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TACHYON GENERALIZATION FOR LORENTZ TRANSFORMS

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Abstract. In the present paper we construct an expansion of the set of Lorentz transforms, which allows for the velocity of the reference frame to be greater than the speed of light. For maximum generality we investigate this tachyon expansion in the case of Minkowski space time over any real Hilbert space.

1. Introduction

The fact that the existence of superlight motions is consistent with the kinematics of Einstein's special theory of relativity at the present time may be considered as generally known. In [1, 2] this fact is proved by means of mathematical logic. It is interesting that the last fact can also be proved in another way [4, 3]. In these papers the kinematics that permits superlight transformations was built explicitly using the theory of changeable sets (see [3, p. 128, example 2.3], [4, p. 41, example 10.3]). Although the existence of tachyons can not be considered as an experimentally verified fact, the theory of tachyons and superluminal motions is intensively developing for more than 50 years [5, 6], and it is very actual in our time. In many previous studies the theory of tachyons was considered in the framework of classical Lorentz transformations, and the superlight speed for the frame of reference was forbidden. But in the paper [7] some extension of classical Lorentz transforms for superlight velocity of reference frames is proposed. These transforms in the Minkowski space $\mathbb{R}^4$ may be expressed by the formula

$$U_v(t,x,y,z) = \left(\frac{s\left(t - \frac{vx}{c^2}\right)}{\sqrt{\left(\frac{v}{c}\right)^2 - 1}}, \frac{s\,(x - vt)}{\sqrt{\left(\frac{v}{c}\right)^2 - 1}}, y, z \right), \quad (t,x,y,z) \in \mathbb{R}^4, \tag{1}$$

where $v \in \mathbb{R}$, $|v| > c$, $s \in \{-1, 1\}$, and c is a positive real constant, which has the physical meaning of the speed of light in vacuum. The work [7] was some sensation, and it has become famous not only among physicists and mathematicians. One of the conclusions of this work is that some aspects of the existing tachyon theories may be revised. In particular the transforms (1) do not involve the need to introduce imaginary masses or complicated physics to construct kinematics and dynamics for tachyons [7]. Note, that the transforms similar to (1) was also obtained earlier (see [8, 9]) but, unfortunately, these works did not became such famous as [7].

It should be emphasized that the papers [7, 8, 9] were written mainly in "physical style". In particular in these papers it is examined only the case when two inertial frames are moving along the x-axis, and there were not investigated the new transforms for arbitrary orientation of axes.

In the present paper the conditions for general transforms of type (1) are formulated with arbitrary orientation of axes, and the properties of transforms satisfying these conditions are investigated. For maximum generality we conduct our research for the case of

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Minkowski space time over any real Hilbert space using methods of the operator theory in Hilbert space. Note that a generalization of the classical Lorentz group (with speeds less then light) for the case of real Hilbert space was investigated in [12, 13, 14].

2. Abstract coordinate transforms in Minkowski space time over Hilbert space and their properties

Let $(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a real Hilbert space, where the $\|\cdot\|$ is the norm and $\langle\cdot,\cdot\rangle$ is the inner product over the space $\mathfrak{H}$. Denote by $\mathcal{M}(\mathfrak{H})$ the Hilbert space

$$\mathcal{M}(\mathfrak{H}) := \mathbb{R} \times \mathfrak{H} = \{(t,x) \mid t \in \mathbb{R},\ x \in \mathfrak{H}\},$$

equipped by the following inner product and norm:

$$\begin{aligned}&\langle \omega_1, \omega_2 \rangle_{\mathcal{M}(\mathfrak{H})} = t_1 t_2 + \langle x_1, x_2 \rangle, \\ &\|\omega_1\|_{\mathcal{M}(\mathfrak{H})} = t_1^2 + \|x_1\|^2 \quad (\omega_i = (t_i, x_i),\ i \in \{1,2\}).\end{aligned}$$

The space $\mathcal{M}(\mathfrak{H})$ is called a ***Minkowski space*** over the Hilbert space $\mathfrak{H}$. In the space $\mathcal{M}(\mathfrak{H})$ we select the subspaces

$$\begin{aligned}\mathfrak{H}_0 &= \{(t, \mathbf{0}) \mid t \in \mathbb{R}\}, \\ \mathfrak{H}_1 &= \{(0, x) \mid x \in \mathfrak{H}\}\end{aligned}$$

with $\mathbf{0}$ being zero vector. Then

$$\mathcal{M}(\mathfrak{H}) = \mathfrak{H}_0 \oplus \mathfrak{H}_1,$$

where $\oplus$ means the orthogonal sum of the subspaces. The space $\mathfrak{H}_0$ is isomorphic to the real field $\mathbb{R}$ and the space $\mathfrak{H}_1$ is isomorphic to the space $\mathfrak{H}$. Hence, the space $\mathfrak{H}$ may be identified with the subspace $\mathfrak{H}_1$ of the space $\mathcal{M}(\mathfrak{H})$, and $\mathcal{M}(\mathfrak{H})$ may be considered as an extension of the space $\mathfrak{H}$. That is why, further we will use the same notations for the inner product and the norm in the spaces $\mathfrak{H}$ and $\mathcal{M}(\mathfrak{H})$ (that is $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, without the index "$\mathcal{M}(\mathfrak{H})$" in the subscript).

Denote by $\mathbf{e}_0$ the vector

$$\mathbf{e}_0 = (1, \mathbf{0}) \in \mathcal{M}(\mathfrak{H}).$$

We introduce the following orthogonal projectors by the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$:

$$\begin{aligned}&\mathbf{T}\omega = t\mathbf{e}_0 = (t, \mathbf{0}) \in \mathfrak{H}_0, \qquad \omega = (t,x) \in \mathcal{M}(\mathfrak{H}); \\ &\mathbf{X}\omega = (0, x) \in \mathfrak{H}_1, \qquad \omega = (t,x) \in \mathcal{M}(\mathfrak{H})\end{aligned}$$

(recall, that an operator $P \in \mathcal{L}(\mathfrak{H})$ is named orthogonal projector if $P^2 = P^* = P$, where P^* is the adjoint operator to the operator P). Also we denote by $\mathcal{T}$ the following linear operator:

$$\mathcal{T}(\omega) = t, \quad \omega = (t,x) \in \mathcal{M}(\mathfrak{H})$$

from $\mathcal{M}(\mathfrak{H})$ to $\mathbb{R}$. Then the following equality apparently holds:

$$\mathbf{T}\omega = \mathcal{T}(\omega)\,\mathbf{e}_0, \quad \omega \in \mathcal{M}(\mathfrak{H}). \tag{2}$$

And any vector $\omega \in \mathcal{M}(\mathfrak{H})$ can be uniquely represented as

$$\omega = t\mathbf{e}_0 + x = \mathcal{T}(\omega)\,\mathbf{e}_0 + \mathbf{X}\omega, \tag{3}$$

where $x = \mathbf{X}\omega \in \mathfrak{H}_1,\ t = \mathcal{T}(\omega) \in \mathbb{R}$.

Denote by $\mathcal{L}(\mathfrak{H})$ the space of linear continuous operators over the space $\mathfrak{H}$.

Definition 1. *The operator $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is referred to as* ***coordinate transform*** *if and only if there exist the continuous inverse operator $S^{-1} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$.*

It is clear, that product (composition) of any two coordinate transforms is coordinate transform and the operator, inverse to coordinate transform again is coordinate transform. Thus the set of all coordinate transforms is the group of operators in the space $\mathcal{M}(\mathfrak{H})$.

Definition 2. *The coordinate transform $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ is called **v-determined** if and only if $\mathcal{T}\left(S^{-1}\mathbf{e}_0\right) \neq 0$. The vector*

$$\mathcal{V}(S) = \frac{\mathbf{X}S^{-1}\mathbf{e}_0}{\mathcal{T}\left(S^{-1}\mathbf{e}_0\right)} \in \mathfrak{H}_1$$

is named the velocity of the v-determined coordinate transform S.

The definition 2 is consistent with the physical understanding of the speed of reference frame. Indeed suppose, that the v-determined coordinate transform $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ maps the coordinates of any point in the fixed frame of reference $\mathfrak{L}$ to coordinates of this point in another frame $\mathfrak{L}'$, moving with constant velocity relative to the frame $\mathfrak{L}$. Consider any stationary relative the frame $\mathfrak{L}'$ point $\omega_t' = x_0 + t\mathbf{e}_0$ (where $x_0 \in \mathfrak{H}_1$ is fixed vector, and variable t runs over all real axis $\mathbb{R}$). Then the point ω_t' in the frame $\mathfrak{L}$ will look like as $\omega_t = S^{-1}\omega_t'$, and using (3) we obtain

$$\begin{aligned}\omega_t = S^{-1}x_0 + tS^{-1}\mathbf{e}_0 &= \mathcal{T}\left(S^{-1}x_0\right)\mathbf{e}_0 + \mathbf{X}S^{-1}x_0 + t\left(\mathcal{T}\left(S^{-1}\mathbf{e}_0\right)\mathbf{e}_0 + \mathbf{X}S^{-1}\mathbf{e}_0\right)\\ &= \mathcal{T}\left(S^{-1}\left(x_0 + t\mathbf{e}_0\right)\right)\mathbf{e}_0 + \mathbf{X}S^{-1}\left(x_0 + t\mathbf{e}_0\right).\end{aligned}$$

Thus, for any $t_1, t_2 \in \mathbb{R}$ such, that $t_1 \neq t_2$ we deliver

$$\frac{\mathbf{X}\omega_{t_2} - \mathbf{X}\omega_{t_1}}{\mathcal{T}\left(\omega_{t_2}\right) - \mathcal{T}\left(\omega_{t_1}\right)} = \frac{\mathbf{X}S^{-1}\left(x_0 + t_2\mathbf{e}_0 - \left(x_0 + t_1\mathbf{e}_0\right)\right)}{\mathcal{T}\left(S^{-1}\left(x_0 + t_2\mathbf{e}_0\right) - S^{-1}\left(x_0 + t_1\mathbf{e}_0\right)\right)} = \mathcal{V}(S).$$

Thus, any stationary relative the frame $\mathfrak{L}'$ point is moving relative the frame $\mathfrak{L}$ with constant velocity $\mathcal{V}(S)$.

For any vector $V \in \mathfrak{H}_1$ generates the subspaces

$$\begin{aligned}\mathfrak{H}_1[V] &= \mathbf{span}\,\{V\},\\ \mathfrak{H}_{1\perp}[V] &= \mathfrak{H}_1 \ominus \mathfrak{H}_1[V] = \{x \in \mathfrak{H}_1 \mid \langle x, V\rangle = 0\},\end{aligned}$$

where $\mathbf{span}\,M$ denotes the linear span of the set $M \subseteq \mathcal{M}(\mathfrak{H})$. The orthogonal projectors for the subspaces $\mathfrak{H}_1[V]$ and $\mathfrak{H}_{1\perp}[V]$ will be denoted by $\mathbf{X}_1[V]$, $\mathbf{X}_1^{\perp}[V]$

$$\mathbf{X}_1[V]\omega = \begin{cases}\frac{\langle V,\omega\rangle}{\|V\|^2}V, & V \neq \mathbf{0}\\ \mathbf{0}, & V = \mathbf{0}\end{cases}, \quad \omega \in \mathcal{M}(\mathfrak{H}), \tag{4}$$
$$\mathbf{X}_1^{\perp}[V] = \mathbf{X} - \mathbf{X}_1[V].$$

It is not hard to verify, that for an arbitrary $V \in \mathfrak{H}_1$ the following equalities are performed:

$$\begin{gathered}\mathbf{T} + \mathbf{X} = \mathbb{I}, \quad \mathbf{X}_1[V] + \mathbf{X}_1^{\perp}[V] = \mathbf{X}, \quad \mathbf{T} + \mathbf{X}_1[V] + \mathbf{X}_1^{\perp}[V] = \mathbb{I},\\ \mathbf{T}\mathbf{X} = \mathbf{X}\mathbf{T} = \mathbb{O}, \quad \mathbf{X}_1[V]\mathbf{X}_1^{\perp}[V] = \mathbf{X}_1^{\perp}[V]\mathbf{X}_1[V] = \mathbb{O},\\ \mathbf{T}\mathbf{X}_1[V] = \mathbf{X}_1[V]\mathbf{T} = \mathbb{O}, \quad \mathbf{T}\mathbf{X}_1^{\perp}[V] = \mathbf{X}_1^{\perp}[V]\mathbf{T} = \mathbb{O},\\ \mathbf{X}\mathbf{X}_1[V] = \mathbf{X}_1[V]\mathbf{X} = \mathbf{X}_1[V], \quad \mathbf{X}\mathbf{X}_1^{\perp}[V] = \mathbf{X}_1^{\perp}[V]\mathbf{X} = \mathbf{X}_1^{\perp}[V],\end{gathered} \tag{5}$$

where $\mathbb{I}$ and $\mathbb{O}$ are identity and zero operators in the space $\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ correspondingly.

