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INDEFINITE MOMENT PROBLEM AS AN ABSTRACT INTERPOLATION PROBLEM

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Abstract. Indefinite moment problem was considered by M. G. Krein and H. Langer in 1979. In the present paper the general indefinite moment problem is associated with an abstract interpolation problem in generalized Nevanlinna classes. To prove the equivalence of these two problems we investigate the structure of de Branges space $\mathcal{H}(m)$ associated with a generalized Nevanlinna function m.

A general formula for description of the set of solutions of indefinite moment problem is found. It is shown that the Kein-Langer description can be derived from this formula by a special choice of biorthogonal system of polynomials.

1. Introduction

The classical moment problem consists in finding a measure σ on the real line $\mathbb{R}$ with given moments $s_k \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} t^k d\sigma(t) = s_k \quad (k = 0, 1, \dots). \tag{1.1}$$

This problem was studied by T. Stieltjes, H. Hamburger, M. Riesz, R. Nevanlinna and others. As is known the necessary condition for the solvability of the moment problem (1.1) is the nonnegativity of Hankel matrices $D_n := (s_{j+k})_{j,k=0}^{n-1}$ for every $n \in \mathbb{N}\cup\{0\}$. A connection of this problem to some multiple interpolation problem at ∞ was found by Hamburger in 1920 where it was shown that σ is a solution to the classical moment problem (1.1) if and only if its associated function

$$m(\lambda) = \int_{\mathbb{R}} \frac{d\sigma(t)}{t - \lambda} \tag{1.2}$$

admits the following asymptotic expansion:

$$m(\lambda) \sim -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \frac{s_2}{\lambda^3} - \cdots, \quad \lambda \widehat{\to} \infty. \tag{1.3}$$

The notation $\lambda \widehat{\to} \infty$ means that λ tends to ∞ nontangentially remaining inside a sector $\delta < \arg \lambda < \pi - \delta$ ($\delta > 0$).

The function m in (1.2) belongs to the class N of functions holomorphic in the upper half-plane $\mathbb{C}_+$ and having there a nonnegative imaginary part. Let us say that a meromorphic function m with the domain of holomorphy $\mathfrak{h}_m^+$ in $\mathbb{C}_+$ belongs to the class N_κ ($\kappa \in \mathbb{Z}_+$) if the kernel

$$\mathsf{N}_\mu^m(\lambda) = \frac{m(\lambda) - m(\mu)^*}{\lambda - \bar{\mu}} \quad (\lambda, \mu \in \mathfrak{h}_m^+) \tag{1.4}$$

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has κ negative squares in $\mathfrak{h}_m^+$, that is for arbitrary set $\mu_j \in \mathfrak{h}_m^+$ $(j = 1, 2, \dots, n)$ the form

$$\sum_{i,j=1}^{n} \mathsf{N}_{\mu_j}^m(\mu_i)\xi_j\overline{\xi}_i$$

has at most κ negative squares and for some choice of the set $\{\mu_j\}_{j=1}^n$ it has exactly κ negative squares ([3]).

Indefinite moment problem was considered in 1979 by M. G. Krein and H. Langer [22]. It was formulated as an interpolation problem:

Problem $MP_\kappa(\mathbf{s})$. Given is a sequence of real numbers $\mathbf{s} = \{s_j\}_{j=0}^\infty$. Find a function $m \in N_\kappa$, such that (1.3) holds.

Denote by H_κ the class of real sequences $\mathbf{s} = \{s_j\}_{j=0}^\infty$, such that the Hankel matrices D_n have exactly κ negative eigenvalues for all n large enough. As was shown in [22] the problem $MP_\kappa(\mathbf{s})$ is solvable for every $\mathbf{s} \in H_\kappa$. This problem is called *determinate*, if it has a unique solution and *indeterminate*, otherwise. In what follows we suppose that

(I) the problem $MP_\kappa(\mathbf{s})$ is indeterminate.

