Methods of Functional Analysis and Topology
Vol. 19 (2013), no. 2, pp. 161–167

REMARKS ON SCHRÖDINGER OPERATORS WITH SINGULAR MATRIX POTENTIALS

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Abstract. In this paper, an asymmetric generalization of the Glazman–Povzner–Wienholtz theorem is proved for one-dimensional Schrödinger operators with strongly singular matrix potentials from the space $H_{loc}^{-1}(\mathbb{R},\mathbb{C}^{m\times m})$. This result is new in the scalar case as well.

1. Introduction and main results

Let us consider, in the complex separable Hilbert space of vector-valued functions $L^2(\mathbb{R},\mathbb{C}^m)$, $m\in\mathbb{N}$, the operators generated by the formal differential expression

$$\mathrm{l}[u] := -u'' + qu, \quad u=(u_1,\ldots,u_m), \tag{1}$$

where the matrix potential $q=\{q_{ij}\}_{i,j=1}^m$ belongs to the Sobolev negative class $H_{\mathrm{loc}}^{-1}(\mathbb{R},\mathbb{C}^{m\times m})$. Without loss of generality, we assume that the potential q in (1) can be presented in the form

$$q = Q' + s, \quad Q\in L_{\mathrm{loc}}^2(\mathbb{R},\mathbb{C}^{m\times m}), \quad s\in L_{\mathrm{loc}}^1(\mathbb{R},\mathbb{C}^{m\times m}),$$

where the derivative is understood in the distribution sense. Then the block Shin–Zettl matrices are defined by

$$A(x) := \begin{pmatrix} Q & I_m \\ -Q^2+s & -Q \end{pmatrix} \in L_{\mathrm{loc}}^1\left(\mathbb{R},\mathbb{C}^{2m\times 2m}\right), \tag{2}$$

where I_m is a unit $(m\times m)$-matrix. Similarly to the scalar case [15, 7], Shin–Zettl matrices define the quasiderivatives [13]

$$u^{[0]} := u, \quad u^{[1]} := u' - Qu, \quad u^{[2]} := \left(u^{[1]}\right)' + Qu^{[1]} + \left(Q^2 - s\right)u.$$

Then formal differential equation (1) is quasidifferential,

$$\mathrm{l}[u] := -u^{[2]}, \quad \mathrm{Dom}(\mathrm{l}) := \left\{ u \,\middle|\, u,\, u^{[1]} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^m) \right\},$$

where by $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^m)$ we denote the class of locally absolutely continuous vector-valued functions. This definition is motivated by the fact that

$$-u^{[2]} = -u'' + qu$$

in the distribution sense, i. e.,

$$\langle -u^{[2]},\varphi\rangle = \langle -u''+qu,\varphi\rangle, \quad u\in\mathrm{Dom}(\mathrm{l}), \quad \varphi\in \mathrm{C}_0^\infty(\mathbb{R},\mathbb{C}^m).$$

We say that a function u solves the Cauchy problem

$$\mathrm{l}[u] = f, \quad f\in L_{\mathrm{loc}}^1(\mathbb{R},\mathbb{C}^m), \tag{3}$$

$$u(x_0) = c_0, \quad u^{[1]}(x_0) = c_1, \quad x_0\in\mathbb{R}, \quad c_0, c_1\in\mathbb{C}^m, \tag{4}$$

2010 *Mathematics Subject Classification.* Primary 34L40; Secondary 47B44, 47A05.

Key words and phrases. Matrix Schrödinger operator, Glazman–Povzner–Wienholtz theorem, m-accretivity, complex-valued potential, distributional potential.

if u is the first coordinate of the vector-valued function solving the Cauchy problem for the associated Cauchy problem with initial conditions (4)

$$\frac{d}{dx}\begin{pmatrix} u \\ u^{[1]} \end{pmatrix} = A(x)\begin{pmatrix} u \\ u^{[1]} \end{pmatrix} + \begin{pmatrix} 0 \\ -f \end{pmatrix}. \tag{5}$$

An existence and uniqueness theorem implies that the Cauchy problem for system (5) has a unique solution (see [14, Theorem 16.1] and [17, Theorem 2.1]). Therefore our definition of a solution of the equation (3) is correct.