Lemma 1. *Let $S \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ be a coordinate transform such, that both coordinate transforms S ans S^{-1} are v-determined. Then S is bijection between the subspaces $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]$ and $\mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]$. Moreover for any $\omega = t\mathbf{e}_0 + \lambda\mathcal{V}(S) \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]$ the following equality is true:*

$$S\left(t\mathbf{e}_0 + \lambda\mathcal{V}(S)\right) = \alpha_S\left(\left(t - \lambda\beta_S\right)\mathbf{e}_0 + (t - \lambda)\,\mathcal{V}\left(S^{-1}\right)\right) \quad (\forall t, \lambda \in \mathbb{R}),$$

where

$$\alpha_S = \mathcal{T}(S\mathbf{e}_0), \quad \beta_S = 1 - \frac{1}{\mathcal{T}(S\mathbf{e}_0)\,\mathcal{T}(S^{-1}\mathbf{e}_0)} = 1 - \frac{1}{\alpha_S \alpha_{S^{-1}}}.$$

Proof. Let S, S^{-1} be v-determined coordinate transforms. Then, by definition 2 and equalities (2), (5), for any $t, \lambda \in \mathbb{R}$ we obtain

$$\begin{aligned}
S\left(t\mathbf{e}_0 + \lambda \mathcal{V}(S)\right) &= tS\mathbf{e}_0 + \lambda S\mathcal{V}(S) = tS\mathbf{e}_0 + \lambda S\frac{\mathbf{X}S^{-1}\mathbf{e}_0}{\mathcal{T}(S^{-1}\mathbf{e}_0)} \\
&= tS\mathbf{e}_0 + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)} S\left(S^{-1}\mathbf{e}_0 - \mathbf{T}S^{-1}\mathbf{e}_0\right) \\
&= tS\mathbf{e}_0 + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)} S\left(S^{-1}\mathbf{e}_0 - \mathcal{T}\left(S^{-1}\mathbf{e}_0\right)\mathbf{e}_0\right) \\
&= tS\mathbf{e}_0 + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)} \left(\mathbf{e}_0 - \mathcal{T}\left(S^{-1}\mathbf{e}_0\right)S\mathbf{e}_0\right) \\
&= (t-\lambda)\,S\mathbf{e}_0 + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0 \\
&= (t-\lambda)\left(\mathbf{T}S\mathbf{e}_0 + \mathbf{X}S\mathbf{e}_0\right) + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0 \\
&= (t-\lambda)\left(\mathcal{T}(S\mathbf{e}_0)\,\mathbf{e}_0 + \mathbf{X}S\mathbf{e}_0\right) + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0 \\
&= (t-\lambda)\,\mathcal{T}(S\mathbf{e}_0)\left(\mathbf{e}_0 + \frac{\mathbf{X}S\mathbf{e}_0}{\mathcal{T}(S\mathbf{e}_0)}\right) + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0 \\
&= (t-\lambda)\,\mathcal{T}(S\mathbf{e}_0)\left(\mathbf{e}_0 + \mathcal{V}\left(S^{-1}\right)\right) + \frac{\lambda}{\mathcal{T}(S^{-1}\mathbf{e}_0)}\mathbf{e}_0 \\
&= \mathcal{T}(S\mathbf{e}_0)\left(\left(t - \lambda\left(1 - \frac{1}{\mathcal{T}(S\mathbf{e}_0)\,\mathcal{T}(S^{-1}\mathbf{e}_0)}\right)\right)\mathbf{e}_0 + (t-\lambda)\,\mathcal{V}\left(S^{-1}\right)\right) \\
&= \alpha_S\left((t - \lambda\beta_S)\,\mathbf{e}_0 + (t-\lambda)\,\mathcal{V}\left(S^{-1}\right)\right).
\end{aligned}$$

Hence, the operator S maps the subspace $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]$ into the subspace $\mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]$. In the case $\mathcal{V}(S) \neq \mathbf{0}$ the subspace $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]$ is two-dimensional ($\dim(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) = 2$). And since S is bijection on $\mathcal{M}(\mathfrak{H})$, dimension of the image $S(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) \subseteq \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]$ also must be equal 2. And since $\dim(\mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]) \leq 2$, we have, that $S(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) = \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]$ and $\dim(\mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]) = 2$. Thus, in the case $\mathcal{V}(S) \neq \mathbf{0}$, the lemma is proved.

Above we have proved, that if $\mathcal{V}(S) \neq \mathbf{0}$, then $\dim\left(\mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]\right) = 2$, and, consequently, $\mathcal{V}\left(S^{-1}\right) \neq 0$. And, conversely, if $\mathcal{V}\left(S^{-1}\right) \neq 0$, then $\mathcal{V}(S) = \mathcal{V}\left(\left(S^{-1}\right)^{-1}\right) \neq 0$. Thus, in the case $\mathcal{V}(S) = \mathbf{0}$, we have $\mathcal{V}\left(S^{-1}\right) = \mathbf{0}$. Therefore, in this case $\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)] = \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right] = \mathfrak{H}_0$, and, consequently, one-dimensional subspace $\mathfrak{H}_0$ is invariant subspace of the operator S. And, since S is one-to-one mapping, we deliver that $S(\mathfrak{H}_0) = \mathfrak{H}_0$, and, hence, $S(\mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(S)]) = \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(S^{-1}\right)\right]$. Thus, in the case $\mathcal{V}(S) = \mathbf{0}$, the lemma also is proved. □

3. General Lorentz group in Hilbert space

Everywhere in this paper c will be a fixed positive real constant, which has the physical content of the speed of light in vacuum. Denote by $\mathsf{M}_c(\cdot)$ Lorentz-Minkowski pseudometric on the space $\mathcal{M}(\mathfrak{H})$

$$\mathsf{M}_c(\omega) = \|\mathbf{X}\omega\|^2 - c^2\mathcal{T}^2(\omega), \quad \omega \in \mathcal{M}(\mathfrak{H}). \tag{6}$$

Pseudo-metric (6) is generated by the quasi-inner product

$$\langle\langle\omega_1,\omega_2\rangle\rangle_c=\langle\mathbf{X}\omega_1,\mathbf{X}\omega_2\rangle-c^2\mathcal{T}(\omega_1)\,\mathcal{T}(\omega_2)\,,\quad \omega_1,\omega_2\in\mathcal{M}(\mathfrak{H})\,, \tag{7}$$

$$\mathsf{M}_c(\omega)=\langle\langle\omega,\omega\rangle\rangle_c\,,\quad \omega\in\mathcal{M}(\mathfrak{H})\,. \tag{8}$$

It is clear, that quasi-inner product $\langle\langle\omega_1,\omega_2\rangle\rangle_c$ $(\omega_1,\omega_2\in\mathcal{M}(\mathfrak{H}))$ is bilinear form relatively the variables ω_1,ω_2. Hence (by (8)), for any $\omega_1,\omega_2\in\mathcal{M}(\mathfrak{H})$ it holds the equality

$$\langle\langle\omega_1,\omega_2\rangle\rangle_c=\frac{1}{2}\left(\mathsf{M}_c(\omega_1+\omega_2)-\mathsf{M}_c(\omega_1)-\mathsf{M}_c(\omega_2)\right). \tag{9}$$

Denote by $\mathfrak{O}(\mathfrak{H},c)$ the set of all coordinate transforms over $\mathcal{M}(\mathfrak{H})$, leaving unchanged values of the functional (6) , that is the set of all coordinate transforms $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ such, that:

$$\mathsf{M}_c(L\omega)=\mathsf{M}_c(\omega)\quad(\forall\omega\in\mathcal{M}(\mathfrak{H}))\,. \tag{10}$$

Using the equality (9) it is easy to verify, that any coordinate transform $L\in\mathfrak{O}(\mathfrak{H},c)$ leaves unchanged the values of the quasi-inner product (7)

$$\langle\langle L\omega_1,L\omega_2\rangle\rangle_c=\langle\langle\omega_1,\omega_2\rangle\rangle_c\,,\quad \omega_1,\omega_2\in\mathcal{M}(\mathfrak{H})\,. \tag{11}$$

It is not hard to see, that product of any two operators from $\mathfrak{O}(\mathfrak{H},c)$ belongs to $\mathfrak{O}(\mathfrak{H},c)$ and the mapping, inverse to any coordinate transform from $\mathfrak{O}(\mathfrak{H},c)$ also belongs to the set $\mathfrak{O}(\mathfrak{H},c)$. Hence, the set $\mathfrak{O}(\mathfrak{H},c)$ is the group of operators in the space $\mathcal{M}(\mathfrak{H})$. According to [15] we name this group the ***general Lorentz group*** over the space $\mathcal{M}(\mathfrak{H})$.

Any general Lorentz transform $L\in\mathfrak{O}(\mathfrak{H},c)$ is v-determined and $\|\mathcal{V}(L)\|<c$. Indeed,

$$\mathsf{M}_c(\mathbf{e}_0)=\|\mathbf{X}\mathbf{e}_0\|^2-c^2\mathcal{T}^2(\mathbf{e}_0)=0-c^2\cdot 1=-c^2.$$

As it was mentioned above, $L^{-1}\in\mathfrak{O}(\mathfrak{H},c)$ for $L\in\mathfrak{O}(\mathfrak{H},c)$. Therefore, by (10),

$$\mathsf{M}_c\left(L^{-1}\mathbf{e}_0\right)=\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|^2-c^2\mathcal{T}^2\left(L^{-1}\mathbf{e}_0\right)=-c^2.$$

Hence, $\left|\mathcal{T}\left(L^{-1}\mathbf{e}_0\right)\right|=\frac{1}{c}\sqrt{\|\mathbf{X}L^{-1}\mathbf{e}_0\|^2+c^2}>0$. Thus the coordinate transform L is v-determined, moreover

$$\|\mathcal{V}(L)\|=\frac{\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|}{\left|\mathcal{T}\left(L^{-1}\mathbf{e}_0\right)\right|}=c\frac{\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|}{\sqrt{\left\|\mathbf{X}L^{-1}\mathbf{e}_0\right\|^2+c^2}}<c. \tag{12}$$

The aim of the next assertion is to emphasize some characteristic properties of the general Lorentz transforms, which may serve as a basis for another definition of the general Lorentz group. And these properties also will become the motivation for definition of the set of extended Lorentz transforms, which allows superlight speed of reference frame.

