

THE INNER STRUCTURE OF THE JACOBI-LAURENT MATRIX RELATED TO THE STRONG HAMBURGER MOMENT PROBLEM

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ABSTRACT. The form of the Jacobi type matrix related to the strong Hamburger moment problem is known [9, 4], i.e., there are known the zero elements of corresponding matrix. We describe the relations between of non-zero elements of such matrices, i.e., we describe "the inner structure" of the Jacobi-Laurent matrices related to the strong Hamburger moment problem.

1. INTRODUCTION

In the article [4] we present a solution of the direct and inverse spectral problems for block Jacobi-Laurent matrices related to the strong Hamburger moment problem. The direct spectral problem was solved under the condition that the corresponding matrix has a special structure and what is more with this connection generates a Hermitian operator connected with the strong Hamburger moment problem. But we did not give any description of the coefficients of such a matrix, because the article [4] is sufficiently lengthy and the corresponding proof is rather long. In some sense this article is an additional part of [4]. The article [4] is similar to [3] where we present a solution of the direct and the inverse spectral problems for the block Jacobi type matrices related to the trigonometric moment problem CMV-matrix [5]. And this paper is similar to [6] where we described relations between non-zero elements of unitary CMV-matrices, i.e., we described its "inner structure".

In general, the strong Hamburger moment problem is very old and was treated in many papers such as [1, 2, 9, 7, 8, 4] and cited in [8]). A generalization to the case of the strong matrix moment problem are known from [10, 11, 12]. In [10], the author has formulated conditions that help to describe the inner structure of the Jacobi-Laurent matrices in case of the strong matrix moment problem (each element of the matrix is some $(n \times n)$ Hermitian matrix). But we give our proof independently from [10].

2. PRELIMINARIES

Let us briefly recall the strong Hamburger moment problem: for a given sequence $s = (s_m)_{m=-\infty}^{\infty}$ of real numbers s_m , to find a measure $d\rho(\lambda)$ on the Borel σ -algebra $\mathfrak{B}(\mathbb{R} \setminus \{0\})$ such that

$$(1) \quad s_n = \int_{\mathbb{R}} \lambda^n d\rho(\lambda), \quad m \in \mathbb{Z} := \{\dots, -1, 0, 1, \dots\}.$$

It follows from [4] that the given sequence s admits representation (1) with some Borel measure $d\rho(\lambda)$ iff it is positive definite, i.e., $\sum_{j,k \in \mathbb{Z}} s_{j+k} f_j \bar{f}_k \geq 0$ for every finite sequences of complex numbers (f_j) , $j \in \mathbb{Z}$, $f_j \in \mathbb{C}$. And, additionally, the measure in representation (1) is unique if for example $\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{s_{2n}}} = \infty$. For simplicity, in [4] we considered the

2000 *Mathematics Subject Classification*. Primary: 44A60, 47A57, 47A70.

Key words and phrases. Hamburger strong moment problems, block three-diagonal matrix.

case where $\text{supp}(d\rho(\lambda))$ is bounded and $0 \notin \text{supp}(d\rho(\lambda))$. Such a case is taken into account here.

Let $d\rho(\lambda)$ be a Borel measure on $\mathbb{R} \setminus \{0\}$ with bounded support and $L^2 = L^2(\mathbb{R}, d\rho(\lambda))$ the space of complex square integrable functions defined on $\mathbb{R} \setminus 0$. We suppose that the Borel measure $d\rho(\lambda)$ is such that all the functions $\mathbb{R} \ni \lambda \mapsto \lambda^m$, $m \in \mathbb{Z}$, belong to L^2 , and all the functions λ^m , $m \in \mathbb{Z}$, are linearly independent.

To find an analog of the usual Jacobi matrix J , we need to choose an order for orthogonalization in L^2 applied to the family of the linear independent functions $\mathbb{R} \ni \lambda \mapsto \lambda^m$, $m \in \mathbb{Z}$. We used the following order for the orthogonalization via the Schmidt procedure:

$$(2) \quad \lambda^0; \quad \lambda^{-1}, \lambda^1; \quad \lambda^{-2}, \lambda^2; \quad \dots; \quad \lambda^{-n}, \lambda^n; \quad \dots$$

Applying the Schmidt orthogonalization procedure to (2) with real coefficients (see, [4]) we obtain a system of orthonormal polynomials in the space L^2 (w.r.t. λ and λ^{-1} , the so-called Laurent polynomials) indexed in the following way:

$$(3) \quad P_{0;0}(\lambda); \quad P_{1;0}(\lambda), P_{1;1}(\lambda); \quad P_{2;0}(\lambda), P_{2;1}(\lambda); \quad \dots; \quad P_{n;0}(\lambda), P_{n;1}(\lambda); \quad \dots$$

where each polynomial has the form $P_{n;\alpha}(\lambda) = k_{n;\alpha} \lambda^{(-1)^{\alpha+1}n} + \dots$, $n \in \mathbb{N}$, $\alpha = 0, 1$, $k_{n;\alpha} > 0$; here $+\dots$ denotes the previous part of the corresponding polynomial; $P_0(\lambda) = P_{0;0}(\lambda) = 1$.