Differential expression (1) gives rise to associated maximal and preminimal operators L and L_{00} in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^m)$,

$$\mathrm{L}u := \mathrm{l}[u], \quad \mathrm{Dom}(\mathrm{L}) := \left\{ u \in L^2(\mathbb{R}, \mathbb{C}^m) \;\middle|\; u,\, u^{[1]} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^m),\ \mathrm{l}[u] \in L^2(\mathbb{R}, \mathbb{C}^m) \right\},$$

and

$$\mathrm{L}_{00}u := \mathrm{l}[u], \quad \mathrm{Dom}(\mathrm{L}_{00}) := \{ u \in \mathrm{Dom}(\mathrm{L}) \mid \mathrm{supp}\, u \Subset \mathbb{R} \}.$$

The block Shin–Zettl matrix (2) defines a Lagrange adjoint quasidifferential expression l^+ in the following way:

$$v^{\{0\}} := v, \quad v^{\{1\}} := v' - Q^* v, \quad v^{\{2\}} := \left(v^{\{1\}}\right)' + Q^* v^{\{1\}} + \left((Q^*)^2 - s^*\right) v,$$

$$\mathrm{l}^+[v] := -v^{\{2\}}, \quad \mathrm{Dom}(\mathrm{l}^+) := \left\{ v \;\middle|\; v,\, v^{\{1\}} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^m) \right\},$$

where the matrix $Q^* := \overline{Q}^T$ is Hermitian conjugate to Q. The matrix s^* has the similar meaning.

The quasidifferential expression l^+ gives rise to associated maximal and preminimal operators L^+ and L^+_{00},

$$\mathrm{L}^+ v := \mathrm{l}^+[v],$$

$$\mathrm{Dom}(\mathrm{L}^+) := \left\{ v \in L^2(\mathbb{R}, \mathbb{C}^m) \;\middle|\; v,\, v^{\{1\}} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^m),\ \mathrm{l}^+[v] \in L^2(\mathbb{R}, \mathbb{C}^m) \right\},$$

and

$$\mathrm{L}^+_{00} v := \mathrm{l}^+[v], \quad \mathrm{Dom}(\mathrm{L}^+_{00}) := \left\{ v \in \mathrm{Dom}(\mathrm{L}^+) \mid \mathrm{supp}\, v \Subset \mathbb{R} \right\}.$$

Below we prove (Proposition 7) that the preminimal operators L_{00}, L^+_{00} are densely defined in the space $L^2(\mathbb{R}, \mathbb{C}^m)$ and have closures L_0 and L^+_0 which are called minimal operators. Maximal operators L and L^+ are closed.

For the case where the potential q is a real-valued symmetric matrix, such operators were earlier considered in [13]. Matrix Schrödinger operators with strongly singular self-adjoint potentials of Miura class were investigated in detail in [2]. These references contain a more detailed review and a more extensive bibliography. For the scalar case of quasidifferential operators generated by Shin–Zettl matrices in a general form, one can find a review of results in [4], see also [8, 18].

Recall that an operator A on a Hilbert space H is called *accretive* if

$$\mathrm{Re}\, \langle Au, u \rangle_H \geq 0, \quad u \in \mathrm{Dom}(A).$$

If, in addition, the left half-plane $\{\lambda \in \mathbb{C} \mid \mathrm{Re}\, \lambda < 0\}$ belongs to the resolvent set of the operator A, then the operator A is called *m-accretive* [10, 16]. This operator is also *maximal accretive* in the sense that it has no accretive extensions in the space H. If an operator A is m-accretive, then the operator $-A$ generates a semigroup of contractions in the space H. The converse is also true.

