigma_a(T) = \sigma_{SBF_+^-}(T) = \mathbf{C}(0, 1) \cup \{0\}$ ; where  $\mathbf{C}(0, 1)$  is the unit circle of  $\mathbb{C}$ . This implies that  $\pi^\alpha(T) = \emptyset$ . Suppose that  $T^*$  has the SVEP at every  $\lambda \notin \sigma_{SBF_+^-}(T)$ . Then  $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$ . But this is impossible. Therefore there exists at least one scalar  $\mu_0 \notin \sigma_{SF_+^-}(T)$  such that  $T^*$  does not have the SVEP at  $\mu_0$ . Hence  $T$  possesses property (gab), but  $T^*$  does not have the SVEP at  $\mu_0 \notin \sigma_{SF_+^-}(T)$ . Now, we consider  $U = T^*$ . Then  $U^* = T$  possesses property (gab). But  $U$  does not have the SVEP at

$$\mu_0 \notin \sigma_{SBF_{\pm}^{-}}(T) = \sigma_{SBF_{\pm}^{+}}(U).$$

2) The converse of the statement (iii) of Lemma 2.3 is not true in general as the following example [4, Example 2.14] shows : Let  $R \in (\ell^2(\mathbb{N}))$  the unilateral right shift and  $S \in (\ell^2(\mathbb{N}))$  the operator defined by

$$S(x_1, x_2, \dots) = (0, x_2, x_3, \dots).$$

Consider the operator  $T = R \oplus S$ , then  $\sigma(T) = \sigma_{BW}(T) = \mathbf{D}(0, 1)$  the closed unit disc in  $\mathbb{C}$ ,  $\text{iso}\sigma(T) = \emptyset$  and  $\sigma_a(T) = \mathbf{C}(0, 1) \cup \{0\}$ , where  $\mathbf{C}(0, 1)$  is the unit circle of  $\mathbb{C}$ . Therefore  $\sigma_a(T)$  has empty interior and  $T$  has the SVEP. On the other hand,  $\sigma_{SBF_{\pm}^{-}}(T) = \mathbf{C}(0, 1)$  and  $\pi^a(T) = \{0\}$ . Hence  $T$  does not possess property (gab).

As a consequence of [18, Corollary 2.7] and [18, Theorem 3.5], we have

**Proposition 2.5.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  satisfies property (gaw). Then  $T$  satisfies generalized Browder's theorem and  $\pi(T) = E^a(T)$ .*

**Definition 2.6.** An operator  $T \in \mathbf{L}(\mathbb{X})$  is said to be polaroid if  $\text{iso}\sigma(T)$  is empty or every isolated point of  $\sigma_a(T)$  is a pole of the resolvent.

**Definition 2.7.** An operator  $T \in \mathbf{L}(\mathbb{X})$  is said to be  $a$ -polaroid if  $\text{iso}\sigma_a(T)$  is empty or every isolated point of  $\sigma(T)$  is a pole of the resolvent.

Clearly,

$$T \text{ } a\text{-polaroid} \Rightarrow T \text{ polaroid.}$$

Observe that if  $T^*$  has SVEP then  $\sigma(T) = \sigma_a(T)$ , see [1, Corollary 2.45], so that

$$T^* \text{ has SVEP and } T \text{ polaroid} \Rightarrow T \text{ } a\text{-polaroid.}$$

If  $T$  is polaroid then  $T^*$  is polaroid [7]. Moreover, if  $T$  has SVEP then  $\sigma(T) = \sigma_a(T^*)$ , see [1, Corollary 2.45], hence

$$T \text{ has SVEP and } T \text{ polaroid} \Rightarrow T^* \text{ } a\text{-polaroid.}$$

**Theorem 2.8.** *Suppose that  $T \in \mathbf{L}(\mathbb{X})$  is  $a$ -polaroid. Then  $T$  satisfies property (ab) if and only if  $T$  satisfies property (aw). Analogously,  $T^*$  satisfies property (ab) if and only if  $T^*$  satisfies property (aw).*

*Proof.* The implication (aw)  $\Rightarrow$  (ab) holds for every  $T \in \mathbf{L}(\mathbb{X})$ , so we have only to show that (ab)  $\Rightarrow$  (aw). Let  $T$  satisfy property (ab). Then  $\Delta(T) = \pi_0^a(T)$ . Now, let  $\lambda \in E_0^a(T)$ . then  $\lambda$  is an isolated point of  $\sigma_a(T)$ , so that  $\lambda$  is a pole and consequently,  $\alpha(T - \lambda I) = (T - \lambda I) < \infty$ . Since  $\alpha(T - \lambda I) < \infty$  by [1, Theorem 3.4] it then follows that  $\beta(T - \lambda I) < \infty$ , hence  $T - \lambda I$  is Browder and consequently  $\lambda \in \pi_0(T)$ . Therefore,  $E_0^a(T) \subset \pi_0(T) \subset \pi_0^a(T)$ . The opposite inclusion holds for every operator, so  $E_0^a(T) = \pi_0^a(T)$  and hence  $\Delta(T) = E_0^a(T)$ , i.e.  $T$  satisfies property (aw).

The second statement is clear: if  $T$  is  $a$ -polaroid then  $T^*$  is polaroid ([7]), so the first part applies.  $\square$

The following two examples show that  $a$ -Weyl's theorem and property (ab) for  $T \in \mathbf{L}(\mathbb{X})$  are independent. The first example shows that Weyl's theorem does not imply property (ab).

**Example 2.9.** Let  $R$  be the canonical unilateral right shift on  $\ell^2(\mathbb{N})$  and let  $P$  denote the projection defined by

$$P(x_1, x_2, \dots) := (0, x_2, \dots) \text{ for all } x := (x_1, x_2, \dots) \in \ell^2(\mathbb{N}).$$

Consider  $T := R \oplus P$  on  $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ . Then  $\sigma(T) = \sigma_W(T) = \mathbf{D}(0, 1)$ , where  $\mathbf{D}(0, 1)$  is the closed unit disc of  $\mathbb{C}$ , so that  $\sigma(T)$  has no isolated points and hence

$\pi_0(T) = \emptyset$ . Furthermore,  $\sigma_a(T) = \mathbf{C}(0, 1) \cup \{0\}$ , where  $\mathbf{C}(0, 1)$  is the closed unit circle, and  $\sigma_{SF_+^-}(T) = \{0\}$ . Therefore,  $T$  does not satisfies property (ab), since

$$\Delta(T) \neq \pi_0^a(T) = \{0\}.$$

On the other hand,  $T$  satisfies  $a$ -Weyl's theorem, since  $\Delta_a(T) = E_0^a(T) = \{0\}$  and hence  $T$  satisfies Weyl's theorem.

The second example shows that property (ab) does not imply  $a$ -Weyl's theorem.

**Example 2.10.** Let  $R \in \underline{\ell}^2(\mathbb{N})$  be right shift and let  $L$  be the weighted unilateral left shift defined by

$$L(x_1, x_2, \dots) = \left( \frac{x_2}{2}, \frac{x_3}{3}, \dots \right) \quad \text{for all } x := (x_1, x_2, \dots) \in \ell^2(\mathbb{N}).$$

If  $T = R \oplus L$ , then  $\sigma(T) = \sigma_W(T) = \mathbf{D}(0, 1)$ , where  $\mathbf{D}(0, 1)$  is the closed unit disc of  $\mathbb{C}$ , so that  $\sigma(T)$  has no isolated points and hence  $\pi_0(T) = \emptyset$ . Moreover,  $\sigma_a(T) = \sigma_{SF_+^-}(T) = \mathbf{C}(0, 1) \cup \{0\}$ , thus

$$\Delta(T) = \pi_0^a(T) = \emptyset,$$

hence  $T$  satisfies property (ab). On the other hand,  $E_0^a(T) = \{0\}$  so  $T$  does not satisfy  $a$ -Weyl's theorem.