As a conclusion in [4] we had that the bounded selfadjoint operator \hat{A} of multiplication by λ in the space L^2 , in the orthonormal basis (3) of polynomials, has the form of a three-diagonal block Jacobi type symmetric matrix $J = (a_{j,k})_{j,k=0}^{\infty}$ that acts on the space

$$(4) \quad \mathcal{H}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots, \quad \mathcal{H}_0 = \mathbb{C}^1, \quad \mathcal{H}_n = \mathbb{C}^2, \quad n \in \mathbb{N}.$$

This matrix has the form

$$(5) \quad J = \begin{bmatrix} *b_0 & * & c_0 & + & & & & & & \dots \\ * & * & & * & 0 & & 0 & & & \\ a_0 & & b_1 & & c_1 & & & & 0 & \dots \\ + & * & & * & * & + & & & & \\ \hline & 0 & & * & * & * & 0 & & 0 & \\ & & a_1 & & b_2 & & c_2 & & & \dots \\ & 0 & & + & * & * & * & + & & \\ \hline & & & & 0 & * & * & * & 0 & 0 \\ & & 0 & & a_2 & & b_3 & & c_3 & \dots \\ & & & & 0 & & + & * & * & + \\ \hline \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In (5), $b_0 = b_{0;0,0}$ is a 1×1 -matrix i.e. a scalar; b_n is a 2×2 -matrix: $b_n = (b_{n;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$, $\forall n \in \mathbb{N}$; a_0 is a 1×2 -matrix: $a_0 = (a_{0;\alpha,\beta})_{\alpha,\beta=0}^{1,0}$; a_n is a 2×2 -matrix: $a_n = (a_{n;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$, $\forall n \in \mathbb{N}$; c_0 is a 2×1 -matrix: $c_0 = (c_{0;\alpha,\beta})_{\alpha,\beta=0}^{0,1}$; c_n is a 2×2 -matrix: $c_n = (c_{n;\alpha,\beta})_{\alpha,\beta=0}^{1,1}$, $\forall n \in \mathbb{N}$. In these matrices a_n and c_n some elements are always equal to zero: $a_{n;0,0} = a_{n;1,0} = 0$, $c_{n;0,0} = c_{n;0,1} = 0$, $\forall n \in \mathbb{N}$. Some other elements are positive, namely $a_{0;1,0}, c_{0;0,1} > 0$, $a_{n;1,1}, c_{n;1,1} > 0$, $n \in \mathbb{N}$. All positive elements in (5) are denoted by $+$.

The matrix (5) in the scalar form is five-diagonal of the indicated structure. It is symmetric in basis (3); $b_{n;\alpha,\beta} = b_{n;\beta,\alpha}$, $c_{n;\alpha,\beta} = a_{n;\beta,\alpha}$, $n \in \mathbb{N}_0$, $\alpha, \beta = 0, 1$.

Since we consider the measure $d\rho(\lambda)$ with compact support on $\mathbb{R} \setminus \{0\}$, the operator of multiplication \hat{A} has an inverse that is a self-adjoint operator \hat{A}^{-1} of multiplication by λ^{-1} in L^2 . Then the bounded operator that is an inverse of \hat{A} is generated, in the

space \mathbf{l}_2 (4), by a three-diagonal block Jacobi type symmetric matrix J^{-1} of the form analogous to (5),

$$(6) \quad J^{-1} = \begin{bmatrix} *q_0 & + & r_0 & 0 & & & & & & \dots \\ + & * & & * & + & 0 & & & & \dots \\ p_0 & & q_1 & & & r_1 & & & & 0 & \dots \\ 0 & * & & * & * & & 0 & & & & \dots \\ & + & & * & * & * & + & 0 & & & \dots \\ & 0 & p_1 & & q_2 & & & r_2 & & & \dots \\ & & 0 & 0 & * & * & * & & 0 & & \dots \\ & & & 0 & + & * & * & * & + & 0 & \dots \\ & & & & 0 & p_2 & & q_3 & & r_3 & \dots \\ & & & & & 0 & 0 & * & * & * & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

So, we have $p_{0;1,0} = r_{0;0,1} = 0$, $p_{n;1,0} = p_{n;1,1} = r_{n;0,1} = r_{n;1,1} = 0$, $n \in \mathbb{N}$, $p_{n;0,0}, r_{n;0,0} > 0$, $n \in \mathbb{N}_0$.

3. THE INNER STRUCTURE OF THE JACOBI-LAURENT MATRIX

Let us redefine the coefficients of the matrix J for the next consideration. Hence the matrix has the form

$$(7) \quad J = \begin{pmatrix} a_1 & b_1 & c_1 & & & & & & & & \\ b_1 & a_2 & b_2 & & & & & & & & 0 \\ c_1 & b_2 & a_3 & b_3 & c_3 & & & & & & \\ & & b_3 & a_4 & b_4 & & & & & & \\ & & c_3 & b_4 & a_5 & b_5 & c_5 & & & & \\ & 0 & & & b_5 & a_6 & b_6 & & & & \\ & & & & c_5 & b_6 & a_7 & b_7 & c_7 & & \\ & & & & & & \ddots & \ddots & \ddots & & \end{pmatrix},$$

where $a_k, b_k \in \mathbb{R}$, $c_{2k-1} > 0$, $k \in \mathbb{N}$.

Consider the algebraic inverse matrix J^{-1} that we understand as a matrix such that the equality

$$(8) \quad JJ^{-1} = E$$

holds on finite vectors, where E is the identity matrix. Together with (8) we don't consider $J^{-1}J = E$ since we suppose that J and J^{-1} generate bounded symmetric operators.