The set of solutions of indeterminate moment problem was described in [22] by the methods of extension theory. In the present paper we will consider an abstract interpolation problem (AIP_κ), associated with the problem $MP_\kappa(\mathbf{s})$ and derive the description of solutions of AIP_κ. The problem AIP_κ in generalized Nevanlinna classes was studied in [25] (see [10] for Nevanlinna classes case). The solution of this problem is based on the ideas of AIP in Schur classes developed in [17]. We will apply methods of AIP_κ to the $MP_\kappa(\mathbf{s})$.

Define a sesquilinear form $K(\cdot,\cdot)$ on the set $\mathcal{X} = \mathbb{C}[\lambda]$ of polynomials $h(\lambda) = \sum_{j=0}^n h_j\lambda^j$ $(n = 0, 1, 2, \dots)$ by the formula

$$K(h, h) = \sum_{j,k=0}^{n} s_{j+k}h_j\overline{h}_k. \tag{1.5}$$

Since the sequence of numbers $\{s_j\}_0^\infty$ belongs to the class H_κ then the form $K(\cdot,\cdot)$ has κ negative squares. A standard procedure of closure of the space $\mathcal{X}$ with respect to the inner product (1.5) leads to a Pontryagin space $\mathcal{H}$ (see [5] for the definition of Pontryagin space).

Let $\{f_k\}_{k=0}^\infty$ be the basis in the space $\mathbb{C}[x]$. The basis $\{g_k\}_{k=0}^\infty$ is called biorthogonal to $\{f_k\}_{k=0}^\infty$ with respect to the form $K(\cdot,\cdot)$, if

$$K(f_j, g_k) = \delta_{jk} \quad (j, k = 0, 1, \dots). \tag{1.6}$$

The system of adjacent polynomial $\{\widetilde{g}_k(\lambda)\}_{k=0}^\infty$ is defined by equalities

$$\widetilde{g}_k(\lambda) = K\left(\frac{g_k(t) - g_k(\lambda)}{t - \lambda}, \mathbf{1}\right) \quad (k = 0, 1, \dots). \tag{1.7}$$

Define the matrix value function $\Theta(\lambda) = \begin{bmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{bmatrix} \in \mathbb{C}^{2\times 2}$ by the formula

$$\begin{aligned} \theta_{11}(\lambda) &= 1 + \lambda\sum_{k=0}^\infty \widetilde{f}_k(\lambda)g_k(0)^*, & \theta_{12}(\lambda) &= \lambda\sum_{k=0}^\infty \widetilde{f}_k(\lambda)\widetilde{g}_k(0)^*, \\ \theta_{21}(\lambda) &= -\lambda\sum_{k=0}^\infty f_k(\lambda)g_k(0)^*, & \theta_{22}(\lambda) &= 1 - \lambda\sum_{k=0}^\infty f_k(\lambda)\widetilde{g}_k(0)^*. \end{aligned} \tag{1.8}$$

Now we can describe the set of solutions of the problem $MP_\kappa(\mathbf{s})$.

Theorem 1.1. *Let the sequence* $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$ *belongs to* H_κ. *Define the inner product* $K(\cdot,\cdot)$ *on the set* $\mathbb{C}[x]$ *by the formula* (1.5). *Let* $\{f_k(\lambda)\}_{k=0}^{\infty}$ *and* $\{g_k(\lambda)\}_{k=0}^{\infty}$ *be biorthogonal bases with respect to the form* K *and let the matrix function* $\Theta(\lambda)$ *be defined by* (1.8). *Then the formula*

$$m(\lambda) = \big(\theta_{11}(\lambda)\varphi(\lambda) + \theta_{12}(\lambda)\big)\big(\theta_{21}(\lambda)\varphi(\lambda) + \theta_{22}(\lambda)\big)^{-1} \tag{1.9}$$

establishes a one-to-one correspondence between the set of all solutions $m(\lambda)$ *of the problem* $MP_\kappa(\mathbf{s})$ *and the set of* $\varphi \in \widetilde{N} = N \cup \{\infty\}$.