**Theorem 2.11.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  is  $a$ -polaroid then the following statements are equivalent:*

- (i)  $T$  satisfies property (aw);
- (ii)  $T$  satisfies Weyl's theorem;
- (iii)  $T$  satisfies property (ab).

*Proof.* We show that if  $T$  is  $a$ -polaroid then  $\pi_0(T) = E_0^a(T)$ . Let  $\lambda \in E_0^a(T)$ . then  $\lambda$  is an isolated point of  $\sigma_a(T)$ , so that  $\lambda$  is a pole and consequently,  $a(T - \lambda I) = (T - \lambda I) < \infty$ . Since  $\alpha(T - \lambda I) < \infty$  by [1, Theorem 3.4] it then follows that  $\beta(T - \lambda I) < \infty$ , hence  $T - \lambda I$  is Browder and consequently  $\lambda \in \pi_0(T)$ . The opposite inclusion holds for every operator, so  $E_0^a(T) = \pi_0(T)$ .

(i) $\Rightarrow$ (ii). If  $T$  satisfies property (aw) then  $\Delta(T) = E_0^a(T) = E_0(T)$ , thus  $T$  satisfies Weyl's theorem.

(ii) $\Rightarrow$ (iii). If  $T$  satisfies Weyl's theorem then  $\Delta(T) = E_0(T) = \pi_0^a(T)$ , so  $T$  satisfies property (ab).

(iii) $\Rightarrow$ (i). If  $T$  satisfies property (ab) then  $\Delta(T) = \pi_0^a(T) = E_0^a(T)$ , thus  $T$  satisfies property (aw).  $\square$

**Theorem 2.12.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  is  $a$ -polaroid then the following statements are equivalent:*

- (i)  $T$  satisfies property (gaw);
- (ii)  $T$  satisfies generalized Weyl's theorem;
- (iii)  $T$  satisfies property (gab).

*Proof.* We show that if  $T$  is  $a$ -polaroid then  $\pi(T) = E^a(T)$ . Let  $\lambda \in E^a(T)$ . then  $\lambda$  is an isolated point of  $\sigma_a(T)$ , so that  $\lambda$  is a pole and consequently,  $a(T - \lambda I) = (T - \lambda I) < \infty$ . Since  $\alpha(T - \lambda I) < \infty$  by [1, Theorem 3.4] it then follows that  $\beta(T - \lambda I) < \infty$ , hence  $T - \lambda I$  is Browder and consequently  $\lambda \in \pi(T)$ . The opposite inclusion holds for every operator, so  $E^a(T) = \pi(T)$ .

(i) $\Rightarrow$ (ii). If  $T$  satisfies property (gaw) then  $\Delta^g(T) = E^a(T) = E(T)$ , thus  $T$  satisfies generalized Weyl's theorem.

(ii) $\Rightarrow$ (iii). If  $T$  satisfies generalized Weyl's theorem then  $\Delta^g(T) = E(T) = \pi^a(T)$ , so  $T$  satisfies property (gab).

(iii) $\Rightarrow$ (i). If  $T$  satisfies property (gab) then  $\Delta^g(T) = \pi^a(T) = E^a(T)$ , thus  $T$  satisfies property (gaw).  $\square$

**Theorem 2.13.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  is  $a$ -polaroid. Then  $T$  possesses property (gw) if and only if*

- (i)  $T$  possesses property (gaw);
- (ii)  $\text{ind}(T - \lambda I) = 0$  for all  $\lambda \in \Delta_a^g(T)$ .

*Proof.* Assume that  $T$  possesses property (gw), then from [9, Theorem 2.4],  $T$  satisfies generalized Weyl's theorem, i.e.  $\Delta^g(T) = E(T)$ , and from [9, Theorem 2.5] we have  $E(T) = \pi^a(T)$ . As  $T$  is  $a$ -polaroid then  $E^a(T) = \pi^a(T)$ . Therefore  $\Delta^g(T) = E^a(T)$  and  $T$  possesses property (gaw), as  $T$  possesses property (gw) then it follows from [9, Theorem 2.4] that  $\text{ind}(T - \lambda I) = 0$  for all  $\lambda \in \Delta_a^g(T)$ . Conversely, assume that  $T$  possesses property (gaw) and  $\text{ind}(T - \lambda I) = 0$  for all  $\lambda \in \Delta_a^g(T)$ . If  $\lambda \in \Delta_a^g(T)$ , then  $T - \lambda I$  is a semi- $B$ -Fredholm such that  $\text{ind}(T - \lambda I) = 0$ . Hence  $T - \lambda I$  is a  $B$ -Weyl operator. Since  $T$  satisfies property (gaw) and  $T$  is  $a$ -polaroid then  $\lambda \in E^a(T) = E(T)$ . To show the opposite inclusion, let  $\lambda \in E^a(T) = E(T)$ , then  $T - \lambda I$  is a  $B$ -Weyl operator and since  $\lambda \in \sigma(T)$ , then  $\alpha(T - \lambda I) > 0$ . Thus  $\lambda \in \Delta_a^g(T)$ . Consequently,  $T$  satisfies property (gw).  $\square$

**Remark 2.14.** If  $T^*$  has SVEP, then it known [27, p. 35] that  $\sigma_a(T) = \sigma(T)$  and from [5, Theorem 2.9] we have  $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$ . Thus  $E^a(T) = E(T)$  and  $\Delta_a^g(T) = \Delta^g(T)$ .

**Theorem 2.15.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T^*$  has the SVEP, then the following conditions are equivalent:*

- (i) property (gw) holds for  $T$ ;
- (ii) generalized Weyl's theorem holds for  $T$ ;
- (iii) generalized  $a$ -Weyl's theorem holds for  $T$ ;
- (iv) property (gaw) holds for  $T$ .

*Proof.* Suppose that  $T^*$  has SVEP, then it follows from Remark 2.14 that

$$\sigma_a(T) = \sigma(T), \sigma_{SBF_+^-}(T) = \sigma_{BW}(T), E^a(T) = E(T) \quad \text{and} \quad \Delta_a^g(T) = \Delta^g(T).$$

Now the equivalence between (i), (ii) and (iii) follows from [9, Theorem 2.7].

(ii) $\Leftrightarrow$ (iv). If  $T$  satisfies generalized Weyl's theorem, then  $\Delta^g(T) = E(T)$  and since  $E(T) = E^a(T)$ . Then  $\Delta^g(T) = E^a(T)$  and so  $T$  satisfies property (gaw). Conversely, if property (gaw) holds for  $T$ , then  $\Delta^g(T) = E^a(T)$ . But  $E(T) = E^a(T)$ . So  $T$  satisfies generalized Weyl's theorem.  $\square$

If  $T$  has the SVEP, then from [27, p. 35] we have  $\sigma(T^*) = \sigma_a(T^*)$ . In the same way as in the previous theorem, we have the following result:

**Theorem 2.16.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  has the SVEP, then the following conditions are equivalent:*

- (i) property (gw) holds for  $T^*$ ;
- (ii) generalized Weyl's theorem holds for  $T^*$ ;
- (iii) generalized  $a$ -Weyl's theorem holds for  $T^*$ ;
- (iv) property (gaw) holds for  $T^*$ .