Assertion 1. *Any coordinate transform $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to $\mathfrak{O}(\mathfrak{H},c)$ if and only if the following conditions are satisfied:*

(1) *Coordinate transforms L and L^{-1} are v-determined;*
(2) $\mathsf{M}_c(L\omega)=\mathsf{M}_c(\omega)$ $(\forall\omega\in\mathfrak{H}_0\oplus\mathfrak{H}_1[\mathcal{V}(L)])$;
(3) *if* $\mathbf{T}\omega=\mathbf{X}_1[\mathcal{V}(L)]\,\omega=\mathbf{0}$, *then* $\mathbf{T}L\omega=\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega=\mathbf{0}$ $(\forall\omega\in\mathcal{M}(\mathfrak{H}))$;
(4) $\left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\,\omega\right\|=\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega\right\|$ $(\forall\omega\in\mathcal{M}(\mathfrak{H}))$.

Proof. **1.** Let $L\in\mathfrak{O}(\mathfrak{H},c)$.

1.1. By (12), L is v-determined. Since $\mathfrak{O}(\mathfrak{H},c)$ is the group of operators, $L^{-1}\in\mathfrak{O}(\mathfrak{H},c)$, and so L^{-1} also is v-determined.

1.2. Performance of the second condition follows from the equality (10).

1.3. Suppose, that $\omega \in \mathcal{M}(\mathfrak{H})$ and $\mathbf{T}\omega = \mathbf{X}_1[\mathcal{V}(L)]\omega = \mathbf{0}$. Then, for any vector $w_{t,\lambda} = t\mathbf{e}_0 + \lambda\mathcal{V}(L) \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$ we obtain

$$\begin{aligned}\langle\langle w_{t,\lambda}, \omega\rangle\rangle_c &= \langle \mathbf{X}w_{t,\lambda}, \mathbf{X}\omega\rangle - c^2\mathcal{T}(w_{t,\lambda})\mathcal{T}(\omega) = \lambda\langle\mathcal{V}(L), \mathbf{X}\omega\rangle - c^2 t\langle\mathbf{T}\omega, \mathbf{e}_0\rangle \\ &= \lambda\langle\mathcal{V}(L), \mathbf{X}\omega\rangle = \lambda\langle\mathbf{X}\mathcal{V}(L), \omega\rangle = \lambda\langle\mathcal{V}(L), \omega\rangle = 0.\end{aligned}$$

Consequently, by the equality (11)

$$\langle\langle Lw_{t,\lambda}, L\omega\rangle\rangle_c = 0 \quad (\forall t, \lambda \in \mathbb{R}).$$

Hence, using the lemma 1, we deliver

$$\left\langle\left\langle \alpha_L\left((t-\lambda\beta_L)\mathbf{e}_0 + (t-\lambda)\mathcal{V}\left(L^{-1}\right)\right), L\omega\right\rangle\right\rangle_c = 0 \quad (\forall t, \lambda \in \mathbb{R}),$$

where (because L, L^{-1} are v-determined), $\alpha_L = \mathcal{T}(L\mathbf{e}_0) \neq 0$, $\alpha_{L^{-1}} \neq 0$, $\beta_L = 1 - \frac{1}{\alpha_L\alpha_{L^{-1}}} \neq 1$. Since $\beta_L \neq 1$, the set of pairs $\{(t-\lambda\beta_L,\, t-\lambda) \mid t, \lambda \in \mathbb{R}\}$ coincides with $\mathbb{R}^2$. Thus, since $\alpha_L \neq 0$, we obtain

$$\left\langle\left\langle t\mathbf{e}_0 + \lambda\mathcal{V}\left(L^{-1}\right), L\omega\right\rangle\right\rangle_c = 0 \quad (\forall t, \lambda \in \mathbb{R}).$$

In particular for $t_1 = -\frac{1}{c^2}$, $\lambda_1 = 0$ and $t_2 = 0$, $\lambda_2 = 1$ we have

$$0 = \left\langle\left\langle -\frac{1}{c^2}\mathbf{e}_0, L\omega\right\rangle\right\rangle_c = -c^2\mathcal{T}\left(-\frac{1}{c^2}\mathbf{e}_0\right)\mathcal{T}(L\omega) = \mathcal{T}(L\omega),$$

$$\mathbf{T}L\omega = \mathcal{T}(L\omega)\mathbf{e}_0 = \mathbf{0}, \tag{13}$$

$$0 = \left\langle\left\langle\mathcal{V}\left(L^{-1}\right), L\omega\right\rangle\right\rangle_c = \left\langle\mathcal{V}\left(L^{-1}\right), L\omega\right\rangle,$$

$$\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega = \left\{\begin{matrix} \frac{\langle\mathcal{V}(L^{-1}), L\omega\rangle}{\|\mathcal{V}(L^{-1})\|^2}\mathcal{V}\left(L^{-1}\right), & \mathcal{V}\left(L^{-1}\right) \neq \mathbf{0} \\ \mathbf{0}, & \mathcal{V}\left(L^{-1}\right) = \mathbf{0}\end{matrix}\right\} = \mathbf{0}. \tag{14}$$

Thus, by (13), (14), $\mathbf{T}L\omega = \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega = \mathbf{0}$.

1.4. Let $\omega \in \mathcal{M}(\mathfrak{H})$. Then, by (11) and (5)

$$\begin{aligned}\left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\|^2 &= \left\langle\left\langle\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, \mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle\right\rangle_c \\ &= \left\langle\left\langle L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle\right\rangle_c \\ &= \left\langle\left\langle\left(\mathbf{T} + \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right] + \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]\right)L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle\right\rangle_c \\ &= \left\langle\left\langle\mathbf{T}L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle\right\rangle_c \\ &\quad + \left\langle\left\langle\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle\right\rangle_c \\ &\quad + \left\langle\left\langle\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle\right\rangle_c.\end{aligned} \tag{15}$$

Since (by (5)) $\mathbf{T}\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega = \mathbf{X}_1[\mathcal{V}(L)]\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega = \mathbf{0}$, using the previous item, we conclude, that $\mathbf{T}L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega = \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega = \mathbf{0}$. Hence, from (15) it follows, that:

$$\begin{aligned}\left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\|^2 &= \left\langle\left\langle\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle\right\rangle_c \\ &= \left\langle\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle \\ &= \left\langle\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega,\, \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\rangle \\ &= \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega\right\|^2.\end{aligned} \tag{16}$$

Note, that by (5), $L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega = L(\omega - \mathbf{T}\omega - \mathbf{X}_1[\mathcal{V}(L)]\omega) = L\omega - Lw$, where $w = \mathbf{T}\omega + \mathbf{X}_1[\mathcal{V}(L)]\omega \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$. By lemma 1, $Lw \in \mathfrak{H}_0 \oplus \mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right]$. Therefore, by (5), $\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]Lw = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]\left(\mathbf{T} + \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\right)Lw = \mathbf{0}$, and

$$\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\omega = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right](L\omega - Lw) = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega.$$

Hence, by (16)

$$\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\omega\right\|^2=\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega\right\|^2,\quad \omega\in\mathcal{M}(\mathfrak{H}).$$

Thus, all conditions 1–4 for any coordinate transform $L\in\mathfrak{O}(\mathfrak{H},c)$ are satisfied.

2. Suppose, that coordinate transform $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ satisfies the conditions 1–4. Chose any $\omega\in\mathcal{M}(\mathfrak{H})$. Vector ω can be represented in the form

$$\begin{aligned}&\omega=\omega_1+\omega_2,\quad \text{where}\\ &\omega_1=\mathcal{T}(\omega)\,\mathbf{e}_0+\mathbf{X}_1\left[\mathcal{V}(L)\right]\omega\in\mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}(L)\right],\\ &\omega_2=\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\omega\in\mathfrak{H}_{1\perp}\left[\mathcal{V}(L)\right].\end{aligned}\tag{17}$$

Note, that by (17) and (5), $\mathbf{T}\omega_2=\mathbf{X}_1\left[\mathcal{V}(L)\right]\omega_2=\mathbf{0}$. Therefore, by the condition 3

$$\mathbf{T}L\omega_2=\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega_2=\mathbf{0}.\tag{18}$$

So

$$\begin{aligned}\mathsf{M}_c(L\omega)&=\mathsf{M}_c(L\omega_1+L\omega_2)=\left\|\mathbf{X}L\omega_1+\mathbf{X}L\omega_2\right\|^2-c^2\left(\mathcal{T}(L\omega_1)+\mathcal{T}(L\omega_2)\right)^2\\ &=\left\|\mathbf{X}L\omega_1+\mathbf{X}L\omega_2+\mathbf{T}L\omega_2\right\|^2-c^2\left(\mathcal{T}(L\omega_1)+0\right)^2\\ &=\left\|\mathbf{X}L\omega_1+L\omega_2\right\|^2-c^2\mathcal{T}^2(L\omega_1).\end{aligned}\tag{19}$$

Since $\omega_1\in\mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}(L)\right]$, by lemma 1, $L\omega_1\in\mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right]$. Hence, by (18),

$$\begin{aligned}\langle\mathbf{X}L\omega_1,L\omega_2\rangle&=\langle L\omega_1,\mathbf{X}L\omega_2\rangle=\langle L\omega_1,(\mathbf{T}+\mathbf{X})L\omega_2\rangle=\langle L\omega_1,L\omega_2\rangle\\ &=\left\langle\left(\mathbf{T}+\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\right)L\omega_1,L\omega_2\right\rangle=\left\langle L\omega_1,\left(\mathbf{T}+\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\right)L\omega_2\right\rangle=0.\end{aligned}$$

Thus, $\left\|\mathbf{X}L\omega_1+L\omega_2\right\|^2=\left\|\mathbf{X}L\omega_1\right\|^2+\left\|L\omega_2\right\|^2$. And using the equalities (19), (18), conditions 2, 4, taking into account, that $\omega_1\in\mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}(L)\right]$ we obtain

$$\begin{aligned}\mathsf{M}_c(L\omega)&=\mathsf{M}_c(L\omega_1)+\left\|L\omega_2\right\|^2=\mathsf{M}_c(L\omega_1)+\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega_2\right\|^2\\ &=\mathsf{M}_c(\omega_1)+\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\omega_2\right\|^2\\ &=\mathsf{M}_c\left(\mathcal{T}(\omega)\,\mathbf{e}_0+\mathbf{X}_1\left[\mathcal{V}(L)\right]\omega\right)+\left\|\left(\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\right)^2\omega\right\|^2\\ &=\left\|\mathbf{X}_1\left[\mathcal{V}(L)\right]\omega\right\|^2-c^2\mathcal{T}^2(\omega)+\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\omega\right\|^2=\mathsf{M}_c(\omega)\quad(\forall\omega\in\mathcal{M}(\mathfrak{H})).\end{aligned}$$

Consequently, $L\in\mathfrak{O}(\mathfrak{H},c)$. □

4. Generalized Lorentz transforms for finite speeds

Denote by $\mathfrak{O}\mathfrak{T}_{\mathbf{fin}}(\mathfrak{H},c)$ the set of all coordinate transforms $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$, satisfying conditions:

1′. Coordinate transforms L and L^{-1} are v-determined;
2′. $\left(\mathsf{M}_c(L\omega)\right)^2=\left(\mathsf{M}_c(\omega)\right)^2$ $(\forall\omega\in\mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}(L)\right])$;
3′. if $\mathbf{T}\omega=\mathbf{X}_1\left[\mathcal{V}(L)\right]\omega=\mathbf{0}$, then $\mathbf{T}L\omega=\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega=\mathbf{0}$ $(\forall\omega\in\mathcal{M}(\mathfrak{H}))$;
4′. $\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}(L)\right]\omega\right\|=\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega\right\|$, $(\forall\omega\in\mathcal{M}(\mathfrak{H}))$.