The matrix valued function $\Theta(\lambda)$ mentioned above turns out to be a resolvent matrix (up to a J-inner factor) of some symmetric operator constructed by the data of AIP_κ. An explicit formula for the resolvent matrix of a symmetric operator in a Pontryagin space can be found in [9] (see also [14] for the Hilbert space case).

We show that the set of solutions of $MP_\kappa(\mathbf{s})$ coincides with the set of solutions of an abstract interpolation problem AIP_κ, associated with $MP_\kappa(\mathbf{s})$. We have used methods of reproducing kernel Pontryagin spaces. In particular, a general form of the space $\mathcal{H}(m)$ with the reproducing kernel $\mathsf{N}_\mu^m(\lambda)$ of the form (1.4) for the function $m \in N_\kappa$ is found (see Theorem 3.3). This representation of the space $\mathcal{H}(m)$ is used in order to reformulate the asymptotic formula (1.3) for $m \in N_\kappa$ in terms of the space $\mathcal{H}(m)$ (see Lemma 3.4). In [18] the classical and truncated moment problem were studied by the method of transformed Potapov's fundamental matrix inequality (see also [17]).

The paper is organized as follows. In Section 2 we recall the main results of the theory of generalized Nevanlinna functions and also present a description of the set of solutions of an abstract interpolation problem. In Section 3 we describe the space $\mathcal{H}(m)$. In Section 4 the problem AIP_κ corresponding to the problem $MP_\kappa(\mathbf{s})$ is constructed. In Sections 5 the equivalence of these problems is shown. In Section 6 the main result of this paper (Theorem 1.1) is proved. Also a specific algorithm for constructing biorthogonal bases with respect to the form $K(\cdot,\cdot)$ is presented. In Appendix an auxiliary statement (Lemma 3.4) is proved.

2. Preliminaries

2.1. **Abstract interpolation problem.** We present a scalar analogue of an abstract interpolation problem (AIP) which was considered by the author in [25].

Let $\mathcal{X}$ be a complex linear space, let B_1, B_2 be linear operators in $\mathcal{X}$, let C_1, C_2 be linear operators from $\mathcal{X}$ to $\mathbb{C}$ and let K be a nondegenerate sesquilinear form on $\mathcal{X}$. Denote by $\nu_-(K)$ the number of negative squares of K. Define the Pontryagin space $\mathcal{H}$ as the completion of $\mathcal{X}$ endowed with the inner product

$$\langle h, g\rangle_{\mathcal{H}} = K(h,g), \quad h, g \in \mathcal{X}. \tag{2.1}$$

We identify the linear operators $B_1,\ B_2 : \mathcal{X} \to \mathcal{X}$ with the linear operators $B_1,\ B_2 : \mathcal{X} \to \mathcal{H}$.

Let $\mathcal{H}(m)$ be the reproducing kernel Pontryagin space (RKPS) with the reproducing kernel $\mathsf{N}_\mu^m(\lambda)$ defined by (1.4) (see [6], [4]). This space is characterized by the properties

(1) $\mathsf{N}_\mu^m(\cdot) \in \mathcal{H}(m)$ for all $\mu \in \mathfrak{h}_m^+$;
(2) for every $f \in \mathcal{H}(m)$ the following identity holds

$$\big\langle f(\cdot), \mathsf{N}_\mu^m(\cdot)\big\rangle_{\mathcal{H}(m)} = f(\mu) \quad (\mu \in \mathfrak{h}_m^+). \tag{2.2}$$

Problem $AIP_\kappa(B_1, B_2, C_1, C_2, K)$. Let the data set (B_1, B_2, C_1, C_2, K) satisfy the assumptions