**Theorem 2.17.** *Let  $T \in \mathbf{L}(\mathbb{X})$  be a polaroid operator.*

- i) If  $T^*$  has the SVEP, then property (gaw) holds for  $T$ .
- ii) If  $T$  has the SVEP, then property (gaw) holds for  $T^*$ .



*Proof.* i) Suppose that  $T$  is a polaroid operator and  $T^*$  has the SVEP. Then from [8, Theorem 2.3],  $T$  satisfies generalized Browder's theorem. As  $T$  is polaroid, then  $E(T) = \pi(T)$ . Hence  $T$  satisfies generalized Weyl's theorem. Since  $T^*$  has the SVEP, from Theorem 2.15, it follows that  $T$  satisfies property (gaw).

ii) As  $T$  has the SVEP, then  $T$  satisfies generalized Browder's theorem. As  $T$  is polaroid, then  $E(T) = \pi(T)$ . Hence  $T$  satisfies generalized Weyl's theorem. As we have  $\sigma(T^*) = \sigma(T)$ ,  $\sigma_{BW}(T^*) = \sigma_{BW}(T)$ ,  $\pi(T) = \pi(T^*)$ ,  $E(T) = E(T^*)$ , then  $T^*$  satisfies generalized Weyl's theorem too. From Theorem 2.16, it follows that  $T^*$  satisfies property (gaw).  $\square$

**Theorem 2.18.** *Let  $T \in \mathbf{L}(\mathbb{X})$  be a polaroid operator. If  $T^*$  has the SVEP, then  $f(T)$  satisfies property (gaw) for every  $f \in \text{Hol}(\sigma(T))$ .*

*Proof.* Suppose that  $T^*$  has the SVEP, then  $f(T)^* = f(T^*)$  has the SVEP, which in turn implies that generalized Browder's theorem holds for  $f(T)$  that is  $\Delta^g(T) = \pi(f(T)) = \pi^\alpha(f(T))$ . As  $T$  is polaroid and  $T^*$  has SVEP, then  $T$  is  $a$ -polaroid. So it follows from [14, Theorem 3.3] that  $E^\alpha(f(T)) = \pi^\alpha(f(T))$ . Hence  $f(T)$  satisfies property (gaw).  $\square$

**Theorem 2.19.** *Let  $T \in \mathbf{L}(\mathbb{X})$ .  $T$  satisfies property (gaw), if and only if  $T$  satisfies property (aw) and  $\pi(T) = E^\alpha(T)$ .*

*Proof.* If  $T$  satisfies property (gaw), then from [18, Theorem 3.3]  $T$  satisfies property (aw). As  $T$  satisfies property (gaw) then  $\Delta^g(T) = E^\alpha(T)$ . Since  $T$  satisfies property (gaw), then  $T$  satisfies property (gab). So from [18, Corollary 2.7] and [18, Theorem 3.5], we conclude that  $\pi^\alpha(T) = \pi(T)$  and  $E^\alpha(T) = \pi^\alpha(T)$ . Conversely, assume that  $T$  satisfies property (aw) and  $\pi(T) = E^\alpha(T)$ , then it follows from [18, Theorem 3.6] that  $T$  satisfies property (ab) and hence from [18, Theorem 2.4] that  $T$  satisfies Browder's theorem and hence by [10, Theorem 2.1]  $T$  satisfies generalized Browder's theorem. That is,  $\Delta^g(T) = \pi(T)$ . Therefore,  $\Delta^g(T) = E^\alpha(T)$ . So  $T$  satisfies property (gaw).  $\square$

**Theorem 2.20.** *Let  $T \in \mathbf{L}(\mathbb{X})$ .  $T$  satisfies property (w) if and only if  $T$  satisfies property (aw) and  $E_0(T) = E_0^\alpha(T)$ .*

*Proof.* Suppose that  $T$  satisfies property (w), then by [4, Theorem 2.8],  $T$  satisfies Weyl's theorem, i.e.,  $\Delta(T) = E_0(T)$ , and from [18, Theorem 3.6] we have  $E_0^\alpha(T) = \pi_0^\alpha(T)$  and from [18, Corollary 2.6] we have  $\pi_0(T) = \pi_0^\alpha(T)$ . Hence  $\Delta(T) = E_0^\alpha(T)$ . So  $T$  satisfies property (aw). Conversely, Assume that  $T$  satisfies property (aw) and  $E_0(T) = E_0^\alpha(T)$ . Then  $T$  satisfies property (ab). So from Corollary 2.6 and Theorem 3.6 of [18] we conclude that  $\pi_0(T) = \pi_0^\alpha(T) = E_0^\alpha(T)$ . Hence  $T$  satisfies  $a$ -Browder's theorem. That is,  $\Delta_a(T) = \pi_0^\alpha(T)$  and so  $\Delta_a(T) = E_0^\alpha(T)$ . Therefore,  $T$  satisfies property (w).  $\square$

**Theorem 2.21.** *Let  $T \in \mathbf{L}(\mathbb{X})$ .  $T$  satisfies property (gw) if and only if  $T$  satisfies property (gaw) and  $E(T) = E^\alpha(T)$ .*

**Theorem 2.22.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  satisfies property (aw). Then  $T$  satisfies Weyl's theorem.*

*Proof.* From [18, Corollary 2.6] and [18, Theorem 3.6] we conclude that  $T$  satisfies property (ab), and  $\pi_0(T) = \pi_0^\alpha(T) = E_0^\alpha(T)$ . But  $\pi_0(T) \subseteq E_0(T) \subseteq E_0^\alpha(T)$  is always verified. Hence  $\Delta(T) = E_0(T)$ . That is,  $T$  satisfies Weyl's theorem.  $\square$

The following example shows the converse of Theorem 2.22 is not true in general.

**Example 2.23.** Let  $R \in \ell^2(\mathbb{N})$  be the unilateral right shift and

$$S(x_1, x_2, \dots) := (0, x_2, x_3, \dots) \quad \text{for all } x_n \in \ell^2(\mathbb{N}).$$

If  $T := R \oplus S$  then  $\sigma(T) = \sigma_W(T) = \mathbf{D}(0, 1)$ . so  $\text{iso}\sigma(T) = E_0(T) = \emptyset$ . Moreover,  $\sigma_a(T) = \mathbf{C}(0, 1) \cup \{0\}$ , where  $\mathbf{C}(0, 1)$  is the unit circle of  $\mathbb{C}$ ,  $\sigma_{SF_+^-}(T) = \mathbf{C}(0, 1)$ . On the other hand we have  $E_0^a(T) = \{0\}$ , then  $\Delta_a(T) = \{0\}$ . So  $T$  satisfies  $a$ -Weyl's theorem and hence  $T$  satisfies Weyl's theorem, while  $\Delta(T) = \emptyset \neq E_0^a(T)$ . So  $T$  does not satisfies property (aw).

**Theorem 2.24.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  satisfies property (gaw). Then  $T$  satisfies generalized Weyl's theorem.*

*Proof.* From [18, Corollary 2.7] and [18, Theorem 3.5] we conclude that  $T$  satisfies property (gab), and  $\pi(T) = \pi^a(T) = E^a(T)$ . But  $\pi(T) \subseteq E(T) \subseteq E^a(T)$  is always verified. Hence  $\Delta^g(T) = E(T)$ . That is,  $T$  satisfies generalized Weyl's theorem.  $\square$

The following example shows the converse of Theorem 2.24 is not true in general.