Suppose the matrix J has an algebraic inverse J^{-1} . We redefine coefficients of (6) as follows:

$$(9) \quad J^{-1} = \begin{pmatrix} \alpha_1 & \beta_1 & & & & & & & & & \\ \beta_1 & \alpha_2 & \beta_2 & \gamma_2 & & & & & & & 0 \\ \beta_2 & \alpha_3 & \beta_3 & & & & & & & & \\ \gamma_2 & \beta_3 & \alpha_4 & \beta_4 & \gamma_4 & & & & & & \\ & & \beta_4 & \alpha_5 & \beta_5 & & & & & & \\ & 0 & & \gamma_4 & \beta_5 & \alpha_6 & \beta_6 & \gamma_6 & & & \\ & & & & & \beta_6 & \alpha_7 & \beta_7 & & & \\ & & & & & & \ddots & \ddots & \ddots & & \end{pmatrix},$$

where $\alpha_k, \beta_k \in \mathbb{R}$, $\gamma_{2k} > 0$, $k \in \mathbb{N}$.

Let us repeat that the forms of matrices (7) and (9) are known from [9, 4]. We investigate the following (in some sense) inverse question. Let us take a matrix of the

form (7) with arbitrary (elements) coefficients. Can we assert that there exist a matrix J^{-1} of the form (9) such that it is an inverse (algebraic) of J , i. e., (8) holds? This question is not trivial, since the operator generated by matrix (7) with coefficients (for example) $a_k = 0$, $c_{2k-1} = 1$, $k \in \mathbb{N}$, $b_1 = 1$, $b_k = -1$, $k > 1$ has zero in the point spectrum and hence does not have a bounded inverse defined on the whole l_2 . It is not difficult to construct an example with some elements $a_k = 2$, $b_k = 1$, $c_{2k-1} = 1$, $k \in \mathbb{N}$, such that the matrix J has an inverse but not of the form (9).

Necessary and sufficient conditions that give an answer to the above question are given in the next theorem.

Theorem 1. *The matrix J of the form (7) has an algebraic inverse of the form (9) if and only if*

$$(10) \quad \det \begin{vmatrix} b_{n-1} & c_{n-1} \\ a_n & b_n \end{vmatrix} = 0,$$

$$(11) \quad \det \begin{vmatrix} a_1 & b_1 \\ b_1 & a_2 \end{vmatrix} \neq 0, \quad \det \begin{vmatrix} a_n & b_n & 0 \\ b_n & a_{n+1} & b_{n+1} \\ 0 & b_{n+1} & a_{n+2} \end{vmatrix} \neq 0, \quad n = 2k, \quad k \in \mathbb{N}.$$

Remark. The condition (11) is natural in the sense that if we have, for the matrix (7), some of the determinants equal to zero, then we obtain linear dependence of the corresponding rows and columns which we can every time exclude from the matrix. In such a case we have linearly dependent corresponding polynomials $P_{k;\alpha}(\lambda)$, $k \in \mathbb{N}$, $\alpha = 0, 1$, and consequently the functions $\lambda^m \in L^2(\mathbb{R}, d\rho(\lambda))$, $m \in \mathbb{Z}$, will be linear dependent. On the another hand, condition (11) guarantees that zero is not in the point spectrum of the operator generated by J .

Proof. We consider the equality (8) and solve the following problem: for given elements of the matrix (7) a_k, b_k, c_{2k-1} , $k \in \mathbb{N}$, to uniquely find (and calculate) elements of the matrix (9) $\alpha_k, \beta_k, \gamma_{2k}$, $k \in \mathbb{N}$.

Denote the positions of elements in the matrix JJ^{-1} by (x, y) , where x is the row and y the column of the matrix JJ^{-1} , also by (x, y) we denote corresponding scalar equations generated by (8).

At the beginning, let us consider the couples of elements of the matrix JJ^{-1} in the positions

$$(n-1, n+1) \text{ and } (n, n+1); \quad ((n-1, n+2) \text{ and } (n, n+2)), \quad n = 2k, \quad k \in \mathbb{N}.$$

Elements of these couple must equal to zero due to (8): for $n = 2k$, $k \in \mathbb{N}$

$$(12) \quad \begin{cases} b_{n-1}\beta_n + c_{n-1}\alpha_{n+1} = 0 \\ a_n\beta_n + b_n\alpha_{n+1} = 0 \end{cases}, \quad \left(\begin{cases} b_{n-1}\gamma_n + c_{n-1}\beta_{n+1} = 0 \\ a_n\gamma_n + b_n\beta_{n+1} = 0 \end{cases} \right).$$

Nonzero elements β_n, α_{n+1} and γ_n, β_{n+1} , $n = 2k$, $k \in \mathbb{N}$, exist as solutions of corresponding homogeneous system (12) iff the corresponding determinants are equal zero, i.e., the condition (10) is fulfilled.

Analogously we have the same conclusion considering the couples of elements of the matrix JJ^{-1} : for $n = 2k$, $k \in \mathbb{N}$

$$(1, 2) \text{ and } (1, 3); \quad (n+2, n) \text{ and } (n+3, n); \quad ((n+2, n+1) \text{ and } (n+3, n+1)).$$

In such a simple way we had showed the necessity of the condition (10). In fact, this condition is also sufficient in the theorem, taking into account (11). But for this fact, we need to consider all equations from (8). We do it inductively. At the beginning of the induction we consider non standard step.