(A1) $K(B_2h, B_1g) - K(B_1h, B_2g) = (C_1h, C_2g)_{\mathbb{C}} - (C_2h, C_1g)_{\mathbb{C}}\ \forall h, g \in \mathcal{X}$;
(A2) $\ker K = \{0\}$, where $\ker K = \{h \in \mathcal{X} :\ K(h,g) = 0\ \forall g \in \mathcal{X}\}$;

(A3) $B_2 = I_{\mathcal{X}}$ and the operators $B_1 : \mathcal{X} \subseteq \mathcal{H} \to \mathcal{H}$, $C_1, C_2 : \mathcal{X} \subseteq \mathcal{H} \to \mathcal{L}$ are bounded;
(A4) for some choice of $\lambda_j \in \mathbb{C}_+$ $(j = 1, \dots, \kappa)$ the following condition holds:

$$\ker \begin{bmatrix} C_2^* & (1 - \lambda_1 B_1^*)^{-1} C_2^* & (1 - \lambda_2 B_1^*)^{-1} C_2^* & \cdots & (1 - \lambda_\kappa B_1^*)^{-1} C_2^* \end{bmatrix} = \{0\}.$$

Find a function $m(\lambda)$ from the class N_κ such that for some linear mapping $F : \mathcal{X} \to \mathcal{H}(m)$ the following conditions hold:

(C1) $(FB_2 h)(\lambda) - \lambda (FB_1 h)(\lambda) = \begin{bmatrix} \mathbf{1} & -m(\lambda) \end{bmatrix} \begin{bmatrix} C_1 h \\ C_2 h \end{bmatrix}$ for all $h \in \mathcal{X}$;
(C2) $\langle Fh, Fh \rangle_{\mathcal{H}(m)} \le K(h, h)$ for all $h \in \mathcal{X}$.

Note that the condition $\nu_-(K) \le \kappa$ is necessary for the solvability of AIP_κ (see [25, Remark 3.1]). Define the 2×2 matrix function $\Theta(\lambda)$ by the formula

$$\Theta(\lambda) = \begin{bmatrix} \theta_{11}(\lambda) & \theta_{12}(\lambda) \\ \theta_{21}(\lambda) & \theta_{22}(\lambda) \end{bmatrix} = I_{\mathbb{C}\oplus\mathbb{C}} - \lambda \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (\mathbf{1} - \lambda B_1)^{-1} \begin{bmatrix} -C_2^* & C_1^* \end{bmatrix}. \tag{2.3}$$

The main result of the paper [25] is the following description of all solutions of the AIP_κ.

Theorem 2.1. *Let the data set (B_1, B_2, C_1, C_2, K) satisfy the assumptions (A1)–(A4), $\kappa = \nu_-(K)$, and let $\Theta(\lambda)$ be defined by (2.3). Then the formula*

$$m(\lambda) = \big(\theta_{11}(\lambda)\varphi(\lambda) + \theta_{12}(\lambda)\big)\big(\theta_{21}(\lambda)\varphi(\lambda) + \theta_{22}(\lambda)\big)^{-1}, \tag{2.4}$$

establishes a one-to-one correspondence between the set of all solutions $m(\lambda)$ of the problem $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ and the set of $\varphi \in \widetilde{N} = N \cup \{\infty\}$, such that the function m defined by the formula (2.4) belongs to the class N_κ.

Remark 2.2. Let the data set (B_1, B_2, C_1, C_2, K) satisfy the assumptions (A1)–(A4). Then the mapping $F : \mathcal{X} \to \mathcal{H}(m)$ in (C1) is uniquely defined by the formula

$$(Fh)(\lambda) = \begin{bmatrix} \mathbf{1} - m(\lambda) \end{bmatrix} G(\lambda) h \quad (\lambda \in \mathcal{O}, \quad h \in \mathcal{X}), \tag{2.5}$$

where $\mathcal{O}$ is a nonempty neighborhood of the point 0 and

$$G(\lambda) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (\mathbf{1} - \lambda B_1)^{-1} \quad (\lambda \in \mathcal{O}). \tag{2.6}$$