**Example 2.25.** Let  $R \in \ell^2(\mathbb{N})$  be the unilateral right shift and

$$S(x_1, x_2, \dots) := (0, x_2, x_3, \dots) \quad \text{for all } x_n \in \ell^2(\mathbb{N}).$$

If  $T := R \oplus S$  then  $\sigma(T) = \sigma_{BW}(T) = \mathbf{D}(0, 1)$ . so  $\text{iso}\sigma(T) = E(T) = \emptyset$ . Moreover,  $\sigma_a(T) = \mathbf{C}(0, 1) \cup \{0\}$ , where  $\mathbf{C}(0, 1)$  is the unit circle of  $\mathbb{C}$ ,  $\sigma_{SBF_+^-}(T) = \mathbf{C}(0, 1)$ . On the other hand we have  $E^a(T) = \{0\}$ , then  $\Delta_a^g(T) = \{0\}$ . So  $T$  satisfies generalized  $a$ -Weyl's theorem and hence  $T$  satisfies generalized Weyl's theorem, while  $\Delta^g(T) = \emptyset \neq E^a(T)$ . So  $T$  does not satisfies property (gaw).

### 3. PERTURBATIONS

We begin this section by the following lemmas in order to give the proof of our main results.

**Lemma 3.1.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  satisfies generalized Browder's theorem and if  $F$  is a finite rank operator commuting with  $T$ , then  $T + F$  satisfies generalized Browder's theorem.*

*Proof.* From the characterization of  $\sigma_{BW}(T)$  it follows that if  $F$  is a finite rank operator, then  $\sigma_{BW}(T + F) = \sigma_{BW}(T)$ . Moreover, if  $F$  commutes with  $T$ , it follows from [13, Theorem 2.7] that  $\sigma_D(T + F) = \sigma_D(T)$ . If  $T$  satisfies generalized Browder's theorem,  $\sigma_{BW}(T) = \sigma_D(T)$ . Hence  $\sigma_{BW}(T + F) = \sigma_D(T + F)$ , and so  $T + F$  satisfies generalized Browder's theorem.  $\square$

**Corollary 3.2.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  satisfies Browder's theorem and if  $F$  is a finite rank operator commuting with  $T$ , then  $T + F$  satisfies Browder's theorem.*

**Theorem 3.3.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $F$  is a finite rank operator commuting with  $T$ . Then the following statements are equivalent:*

- (i)  $T + F$  satisfies generalized Browder's theorem;
- (ii)  $\sigma_{BW}(T + F) = \sigma_D(T + F)$ ;
- (iii)  $\sigma(T + F) = \sigma_{BW}(T + F) \cup E(T + F)$ ;
- (iv)  $\text{acc}(\sigma(T + F)) \subseteq \sigma_{BW}(T + F)$ ;
- (v)  $\sigma(T + F) \setminus \sigma_{BW}(T + F) \subseteq E(T + F)$ .

*Proof.* (i) $\Rightarrow$ (ii). Suppose that  $T + F$  satisfies generalized Browder's theorem. Then  $\sigma(T + F) \setminus \sigma_{BW}(T + F) = \pi(T + F)$ . Let  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Then  $\lambda \in \pi(T + F)$ , and so  $T + F - \lambda I$  is Drazin invertible. Therefore  $\lambda \in \sigma(T + F) \setminus \sigma_D(T + F)$ , and hence,  $\sigma_D(T + F) \subseteq \sigma_{BW}(T + F)$ . On the other hand, since  $\sigma_{BW}(T) \subseteq \sigma_D(T)$  is always verified for any operator [15, Lemma 2.12].

(ii) $\Rightarrow$ (i). We assume that  $\sigma_{BW}(T + F) = \sigma_D(T + F)$  and we will establish that

$\sigma(T + F) \setminus \sigma_{BW}(T) = \pi(T + F)$ . Suppose first that  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Then  $\lambda \in \sigma(T + F) \setminus \sigma_D(T + F)$ , and so  $T + F - \lambda I$  is Drazin invertible. Therefore,  $T + F - \lambda I$  has finite ascent and descent. Since  $\lambda \in \sigma(T + F)$ , we have  $\lambda \in \pi(T + F)$ . Thus  $\sigma(T + F) \setminus \sigma_{BW}(T + F) \subseteq \pi(T + F)$ .

Conversely, suppose that  $\lambda \in \pi(T + F)$ . Then  $T + F - \lambda I$  is Drazin invertible but not invertible. Since  $\lambda$  is an isolated point of  $\sigma(T)$ , then  $T + F - \lambda I$  is  $B$ -Weyl. Therefore  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Thus  $\pi(T + F) \supseteq \sigma(T + F) \setminus \sigma_{BW}(T + F)$ .

(ii) $\Rightarrow$ (iii). Let  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Then  $\lambda \in \sigma(T + F) \setminus \sigma_D(T + F)$ , and so  $T + F - \lambda I$  is Drazin invertible but not invertible. Therefore  $\lambda \in E(T + F)$ . Thus  $\sigma(T + F) \subseteq \sigma_{BW}(T + F) \cup E(T + F)$ . Since the other inclusion is always true, we must have  $\sigma(T + F) = \sigma_{BW}(T + F) \cup E(T + F)$ .

(iii) $\Rightarrow$ (ii). Suppose  $\sigma(T + F) = \sigma_{BW}(T + F) \cup E(T + F)$ . To show that  $\sigma_{BW}(T + F) = \sigma_D(T + F)$ . it suffices to show that  $\sigma_{BW}(T + F) \subseteq \sigma_D(T + F)$ . Suppose that  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Then  $T + F - \lambda I$  is  $B$ -Weyl but not invertible. Since  $\sigma(T + F) = \sigma_{BW}(T + F) \cup E(T + F)$ , we see that  $\lambda \in E(T + F)$ . In particular,  $\lambda$  is an isolated point of  $\sigma(T + F)$ . Hence,  $T + F - \lambda I$  is Drazin invertible, and so  $\sigma_{BW}(T + F) = \sigma_D(T + F)$ .

(i) $\Leftrightarrow$ (iv). Suppose  $T + F$  satisfies generalized Browder's theorem. Then  $\sigma_{BW}(T + F) = \sigma(T + F) \setminus \pi(T + F)$ . Let  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Then  $\lambda \in \pi(T + F)$ , and so  $\lambda$  is an isolated point of  $\sigma(T + F)$ . Therefore  $\lambda \in \sigma(T + F) \setminus \text{acc}(\sigma(T + F))$ , and hence,  $\text{acc}(\sigma(T + F)) \subseteq \sigma_{BW}(T + F)$ .

Conversely, let  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Since  $\text{acc}(\sigma(T + F)) \subseteq \sigma_{BW}(T + F)$ , it follows that  $\lambda \in \text{iso}(\sigma(T + F))$  and  $T + F - \lambda I$  is  $B$ -Weyl. It follows from [13, Theorem 2.3] that  $\lambda \in \pi(T + F)$ . Therefore  $\sigma(T + F) \setminus \sigma_{BW}(T + F) \subseteq \pi(T + F)$ . For the converse, suppose  $\lambda \in \pi(T + F)$ . Then  $\lambda$  is a pole of the resolvent of  $T$ , and so  $\lambda$  is an isolated point of  $\sigma(T + F)$ . Therefore  $\lambda \in \sigma(T + F) \setminus \text{acc}(\sigma(T + F))$ . It follows from [13, Theorem 2.3] that  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Thus  $\pi(T + F) \subseteq \sigma(T + F) \setminus \sigma_{BW}(T + F)$ , and so  $T + F$  satisfies generalized Browder's theorem.