In comparison with the conditions 1–4 of the assertion 1, only the condition 2 is modified. It is evidently, that the condition condition 2 of the assertion 1 implies the condition 2′. Thus

$$\mathfrak{O}(\mathfrak{H},c)\subseteq\mathfrak{O}\mathfrak{T}_{\mathbf{fin}}(\mathfrak{H},c).\tag{20}$$

And, as it will be proved below, in the theorem 1, this small modification of the second condition leads to permission of superlight speed for reference frame (that is to the possibility of $\left\|\mathcal{V}(L)\right\|>c$ for $L\in\mathfrak{O}\mathfrak{T}_{\mathbf{fin}}(\mathfrak{H},c)$). This, means, that the inclusion, inverse to (20) can not be true.

From the condition $3'$ it follows, that for any operator $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$

$$L\mathfrak{H}_{1\perp}[\mathcal{V}(L)] \subseteq \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{21}$$

Indeed, for any $\omega \in \mathfrak{H}_{1\perp}[\mathcal{V}(L)]$ we have, $\mathbf{T}\omega = \mathbf{X}_1[\mathcal{V}(L)]\omega = \mathbf{0}$. Thus, by condition $3'$, $\mathbf{T}L\omega = \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega = \mathbf{0}$, and, by equalities (5), $L\omega = \left(\mathbf{T} + \mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right] + \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]\right)L\omega = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega \in \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$.

Denote by $\mathfrak{U}(\mathfrak{H}_1)$ the set of all unitary operators over the space $\mathfrak{H}_1$. That is the set of all linear operators $J : \mathfrak{H}_1 \mapsto \mathfrak{H}_1$ $(J \in \mathcal{L}(\mathfrak{H}_1))$, such, that

$$\|Jx\| = \|x\| \quad (\forall x \in \mathfrak{H}_1) \quad \text{and} \quad J\mathfrak{H}_1 = \mathfrak{H}_1.$$

For any operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ we introduce the operator $\widetilde{J} \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$

$$\widetilde{J}\omega := \mathbf{T}\omega + J\mathbf{X}\omega = \mathcal{T}(\omega)\mathbf{e}_0 + J\mathbf{X}\omega, \quad \omega \in \mathcal{M}(\mathfrak{H}). \tag{22}$$

From (22) it follows, that

$$\forall J \in \mathfrak{U}(\mathfrak{H}_1) \quad \widetilde{J} \in \mathfrak{U}(\mathcal{M}(\mathfrak{H})), \tag{23}$$

where $\mathfrak{U}(\mathcal{M}(\mathfrak{H}))$ is the set of all unitary operators over the space $\mathcal{M}(\mathfrak{H})$.

Theorem 1. *Operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$ if and only if there exist number $s \in \{-1, 1\}$, vector $V \in \mathfrak{H}_1$, $\|V\| \neq c$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any $\omega \in \mathcal{M}(\mathfrak{H})$ vector $L\omega$ can be represented by the formula*

$$L\omega = \frac{s\left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + J\left(\frac{s\left(\mathcal{T}(\omega)V - \mathbf{X}_1[V]\omega\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[V]\omega\right), \tag{24}$$

moreover,

$$\mathcal{V}(L) = V.$$

Note, that in the case $\mathfrak{H} = \mathbb{R}^3$, $\mathcal{M}(\mathfrak{H}) = \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$, $V = (0, v, 0, 0)$ (where $v \in \mathbb{R}$, $|v| > c$), and

$$J(0, x, y, z) = (0, -x, y, z), \quad x, y, z \in \mathbb{R}$$

we obtain the transforms (1) from the formula (24).

To prove the theorem we need the following lemma.

Lemma 2. *If for operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ there exist number $s \in \{-1, 1\}$, vector $V \in \mathfrak{H}_1$, $\|V\| \neq c$ and operator $J \in \mathfrak{U}(\mathfrak{H}_1)$ such, that for any $\omega \in \mathcal{M}(\mathfrak{H})$ vector $L\omega$ can be represented by the formula (24), then L is a coordinate transform, moreover*

$$L^{-1} = L_0[\operatorname{sign}(c - \|V\|)s, V]\,\widetilde{J^{-1}}, \quad \textit{where}$$

$$L_0[s, V]\omega = \frac{s\left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + \frac{s\left(\mathcal{T}(\omega)V - \mathbf{X}_1[V]\omega\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[V]\omega. \tag{25}$$

Proof. Let the operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ satisfy the conditions of the lemma. We need to prove, that the operator L have the inverse L^{-1}. By (24), operator L can be represented in the form

$$L = \widetilde{J}L_0[s, V].$$

Since $\widetilde{J}$ is unitary operator over $\mathcal{M}(\mathfrak{H})$, it is sufficient to prove that the inverse operator exist for the operator L_0. It is obvious that

$$\widetilde{J}^{-1} = \widetilde{J^{-1}}. \tag{26}$$

Hence, the lemma will be fully proved, if we will be be able to verify the equality:

$$L_0\left[s,V\right]L_0\left[\mathrm{sign}\ \left(c-\|V\|\right)s,V\right]=\mathbb{I} \tag{27}$$

(then the equality $L_0\left[\mathrm{sign}\ \left(c-\|V\|\right)s,V\right]L_0\left[s,V\right]=\mathbb{I}$ will be follow by applying the equality (27) to the operator $L_0\left[s',V\right]$, where $s'=\mathrm{sign}\ \left(c-\|V\|\right)s$).

In the case $V=\mathbf{0}$, using (25) and (4), we obtain

$$L_0\left[s,V\right]\omega=s\mathcal{T}\left(\omega\right)\mathbf{e}_0+\mathbf{X}_1^{\perp}\left[V\right]\omega=s\mathcal{T}\left(\omega\right)\mathbf{e}_0+\left(\mathbf{X}-\mathbf{X}_1\left[V\right]\right)\omega=s\mathcal{T}\left(\omega\right)\mathbf{e}_0+\mathbf{X}\omega.$$

Thus, in this case equality (27) is clear.

So, one can be restricted by the case $V\neq\mathbf{0}$. Applying equalities (25) and (4) we deliver

$$L_0\left[s,V\right]\omega=\frac{s\left(\mathcal{T}\left(\omega\right)-\frac{\langle V,\omega\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0+\frac{s\left(\mathcal{T}\left(\omega\right)-\frac{\langle V,\omega\rangle}{\|V\|^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}V+\mathbf{X}_1^{\perp}\left[V\right]\omega,\quad \omega\in\mathcal{M}\left(\mathfrak{H}\right). \tag{28}$$

Denote $s':=\mathrm{sign}\ \left(c-\|V\|\right)s$. Then for an arbitrary $\omega\in\mathcal{M}\left(\mathfrak{H}\right)$ we have

$$L_0\left[s,V\right]L_0\left[\mathrm{sign}\ \left(c-\|V\|\right)s,V\right]\omega=L_0\left[s,V\right]\widetilde{\omega},\quad \text{where}\quad \widetilde{\omega}=L_0\left[s',V\right]\omega. \tag{29}$$

By (28),

$$\begin{gathered}\mathcal{T}\left(\widetilde{\omega}\right)=\mathcal{T}\left(L_0\left[s',V\right]\omega\right)=\frac{s'\left(\mathcal{T}\left(\omega\right)-\frac{\langle V,\omega\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}},\quad \langle V,\widetilde{\omega}\rangle=\frac{s'\left(\mathcal{T}\left(\omega\right)\|V\|^2-\langle V,\omega\rangle\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}},\\ \mathbf{X}_1^{\perp}\left[V\right]\widetilde{\omega}=\mathbf{X}_1^{\perp}\left[V\right]\omega.\end{gathered} \tag{30}$$

Applying equality (28) for vector $\widetilde{\omega}$ and using (30), we deduce

$$\begin{aligned}L_0\left[s,V\right]\widetilde{\omega}&=\frac{s\left(\mathcal{T}\left(\widetilde{\omega}\right)-\frac{\langle V,\widetilde{\omega}\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0+\frac{s\left(\mathcal{T}\left(\widetilde{\omega}\right)-\frac{\langle V,\widetilde{\omega}\rangle}{\|V\|^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}V+\mathbf{X}_1^{\perp}\left[V\right]\widetilde{\omega}\\ &=ss'\left(\frac{\mathcal{T}\left(\omega\right)\left(1-\frac{\|V\|^2}{c^2}\right)}{\left|1-\frac{\|V\|^2}{c^2}\right|}\mathbf{e}_0+\frac{\frac{\langle V,\omega\rangle}{\|V\|^2}\left(1-\frac{\|V\|^2}{c^2}\right)}{\left|1-\frac{\|V\|^2}{c^2}\right|}V\right)+\mathbf{X}_1^{\perp}\left[V\right]\omega\\ &=ss'\mathrm{sign}\ \left(c-\|V\|\right)\left(\mathcal{T}\left(\omega\right)\mathbf{e}_0+\frac{\langle V,\omega\rangle}{\|V\|^2}V\right)+\mathbf{X}_1^{\perp}\left[V\right]\omega=\omega.\end{aligned}$$

Thus, using (29), we obtain (27). □

Proof of the theorem 1. **I.** Suppose, that $L\in\mathfrak{OT}_{\mathbf{fin}}\left(\mathfrak{H},c\right)$. Then, L is coordinate transform, which satisfies the conditions $1'$–$4'$. Denote

$$V:=\mathcal{V}\left(L\right). \tag{31}$$

First we prove the formula (24) in the case $V\neq 0$. By equalities (5),(2) and (4), for any $\omega\in\mathcal{M}\left(\mathfrak{H}\right)$ we have

$$\begin{aligned}L\omega=L\left(\mathbf{T}+\mathbf{X}_1\left[V\right]+\mathbf{X}_1^{\perp}\left[V\right]\right)\omega&=L\left(\mathcal{T}\left(\omega\right)\mathbf{e}_0+\mathbf{X}_1\left[V\right]\omega\right)+L\mathbf{X}_1^{\perp}\left[V\right]\omega\\ &=L\left(\mathcal{T}\left(\omega\right)\mathbf{e}_0+\frac{\langle V,\omega\rangle}{\|V\|^2}V\right)+L\mathbf{X}_1^{\perp}\left[V\right]\omega.\end{aligned}$$

Hence, by lemma 1

$$L\omega = \alpha_L\left(\left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{\|V\|^2}\beta_L\right)\mathbf{e}_0 + \left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{\|V\|^2}\right)\mathcal{V}\left(L^{-1}\right)\right) + L\mathbf{X}_1^{\perp}[V]\,\omega \qquad (\omega\in\mathcal{M}(\mathfrak{H})) \tag{32}$$

Now, introduce the linear operator J_1 on the subspace $\mathfrak{H}_{1\perp}[V] = \mathfrak{H}_{1\perp}[\mathcal{V}(L)]$. Denote

$$J_1 x := Lx, \quad x\in\mathfrak{H}_{1\perp}[V]. \tag{33}$$

According to the formula (21), operator J_1 maps the subspace $\mathfrak{H}_{1\perp}[V]$ into the subspace $\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$. By the formula (21) and condition $4'$, for any $x\in\mathfrak{H}_{1\perp}[V]$ we obtain

$$\|J_1x\| = \|Lx\| = \left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]Lx\right\| = \left\|\mathbf{X}_1^{\perp}[\mathcal{V}(L)]\,x\right\| = \left\|\mathbf{X}_1^{\perp}[V]\,x\right\| = \|x\|. \tag{34}$$

Hence, J_1 is isometric operator from the subspace $\mathfrak{H}_{1\perp}[V]$ to $\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$. Now the aim is to prove, that operator J_1 is unitary operator from $\mathfrak{H}_{1\perp}[V]$ to $\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$, that is

$$J_1\mathfrak{H}_{1\perp}[V] = \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{35}$$