Step 0. In this step we consider the following part of the equality (8):

$$(13) \quad \begin{pmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\beta_1 + b_1\alpha_2 + c_1\beta_2 & * \\ b_1\alpha_1 + a_2\beta_1 & b_1\beta_1 + a_2\alpha_2 + b_2\beta_2 & * \\ c_1\alpha_1 + b_2\beta_1 & * & * \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & * & * \end{pmatrix},$$

where * denotes elements not considered in this step, i.e., we consider positions (1,1), (2,1), (3,1), (1,2), (2,2). From the earlier considerations we have

$$\begin{cases} b_1\alpha_1 + a_2\beta_1 = 0 \\ c_1\alpha_1 + b_2\beta_1 = 0 \end{cases}$$

and, taking into account the condition (11), we can consider the system generated by the elements (1,1) and (2,1),

$$\begin{cases} a_1\alpha_1 + b_1\beta_1 = 1 \\ b_1\alpha_1 + a_2\beta_1 = 0 \end{cases}.$$

The mean determinant of this non-homogeneous system due to (11) is not equal zero and hence this system has a unique solution α_1, β_1 ,

$$(14) \quad \alpha_1 = \frac{a_2}{a_1a_2 - b_1^2}, \quad \beta_1 = \frac{-b_1}{a_1a_2 - b_1^2}.$$

Let us show that the elements (1,2) and (2,2) of JJ^{-1} form a system of equations in (8) with linear dependent coefficients taking into account that α_1 and β_1 are defined in (14). Indeed, for the coefficients of the system

$$\begin{cases} a_1\beta_1 + b_1\alpha_2 + c_1\beta_2 = 0 \\ b_1\beta_1 + a_2\alpha_2 + b_2\beta_2 = 1 \end{cases}$$

we have, using condition (11) and (14), that

$$\frac{b_1}{a_2} = \frac{c_1}{b_2} = \frac{-a_1\beta_1}{1 - b_1\beta_1} = \frac{a_1 \frac{b_1}{a_1a_2 - b_1^2}}{1 + b_1 \frac{b_1}{a_1a_2 - b_1^2}} = \frac{a_1b_1}{a_1a_2 - b_1^2 + b_1^1} = \frac{a_1b_1}{a_1a_2} = \frac{b_1}{a_2}.$$

Hence in the next step we can consider only the position (2,2) instead of considering both (1,2) and (2,2) together.

Step 1. In this step we consider the following positions of elements of the matrix JJ^{-1} :

$$(2,2), (3,2), (4,2), (5,2), (1,3), (2,3), (3,3), (4,3), (5,3), (1,4), (2,4), (3,4), (4,4),$$

i.e., the corresponding portion of (8) has the form

$$(15) \quad \begin{pmatrix} ** & b_1\beta_2 + c_1\alpha_3 & b_1\gamma_2 + c_1\beta_3 & * \\ *b_1\beta_1 + a_2\alpha_2 + b_2\beta_2 & a_2\beta_2 + b_2\alpha_3 & a_2\gamma_2 + b_2\beta_3 & * \\ *c_1\beta_1 + b_2\alpha_2 + a_3\beta_2 + b_3\gamma_2 & b_2\beta_2 + a_3\alpha_3 + b_3\beta_3 & b_2\gamma_2 + a_3\beta_3 + b_3\alpha_4 + c_3\beta_4 & * \\ *b_3\beta_2 + a_4\gamma_2 & b_3\alpha_3 + a_4\beta_3 & b_3\beta_3 + a_4\alpha_4 + b_4\beta_4 & * \\ *c_3\beta_2 + b_4\gamma_2 & c_3\alpha_3 + b_4\beta_3 & * & * \\ *0 & 0 & * & * \end{pmatrix} = \begin{pmatrix} * & * & 0 & 0 & * \\ * & 1 & 0 & 0 & * \\ * & 0 & 1 & 0 & * \\ * & 0 & 0 & 1 & * \\ * & 0 & 0 & * & * \\ * & 0 & 0 & * & * \end{pmatrix},$$

where as usual * denotes elements that are not considered.

Since the elements (4,2) and (5,2) form a linear dependent system (condition (11)), let us consider the elements (2,2), (3,2) and (4,2) of JJ^{-1} and show that the corresponding equations of (8) form a linear independent system that gives a unique solution $\alpha_2, \beta_2, \gamma_2$, taking in to account that β_1 are defined in (14),

$$\begin{cases} b_1\beta_1 + a_2\alpha_2 + b_2\beta_2 = 1 \\ c_1\beta_1 + b_2\alpha_2 + a_3\beta_2 + b_3\gamma_2 = 0 \\ b_3\beta_2 + a_4\gamma_2 = 0 \end{cases}$$

and

$$(16) \quad \begin{cases} a_2\alpha_2 + b_2\beta_2 = 1 - b_1\beta_1 \\ b_2\alpha_2 + a_3\beta_2 + b_3\gamma_2 = -c_1\beta_1 \\ b_3\beta_2 + a_4\gamma_2 = 0 \end{cases}.$$

Here we use the condition (11)

$$\det \begin{vmatrix} a_2 & b_2 & 0 \\ b_2 & a_3 & b_3 \\ 0 & b_3 & a_4 \end{vmatrix} \neq 0$$

that gives uniqueness of the solution $\alpha_2, \beta_2, \gamma_2$. Substitute γ_2 from the third equation in to the second one in (16),

$$\begin{cases} a_2\alpha_2 + b_2\beta_2 = 1 - b_1\beta_1 \\ b_2\alpha_2 + (a_3 - \frac{b_3^2}{a_4})\beta_2 = -c_1\beta_1 \end{cases}.$$