(iv) $\Leftrightarrow$ (v). Suppose that  $\text{acc}(\sigma(T + F)) \subseteq \sigma_{BW}(T + F)$ , and let  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ . Then  $T + F - \lambda I$  is  $B$ -Weyl but not invertible. Since  $\text{acc}(\sigma(T + F)) \subseteq \sigma_{BW}(T + F)$ ,  $\lambda$  is an isolated point of  $\sigma(T + F)$ . It follows from [13, Theorem 2.3] that  $\lambda$  is a pole of the resolvent of  $T + F$ . Therefore  $\lambda \in \pi(T + F)$ , and hence,  $\sigma(T + F) \setminus \sigma_{BW}(T + F) \subseteq E(T + F)$ .

Conversely, suppose that  $\sigma(T + F) \setminus \sigma_{BW}(T + F) \subseteq E(T + F)$  and let  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F) \subseteq E(T + F)$ . Then  $\lambda \in E(T + F)$ , and so  $\lambda$  is an isolated point of  $\sigma(T + F)$ . Therefore  $\lambda \in \sigma(T + F) \setminus \text{acc}(\sigma(T + F))$ , which implies that  $\text{acc}(\sigma(T + F)) \subseteq \sigma_{BW}(T + F)$ .  $\square$

**Definition 3.4.** A bounded linear operator  $T$  is said to be *algebraic* if there exists a non-trivial polynomial  $h$  such that  $h(T) = 0$ .

From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators  $K$  are algebraic; more generally, if  $K^n$  is a finite rank operator for some  $n \in \mathbb{N}$  then  $K$  is algebraic. Clearly, if  $T$  is algebraic then its dual  $T^*$  is algebraic, as well as  $T'$  in the case of Hilbert space operators.

**Corollary 3.5.** *Let  $T$  be quasinilpotent or algebraic. If  $F$  is a finite rank operator commuting with  $T$ . Then  $T + F$  satisfies generalized Browder's theorem.*

*Proof.* Straightforward from Theorem 3.3 and the facts that  $\text{acc}\sigma(T) = \emptyset$  whenever  $T$  is quasinilpotent or algebraic and  $\lambda \in \text{acc}\sigma(T) \Leftrightarrow \lambda \in \text{acc}\sigma(T + F)$ .  $\square$

**Theorem 3.6.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . If  $T$  satisfies generalized Browder's theorem and  $F$  is a finite rank operator commuting with  $T$ . The following statements are equivalent:*

- (i)  $T + F$  satisfies generalized Weyl's theorem;
- (ii)  $\sigma_{BW}(T + F) \cap E(T + F) = \emptyset$ ;
- (iii)  $E(T + F) = \pi(T + F)$ .

*Proof.* (i) $\Rightarrow$ (ii). Assume that  $T + F$  satisfies generalized Weyl's theorem, that is,  $\sigma(T + F) \setminus \sigma_{BW}(T + F) = E(T + F)$ . It then easily that  $\sigma_{BW}(T + F) \cap E(T + F) = \emptyset$ , as required for (ii).

(ii) $\Rightarrow$ (iii). Let  $\lambda \in E(T + F)$ . The condition in (ii) implies that  $\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F)$ , and since  $T + F$  satisfies generalized Browder's theorem, we must have  $\lambda \in \pi(T + F)$ . It follows that  $E(T + F) \subseteq \pi(T + F)$ , and since the reverse inclusion always holds, we obtain (iii).

(iii) $\Rightarrow$ (i). Since  $T + F$  satisfies generalized Browder's theorem, we know that  $\sigma(T + F) \setminus \sigma_{BW}(T + F) = \pi(T + F)$ , and since we are assuming  $E(T + F) = \pi(T + F)$ , it follows that  $\sigma(T + F) \setminus \sigma_{BW}(T + F) = E(T + F)$ , that is,  $T + F$  satisfies generalized Weyl's theorem.  $\square$

**Theorem 3.7.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . Let  $F$  be a finite rank operator such that  $TF = FT$ . If  $T$  satisfies property (gab), then the following are equivalent:*

- (i)  $T + F$  satisfies property (gab);
- (ii)  $T + F$  satisfies generalized Browder's theorem and  $\pi(T + F) = \pi^a(T + F)$ .

*Proof.* (i) $\Rightarrow$ (ii). Suppose that  $T + F$  satisfies property (gab). Then it follows from [18, Corollary 2.7] that  $T + F$  satisfies generalized Browder's theorem and  $\pi(T + F) = \pi^a(T + F)$ .

(ii) $\Rightarrow$ (i). If  $T + F$  satisfies generalized Browder's theorem and  $\pi(T + F) = \pi^a(T + F)$ . Then  $\Delta^g(T + F) = \pi(T + F) = \pi^a(T + F)$ . That is,  $T + F$  satisfies property (gab).  $\square$

Similarly to Theorem 3.7 we have the following result in the case of property (ab), which we give without proof.

**Theorem 3.8.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . Let  $F$  be a finite rank operator such that  $TF = FT$ . If  $T$  satisfies property (ab), then the following are equivalent:*

- (i)  $T + F$  satisfies property (ab);
- (ii)  $T + F$  satisfies Browder's theorem and  $\pi_0(T + F) = \pi_0^a(T + F)$ .

**Theorem 3.9.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . Let  $F$  be a finite rank operator such that  $TF = FT$ . If  $T$  satisfies property (gaw), then the following are equivalent:*

- (i)  $T + F$  satisfies property (gaw);
- (ii)  $E^a(T + F) = \pi(T + F)$ .

*Proof.* (i) $\Rightarrow$ (ii). Suppose that  $T + F$  satisfies property (gaw). From [18, Theorem 3.5] we have  $T + F$  satisfies property (gab) and  $\pi^a(T + F) = E^a(T + F)$ . It follows from Theorem 3.7 that  $\pi^a(T + F) = \pi(T + F)$ . Hence (ii) follows.

(ii) $\Rightarrow$ (i). Assume that  $E^a(T + F) = \pi(T + F)$ . Since  $T$  satisfies property (gaw), then  $T$  satisfies generalized theorem and so  $T + F$  satisfies generalized Browder's theorem. That is,  $\Delta^g(T + F) = \pi(T + F)$ . Therefore,  $\Delta^g(T + F) = E^a(T + F)$ . So  $T + F$  satisfies property (gaw).  $\square$

Similarly to Theorem 3.9 we have the following result in the case of property (aw), which we give without proof.

**Theorem 3.10.** *Let  $T \in \mathbf{L}(\mathbb{X})$ . Let  $F$  be a finite rank operator such that  $TF = FT$ . If  $T$  satisfies property (aw), then the following are equivalent:*

- (i)  $T + F$  satisfies property (aw);
- (ii)  $E_0^a(T + F) = \pi_0(T + F)$ .

**Example 3.11.** In general properties  $(gw)$ ,  $(w)$ ,  $(gaw)$  and  $(aw)$  are not transmitted from an operator to a commuting finite rank perturbation as the following example shows. Let  $S : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  be an injective quasinilpotent operator which is not nilpotent. We define  $T$  on the Banach space  $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$  by  $T = I \oplus S$  where  $I$  is the identity operator on  $\ell^2(\mathbb{N})$ . Then  $\sigma(T) = \sigma_a(T) = \{0, 1\}$  and  $E^a(T) = \{1\}$ . It follows from [16, Example 2] that  $\sigma_{BW}(T) = \{0\}$ . This implies that  $\sigma_{SBF_+^-}(T) = \{0\}$ . Hence  $\Delta_a^g(T) = \Delta^g(T) = \{1\} = E^a(T) = E(T)$  and  $T$  satisfies property  $(gw)$ , so it satisfies property  $(w)$ , property  $(gaw)$  and property  $(aw)$ .