The main result of this paper is a non-symmetric generalization of the Glazman–Povzner–Wienholtz theorem for the operators generated by differential expression (1).

Theorem 1. *The operator* L_0 *is m-accretive if and only if preminimal operators* L_{00} *and* L_{00}^{+} *are accretive. In this case,* $\mathrm{L}_0 = \mathrm{L}$.

Note that in this theorem we assume both preminimal operators L_{00} and L_{00}^{+} to be accretive. In the scalar case, one of these operators being accretive implies that other is also accretive.

Corollary 2. (Cf. [5]). *If the matrix potential q is self-adjoint,* $Q = Q^*$ *and* $s = s^*$*, then the operator* L_0 *is symmetric. Moreover, if the operator* L_0 *is bounded from below, then it is self-adjoint and* $\mathrm{L}_0 = \mathrm{L}$.

For $m = 1$ this is known [1, Remark III.2], see also [3, 9, 11].

Remark 3. If the complex matrices Q and s are symmetric, i. e., $Q = Q^T$, $s = s^T$, then Theorem 1 can be strengthened. Since the operator L_{00} is accretive, the operator L_0 is maximal accretive and its residual spectrum is empty.

In particular, this condition is satisfied in the scalar case, when $m = 1$. In this case, the operators L_{00} and L_{00}^{+} are obviously accretive if the real part of the potential q is positive in the sense of distributions. This condition is equivalent to

$$q = \mu + i\nu,$$

where μ is a nonnegative Radon measure on a locally compact space $\mathbb{R}$ and ν is a real-valued distribution from $H^{-1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{m\times m})$.

The paper is organized as follows. In Section 2, we introduce notations used in the paper and thoroughly investigate properties of the operators L, L_0 and L^+, L_0^+ (Proposition 7). Section 3 contains proofs of the main Theorem 1, Corollary 2 and Remark 3.

2. Properties of the minimal and maximal operators

In this paper, we use the following notations. We denote by $(\cdot\,,\cdot)_{\mathbb{C}^m}$ the inner product in the space $\mathbb{C}^m$,

$$(u, v)_{\mathbb{C}^m} := \sum_{i=1}^{m} u_i \overline{v_i}, \quad u = (u_1, \dots, u_m), \quad v = (v_1, \dots, v_m) \in \mathbb{C}^m.$$

We denote by $\langle\cdot\,,\cdot\rangle_{L^2(\mathbb{R},\mathbb{C}^m)}$ the inner product in the Hilbert space of square-integrable vector-valued functions $L^2(\mathbb{R}, \mathbb{C}^m)$,

$$\langle u, v\rangle_{L^2(\mathbb{R},\mathbb{C}^m)} := \int_{\mathbb{R}} (u, v)_{\mathbb{C}^m}\, d\,x.$$

For an arbitrary matrix $A = \{a_{ij}\}_{i,j=1}^m \in \mathbb{C}^{m\times m}$, we denote the transposed matrix by $A^T = \{a_{ij}^T\}_{i,j=1}^m$ and the Hermitian conjugate matrix by $A^* = \{a_{ij}^*\}_{i,j=1}^m$: $a_{ij}^* = \overline{a_{ji}}$. For an arbitrary complex number $a \in \mathbb{C}$, we denote the corresponding complex conjugate number by $\overline{a}$.

We say that a matrix-valued function $A(x) = \{a_{ij}(x)\}_{i,j=1}^m$ belongs to the space $L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{m\times m})$, if each element of this matrix $a_{ij}(x)$ belongs to the space $L^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{C})$, $p \in [1, \infty)$.