From the last system we have

$$\alpha_2 = \frac{(1 - b_1\beta_1)(a_3 - \frac{b_3^2}{a_4}) + c_1\beta_1 b_2}{a_2(a_3 - \frac{b_3^2}{a_4}) - b_2^2},$$

$$\beta_2 = \frac{-a_2 c_1 \beta_1 - b_2(1 - b_1\beta_1)}{a_2(a_3 - \frac{b_3^2}{a_4}) - b_2^2} = \frac{(-a_2 c_1 + b_1 b_2)\beta_1 - b_2}{a_2(a_3 - \frac{b_3^2}{a_4}) - b_2^2}.$$

Taking into account condition (10) in the form $a_2 c_1 - b_1 b_2 = 0$ we obtain

$$(17) \quad \beta_2 = \frac{-b_2}{a_2(a_3 - \frac{b_3^2}{a_4}) - b_2^2}.$$

From the position (4.2) we obtain

$$(18) \quad \gamma_2 = -\frac{b_3}{a_4}\beta_2.$$

Using positions, for example, (1,3) and (1,4) (also we can the consider position (2,3), (2,4)) we obtain

$$(19) \quad \alpha_3 = -\frac{b_1}{c_1}\beta_2,$$

$$(20) \quad \beta_3 = -\frac{b_1}{c_1}\gamma_2.$$

Let us show that β_3 obtained in (20) is the same as in the position (4,3) (also we can consider (5.3)). Indeed, substituting γ_2 from (18) into (20) gives that

$$(21) \quad \beta_3 = \frac{b_1}{c_1} \frac{b_3}{a_4} \beta_2$$

is the same as β_3 given from the position (4,3), and we substitute α_3 from (19),

$$\beta_3 = -\frac{b_3}{a_4}\alpha_3 = \frac{b_3}{a_4} \frac{b_1}{c_1} \beta_2.$$

In the next local step we show that the position (3,3) is fulfilled with the obtained α_3 and β_3 in (19) and (21) correspondingly, that is, we must to verify the equality

$$b_2\beta_2 + a_3\alpha_3 + b_3\beta_3 = 1.$$

Substituting α_3 and β_3 from (19) and (21) gives

$$\begin{aligned} b_2\beta_2 - a_3\frac{b_1}{c_1}\beta_2 + b_3\frac{b_1}{c_1}\frac{b_3}{a_4}\beta_2 &= 1, \quad \text{i.e.} \quad \beta_2(b_2 - a_3\frac{b_1}{c_1} + b_3\frac{b_1}{c_1}\frac{b_3}{a_4}) = 1, \\ \beta_2(b_2 - \frac{b_1}{c_1}(a_3 - \frac{b_3^2}{a_4})) &= 1, \quad \text{i.e.} \quad \beta_2 = \frac{1}{b_2 - \frac{b_1}{c_1}(a_3 - \frac{b_3^2}{a_4})}. \end{aligned}$$

We use the condition (10) in the form $\frac{b_1}{c_1} = \frac{a_2}{b_2}$,

$$\beta_2 = \frac{1}{b_2 - \frac{a_2}{b_2}(a_3 - \frac{b_3^2}{a_4})} = \frac{b_2}{b_2^2 - a_2(a_3 - \frac{b_3^2}{a_4})}$$

that is equal to β_2 obtained in (17).

Let us consider position (3,4) and (4,4) and show that elements in the positions (3,4) and (4,4) of JJ^{-1} form a system with linearly dependent coefficients in (8) taking into account that γ_2 and β_3 are defined in (18) and (21). Indeed, for the coefficients of the system

$$\begin{cases} b_2\gamma_2 + a_3\beta_3 + b_3\alpha_4 + c_3\beta_4 = 0 \\ b_3\beta_3 + a_4\alpha_4 + b_4\beta_4 = 1 \end{cases}$$

in the form

$$\begin{cases} b_3\alpha_4 + c_3\beta_4 = -b_2\gamma_2 - a_3\beta_3 \\ a_4\alpha_4 + b_4\beta_4 = 1 - b_3\beta_3 \end{cases}$$

using condition (10), (17), (18) and (21) we have

$$\begin{aligned} \frac{b_3}{a_4} &= \frac{c_3}{b_4} = \frac{-b_2\gamma_2 - a_3\beta_3}{1 - b_3\beta_3} = \frac{b_2\frac{b_3}{a_4}\beta_2 - a_3\frac{b_1}{c_1}\frac{b_3}{a_4}\beta_2}{1 - b_3\frac{b_1}{c_1}\frac{b_3}{a_4}\beta_2} = \frac{\frac{b_3}{a_4}\beta_2(b_2 - a_3\frac{b_1}{c_1})}{1 - \frac{b_1}{c_1}\frac{b_3^2}{a_4}\beta_2} \\ &= \frac{b_3}{a_4} \frac{(b_2 - a_3\frac{b_1}{c_1}) \frac{-b_2}{a_2(a_3 - \frac{b_3^2}{a_4}) - b_2^2}}{1 + \frac{b_1}{c_1}\frac{b_3^2}{a_4} \frac{b_2}{a_2(a_3 - \frac{b_3^2}{a_4}) - b_2^2}} = \frac{b_3}{a_4} \frac{-b_2(b_2 - a_3\frac{b_1}{c_1})c_1a_4}{-c_1a_4b_2^2 + c_1a_4a_2(a_3 - \frac{b_3^2}{a_4}) + b_1b_3^2b_2} \\ &= \frac{b_3}{a_4} \frac{-b_2^2c_1a_4 + b_1b_2a_3a_4}{-c_1a_4b_2^2 + c_1a_2a_3a_4 - c_1a_2b_3^2 + b_1b_2b_3^2} \\ &= \frac{b_3}{a_4} \frac{-b_2^2c_1a_4 + b_1b_2a_3a_4}{-b_2^2c_1a_4 + b_1b_2a_3a_4 + b_3^2(b_1b_2 - c_1a_2)} = \frac{b_3}{a_4}, \end{aligned}$$

where we used the condition (10) in the form $(b_1b_2 - c_1a_2) = 0$ two times.