Let us consider any vector $y\in\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$. Since L is coordinate transform, there exist vector $x = L^{-1}y$. By equalities (5) vector x can be represented as

$$x = (\mathbf{T} + \mathbf{X}_1[V])\,x + \mathbf{X}_1^{\perp}[V]\,x, \tag{36}$$

where $(\mathbf{T} + \mathbf{X}_1[V])\,x \in \mathfrak{H}_1\oplus\mathfrak{H}_1[V] = \mathfrak{H}_1\oplus\mathfrak{H}_1[\mathcal{V}(L)]$, $\mathbf{X}_1^{\perp}[V]\,x\in\mathfrak{H}_{1\perp}[V]$. Therefore, $Lx = L(\mathbf{T} + \mathbf{X}_1[V])\,x + L\mathbf{X}_1^{\perp}[V]\,x$. Hence

$$L(\mathbf{T} + \mathbf{X}_1[V])\,x + L\mathbf{X}_1^{\perp}[V]\,x = Lx = LL^{-1}y = y\in\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right], \tag{37}$$

where, by lemma 1 and formula (21)

$$\begin{gathered} L(\mathbf{T} + \mathbf{X}_1[V])\,x \in \mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right], \\ L\mathbf{X}_1^{\perp}[V]\,x\in\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \end{gathered} \tag{38}$$

Since $\mathfrak{H}_0\oplus\mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right]\oplus\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] = \mathcal{M}(\mathfrak{H})$, from the equalities (37), (38) we conclude, that

$$L\mathbf{X}_1^{\perp}[V]\,x = y \quad\text{and}\quad L(\mathbf{T} + \mathbf{X}_1[V])\,x = \mathbf{0}.$$

Since L is coordinate transform, from the equality $L(\mathbf{T} + \mathbf{X}_1[V])\,x = \mathbf{0}$ it follows, that $(\mathbf{T} + \mathbf{X}_1[V])\,x = \mathbf{0}$. Hence, by (36), $x = \mathbf{X}_1^{\perp}[V]\,x\in\mathfrak{H}_{1\perp}[V]$, and, by definition of the operator J_1, we deliver

$$J_1x = Lx = y.$$

Thus, we have proved, that for any $y\in\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$ there exists the element $x\in\mathfrak{H}_{1\perp}[V]$ such, that $J_1x = y$. This means, that the operator $J_1:\mathfrak{H}_{1\perp}[V]\mapsto\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$ truly is unitary. Applying the operator J_1 we can write

$$L\mathbf{X}_1^{\perp}[V]\,\omega = J_1\mathbf{X}_1^{\perp}[V]\,\omega, \quad \omega\in\mathcal{M}(\mathfrak{H}). \tag{39}$$

Next, using the lemma 1, for any $t,\lambda\in\mathbb{R}$ we obtain

$$L(t\mathbf{e}_0 + \lambda\mathcal{V}(L)) = \alpha_L\left((t-\lambda\beta_L)\,\mathbf{e}_0 + (t-\lambda)\,\mathcal{V}\left(L^{-1}\right)\right). \tag{40}$$

Using the formulas (6) and (40) we deliver

$$\begin{aligned} &\mathsf{M}_c(t\mathbf{e}_0 + \lambda\mathcal{V}(L)) = \lambda^2\|\mathcal{V}(L)\|^2 - c^2t^2 = \lambda^2\|V\|^2 - c^2t^2, \\ &\quad\mathsf{M}_c(L(t\mathbf{e}_0 + \lambda\mathcal{V}(L))) = \alpha_L^2\mathsf{M}_c\left((t-\lambda\beta_L)\,\mathbf{e}_0 + (t-\lambda)\,\mathcal{V}\left(L^{-1}\right)\right) \\ &\qquad = \alpha_L^2\left((t-\lambda)^2\left\|\mathcal{V}\left(L^{-1}\right)\right\|^2 - c^2(t-\lambda\beta_L)^2\right) = \alpha_L^2\left((t-\lambda)^2\gamma_L - c^2(t-\lambda\beta_L)^2\right), \end{aligned}$$

where $\gamma_L = \left\|\mathcal{V}\left(L^{-1}\right)\right\|^2$. Since $t\mathbf{e}_0 + \lambda\mathcal{V}(L) \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$, by the condition $2'$, $(\mathsf{M}_c(L(t\mathbf{e}_0 + \lambda\mathcal{V}(L))))^2 = (\mathsf{M}_c(t\mathbf{e}_0 + \lambda\mathcal{V}(L)))^2$, $t,\lambda \in \mathbb{R}$. Thus

$$\left(\lambda^2\|V\|^2 - c^2t^2\right)^2 = \left(\alpha_L^2\left((t-\lambda)^2\gamma_L - c^2(t-\lambda\beta_L)^2\right)\right)^2, \quad \text{or}$$

$$\lambda^2\|V\|^2 - c^2t^2 = \pm\alpha_L^2\left((t-\lambda)^2\gamma_L - c^2(t-\lambda\beta_L)^2\right) \quad (t,\lambda \in \mathbb{R}).$$

And after simple transformations the last formula takes the form

$$\lambda^2\|V\|^2 - c^2t^2 = \pm\alpha_L^2\left(t^2\left(\gamma_L - c^2\right) - 2t\lambda\left(\gamma_L - c^2\beta_L\right) + \lambda^2\left(\gamma_L - c^2\beta_L^2\right)\right) \quad (t,\lambda \in \mathbb{R}).$$

Thus, we obtain two systems of equations

$$\begin{cases}\alpha_L^2\left(\gamma_L - c^2\right) = -c^2\\ \gamma_L - c^2\beta_L = 0\\ \alpha_L^2\left(\gamma_L - c^2\beta_L^2\right) = \|V\|^2\end{cases}, \qquad \begin{cases}\alpha_L^2\left(\gamma_L - c^2\right) = c^2\\ \gamma_L - c^2\beta_L = 0\\ \alpha_L^2\left(\gamma_L - c^2\beta_L^2\right) = -\|V\|^2\end{cases}.$$

By means of simple transformations, these two systems can be reduced to the form

$$\begin{cases}\alpha_L^2\left(1 - \frac{\|V\|^2}{c^2}\right) = 1\\ \gamma_L = \|V\|^2\\ \beta_L = \frac{\|V\|^2}{c^2}\end{cases}, \qquad \begin{cases}\alpha_L^2\left(1 - \frac{\|V\|^2}{c^2}\right) = -1\\ \gamma_L = \|V\|^2\\ \beta_L = \frac{\|V\|^2}{c^2}\end{cases}.$$

The first system has (real) solutions only for $\|V\| < c$, and the second system has solutions only for $\|V\| > c$. Thus, the solutions exist only for $\|V\| \neq c$. Solving the last systems and taking into account, that $\gamma_L = \left\|\mathcal{V}\left(L^{-1}\right)\right\|^2$, in the both cases we obtain

$$\alpha_L = \frac{s}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}, \quad \beta_L = \frac{\|V\|^2}{c^2}, \quad \left\|\mathcal{V}\left(L^{-1}\right)\right\|^2 = \|V\|^2 \quad (\|V\| \neq c), \tag{41}$$

where $s \in \{-1, 1\}$.

Substituting the values of $L\mathbf{X}_1^{\perp}[V]\omega$ from the formula (39) and the values of α_L,β_L from the formula (41) into (32), we deliver

$$\begin{aligned} L\omega &= \frac{s}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\left(\left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{\|V\|^2}\frac{\|V\|^2}{c^2}\right)\mathbf{e}_0\right.\\ &\qquad \left.+\left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{\|V\|^2}\right)\mathcal{V}\left(L^{-1}\right)\right) + J_1\mathbf{X}_1^{\perp}[V]\omega \\ &= \frac{s\left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + \frac{s\left(\mathcal{T}(\omega)\mathcal{V}\left(L^{-1}\right) - \frac{\langle V,\omega\rangle}{\|V\|^2}\mathcal{V}\left(L^{-1}\right)\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} + J_1\mathbf{X}_1^{\perp}[V]\omega. \end{aligned} \tag{42}$$

Introduce the following operator on the subspace $\mathfrak{H}_1$:

$$Jx := \frac{\langle V,x\rangle}{\|V\|^2}\mathcal{V}\left(L^{-1}\right) + J_1\mathbf{X}_1^{\perp}[V]x, \quad x \in \mathfrak{H}_1. \tag{43}$$

Since J_1 maps subspace $\mathfrak{H}_{1\perp}[\mathcal{V}(L)]$ to subspace $\mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]$,

$$\left\langle \mathcal{V}\left(L^{-1}\right), J_1\mathbf{X}_1^{\perp}[V]x\right\rangle = 0.$$

Hence, using (34), (41) and (4), we obtain

$$\|Jx\|^2 = \left(\frac{\langle V, x\rangle}{\|V\|^2}\left\|\mathcal{V}\left(L^{-1}\right)\right\|\right)^2 + \left\|J_1\mathbf{X}_1^{\perp}[V]\,x\right\|^2 = \left(\frac{\langle V, x\rangle}{\|V\|}\right)^2 + \left\|\mathbf{X}_1^{\perp}[V]\,x\right\|^2$$
$$= \left\|\frac{\langle V, x\rangle}{\|V\|^2}V\right\|^2 + \left\|\mathbf{X}_1^{\perp}[V]\,x\right\|^2 = \|\mathbf{X}_1[V]\,x\|^2 + \left\|\mathbf{X}_1^{\perp}[V]\,x\right\|^2 = \|x\|^2.$$

Thus, operator J is isometric on $\mathfrak{H}_1$.

For $x = \lambda V \in \mathfrak{H}_1[V]$ by (43) we have $J(\lambda V) = \lambda\mathcal{V}\left(L^{-1}\right)$. Hence

$$J\mathfrak{H}_1[V] = \mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{44}$$

And for $x \in \mathfrak{H}_{1\perp}[V]$ according to (43) we obtain

$$Jx = J_1\mathbf{X}_1^{\perp}[V]\,x = J_1 x \quad (x \in \mathfrak{H}_{1\perp}[V]). \tag{45}$$

Hence, by (35)

$$J\mathfrak{H}_{1\perp}[V] = J_1\mathfrak{H}_{1\perp}[V] = \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right]. \tag{46}$$

From (44) and (46) it follows, that

$$J\mathfrak{H}_1 = J\left(\mathfrak{H}_1[V] \oplus \mathfrak{H}_{1\perp}[V]\right) \supseteq \mathfrak{H}_1\left[\mathcal{V}\left(L^{-1}\right)\right] \oplus \mathfrak{H}_{1\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] = \mathfrak{H}_1.$$

Thus, $J\mathfrak{H}_1 = \mathfrak{H}_1$. And so operator J is unitary on $\mathfrak{H}_1$, that is

$$J \in \mathfrak{U}(\mathfrak{H}_1).$$

In accordance with (43), $JV = \mathcal{V}\left(L^{-1}\right)$. Hence, using (42), (45) and (4), we deliver the formula (24). So, for the case $V \neq \mathbf{0}$ formula (24) is proved.