J. Weidmann [17] previously studied in detail the quasidifferential matrix-valued Sturm–Liouville operators generated by quasidifferential expressions τ,

$$\tau[u] := -(u' - Qu)' - Q^*(u' - Qu) - (Q^*Q - s)u,$$
$$Q \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{m\times m}), \quad s \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{C}^{m\times m}), \quad s = s^*.$$

In this case, the preminimal operators generated by the quasidifferential expressions τ are symmetric [17, Theorem 3.1].

Obviously, if the matrices $Q = Q^*$ and $s = s^*$ are self-adjoint, then the operators generated by the quasidifferential expressions τ and the operators generated by the quasidifferential expressions l and l^+ coincide.

The following properties of the operators L, L_0, L_{00} and L^+, L_0^+, L_{00}^+ we state without a proof, since they are proved in the same way as the properties of operators generated by the quasidifferential expressions τ [17].

Lemma 4. *For arbitrary vector-valued functions $u \in \mathrm{Dom}(\mathrm{L})$, $v \in \mathrm{Dom}(\mathrm{L}^+)$ and an arbitrary bounded line segment $[a,b]$, we have*

$$\int_a^b (l[u], v)_{\mathbb{C}^m}\, d\,x - \int_a^b \left(u, \mathrm{l}^+[v]\right)_{\mathbb{C}^m} d\,x = [u,v]_a^b,$$

where

$$[u,v](t) \equiv [u,v] := \left(u, v^{\{1\}}\right)_{\mathbb{C}^m} - \left(u^{[1]}, v\right)_{\mathbb{C}^m},$$
$$[u,v]_a^b := [u,v](b) - [u,v](a), \quad -\infty \le a \le b \le \infty.$$

Lemma 5. *For arbitrary vector-valued functions $u \in \mathrm{Dom}(\mathrm{L})$ and $v \in \mathrm{Dom}(\mathrm{L}^+)$, the following limits exist and are finite:*

$$[u,v](-\infty) := \lim_{t\to-\infty}[u,v](t), \quad [u,v](\infty) := \lim_{t\to\infty}[u,v](t).$$

Lemma 6. (Generalized Lagrange identity). *For arbitrary vector functions $u \in \mathrm{Dom}(\mathrm{L})$ and $v \in \mathrm{Dom}(\mathrm{L}^+)$, the following relation holds:*

$$\int_{-\infty}^{\infty} (l[u], v)_{\mathbb{C}^m}\, d\,x - \int_{-\infty}^{\infty} (l[u], v)_{\mathbb{C}^m}\, d\,x = [u,v]_{-\infty}^{\infty}.$$

Proposition 7. *The operators L, L_{00} and L^+, L_{00}^+ have the following properties:*

1^0. *The operators L_{00} and L_{00}^+ are densely defined in the Hilbert space $L^2(\mathbb{R},\mathbb{C}^m)$.*

2^0. *The equalities*

$$(\mathrm{L}_{00})^* = \mathrm{L}^+, \quad \left(\mathrm{L}_{00}^+\right)^* = \mathrm{L}$$

hold. In particular, the operators L, L^+ are closed and the operators L_{00}, L_{00}^+ are closable.

3^0. *Domains of the operators L_0, L_0^+ may be described in the following way:*

$$\mathrm{Dom}(\mathrm{L}_0) = \left\{u \in \mathrm{Dom}(\mathrm{L}) \,\middle|\, [u,v]_{-\infty}^{\infty} = 0 \ \forall v \in \mathrm{Dom}(\mathrm{L}^+)\right\},$$
$$\mathrm{Dom}(\mathrm{L}_0^+) = \left\{v \in \mathrm{Dom}(\mathrm{L}^+) \,\middle|\, [u,v]_{-\infty}^{\infty} = 0 \ \forall u \in \mathrm{Dom}(\mathrm{L})\right\}.$$

4^0. *The following inclusions take place:*

$$\mathrm{Dom}(\mathrm{L}) \subset H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^m), \quad \mathrm{Dom}(\mathrm{L}^+) \subset H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{C}^m).$$

For the case $m = 1$ the results of this section are established in [12].