We define the operator  $U$  on  $\ell^2(\mathbb{N})$  by

$$U(x_1, x_2, \dots) = (-x_1, 0, \dots)$$

and  $F = U \oplus 0$  on the Banach space  $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ . Then  $F$  is a finite rank operator commuting with  $T$ . On the other hand,  $\sigma(T+F) = \sigma_a(T+F) = \{0, 1\}$  and  $E^a(T+F) = \{0, 1\}$ . As  $\sigma_{SBF_+^-}(T+F) = \sigma_{SBF_+^-}(T) = \{0\}$ , then  $\Delta_a^g(T+F) = \{1\} \neq E^a(T+F)$  and  $T+F$  does not satisfy property  $(gw)$ . Moreover,  $E(T+F) = \{0, 1\}$ , and as by [12, Theorem 4.3] we have  $\sigma_{BW}(T+F) = \sigma_{BW}(T) = \{0\}$ , then  $\Delta^g(T+F) = \{1\} \neq E^a(T+F)$  and hence  $T+F$  does not satisfy property  $(gaw)$ . Moreover we have  $\sigma_W(T+F) = \{0, 1\}$  and  $E_0(T+F) = \{0\} = E_0^a(T+F)$ . As  $\sigma(T+F) = \{0, 1\}$  then  $\Delta(T+F) \neq E^0(T+F)$  and  $T+F$  does not satisfy Weyl's theorem. So  $T+F$  does not satisfy property  $(w)$ . Note that  $\Delta(T+F) \neq E_0^a(T+F)$ . Hence  $T+F$  does not satisfy property  $(aw)$ .

**Theorem 3.12.** *Suppose that  $T \in \mathbf{L}(\mathbb{X})$  is polaroid,  $N \in \mathbf{L}(\mathbb{X})$  a nilpotent operator commuting with  $T$ .*

- (i) *If  $T^*$  has SVEP then  $T + N$  satisfies property  $(gaw)$ .*
- (ii) *If  $T$  has SVEP then  $T^* + N^*$  satisfies property  $(gaw)$ .*

*Proof.* (i) If  $T$  is polaroid then by [7, Theorem 2.5]  $T^*$  is polaroid. Clearly,  $N^*$  is nilpotent, since  $(N^*)^n = (N^n)^* = 0$  for some  $n \in \mathbb{N}$ . Therefore  $T^* + N^*$  is polaroid, by [7, Theorem 2.10]. Since  $T^* + N^*$  has SVEP, by [1, Corollary 2.12], it then follows, by Theorem 2.17, that  $T + N$  satisfies property  $(gaw)$ .

(ii) If  $T$  has SVEP then  $T + N$  has SVEP, see [1, Corollary 2.12]. Moreover, by [7, Lemma 2.10]  $T + N$  is polaroid. By Theorem 2.17 it then follows property  $(gaw)$  holds for  $T^* + N^*$ .  $\square$

**Theorem 3.13.** *Suppose that  $T$  is polaroid and  $N \in \mathbf{L}(\mathbb{X})$  a nilpotent operator commuting with  $T$ . If  $T^*$  has SVEP and  $f \in \text{Hol}(\sigma(T))$  then property  $(gaw)$  holds for  $f(T) + N$ .*

*Proof.* By Theorem 2.17,  $T$  satisfies property  $(gaw)$ . The SVEP for  $T^*$  implies that  $\sigma(T) = \sigma_a(T)$ , see [1, Corollary 2.45], so every isolated point of  $\sigma_a(T)$  is a pole of the resolvent of  $T$ . By Theorem 2.18 then  $f(T)$  satisfies property  $(gaw)$ . Finally, by 3.12  $f(T) + N$  satisfies property  $(gaw)$ .  $\square$

A bounded operator  $T \in \mathbf{L}(\mathbb{X})$  is said to be  $a$ -isoloid if every isolated point of  $\sigma_a(T)$  is an eigenvalue of  $T$ .

**Theorem 3.14.** *Suppose that  $T \in \mathbf{L}(\mathbb{X})$  is  $a$ -isoloid and  $F$  is a finite rank operator commuting with  $T$  such that  $\sigma_a(T+F) = \sigma_a(T)$ . If  $T$  satisfies property  $(gaw)$ , then  $T+F$  satisfies property  $(gaw)$ .*

*Proof.* Suppose that  $T$  satisfies property  $(gaw)$ . Then, by Proposition 2.5,  $T$  satisfies generalized Browder's theorem, and hence also  $T+F$  satisfies generalized Browder's theorem. By Theorem 3.9, in order to show that  $T+F$  satisfies property  $(gaw)$  it suffices only to prove the equality  $E^a(T+F) = \pi(T+F)$ . The inclusion  $\pi(T+F) \subseteq E^a(T+F)$

holds for every operator, so we need only to show the opposite inclusion  $E^a(T + F) \subseteq \pi(T + F)$ . We first show the inclusion

$$(3.1) \quad E^a(T + F) \subseteq \pi(T).$$

Let  $\lambda \in E^a(T + F)$ . By assumption  $\lambda \in \text{iso}\sigma_a(T + F)$  and  $\alpha(T + F - \lambda I) > 0$  so  $\lambda \in \text{iso}\sigma_a(T + F)$ , and hence  $\lambda \in \text{iso}\sigma_a(T)$ . Furthermore, since  $T$  is  $a$ -isoloid, we have also  $0 < \alpha(T - \lambda I)$ . Therefore, the inclusion  $E^a(T + F) \subseteq E^a(T)$  is proved. Now, since property (gaw) entails that  $T$  satisfies generalized Browder's theorem and  $E^a(T) = \pi(T)$ , we then have  $E^a(T + F) \subseteq \pi(T)$  and hence the inclusion 3.1 is established. Consequently, if  $\lambda \in E^a(T + F)$ , then  $T - \lambda I$  is generalized Browder. By Lemma 3.1 it then follows that  $T + F - \lambda I$  is also generalized Browder, hence

$$\lambda \in \Delta^g(T + F) = \pi(T),$$

as desired.  $\square$

**Theorem 3.15.** *Suppose that  $T \in \mathbf{L}(\mathbb{X})$  is  $a$ -isoloid and suppose that  $N$  is a nilpotent operator that commutes with  $T$ . If  $T$  satisfies property (gaw), then  $T + N$  satisfies property (gaw).*

*Proof.* Suppose that  $T$  satisfies property (gaw). Then, by Proposition 2.5,  $T$  satisfies generalized Browder's theorem,  $E^a(T) = \pi(T)$  and hence also  $T + N$  satisfies generalized Browder's theorem. By Theorem 3.9, in order to show that  $T + N$  satisfies property (gaw) it suffices only to prove the equality  $E^a(T + N) = \pi(T + N)$ . From [20, Lemma 3.1], we have  $E^a(T + N) = E^a(T)$ . Consequently,  $E^a(T + N) = \pi(T)$ . If  $\lambda \in E^a(T + F)$ , then  $T - \lambda I$  is generalized Browder. By Lemma 3.1 it then follows that  $T + F - \lambda I$  is also generalized Browder, hence

$$\lambda \in \Delta^g(T + N) = \pi(T + N),$$

and so  $E^a(T + N) \subseteq \pi(T + N)$ . Since the other inclusion is always verified. Therefore,  $E^a(T + N) = \pi(T + N)$ , as desired.  $\square$

**Example 3.16.** The following example shows that both Theorem 3.14 and Theorem 3.15 fail if we do not assume that the nilpotent operator  $N$ , and the finite rank operator  $F$  do not commute with  $T$ . Let  $\mathbb{X} = \ell^2(\mathbb{N})$  and  $T$  and  $N$  be defined by

$$T(x_1, x_2, \dots) := \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \dots\right), \quad (x_n) \in \ell^2(\mathbb{N}),$$

and

$$N(x_1, x_2, \dots) := \left(0, -\frac{x_1}{2}, 0, \dots\right), \quad (x_n) \in \ell^2(\mathbb{N}),$$

Clearly,  $N$  is a nilpotent finite rank operator,  $T$  is a quasi-nilpotent operator satisfying generalized Weyl's theorem. Since  $T$  is decomposable, then  $T$  satisfies property (gaw). On the other hand, it is easily seen that  $0 \in E^a(T + N)$  and  $0 \notin \Delta^g(T + N)$ , so that  $T + N$  does not satisfies property (gaw). Note that  $\sigma_a(T + N) = \sigma_a(T)$ .