Now consider the case $V = \mathbf{0}$, that is $\mathcal{V}(L) = \mathbf{0}$. In this case, by the formula (4)

$$\mathbf{X}_1[V] = \mathbf{X}_1[\mathbf{0}] = \mathbb{O}, \quad \mathbf{X}_1^{\perp}[V] = \mathbf{X}. \tag{47}$$

Since, by condition $1'$, transforms L and L^{-1} are v-determined, by lemma 1, the following equality must hold:

$$tL\mathbf{e}_0 = L\left(t\mathbf{e}_0 + \lambda\mathcal{V}(L)\right) = \alpha_L\left((t - \lambda\beta_L)\,\mathbf{e}_0 + (t-\lambda)\,\mathcal{V}\left(L^{-1}\right)\right) \quad (\forall t, \lambda \in \mathbb{R}) \tag{48}$$

with $\alpha_L = \mathcal{T}(L\mathbf{e}_0) \neq 0$, $\beta_L = 1 - \frac{1}{\alpha_L\alpha_{L^{-1}}} \neq 1$. Since the left-hand side of the equality (48) does not depend of λ, the coefficient of the variable λ in the right-hand side of the equality must be zero. Hence, $\beta_L\mathbf{e}_0 + \mathcal{V}\left(L^{-1}\right) = \mathbf{0}$, and so

$$\beta_L = 0, \quad \mathcal{V}\left(L^{-1}\right) = \mathbf{0}. \tag{49}$$

Thus, the formula (48) takes the form $L\mathbf{e}_0 = \alpha_L\mathbf{e}_0$. And, applying the condition $2'$ to the vector $\mathbf{e}_0 \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$, we obtain $\alpha_L = s$, where $s \in \{-1, 1\}$. Consequently

$$L\mathbf{e}_0 = s\mathbf{e}_0, \quad \text{where} \quad s \in \{-1, 1\}. \tag{50}$$

Using (47), (50) for any vector $\omega \in \mathcal{M}(\mathfrak{H})$ we obtain

$$L\omega = L\left(\mathcal{T}(\omega)\,\mathbf{e}_0 + \mathbf{X}\omega\right) = s\mathcal{T}(\omega)\,\mathbf{e}_0 + L\mathbf{X}\omega = s\mathcal{T}(\omega)\,\mathbf{e}_0 + J\mathbf{X}_1^{\perp}[V]\,\omega, \tag{51}$$

where

$$Jx = Lx, \quad x \in \mathfrak{H}_1 = \mathbf{X}\mathcal{M}(\mathfrak{H}) = \mathbf{X}_1^{\perp}[V]\,\mathcal{M}(\mathfrak{H}) = \mathfrak{H}_{1\perp}[V]. \tag{52}$$

By condition $3'$ and formula (47), the subspace $\mathfrak{H}_1 = \{\omega \in \mathcal{M}(\mathfrak{H}) \mid \mathbf{T}\omega = \mathbf{0}\}$ is invariant for the operator L. Hence, the operator J from (52) maps the subspace $\mathfrak{H}_1$ into the subspace $\mathfrak{H}_1$.

According to the formula (49), $\mathcal{V}\left(L^{-1}\right) = \mathbf{0}$. Consequently, by the formula (47) $\mathbf{X}_1^{\perp}[V] = \mathbf{X}_1^{\perp}[\mathcal{V}(L)] = \mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] = \mathbf{X}$. So, by the condition $4'$ operator J is isometric on the subspace $\mathfrak{H}_1$. Now, we have to prove, that operator J is unitary. Consider any vector $y \in \mathfrak{H}_1$. Denote $x := L^{-1}y$. Then, by (51), $Lx = s\mathcal{T}(x)\,\mathbf{e}_0 + J\mathbf{X}x = y \in \mathfrak{H}_1$.

Hence, $\mathcal{T}(x) = 0$ and $J\mathbf{X}x = y$. This means, that $y \in J\mathfrak{H}_1$. Therefore, we have seen, that $J\mathfrak{H}_1 = \mathfrak{H}_1$, and the operator J really is unitary on the $\mathfrak{H}_1 = \mathfrak{H}_{1\perp}[V]$. Thus for the case $V = \mathbf{0}$ the formula (24) also is proved.

II. Inversely, suppose, that the operator $L \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$ can be represented in the form (24). Then, by the lemma 2, L is coordinate transform.

1. By the formula (24) we deliver

$$L\mathbf{e}_0 = \chi_V(\mathbf{e}_0 + JV), \quad \text{where} \quad \chi_V = \frac{s}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}} \neq 0,$$

$$L(\mathbf{e}_0 + V) = \frac{s\left(1 - \frac{\|V\|^2}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 = \frac{\text{sign }(c - \|V\|)}{\chi_V}\mathbf{e}_0, \quad L^{-1}\mathbf{e}_0 = \frac{\chi_V(\mathbf{e}_0 + V)}{\text{sign }(c - \|V\|)}.$$

Hence $\mathcal{T}(L\mathbf{e}_0) = \chi_V \neq 0$, $\mathcal{T}\left(L^{-1}\mathbf{e}_0\right) = \frac{\chi_V}{\text{sign }(c - \|V\|)} \neq 0$. Thus, coordinate transforms L and L^{-1} are v-determined, moreover

$$\mathcal{V}(L) = \frac{\mathbf{X}L^{-1}\mathbf{e}_0}{\mathcal{T}(L^{-1}\mathbf{e}_0)} = V, \quad \mathcal{V}\left(L^{-1}\right) = \frac{\mathbf{X}L\mathbf{e}_0}{\mathcal{T}(L\mathbf{e}_0)} = JV. \tag{53}$$

2. In accordance with (24), for $\omega = t\mathbf{e}_0 + \lambda V \in \mathfrak{H}_0 \oplus \mathfrak{H}_1[V] = \mathfrak{H}_0 \oplus \mathfrak{H}_1[\mathcal{V}(L)]$, we obtain

$$L\omega = \frac{s\left(t - \lambda\frac{\|V\|^2}{c^2}\right)}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + \frac{s(t - \lambda)JV}{\sqrt{\left|1 - \frac{\|V\|^2}{c^2}\right|}}.$$

Hence, since J is isometric operator, we obtain

$$\begin{aligned}
(\mathsf{M}_c(L\omega))^2 &= \left(\frac{1}{\left|1 - \frac{\|V\|^2}{c^2}\right|}\left((t - \lambda)^2\|JV\|^2 - c^2\left(t - \lambda\frac{\|V\|^2}{c^2}\right)^2\right)\right)^2 \\
&= \left(\frac{1}{1 - \frac{\|V\|^2}{c^2}}\left((t - \lambda)^2\|V\|^2 - c^2\left(t - \lambda\frac{\|V\|^2}{c^2}\right)^2\right)\right)^2 \\
&= \left(\lambda^2\|V\|^2 - c^2t^2\right)^2 = (\mathsf{M}_c(\omega))^2.
\end{aligned}$$

Thus, the condition $2'$ for the operator L also is satisfied.

3. Suppose, that $\omega \in \mathcal{M}(\mathfrak{H})$, $\mathbf{T}\omega = \mathbf{X}_1[\mathcal{V}(L)]\omega = \mathbf{0}$. Then, $\mathcal{T}(\omega) = 0$, and (since $\mathcal{V}(L) = V$, by (53)), we have, $\langle V, \omega\rangle = 0$, $\mathbf{X}_1^{\perp}[V]\omega = (\mathbf{X} - \mathbf{X}_1[\mathcal{V}(L)])\omega = \mathbf{X}\omega = \omega$. So, by (24)

$$L\omega = J\mathbf{X}_1^{\perp}[V]\omega = J\omega.$$

And, taking into account, that J is unitary operator on $\mathfrak{H}_1$, using (53) and (4) we obtain

$$\begin{aligned}
\mathbf{T}L\omega &= \mathbf{T}J\omega = \mathbf{0}, \\
\mathbf{X}_1\left[\mathcal{V}\left(L^{-1}\right)\right]L\omega = \mathbf{X}_1[JV]J\omega &= \begin{cases} \frac{\langle JV, J\omega\rangle}{\|V\|^2}JV, & JV \neq \mathbf{0} \\ \mathbf{0}, & JV = \mathbf{0} \end{cases} \\
&= J\begin{cases} \frac{\langle V, \omega\rangle}{\|V\|^2}V, & V \neq \mathbf{0} \\ \mathbf{0}, & V = \mathbf{0} \end{cases} = J\mathbf{X}_1[V]\omega = \mathbf{0}.
\end{aligned}$$

Hence, we have checked the condition $3'$ for the operator L.

4. Using the unitarity of the operator J ($\langle J\omega, Jw\rangle = \langle \omega, w\rangle,\ \omega, w \in \mathcal{M}(\mathfrak{H})$) and equalities (4),(5) one can easy verify the following formulas:

$$J\mathbf{X}_1[V]\,\omega = \mathbf{X}_1[JV]\,J\mathbf{X}\omega, \qquad \omega \in \mathcal{M}(\mathfrak{H}),$$
$$J\mathbf{X}_1^{\perp}[V]\,\omega = \mathbf{X}_1^{\perp}[JV]\,J\mathbf{X}\omega, \qquad \omega \in \mathcal{M}(\mathfrak{H}).$$

So, by the formula (24) for any $\omega \in \mathcal{M}(\mathfrak{H})$ we deliver

$$\begin{aligned}
&\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\omega = \mathbf{X}_1^{\perp}[JV]\,L\omega \\
&\quad= \mathbf{X}_1^{\perp}[JV]\left(\frac{s\left(\mathcal{T}(\omega) - \frac{\langle V,\omega\rangle}{c^2}\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}}\mathbf{e}_0 + \frac{s\left(\mathcal{T}(\omega)\,JV - \mathbf{X}_1[JV]\,J\mathbf{X}\omega\right)}{\sqrt{\left|1-\frac{\|V\|^2}{c^2}\right|}} + J\mathbf{X}_1^{\perp}[V]\,\omega\right) \\
&\quad= \mathbf{X}_1^{\perp}[JV]\ J\mathbf{X}_1^{\perp}[V]\,\omega = \mathbf{X}_1^{\perp}[JV]\ \mathbf{X}_1^{\perp}[JV]\,J\mathbf{X}\omega \\
&\quad= \mathbf{X}_1^{\perp}[JV]\,J\mathbf{X}\omega = J\mathbf{X}_1^{\perp}[V]\,\omega,
\end{aligned}$$

$$\left\|\mathbf{X}_1^{\perp}\left[\mathcal{V}\left(L^{-1}\right)\right] L\omega\right\| = \left\|J\mathbf{X}_1^{\perp}[V]\,\omega\right\| = \left\|\mathbf{X}_1^{\perp}[V]\,\omega\right\|.$$

Thus, all conditions $1'$–$4'$ for the coordinate transform L are satisfied. Hence $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$. □

5. Generalized Lorentz transforms for infinite speeds

Now we investigate the behavior of coordinate transforms from the class $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$, when the norm of the rate of reference frame ($\|V\|$) tends to infinity. For this purpose we will substitute

(54) $\qquad V = \lambda s\mathbf{n}, \quad \text{where} \quad \lambda > 0, \quad \lambda \neq c; \quad \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1) = \{x \in \mathfrak{H}_1 \mid \|x\| = 1\}$

to the formula (24) and then we will take the limit while $\lambda \to \infty$. Note, that, by (4)

(55)
$$\begin{aligned}
\mathbf{X}_1[\lambda s\mathbf{n}]\,\omega &= \frac{\langle \lambda s\mathbf{n}, \omega\rangle}{\|\lambda s\mathbf{n}\|^2}\lambda s\mathbf{n} = \langle \mathbf{n}, \omega\rangle\,\mathbf{n} = \mathbf{X}_1[\mathbf{n}]\,\omega, \\
\mathbf{X}_1^{\perp}[\lambda s\mathbf{n}]\,\omega &= \mathbf{X}\omega - \mathbf{X}_1[\lambda s\mathbf{n}]\,\omega = \mathbf{X}\omega - \mathbf{X}_1[\mathbf{n}]\,\omega = \mathbf{X}_1^{\perp}[\mathbf{n}]\,\omega \quad (\omega \in \mathcal{M}(\mathfrak{H})).
\end{aligned}$$

Hence, substitution the velocity (54) to the formula (24) lead us to the following representation for operators $L \in \mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H}, c)$ (with $\mathcal{V}(L) \neq \mathbf{0}$):

(56)
$$\begin{aligned}
L\omega &= \mathbf{W}_\lambda[s, \mathbf{n}, J]\,\omega \\
&= \frac{\left(s\mathcal{T}(\omega) - \frac{\lambda}{c^2}\langle \mathbf{n}, \omega\rangle\right)}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}}\mathbf{e}_0 + J\left(\frac{\lambda\mathcal{T}(\omega)\,\mathbf{n} - s\mathbf{X}_1[\mathbf{n}]\,\omega}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\omega\right), \ \omega \in \mathcal{M}(\mathfrak{H}),
\end{aligned}$$

where $s \in \{-1, 1\}$, $J \in \mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$, $\lambda > 0$.