3. Proofs

The following lemma is proved by a direct calculation.

Lemma 8. *For arbitrary vector-valued functions $u \in \mathrm{Dom}(\mathrm{L})$, $v \in \mathrm{Dom}(\mathrm{L}^+)$, and functions $\varphi \in C_0^\infty(\mathbb{R},\mathbb{C})$ we have*

i) $\mathrm{l}[\varphi I_m u] = \varphi I_m \mathrm{l}[u] - \varphi'' I_m u - 2\varphi' I_m u', \quad \varphi I_m u \in \mathrm{Dom}(\mathrm{L}_{00});$

ii) $\mathrm{l}^+[\varphi I_m v] = \varphi I_m \mathrm{l}^+[v] - \varphi'' I_m v - 2\varphi' I_m v', \quad \varphi I_m v \in \mathrm{Dom}(\mathrm{L}_{00}^+).$

Proof of Theorem 1. Sufficiency. Due to the assumptions of the theorem, the minimal operators L_0 and L_0^+ are accretive. Without loss of generality we assume that the following inequalities hold:

$$\operatorname{Re}\left\langle \mathrm{L}_0 u, u\right\rangle_{L^2(\mathbb{R},\mathbb{C}^m)} \geq \langle u, u\rangle_{L^2(\mathbb{R},\mathbb{C}^m)}, \quad u \in \operatorname{Dom}(\mathrm{L}_0),$$

and

$$\operatorname{Re}\left\langle \mathrm{L}_0^+ v, v\right\rangle_{L^2(\mathbb{R},\mathbb{C}^m)} \geq \langle v, v\rangle_{L^2(\mathbb{R},\mathbb{C}^m)}, \quad v \in \operatorname{Dom}(\mathrm{L}_0^+). \tag{6}$$

To prove that the minimal operator L_0 id m-accretive, it suffices to show that the kernel of the operator L^+ contains only the zero element.

Let v be a solution to the equation

$$\mathrm{L}^+ v = 0.$$

We will show that $v \equiv 0$.

For an arbitrary function $\varphi \in \mathrm{C}_0^\infty(\mathbb{R},\mathbb{R})$, due to Lemma 8, we have $\varphi I_m v \in \operatorname{Dom}(\mathrm{L}_{00}^+)$. Therefore, taking into account that $\mathrm{l}^+[v] = 0$, after some simple calculations we obtain

$$\left\langle \mathrm{L}_0^+ \varphi I_m v, \varphi I_m v\right\rangle_{L^2(\mathbb{R},\mathbb{C}^m)} = \int_{\mathbb{R}} (\varphi')^2 (v,v)_{\mathbb{C}^m}\, dx + \int_{\mathbb{R}} \varphi\varphi' \left((v,v')_{\mathbb{C}^m} - (v',v)_{\mathbb{C}^m}\right) dx. \tag{7}$$

Since

$$\operatorname{Re} \int_{\mathbb{R}} \varphi\varphi' \left((v,v')_{\mathbb{C}^m} - (v',v)_{\mathbb{C}^m}\right) dx = 0,$$

we obtain from (7), taking into account (6), that

$$\int_{\mathbb{R}} (\varphi')^2 (v,v)_{\mathbb{C}^m}\, dx \geq \int_{\mathbb{R}} (\varphi)^2 (v,v)_{\mathbb{C}^m}\, dx \quad \forall \varphi \in \mathrm{C}_0^\infty(\mathbb{R},\mathbb{R}). \tag{8}$$

Furthermore, let us take a sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ which has the following properties:

i) $\varphi_n \in C_0^\infty(\mathbb{R},\mathbb{R})$;
ii) $\operatorname{supp}\varphi_n \subset [-n-1, n+1]$;
iii) $\varphi_n(x) = 1$, $x \in [-n, n]$;
iv) $|\varphi_n'(x)| \leq C$ where $C > 0$ is an absolute constant.