Remark 2.3. It follows from (A1) that the linear relation

$$A : \begin{bmatrix} B_1 h \\ C_1 h \end{bmatrix} \to \begin{bmatrix} B_2 h \\ C_2 h \end{bmatrix} \quad (h \in \mathcal{X}) \tag{2.7}$$

is symmetric in $\mathcal{H} \oplus \mathbb{C}$. The statement of Theorem 2.1 was obtained in [25] by the methods of extension theory of isometric operators. Namely, the problem of describing the set of solutions of $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ was reduced to the description of all $\mathbb{C}$-resolvents of a symmetric linear relation A. The formula (2.3) for the solution matrix was derived from the results of works [14], [15] and [9], where the formula for the $\mathbb{C}$-resolvent matrix was obtained in terms of the boundary triplet for the symmetric lineal relation.

2.2. **Classes of Nevanlinna functions.** As is known (see [2, §69, Theorem 2]) every function from the class N admits the integral representation

$$m(\lambda) = a + b\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\sigma(t), \tag{2.8}$$

where a, b are real constants such that $b \ge 0$ and $\sigma(t)$ is a right continuous non-decreasing function such that

$$\int_{-\infty}^{\infty} \frac{d\sigma(t)}{1 + t^2} < \infty. \tag{2.9}$$

Define a subclass N^0 of functions $m_0 \in N$, which admit the integral representation

$$m_0(\lambda) = \int_{\mathbb{R}} \frac{d\sigma(t)}{t-\lambda} \tag{2.10}$$

with a bounded non-decreasing function $\sigma(t)$. Note that each function m of the class N satisfies the condition $m(\lambda) = O(\lambda)$ for $\lambda \widehat{\to} \infty$. As is known (see [2, §69 Theorem 3]) the function $m_0 \in N$ belongs to the class N^0, if and only if $m_0(\lambda) = O(\frac{1}{\lambda})$ for $\lambda \widehat{\to} \infty$.

It follows from Hamburger-Nevanlinna Theorem (see [1, Theorem 3.2.1]) that the function $m_0 \in N^0$ of the form (2.10) admits an asymptotic expansion

$$m_0(\lambda) = -\frac{s_0^0}{\lambda} - \frac{s_1^0}{\lambda^2} - \frac{s_2^0}{\lambda^3} - \cdots, \quad \lambda \widehat{\to} \infty, \tag{2.11}$$

where $s_j^0 \in \mathbb{R}$, if and only if the next relations hold

$$\int_{-\infty}^{\infty} t^j \, d\sigma(t) = s_j^0 \quad (j = 0, 1, \ldots). \tag{2.12}$$

Remark 2.4. It follows from the relations (2.12) that any polynomial belongs to the space $L_2(d\sigma)$.

3. Description of the space $\mathcal{H}(m)$

Recall the known de Branges result about description of the space $\mathcal{H}(m_0)$, where $m_0 \in N$.

Theorem 3.1. ([6, Theorem 5]). *Let the function m_0 belong to N and admit the integral representation* (2.8)*. Then*

(i) *the space $\mathcal{H}(m_0)$ coincides with the set of functions*

$$\mathcal{H}(m_0) = \left\{ c + \int_{-\infty}^{\infty} \frac{f(t)}{t-\lambda} d\sigma(t), \quad c \in \mathbb{C}, \quad f(t) \in L_2(d\sigma) \right\}.$$

(ii) *If, in addition, the function m_0 belongs to N^0 and admits the integral representation* (2.10)*, then*

$$\mathcal{H}(m_0) = \left\{ \int_{-\infty}^{\infty} \frac{f(t)}{t-\lambda} d\sigma(t), \quad f(t) \in L_2(d\sigma) \right\}. \tag{3.1}$$

Let $p(\lambda) = p_0\lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ be a polynomial of degree n. Define $p^{\#}(\lambda)$ by

$$p^{\#}(\lambda) := p(\overline{\lambda})^* = \overline{p}_0\lambda^n + \overline{p}_1\lambda^{n-1} + \cdots + \overline{p}_{n-1}\lambda + \overline{p}_n.$$