Taking in (56) limit while $\lambda \to \infty$, we get the following linear operators in the space $\mathcal{M}(\mathfrak{H})$:

(57) $$\mathbf{W}_\infty[\mathbf{n}, J]\,\omega = \lim_{\lambda\to+\infty} \mathbf{W}_\lambda[s, \mathbf{n}, J]\,\omega = -\frac{\langle \mathbf{n}, \omega\rangle}{c}\mathbf{e}_0 + J\left(c\mathcal{T}(\omega)\,\mathbf{n} + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\omega\right),$$

where limit exists in the sense of norm of the space $\mathcal{M}(\mathfrak{H})$. Note, that limit in (57) does not depend of the number s. It is not hard to verify, that $\mathbf{W}_\infty[\mathbf{n}, J] \in \mathcal{L}(\mathcal{M}(\mathfrak{H}))$.

Now we introduce the following class of linear bounded operators in the space $\mathcal{M}(\mathfrak{H})$:

$$\mathfrak{OT}_\infty(\mathfrak{H}, c) := \{\mathbf{W}_\infty[\mathbf{n}, J] \mid \mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1),\ J \in \mathfrak{U}(\mathfrak{H}_1)\}.$$

Lemma 3. *For any* $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ *and* $J\in\mathfrak{U}(\mathfrak{H}_1)$ *the following equalities holds:*

$$\widetilde{J}\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]=\mathbf{W}_\infty[\mathbf{n},J],\quad \mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]\widetilde{J}=\mathbf{W}_\infty\left[J^{-1}\mathbf{n},J\right], \tag{58}$$

where the operator $\widetilde{J}$ *is defined in (22), and* $\mathbb{I}_1$ *denotes the identity operator on the subspace* $\mathfrak{H}_1$.

Proof. The first equality (58) immediately follows from (22) and (57). Hence, we prove only the second equality (58). Using (22) and (57) we obtain for any $\omega\in\mathcal{M}(\mathfrak{H})$

$$\begin{aligned}\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]\widetilde{J}\omega&=\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1](\mathcal{T}(\omega)\mathbf{e}_0+J\mathbf{X}\omega)=-\frac{\langle\mathbf{n},\mathcal{T}(\omega)\mathbf{e}_0+J\mathbf{X}\omega\rangle}{c}\mathbf{e}_0\\&\quad+c\mathcal{T}(\mathcal{T}(\omega)\mathbf{e}_0+J\mathbf{X}\omega)\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}](\mathcal{T}(\omega)\mathbf{e}_0+J\mathbf{X}\omega)\\&=-\frac{\langle\mathbf{n},J\mathbf{X}\omega\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\omega)\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}]J\mathbf{X}\omega\\&=-\frac{\left\langle J^{-1}\mathbf{n},\omega\right\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\omega)\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}]J\mathbf{X}\omega.\end{aligned} \tag{59}$$

Note, that, by definition of class $\mathfrak{OT}_\infty(\mathfrak{H},c)$, $\mathbf{n}\neq 0$. So, applying (4),(5), and using the fact that the operator J maps $\mathfrak{H}_1$ into $\mathfrak{H}_1$, we obtain

$$\begin{aligned}\mathbf{X}_1^\perp[\mathbf{n}]J\mathbf{X}\omega&=(\mathbf{X}-\mathbf{X}_1[\mathbf{n}])J\mathbf{X}\omega=\mathbf{X}J\mathbf{X}\omega-\langle\mathbf{n},J\mathbf{X}\omega\rangle\mathbf{n}\\&=\mathbf{X}J\mathbf{X}\omega-\left\langle\mathbf{X}J^{-1}\mathbf{n},\omega\right\rangle\mathbf{n}=J\mathbf{X}\omega-\left\langle J^{-1}\mathbf{n},\omega\right\rangle\mathbf{n}\\&=J\left(\mathbf{X}\omega-\left\langle J^{-1}\mathbf{n},\omega\right\rangle J^{-1}\mathbf{n}\right)=J\left(\mathbf{X}-\mathbf{X}_1\left[J^{-1}\mathbf{n}\right]\right)\omega=J\mathbf{X}_1^\perp\left[J^{-1}\mathbf{n}\right]\omega.\end{aligned}$$

Thus, according to (59), we deduce

$$\begin{aligned}\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]\widetilde{J}\omega&=-\frac{\left\langle J^{-1}\mathbf{n},\omega\right\rangle}{c}\mathbf{e}_0+c\mathcal{T}(\omega)\mathbf{n}+J\mathbf{X}_1^\perp\left[J^{-1}\mathbf{n}\right]\omega\\&=-\frac{\left\langle J^{-1}\mathbf{n},\omega\right\rangle}{c}\mathbf{e}_0+J\left(c\mathcal{T}(\omega)J^{-1}\mathbf{n}+\mathbf{X}_1^\perp\left[J^{-1}\mathbf{n}\right]\omega\right)=\mathbf{W}_\infty\left[J^{-1}\mathbf{n},J\right]\omega.\end{aligned}$$

□

Lemma 4. *For any vector* $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$ *it is true the following equality:*

$$\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]\mathbf{W}_\infty[-\mathbf{n},\mathbb{I}_1]=\mathbb{I}.$$

Proof. Consider an arbitrary vector $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$. For vector $\omega\in\mathcal{M}(\mathfrak{H})$, using (57), (55), (5), we get

$$\begin{aligned}\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]\mathbf{W}_\infty[-\mathbf{n},\mathbb{I}_1]\omega&=\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]\left(-\frac{\langle-\mathbf{n},\omega\rangle}{c}\mathbf{e}_0-c\mathcal{T}(\omega)\mathbf{n}+\mathbf{X}_1^\perp[-\mathbf{n}]\omega\right)\\&=\mathbf{W}_\infty[\mathbf{n},\mathbb{I}_1]\left(\frac{\langle\mathbf{n},\omega\rangle}{c}\mathbf{e}_0-c\mathcal{T}(\omega)\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}]\omega\right)\\&=-\frac{\left\langle\mathbf{n},\left(\frac{\langle\mathbf{n},\omega\rangle}{c}\mathbf{e}_0-c\mathcal{T}(\omega)\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}]\omega\right)\right\rangle}{c}\mathbf{e}_0\\&\quad+c\mathcal{T}\left(\frac{\langle\mathbf{n},\omega\rangle}{c}\mathbf{e}_0-c\mathcal{T}(\omega)\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}]\omega\right)\mathbf{n}\\&\quad+\mathbf{X}_1^\perp[\mathbf{n}]\left(\frac{\langle\mathbf{n},\omega\rangle}{c}\mathbf{e}_0-c\mathcal{T}(\omega)\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}]\omega\right)\\&=\frac{c\mathcal{T}(\omega)\mathbf{e}_0}{c}+c\frac{\langle\mathbf{n},\omega\rangle}{c}\mathbf{n}+\mathbf{X}_1^\perp[\mathbf{n}]\mathbf{X}_1^\perp[\mathbf{n}]\omega=\omega.\end{aligned}$$

□

From the lemmas 4, 3 and formula (26), we immediately deduce the following theorem.

Theorem 2. *Any operator* $\mathbf{W}_\infty[\mathbf{n},J]\in\mathfrak{OT}_\infty(\mathfrak{H},c)$ *is a coordinate transform, moreover*

$$(\mathbf{W}_\infty[\mathbf{n},J])^{-1}=\mathbf{W}_\infty\left[-J\mathbf{n},J^{-1}\right].$$

Coordinate transforms, which belong to the class $\mathfrak{OT}_\infty(\mathfrak{H},c)$ will be named ***generalized Lorentz transforms for infinite speeds*** of reference frames.

Remark 1. Note, that any generalized Lorentz transform with infinite speed $\mathbf{W}_\infty[\mathbf{n},J]\in\mathfrak{OT}_\infty(\mathfrak{H},c)$ is not v-determined, because, by (57), $\mathcal{T}(\mathbf{W}_\infty[\mathbf{n},J]\mathbf{e}_0)=0$.

Denote

$$\mathfrak{OT}(\mathfrak{H},c):=\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)\cup\mathfrak{OT}_\infty(\mathfrak{H},c).$$

Coordinate transforms, which belong to the class $\mathfrak{OT}(\mathfrak{H},c)$ will be named generalized ***tachyon Lorentz transforms***.

6. General representation for tachyon Lorentz transforms

The aim of this section is to give general representation for coordinate transforms, from the class $\mathfrak{OT}(\mathfrak{H},c)$, which would be true for finite as well as for infinite velocities of reference frames.

Since any velocity $V\in\mathfrak{H}_1$, $\|V\|\notin\{0,c\}$ can be represented by the form (54), where

$$\mathbf{n}=s\frac{V}{\|V\|},\quad \lambda=\|V\|\quad(\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1),\ \lambda>0)$$

the formula (56) may be considered as general representation for operators from $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$, with nonzero velocity, that is any operator $L\in\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$, such, that $\mathcal{V}(L)\neq\mathbf{0}$ can be represented in the form (56).

Consider the case $\mathcal{V}(L)=\mathbf{0}$. By the formula (24), we have, that any operator $L\in\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$ with zero velocity can be represented in the form

$$L\omega=s\mathcal{T}(\omega)\mathbf{e}_0+J\left(\mathbf{X}_1^\perp[\mathbf{0}]\omega\right)=s\mathcal{T}(\omega)\mathbf{e}_0+J(\mathbf{X}\omega)\quad(\omega\in\mathcal{M}(\mathfrak{H})).\tag{60}$$

From the other hand, substituting $\lambda=0$ ($s\in\{-1,1\}$, $J\in\mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$) into the formula (56), we can define the following operators:

$$\begin{aligned}\mathbf{W}_0[s,\mathbf{n},J]\omega&=s\mathcal{T}(\omega)\mathbf{e}_0+J\left(-s\mathbf{X}_1[\mathbf{n}]\omega+\mathbf{X}_1^\perp[\mathbf{n}]\omega\right)\\&=s\mathcal{T}(\omega)\mathbf{e}_0+J(-s\mathbb{I}_{1,-s}[\mathbf{n}])\mathbf{X}\omega\quad(\omega\in\mathcal{M}(\mathfrak{H})),\end{aligned}\tag{61}$$

where

$$\mathbb{I}_{1,\sigma}[\mathbf{n}]x=\mathbf{X}_1[\mathbf{n}]x+\sigma\mathbf{X}_1^\perp[\mathbf{n}]x,\quad x\in\mathfrak{H}_1,\quad\sigma\in\{-1,1\}.$$

Since, $-s\mathbb{I}_{1,-s}[\mathbf{n}]\in\mathfrak{U}(\mathfrak{H}_1)$, the set of operators, which can be defined by the formula (61) coincides with the set of operators, which can be defined by the formula (60).

Hence, we have seen, that (in the both cases $\mathcal{V}(L)\neq\mathbf{0}$ and $\mathcal{V}(L)=\mathbf{0}$) operator $L\in\mathcal{L}(\mathcal{M}(\mathfrak{H}))$ belongs to the class $\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$ if and only if it can be represented by the formula (56) with $\lambda\geq0$, $s\in\{-1,1\}$, $J\in\mathfrak{U}(\mathfrak{H}_1)$, $\mathbf{n}\in\mathbf{B}_1(\mathfrak{H}_1)$.