**Example 3.17.** The following example shows that and Theorem 3.14 fails if we do not assume that  $T$  is  $a$ -isoloid. Let  $S : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  be an injective quasi-nilpotent operator, and let  $U : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$  be defined:

$$U(x_1, x_2, \dots) := (-x_1, 0, 0, \dots), \quad (x_n) \in \ell^2(\mathbb{N}).$$

Define on  $\mathbb{X} = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$  the operators  $T$  and  $F$  by  $T := I \oplus S$  and  $F := U \oplus 0$ . Clearly,  $F$  is a finite-rank operator and  $TF = FT$ . It is easy to check that

$$\sigma(T) = \sigma_a(T) = \sigma_W(T) = \{0, 1\}.$$

Since  $\alpha(T) = 0$ , then  $T$  is not  $a$ -isoloid. Now, both  $T$  and  $T^*$  have SVEP, since  $\sigma(T) = \sigma(T^*)$  is finite. Moreover,  $\Delta^g(T) = \{1\} = E^a(T)$ , so  $T$  satisfies property (gaw). On the

other hand,  $\sigma(T + F) = \sigma_a(T + F) = \{0, 1\}$ , and  $E^a(T + F) = \{0, 1\}$ . As  $\sigma_{BW}(T + F) = \sigma_{BW}(T) = \{0\}$ , then  $\Delta^g(T + F) = \{1\} \neq E^a(T + F)$ , so that property (gaw) does not hold for  $T + F$ .

An operator  $R \in \mathbf{L}(\mathbb{X})$  is a Riesz operator if  $R - \lambda I$  is a Fredholm operator for all  $\lambda \in \mathbb{C} \setminus \{0\}$ . Evidently, every quasi-nilpotent operator is a Riesz operator. It is well-known that if  $Q$  is a quasi-nilpotent operator commuting with  $T$  then

$$\sigma(T + Q) = \sigma(T) \quad \text{and} \quad \sigma_a(T + Q) = \sigma_a(T).$$

It is well known that if  $R$  is a Riesz operator which commutes with an operator  $T \in \mathbf{L}(\mathbb{X})$ , then  $\sigma_W(T + R) = \sigma_W(T)$  [28, Lemma 2.2]. In [23] the author proved that if  $R$  is a Riesz operator which commutes with an operator  $T \in \mathbf{L}(\mathbb{X})$ , then  $\sigma_{SF_+^-}(T + R) = \sigma_{SF_+^-}(T)$  and  $a$ -Browder's theorem holds for  $T + R$ .

Generally, property (gaw) is not transmitted from  $T$  to a quasi-nilpotent perturbation  $T + Q$ . In fact, if we consider on the Hilbert space  $\ell^2(\mathbb{N})$  the operators  $T = 0$  and  $Q$  defined by  $Q(x_1, x_2, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \dots)$ . Then  $Q$  is a quasi-nilpotent operator commuting with  $T$ . Moreover, we have  $\sigma(T) = \sigma_a(T) = \{0\}$ ,  $\sigma_{SBF_+^-}(T) = \emptyset = \sigma_{BW}(T)$  and  $E^a(T) = \{0\}$ . Hence  $T$  satisfies property (gaw). But property (gaw) fails for  $T + Q = Q$ . Indeed

$$\sigma(T + Q) = \sigma_a(T + Q) = \sigma_{BW}(T + Q) = \sigma_{SBF_+^-}(T + Q) = E^a(T + Q) = E(T + Q) = \{0\}.$$

A bounded operator  $T \in \mathbf{L}(\mathbb{X})$  is said to have property  $H(p)$  if for all  $\lambda \in \mathbb{C}$  there exists a  $p := p(\lambda) \in \mathbb{N}$  such that

$$H_0(T - \lambda I) = \ker(T - \lambda I)^p.$$

This class of operators has been introduced in [28] and for the constant function  $p(\lambda) := 1$  has been also studied in [6]. Clearly, from the implication

$$H_0(T - \lambda I) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda,$$

if  $T$  has property  $H(p)$  then  $T$  has SVEP. Moreover  $T$  is polaroid, see [2, Lemma 3.3].

**Theorem 3.18.** *Suppose that  $T \in \mathbf{L}(\mathbb{X})$  and  $K$  is an algebraic operator commuting with  $T$ . If  $T \in H(p)$ , then  $T + K$  is polaroid.*

*Proof.* Let  $\sigma(K) = \{\nu_1, \nu_2, \dots, \nu_n\}$ . Denote by  $P_i$  the spectral projection associated with  $K$  and with the spectral set  $\nu_i$ . If  $Y_i := P_i(H)$  and  $Z_i := \ker(P_i)$ , then  $\mathbb{X} = Y_i \oplus Z_i$ , the closed subspaces  $Y_i$  and  $Z_i$  are invariant under  $T$  and  $K$ , and  $\sigma(K|_{Y_i}) = \{\nu_i\}$ . Define  $K_i := K|_{Y_i}$  and  $T_i := T|_{Y_i}$ . Clearly, the restrictions  $T_i$  and  $K_i$  commute for every  $i = 1, 2, \dots, n$ . Let  $h$  be a polynomial such that  $h(K) = 0$ . Then  $h(K_i) = h(K)|_{Y_i} = 0$ , and the equalities

$$\{0\} = \sigma(h(K_i)) = h(\sigma(K_i)) = h(\{\nu_i\})$$

entail that  $h(nu_i) = 0$ . Write

$$h(\nu) = (\nu - \nu_i)^\mu g(\nu) \quad \text{with} \quad g(\nu_i) \neq 0.$$

Then  $0 = h(K_i) = (K_i - \nu_i I)^\mu g(K_i)$  where the operators  $g(K_i)$  is invertible. Therefore  $(K_i - \nu_i I)^\mu = 0$ , hence the operators  $N_i := K_i - \nu_i I$  are nilpotent for all  $i = 1, 2, \dots, n$ . Note that

$$T_i + K_i = (T_i + \nu_i I) + (K_i - \nu_i I) = T_i - N_i + \nu_i I.$$

Since SVEP is inherited by restrictions to closed invariant subspaces,  $T_i$  has SVEP and hence, by [1, Corollary 2.12] also  $T_i + K_i = T_i - N_i + \nu_i I$  has SVEP for all  $i = 1, 2, \dots, n$ .

By [1, Theorem 2.9] it then follows that

$$T + K = \bigoplus_{i=1}^n (T_i + K_i)$$

has SVEP.