Hence in the next step we can consider only the position (4,4) instead of considering both (3,4) and (4,4) together.

From the position (4,3) we have $b_3\alpha_3 + a_4\beta_3 = 0$ i.e. $\beta_3 = -\frac{b_3}{a_4}\alpha_3$. Since (1,3) gives $\alpha_3 = -\frac{b_1}{c_1}\beta_2$, we have $\beta_3 = \frac{b_3}{a_4}\frac{b_1}{c_1}\beta_2$. Also from the position (1,4) we have $\beta_3 = -\frac{b_1}{c_1}\gamma_2$. Since (4,2) gives $\gamma_2 = -\frac{b_3}{a_4}\beta_2$ we have the same for β_3 , $\beta_3 = \frac{b_1}{c_1}\frac{b_3}{a_4}\beta_2$. This completes the first inductive step.

Step 2. In fact, we will consider the n -th inductive step. Further in this step we use the indexes $n = 2k$, $k \in \mathbb{N}$.

In this step we consider the following elements of the matrix JJ^{-1} :

$$\begin{aligned} &(n, n), (n+1, n), (n+2, n), (n+3, n), (n-1, n+1), (n, n+1), (n+1, n+1), \\ &(n+2, n+1), (n+3, n+1), (n-1, n+2), (n, n+2), (n+1, n+2), (n+2, n+2), \end{aligned}$$

i.e., the corresponding portion of (8) has the form

$$(22) \quad \begin{pmatrix} * & b_{n-1}\beta_n + c_{n-1}\alpha_{n+1} \\ b_{n-1}\beta_{n-1} + a_n\alpha_n + b_n\beta_n & a_n\beta_n + b_n\alpha_{n+1} \\ c_{n-1}\beta_{n-1} + b_n\alpha_n + a_{n+1}\beta_n + b_{n+1}\gamma_n & b_n\beta_n + a_{n+1}\alpha_{n+1} + b_{n+1}\beta_{n+1} \\ b_{n+1}\beta_n + a_{n+2}\gamma_n & b_{n+1}\alpha_{n+1} + a_{n+2}\beta_{n+1} \\ c_{n+1}\beta_n + b_{n+2}\gamma_n & c_{n+1}\alpha_{n+1} + b_{n+2}\beta_{n+1} \\ 0 & 0 \end{pmatrix} \dots \begin{pmatrix} b_{n-1}\gamma_n + c_{n-1}\beta_{n+1} \\ a_n\gamma_n + b_n\beta_{n+1} \\ b_n\gamma_n + a_{n+1}\beta_{n+1} + b_{n+1}\alpha_{n+2} + c_{n+1}\beta_{n+2} \\ b_{n+1}\beta_{n+1} + a_{n+2}\alpha_{n+2} + b_{n+2}\beta_{n+2} \\ * \\ * \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}, \quad n \in \mathbb{N},$$

where as usual * denotes elements we do not consider.

Since the elements $(n+2, n)$ and $(n+3, n)$ form a linearly dependent system (condition (11)), let us consider the elements (n, n) , $(n+1, n)$ and $(n+2, n)$ of JJ^{-1} and show that these elements form in (8) a linearly independent system that gives a unique solution $\alpha_n, \beta_n, \gamma_n$, taking into account that β_{n-1} are defined earlier in the inductive hypothesis,

$$\begin{cases} b_{n-1}\beta_{n-1} + a_n\alpha_n + b_n\beta_n = 1 \\ c_{n-1}\beta_{n-1} + b_n\alpha_n + a_{n+1}\beta_n + b_{n+1}\gamma_n = 0 \\ b_{n+1}\beta_n + a_{n+2}\gamma_n = 0 \end{cases}$$

and

$$(23) \quad \begin{cases} a_n\alpha_n + b_n\beta_n = 1 - b_{n-1}\beta_{n-1} \\ b_n\alpha_n + a_{n+1}\beta_n + b_{n+1}\gamma_n = -c_{n-1}\beta_{n-1} \\ b_{n+1}\beta_n + a_{n+2}\gamma_n = 0 \end{cases}.$$

Here we use the condition (11),

$$\det \begin{vmatrix} a_n & b_n & 0 \\ b_n & a_{n+1} & b_{n+1} \\ 0 & b_{n+1} & a_{n+2} \end{vmatrix} \neq 0,$$

which gives uniqueness of the solution $\alpha_n, \beta_n, \gamma_n$. Substitute γ_n from the third equation into the second one in (23),

$$\begin{cases} a_n\alpha_n + b_n\beta_n = 1 - b_{n-1}\beta_{n-1} \\ b_n\alpha_n + (a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}})\beta_n = -c_{n-1}\beta_{n-1} \end{cases}.$$