Substituting in (8) we get

$$\int_{-n}^{n} (v,v)_{\mathbb{C}^m}\, dx \leq \int_{\mathbb{R}} \varphi_n^2 (v,v)_{\mathbb{C}^m}\, dx \leq \int_{\mathbb{R}} (\varphi_n')^2 (v,v)_{\mathbb{C}^m}\, dx \leq C^2 \int\limits_{n\leq|x|\leq n+1} (v,v)_{\mathbb{C}^m}\, dx,$$

i. e.

$$\int_{-n}^{n} (v,v)_{\mathbb{C}^m}\, dx \leq C^2 \int\limits_{n\leq|x|\leq n+1} (v,v)_{\mathbb{C}^m}\, dx. \tag{9}$$

Since $v \in L^2(\mathbb{R},\mathbb{C}^m)$ passing in (9) to the limit as $n \to \infty$, we obtain $v \equiv 0$.

Thus we have proved that the operator L_0 is m-accretive.

In a similar way, one can prove that the operator L_0^+ is m-accretive. Then taking into account that an adjoint operator to an m-accretive operator is m-accretive [16, Proposition 3.20], from property 2^0 of Proposition 7 we get that the maximal operator L is also m-accretive. By the definition of maximal accretivity and [16, Proposition 3.24], we have that $\mathrm{L}_0 = \mathrm{L}$ as $\mathrm{L}_0 \subset \mathrm{L}$. Sufficiency is proved.

Necessity. Let us suppose that the operator L_0 is m-accretive. Then taking into account that an adjoint operator to an m-accretive operator is m-accretive [16, Proposition 3.20], from property 2^0 of Proposition 7 we get that the operator L_0^+ is m-accretive. Therefore the operators L_{00} and L_{00}^+ are accretive. Necessity is proved.

The theorem is proved completely. □

Proof of Corollary 2. One only needs to note that in the case of the self-adjoint potential q, the preminimal operators L_{00} and L_{00}^{+} coincide and are symmetric, due to property 2^0 of Proposition 7 (see also [17, Theorem 3.1]). □

Proof of Remark 3. Note that in the case of complex symmetric matrix potentials, we have

$$Q^* = \overline{Q} = \{\overline{Q}_{ij}\}_{i,j=1}^{m}, \quad s^* = \overline{s} = \{\overline{s}_{ij}\}_{i,j=1}^{m}.$$

Then domains of the preminimal operators L_{00} and L_{00}^{+} are related by

$$u \in \mathrm{Dom}(\mathrm{L}_{00}) \Leftrightarrow \overline{u} \in \mathrm{Dom}(\mathrm{L}_{00}^{+}).$$

Therefore accretivity of the operator L_{00} implies accretivity of the operator L_{00}^{+} and vice versa.

Moreover, let J be an antilinear operator of complex conjugation. Then one may easy verify that the following inclusion takes place:

$$\mathrm{J}\mathrm{L}_0\mathrm{J} = \mathrm{L}_0^{+} \subset \mathrm{L}^{+} = \mathrm{L}_0^{*},$$

that is, the operator L_0 is J-symmetric [6]. If the operators L_{00} are accretive, then due to Theorem 1 and property 2^0 of Proposition 7, the operator L_0 is J-self-adjoint,

$$\mathrm{J}\mathrm{L}_0\mathrm{J} = \mathrm{L}_0^{*}.$$

Therefore its residual spectrum is empty. □

Acknowledgments. The first author was partially supported by the grant no. 03-01-12 of the National Academy of Sciences of Ukraine (under the joint Ukrainian–Russian project of NAS of Ukraine and Siberian Branch of Russian Academy of Sciences) and the second author was partially supported by the grant no. 01-01-12 of National Academy of Sciences of Ukraine (under the joint Ukrainian–Russian project of NAS of Ukraine and Russian Foundation of Basic Research).

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Received 17/01/2013