Suppose that  $T \in H(p)$ . Let  $\lambda \in \text{iso}\sigma(T + K)$ . Since  $\sigma(T + K) = \bigcup_{i=1}^n \sigma(T_i + K_i)$  then  $\lambda \in \sigma(T_j + K_j)$  for some positive integer  $1 \leq j \leq n$  and hence  $\lambda - \nu_j \in \text{iso}\sigma(T_j + K_j - \nu_j I)$ . The restriction to a closed invariant subspace of  $H(p)$  is  $H(p)$ , so  $T_j$  is polaroid. Since, as observed before,  $K_j - \nu_j I$  is nilpotent then, by [7, Theorem 2.10], also  $T_j + K_j - \nu_j I$  is polaroid. Therefore  $\lambda - \nu_j$  is a pole of the resolvent of  $T_j + K_j - \nu_j I$ , so there exists by [1, Theorem 3.74] a positive integer  $m_j$  such that

$$H_0[(T_j + K_j - \nu_j I - (\lambda - \nu_j)I)] = H_0(T_j + K_j - \lambda I) = \ker(T_j + K_j - \lambda I)^{m_j}.$$

Therefore, taking in to account that  $H_0(T_j + K_j - \lambda I) = \{0\}$  if  $\lambda \notin \sigma(T_j + K_j)$ , we have

$$\begin{aligned} H_0(T + K - \lambda I) &= \bigoplus_{j=1}^n H_0(T_j + K_j - \lambda I) \\ &= \bigoplus_{j=1}^n \ker(T_j + K_j - \lambda I)^{m_j} = \ker(T + K - \lambda I)^m, \end{aligned}$$

where  $m := \max\{m_1, m_2, \dots, m_n\}$ . Arguing as in the proof of [7, Theorem 2.10] it then follows that  $\lambda$  is a pole of the resolvent of  $T + K$ .  $\square$

**Theorem 3.19.** *Suppose that  $T \in \mathbf{L}(\mathbb{X})$  and  $K$  is an algebraic operator commuting with  $T$ .*

- (i) *If  $T \in H(p)$  then property (gaw) holds for  $T^* + K^*$ .*
- (ii) *If  $T^* \in H(p)$  then property (gaw) holds for  $T + K$ .*

*Proof.* (i) If  $T \in H(p)$  then  $T$  has SVEP and hence  $T + K$  has SVEP by Theorem 3.18. Moreover,  $T$  is polaroid so also  $T + K$  is polaroid. By Theorem 2.17, then property (gaw) holds for  $T^* + K^*$ .

(ii) If  $T^* \in H(p)$  then  $T^*$  has SVEP and hence  $T^* + K^*$  has SVEP. Moreover,  $T^* + K^*$  is polaroid, so, by [7, Theorem 2.5],  $T + K$  is polaroid. By Theorem 2.17 it then follows that property (gaw) holds for  $T + K$ .  $\square$

## REFERENCES

1. P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer Academic Publishers, Dordrecht—Boston—London, 2004.
2. P. Aiena, *Classes of operators satisfying  $a$ -Weyl's theorem*, *Studia Math.* **169** (2005), 105–122.
3. P. Aiena and O. Monsalve, *The single valued extension property and the generalized Kato decomposition property*, *Acta Sci. Math. (Szeged)* **67** (2001), 461–477.
4. P. Aiena and P. Peña, *Variations on Weyl's theorem*, *J. Math. Anal. Appl.* **324** (2006), no. 1, 566–579.
5. P. Aiena and T. L. Miller, *On generalized  $a$ -Browder's theorem*, *Studia Math.* **180** (2007), no. 3, 285–300.
6. P. Aiena and F. Villafañe, *Weyl's theorem for some classes of operators*, *Integr. Equ. Oper. Theory* **53** (2005), 453–466.
7. P. Aiena, J. Guillen, and P. Peña, *Property (w) for perturbations of polaroid operators*, *Linear Algebra Appl.* **428** (2008), 1791–1802.
8. M. Amouch, *Weyl type theorems for operators satisfying the single-valued extension property*, *J. Math. Anal. Appl.* **326** (2007), 1476–1484.
9. M. Amouch, M. Berkani, *On the property (gw)*, *Mediterr. J. Math.* **5** (2008), 371–378.
10. M. Amouch, H. Zguitti, *On the equivalence of Browder's and generalized Browder's theorem*, *Glasgow Math. J.* **48** (2006), 179–185.
11. M. Berkani and M. Sarih, *On semi  $B$ -Fredholm operators*, *Glasgow Math. J.* **43** (2001), 457–465.



12. M. Berkani, *Index of B-Fredholm operators and generalization of a Weyl theorem*, Proc. Amer. Math. Soc. **130** (2001), 1717–1723.
13. M. Berkani, *B-Weyl spectrum and poles of the resolvent*, J. Math. Anal. Appl. **272** (2002), 596–603.
14. M. Berkani, *On the equivalence of Weyl theorem and generalized Weyl theorem*, Acta Math. Sinica **23** (2007), no. 1, 103–110.
15. M. Berkani, J. Koliha, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. (Szeged) **69** (2003), no. 1–2, 359–376.
16. M. Berkani, A. Arroud, *Generalized Weyl's theorem and hyponormal operators*, J. Austral. Math. Soc. **76** (2004), 1–12.
17. M. Berkani, H. Zariouh, *Extended Weyl type theorems*, Mathematica Bohemica **134** (2009), no. 4, 369–378.
18. M. Berkani and H. Zariouh, *New extended Weyl type theorems*, Mat. Vesnik **62** (2010), no. 2, 145–154.
19. M. Berkani, M. Sarik, and H. Zariouh, *Browder-type theorems and SVEP*, Mediterr. J. Math. **6** (2010), 139–150.
20. M. Berkani and H. Zariouh, *Generalized  $\alpha$ -Weyl's theorem and perturbations*, Functional Analysis, Approximation and Computation **2** (2010), no. 1, 7–18.
21. L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.
22. H. R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, London, 1973.
23. B. P. Duggal, *Perturbations of operators satisfying a local growth condition*, Extracta Mathematicae **23** (2008), no. 1, 29–42.
24. B. P. Duggal and S. V. Djordjevic, *Generalized Weyl's theorem for a class of operators satisfying a norm condition II*, Math. Proc. Royal Irish Acad. **104A** (2006), 1–9.
25. J. K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.
26. K. B. Laursen, *Operators with finite ascent*, Pacific J. Math. **152** (1992), no. 2, 323–336.
27. K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford Science Publications, Oxford, 2000.
28. M. Oudghiri, *Weyl's and Browder's theorem for operators satisfying the SVEP*, Studia Math. **163** (2004), 85–101.
29. V. Rakočević, *On a class of operators*, Mat. Vesnik **37** (1985), 423–426.
30. V. Rakočević, *Operators obeying  $\alpha$ -Weyl's theorem*, Rev. Roumaine Math. Pures Appl. **10** (1986), 915–919.
31. M. H. M. Rashid, *Property  $(gb)$  and perturbations*, J. Math. Anal. Appl. **383** (2011), 82–94.
32. M. H. M. Rashid, *Property  $(w)$  and quasi-class  $(A, k)$  operators*, Rev. Un. Mat. Argentina **52** (2011), no. 1, 133–142.
33. M. H. M. Rashid, *Weyl's type theorems and hypercyclic operators*, Acta Mathematica Scientia **32** (2012), no. 2, 539–551.
34. M. H. M. Rashid, *Property  $(gw)$  and perturbations*, Bull. Belg. Math. Soc. - Simon Stevin **18** (2011), no. 4, 635–654.

DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF SCIENCE, P.O.Box(7), MU'TAH UNIVERSITY, I-KARAK-JORDAN  
*E-mail address:* malik\_okasha@yahoo.com

Received 26/08/2012