From the latter system we have

$$\alpha_n = \frac{(1 - b_{n-1}\beta_{n-1})(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}) + c_{n-1}\beta_{n-1}b_n}{a_n(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}) - b_n^2},$$

$$\beta_n = \frac{-a_n c_{n-1} \beta_{n-1} - b_n(1 - b_{n-1} \beta_{n-1})}{a_n(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}) - b_n^2} = \frac{(-a_n c_{n-1} + b_{n-1} b_n) \beta_{n-1} - b_n}{a_n(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}) - b_n^2}.$$

Taking into account condition (10) in the form $a_n c_{n-1} - b_{n-1} b_n = 0$ we obtain that

$$(24) \quad \beta_n = \frac{-b_n}{a_n(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}) - b_n^2}.$$

From the position $(n+2, n)$ we obtain

$$(25) \quad \gamma_n = -\frac{b_{n+1}}{a_{n+2}}\beta_n.$$

Using positions, for example, $(n-1, n+1)$ and $(n-1, n+2)$ (also we can consider position $(n, n+1)$, $(n, n+2)$) we obtain

$$(26) \quad \alpha_{n+1} = -\frac{b_{n-1}}{c_{n-1}}\beta_n,$$

$$(27) \quad \beta_{n+1} = -\frac{b_{n-1}}{c_{n-1}}\gamma_n.$$

Let us show that β_{n+1} obtained in (27) is the same as in the position $(n+2, n+1)$ (also we can consider $(n+3, n+1)$). Indeed, the substitution of γ_n from (25) into (27)

$$(28) \quad \beta_{n+1} = \frac{b_{n-1}}{c_{n-1}}\frac{b_{n+1}}{a_{n+2}}\beta_n$$

and this is the same as β_{n+1} given from the position $(n+2, n+1)$ if we substitute α_{n+1} from (26)

$$\beta_{n+1} = -\frac{b_{n+1}}{a_{n+2}}\alpha_{n+1} = \frac{b_{n+1}}{a_{n+2}}\frac{b_{n-1}}{c_{n-1}}\beta_n.$$

In the next local step we show that the position $(n+1, n+1)$ contains the obtained α_{n+1} and β_{n+1} in (26) and (28) correspondingly, i.e., we must verify the equality

$$b_n\beta_n + a_{n+1}\alpha_{n+1} + b_{n+1}\beta_{n+1} = 1.$$

Substitutions α_{n+1} and β_{n+1} from (26) and (28) give

$$\begin{aligned} b_n\beta_n - a_{n+1}\frac{b_{n-1}}{c_{n-1}}\beta_n + b_{n+1}\frac{b_{n-1}}{c_{n-1}}\frac{b_{n+1}}{a_{n+2}}\beta_n &= 1, \\ \beta_n\left(b_n - a_{n+1}\frac{b_{n-1}}{c_{n-1}} + b_{n+1}\frac{b_{n-1}}{c_{n-1}}\frac{b_{n+1}}{a_{n+2}}\right) &= 1, \\ \beta_n\left(b_n - \frac{b_{n-1}}{c_{n-1}}\left(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}\right)\right) &= 1, \quad \text{i.e.} \quad \beta_n = \frac{1}{b_n - \frac{b_{n-1}}{c_{n-1}}\left(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}\right)}. \end{aligned}$$

We use the condition (10) in the form $\frac{b_{n-1}}{c_{n-1}} = \frac{a_n}{b_n}$

$$\beta_n = \frac{1}{b_n - \frac{a_n}{b_n}\left(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}\right)} = \frac{b_n}{b_n^2 - a_n\left(a_{n+1} - \frac{b_{n+1}^2}{a_{n+2}}\right)}$$

that is equal to β_n obtained in (24).

Let us consider the positions $(n+1, n+2)$ and $(n+2, n+2)$, and show that the elements in the positions $(n+1, n+2)$ and $(n+2, n+2)$ in JJ^{-1} form a system with linearly dependent coefficients in (8), taking into account that γ_n and β_{n+1} are defined in (25) and (28).

Indeed, for the coefficients of the system

$$\begin{cases} b_n\gamma_n + a_{n+1}\beta_{n+1} + b_{n+1}\alpha_{n+2} + c_{n+1}\beta_{n+2} = 0 \\ b_{n+1}\beta_{n+1} + a_{n+2}\alpha_{n+2} + b_{n+2}\beta_{n+2} = 1 \end{cases}$$

in the form

$$\begin{cases} b_{n+1}\alpha_{n+2} + c_{n+1}\beta_{n+2} = -b_n\gamma_n - a_{n+1}\beta_{n+1} \\ a_{n+2}\alpha_{n+2} + b_{n+2}\beta_{n+2} = 1 - b_{n+1}\beta_{n+1} \end{cases}$$