Now we introduce the new parameter

$$\theta:=\frac{1-\frac{\lambda}{c}}{\sqrt{\left|1-\frac{\lambda^2}{c^2}\right|}}.\tag{62}$$

Using simple calculations formula (62) can be reduced to the form

$$\theta=-\operatorname{sign}\left(1-\frac{2}{1+\frac{\lambda}{c}}\right)\sqrt{\left|1-\frac{2}{1+\frac{\lambda}{c}}\right|}.\tag{63}$$

Since function $f(\lambda)=-\operatorname{sign}\left(1-\frac{2}{1+\frac{\lambda}{c}}\right)\sqrt{\left|1-\frac{2}{1+\frac{\lambda}{c}}\right|}$, is decreasing on $[0,+\infty)$, it maps the interval $[0,\infty)$ into the interval $=(-1,1]$, and any value $\lambda\geq 0$ can be uniquely determined by the parameter $\theta\in(-1,1]$. Using simple calculation, one can ensure, that parameter λ can be determined by the parameter θ by means of the formula

$$\lambda=c\frac{1-\theta\left|\theta\right|}{1+\theta\left|\theta\right|},\quad \theta\in(-1,1], \tag{64}$$

and the case $\lambda=c$ corresponds the case $\theta=0$.

By means of substitution the value of parameter λ from the formula (64) to the correlation (56), we obtain the following representation of the operators $L\in\mathfrak{OT}_{\mathbf{fin}}(\mathfrak{H},c)$:

$$\begin{gathered}L\omega=\mathbf{W}_{c\frac{1-\theta|\theta|}{1+\theta|\theta|}}\left[s,\mathbf{n},J\right]\omega=\left(s\varphi_0\left(\theta\right)\mathcal{T}\left(\omega\right)-\varphi_1\left(\theta\right)\frac{\langle\mathbf{n},\omega\rangle}{c}\right)\mathbf{e}_0\\ +J\left(c\varphi_1\left(\theta\right)\mathcal{T}\left(\omega\right)\mathbf{n}-s\varphi_0\left(\theta\right)\mathbf{X}_1\left[\mathbf{n}\right]\omega+\mathbf{X}_1^{\perp}\left[\mathbf{n}\right]\omega\right),\quad \omega\in\mathcal{M}\left(\mathfrak{H}\right),\\ s\in\{-1,1\},\quad J\in\mathfrak{U}\left(\mathfrak{H}_1\right),\quad \mathbf{n}\in\mathbf{B}_1\left(\mathfrak{H}_1\right),\quad \theta\in(-1,1]\setminus\{0\},\end{gathered} \tag{65}$$

where

$$\varphi_0\left(\theta\right)=\frac{1+\theta\left|\theta\right|}{2\left|\theta\right|},\quad \varphi_1\left(\theta\right)=\frac{1-\theta\left|\theta\right|}{2\left|\theta\right|}\quad (\theta\in\mathbb{R},\ \theta\neq 0). \tag{66}$$

Note, that the case $\theta=0$ must be excluded, because in this case we have $\lambda=c$, and the norm of velocity $\mathcal{V}(L)$ is equal to the speed of light c (note, that in the case $\|\mathcal{V}(L)\|=c$ the transforms (24), and, hence, (65) are undefined). From the equality (64) it follows, that in the case $\theta\in(0,1)$ we have, $\lambda=\|\mathcal{V}(L)\|\in(0,c)$. So, in this case, the norm of the velocity of reference frame $\|\mathcal{V}(L)\|$ frame is less then the speed of light c. Similarly, in the case $\theta\in(-1,0)$, we have $\lambda\in(c,+\infty)$. Hence, in this case the norm of frame velocity is greater, then c.

It is easy to verify, that for any $\theta\in\mathbb{R}\setminus\{0\}$ the following equalities are true:

$$\begin{gathered}\varphi_0\left(\theta\right)\varphi_1\left(\theta\right)=-\frac{1}{4}\left(\theta^2-\frac{1}{\theta^2}\right),\quad c\frac{\varphi_1\left(\theta\right)}{\varphi_0\left(\theta\right)}=\lambda=c\frac{1-\theta\left|\theta\right|}{1+\theta\left|\theta\right|},\\ \varphi_0\left(\theta\right)+\varphi_1\left(\theta\right)=\frac{1}{\left|\theta\right|},\quad \varphi_0\left(\theta\right)-\varphi_1\left(\theta\right)=\theta,\quad \varphi_0\left(\theta\right)^2-\varphi_1\left(\theta\right)^2=\operatorname{sign}\theta.\end{gathered} \tag{67}$$

Denote

$$\begin{gathered}\mathbf{U}_\theta\left(s,\mathbf{n},J\right):=\mathbf{W}_{c\frac{1-\theta|\theta|}{1+\theta|\theta|}}\left[s,\mathbf{n},J\right],\\ s\in\{-1,1\},\quad \mathbf{n}\in\mathbf{B}_1\left(\mathfrak{H}_1\right),\quad J\in\mathfrak{U}\left(\mathfrak{H}_1\right),\quad \theta\in(-1,1],\quad \theta\neq 0.\end{gathered}$$

From (66) it follows, that for $\theta=-1$ the functions $\varphi_0\left(\theta\right)$ and $\varphi_1\left(\theta\right)$ also are defined

$$\varphi_0\left(-1\right)=0,\quad \varphi_1\left(-1\right)=1.$$

And substitution $\theta=-1$ to the formula (65) lead us to the following linear operators:

$$\mathbf{U}_{-1}\left(s,\mathbf{n},J\right)=\mathbf{W}_\infty\left[\mathbf{n},J\right],$$

which do not depend on the number $s\in\{-1,1\}$, because terms, which contain variable s are zero (where the operators $\mathbf{W}_\infty\left[\mathbf{n},J\right]$ are defined in (57)).

Hence, for $\theta=-1$ we obtain the generalized Lorentz transforms for infinite speeds $\mathbf{W}_\infty\left[\mathbf{n},J\right]$, which, by remark 1, are not v-determined.

Above we have proved the following theorem.

Theorem 3. *Operator $L\in\mathcal{L}\left(\mathcal{M}\left(\mathfrak{H}\right)\right)$ belongs to the class $\mathfrak{OT}\left(\mathfrak{H},c\right)$ if and only if there exist numbers $s\in\{-1,1\}$, $\theta\in[-1,1]\setminus\{0\}$, vector $\mathbf{n}\in\mathbf{B}_1\left(\mathfrak{H}_1\right)$ and operator $J\in\mathfrak{U}\left(\mathfrak{H}_1\right)$*

such, that for any $\omega \in \mathcal{M}(\mathfrak{H})$ *vector* $L\omega$ *can be represented by the formula*

$$L\omega = \mathbf{U}_{\theta}(s, \mathbf{n}, J)\,\omega = \left(s\varphi_0(\theta)\,\mathcal{T}(\omega) - \varphi_1(\theta)\frac{\langle \mathbf{n}, \omega\rangle}{c}\right)\mathbf{e}_0 + J\left(c\varphi_1(\theta)\,\mathcal{T}(\omega)\,\mathbf{n} - s\varphi_0(\theta)\,\mathbf{X}_1[\mathbf{n}]\,\omega + \mathbf{X}_1^{\perp}[\mathbf{n}]\,\omega\right). \tag{68}$$

Coordinate transform $L = \mathbf{U}_{\theta}(s, \mathbf{n}, J)$ *is v-determined if and only if* $\theta \neq -1$, *and in this case*

$$\mathcal{V}(L) = cs\frac{1-\theta|\theta|}{1+\theta|\theta|}\mathbf{n}.$$

7. Notes about relativity principle in tachyon kinematics

Using the results of the theory of changeable sets [16, 3, 4] the kinematics, including the superlight motion of reference frames can be constructed mathematically strictly. For this purpose we consider the subset $\mathfrak{OT}_{+}(\mathfrak{H}, c) \subseteq \mathfrak{OT}(\mathfrak{H}, c)$ of coordinate transforms

$$\mathfrak{OT}_{+}(\mathfrak{H}, c) = \{\mathbf{U}_{\theta}(s, \mathbf{n}, J) \in \mathfrak{OT}(\mathfrak{H}, c) \mid s = 1\}. \tag{69}$$

Note, that the subset $\mathfrak{O}_{+}(\mathfrak{H}, c) = \mathfrak{OT}_{+}(\mathfrak{H}, c) \cap \mathfrak{O}(\mathfrak{H}, c) \subseteq \mathfrak{OT}_{+}(\mathfrak{H}, c)$ in the case $\mathfrak{H} = \mathbb{R}^3$ coincides with full Lorentz group, defined in [15].

The reason of the selection of subset $\mathfrak{OT}_{+}(\mathfrak{H}, c) \subseteq \mathfrak{OT}(\mathfrak{H}, c)$ (69) is, that in the case, where the velocity of frame is less, then light ($\theta > 0$), we exclude the coordinate transforms with negative direction of time. The most elementary example of coordinate transform which ensures the negative direction of time for corresponding reference frame is as follows:

$$\mathbf{U}_1(-1, \mathbf{n}, -\mathbb{I}_1) = -\mathbb{I} \quad (\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)). \tag{70}$$

The transform (70) has zero velocity, but negative direction of time. And it is easy to see, that $-\mathbb{I} \in \mathfrak{OT}(\mathfrak{H}, c)$, but $-\mathbb{I} \notin \mathfrak{OT}_{+}(\mathfrak{H}, c)$.

Let $\mathcal{B}$ be a basic changeable set (in the sense of [4, definition 7.4], [16, definition 8]) such, that $\mathfrak{Bs}(\mathcal{B}) \subseteq \mathfrak{H}$. For example it may be $\mathcal{B} = \mathcal{A}t(\mathcal{R})$, where $\mathcal{R}$ is a system of abstract trajectories from $\mathbb{R}$ to M, with $M \subseteq \mathfrak{H}$ (note, that trajectories $r \in \mathcal{R}$ may be superlight). Using [4, example 10.2], [3, example 3.2], we can construct the changeable set

$$\mathcal{K}\mathfrak{L}_{\mathcal{T}}(\mathfrak{H}, \mathcal{B}, c) = \mathcal{Z}\mathsf{im}\left(\mathfrak{OT}_{+}(\mathfrak{H}, c), \mathcal{B}\right), \tag{71}$$

which represents the mathematically strict model of kinematics for inertial reference frames, allowing superlight motion of frames, as well, as superlight transformations of objects (that is elementary or elementary-time states).

From the other hand, despite the fact, that the subset $\mathfrak{O}_{+}(\mathfrak{H}, c) \subseteq \mathfrak{OT}_{+}(\mathfrak{H}, c)$ is the group of operators over the space $\mathcal{M}(\mathfrak{H})$, it can be proved, that the whole set $\mathfrak{OT}_{+}(\mathfrak{H}, c)$ does not form a group. Moreover, the composition of transforms from $\mathfrak{OT}_{+}(\mathfrak{H}, c)$ with superlight speeds may give transforms with negative time direction like (70) (for example, in the case $\dim \mathfrak{H} = 1$ we have $\mathbf{W}_{\infty}[\mathbf{n}, \mathbb{I}_1]^2 = -\mathbb{I}$, $\mathbf{n} \in \mathbf{B}_1(\mathfrak{H}_1)$). This means, that kinematics (71) does not satisfy the relativity principle in superlight diapason, because the set of coordinate transforms, providing transition from one frame to all other, is different for different frames. Although, if we restrict ourselves to the sublight speeds (that is if we consider the kinematics $\mathcal{K}\mathfrak{L}_0(\mathfrak{H}, \mathcal{B}, c) = \mathcal{Z}\mathsf{im}\left(\mathfrak{O}_{+}(\mathfrak{H}, c), \mathcal{B}\right)$, which is the "subkinematics" of kinematics (71)), then the relativity principle will be satisfied.

At the present time we do not know, whether is it possible to construct a tachyon kinematics based on some subset of transforms $\mathcal{G} \subseteq \mathfrak{OT}(\mathfrak{H}, c)$, which satisfies the relativity principle and does not contain sublight transforms with negative time direction. But even in the case, when it is impossible, there is no need to enter into conflict with the classical theory of relativity, because kinematics (71) contradicts with the relativity principle only in the superlight diapason, where classical theories must not be true fully.

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