using condition (10), (24), (25) and (28) we have

$$\begin{aligned}
\frac{b_{n+1}}{a_{n+2}} &= \frac{c_{n+1}}{b_{n+2}} = \frac{-b_n \gamma_n - a_{n+1} \beta_{n+1}}{1 - b_{n+1} \beta_{n+1}} = \frac{b_n \frac{b_{n+1}}{a_{n+2}} \beta_n - a_{n+1} \frac{b_{n-1}}{c_{n-1}} \frac{b_{n+1}}{a_{n+2}} \beta_n}{1 - b_{n+1} \frac{b_{n-1}}{c_{n-1}} \frac{b_{n+1}}{a_{n+2}} \beta_n} \\
&= \frac{\frac{b_{n+1}}{a_{n+2}} \beta_n (b_n - a_{n+1} \frac{b_{n-1}}{c_{n-1}})}{1 - \frac{b_{n-1}}{c_{n-1}} \frac{b_{n+1}}{a_{n+2}} \beta_n} = \left(\frac{b_{n+1}}{a_{n+2}} \right) \frac{(b_n - a_{n+1} \frac{b_{n-1}}{c_{n-1}}) \frac{-b_n}{a_n (a_{n+1} - \frac{b_{n+1}}{a_{n+2}}) - b_n^2}}{1 + \frac{b_{n-1}}{c_{n-1}} \frac{b_{n+1}}{a_{n+2}} \frac{b_n}{a_n (a_{n+1} - \frac{b_{n+1}}{a_{n+2}}) - b_n^2}} \\
&= \left(\frac{b_{n+1}}{a_{n+2}} \right) \frac{-b_n (b_n - a_{n+1} \frac{b_{n-1}}{c_{n-1}}) c_{n-1} a_{n+2}}{-c_{n-1} a_{n+2} b_n^2 + c_{n-1} a_{n+2} a_n (a_{n+1} - \frac{b_{n+1}}{a_{n+2}}) + b_{n-1} b_{n+1}^2 b_n} \\
&= \left(\frac{b_{n+1}}{a_{n+2}} \right) \frac{-b_n^2 c_{n-1} a_{n+2} + b_{n-1} b_n a_{n+1} a_{n+2}}{-c_{n-1} a_{n+2} b_n^2 + c_{n-1} a_n a_{n+1} a_{n+2} - c_{n-1} a_n b_{n+1}^2 + b_{n-1} b_n b_{n+1}^2} \\
&= \left(\frac{b_{n+1}}{a_{n+2}} \right) \frac{-b_n^2 c_{n-1} a_{n+2} + b_{n-1} b_n a_{n+1} a_{n+2}}{-b_n^2 c_{n-1} a_{n+2} + b_{n-1} b_n a_{n+1} a_{n+2} + b_{n+1}^2 (b_{n-1} b_n - c_{n-1} a_n)} = \frac{b_{n+1}}{a_{n+2}},
\end{aligned}$$

where we used the condition (10) in the form $(b_{n-1} b_n - c_{n-1} a_n) = 0$ two times.

Hence in the next step we can consider only the position $(n+2, n+2)$ instead of considering both $(n+1, n+2)$ and $(n+2, n+2)$ together (if needed).

From the position $(n+2, n+1)$ we have $b_{n+1} \alpha_{n+1} + a_{n+2} \beta_{n+1} = 0$, i.e., $\beta_{n+1} = -\frac{b_{n+1}}{a_{n+2}} \alpha_{n+1}$. Since $(n-1, n+1)$ gives $\alpha_{n+1} = -\frac{b_{n-1}}{c_{n-1}} \beta_n$, we have $\beta_{n+1} = \frac{b_{n+1}}{a_{n+2}} \frac{b_{n-1}}{c_{n-1}} \beta_n$. Also from the position $(n-1, n+2)$ we have $\beta_{n+1} = -\frac{b_{n-1}}{c_{n-1}} \gamma_n$. Since $(n+2, n)$ gives $\gamma_n = -\frac{b_{n+1}}{a_{n+2}} \beta_n$, we have the same for β_{n+1} , i.e., $\beta_{n+1} = \frac{b_{n-1}}{c_{n-1}} \frac{b_{n+1}}{a_{n+2}} \beta_n$.

This completes the n -th step of the induction and the proof of the theorem. \square

The next corollary of theorem 1 gives a recommendation how to construct a Jacobi type matrix corresponding to the strong Hamburger moment problem, so that the corresponding operators are bounded.

Corollary 1. *If the sequences of real bounded numbers $\{a_k\}$, $\{b_k\}$, and $\{c_k\}$, $k \in \mathbb{N}$ satisfy the conditions*

$$\det \begin{vmatrix} b_{n-1} & c_{n-1} \\ a_n & b_n \end{vmatrix} = 0,$$

$$b_1 \det \begin{vmatrix} a_1 & b_1 \\ b_1 & a_2 \end{vmatrix} < 0, \quad b_n b_{n+1} \det \begin{vmatrix} a_n & b_n & 0 \\ b_n & a_{n+1} & b_{n+1} \\ 0 & b_{n+1} & a_{n+2} \end{vmatrix} > 0, \quad n = 2k, \quad k \in \mathbb{N}.$$

then the Jacobi type matrix constructed in (7) corresponds to the strong Hamburger moment problem.

Inequalities in the corollary guarantee that γ_n in the matrix J^{-1} are positive.

For Theorem 1 we have the following (in some sense) inverse theorem.

Theorem 2. *The matrix J^{-1} in the form (9) has an algebraic inverse one J on the form (7) iff*

$$\det \begin{vmatrix} \beta_n & \gamma_n \\ \alpha_{n+1} & \beta_{n+1} \end{vmatrix} = 0, \quad n = 2k, \quad k \in \mathbb{N},$$

$$\det \begin{vmatrix} \alpha_{n-1} & \beta_{n-1} & 0 \\ \beta_{n-1} & \alpha_n & \beta_n \\ 0 & \beta_n & \alpha_{n+1} \end{vmatrix} \neq 0, \quad n = 2k, \quad k \in \mathbb{N}.$$

Proof. The proof of this theorem is analogous to the proof of Theorem 1. \square

Acknowledgments. The author are grateful to professors Yu. M. Berezansky and L. B. Golinskii for stimulating discussions.

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Received 12/11/2012; Revised 05/03/2013