Theorem 3.2. ([16], [11]). *Any function $m \in N_\kappa$ admits a factorization*

$$m(\lambda) = \frac{p(\lambda)p^{\#}(\lambda)}{q(\lambda)q^{\#}(\lambda)} m_0(\lambda), \tag{3.2}$$

where p and q are uniquely defined coprime monic polynomials and $m_0 \in N$. In this case

$$\max\{\deg(p), \deg(q)\} = \kappa.$$

Let a rational function r be defined by

$$r(\lambda) := \frac{p(\lambda)}{q(\lambda)}, \quad r^{\#}(\lambda) := \frac{p^{\#}(\lambda)}{q^{\#}(\lambda)}. \tag{3.3}$$

Then the formula (3.2) can be rewritten as

$$m(\lambda) = r(\lambda)m_0(\lambda)r^{\#}(\lambda). \tag{3.4}$$

Let p, q be polynomials, let $\kappa = \max\{\deg p, \deg q\}$ and let the coefficients b_{ij} be defined by the expansion

$$\frac{p(x)q(y) - p(y)q(x)}{x - y} = \sum_{i,j=0}^{\kappa-1} b_{ij} x^i y^j.$$

The matrix $B_{p,q} := \big(b_{ij}\big)_{i,j=0}^{\kappa-1}$ is called *the bezutiant* of polynomials p and q (see [23]). The bezutiant $B_{p,q}$ is an invertible matrix, if p and q are coprime polynomials.

Denote the vector-functions

$$\Lambda := \Lambda_\kappa = (1, \lambda, \lambda^2, \dots, \lambda^{\kappa-1}), \quad M := M_\kappa = (1, \mu, \mu^2, \dots, \mu^{\kappa-1}).$$

Let $\mathcal{H}(m)$ be a reproducing kernel Pontryagin space with the reproducing kernel $\mathsf{N}_\mu^m(\lambda)$ where $m \in N_\kappa$. Next we will describe the space $\mathcal{H}(m)$ (compare with [11, Proposition 3.1]).

Theorem 3.3. *Let $m \in N_\kappa$ have the factorization* (3.2). *Then the space $\mathcal{H}(m)$ coincides with the space $\mathcal{H}$ of functions*

$$f(\lambda) = r(\lambda) f_0(\lambda) + \frac{1}{q(\lambda)} \varphi_1(\lambda) + \frac{r(\lambda)}{q^\#(\lambda)} m_0(\lambda) \varphi_2(\lambda), \tag{3.5}$$

where $f_0 \in \mathcal{H}(m_0)$ and φ_1, φ_2 are arbitrary polynomials of formal degree $\kappa - 1$.

Proof. Every function f of the form (3.5) can be represented as

$$f(\lambda) = \begin{bmatrix} r(\lambda) & \frac{1}{q(\lambda)}\Lambda & \frac{r(\lambda)}{q^\#(\lambda)} m_0(\lambda)\Lambda \end{bmatrix} \begin{bmatrix} f_0 \\ f^{(1)} \\ f^{(2)} \end{bmatrix}, \tag{3.6}$$

where $f_0 \in \mathcal{H}(m_0)$ and $f^{(1)}, f^{(2)} \in \mathbb{C}^\kappa$. Let the inner product in $\mathcal{H}$ be defined by

$$\langle f, f \rangle_{\mathcal{H}} = (f_0, f_0)_{\mathcal{H}(m_0)} + \mathcal{F}^* \mathcal{B}^{-1} \mathcal{F}, \tag{3.7}$$

where

$$\mathcal{F} = \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix} \in \mathbb{C}^{2\kappa}, \quad \mathcal{B} = \begin{bmatrix} 0 & B_{p,q} \\ B_{p,q}^* & 0 \end{bmatrix}.$$

In particular, the function $\mathsf{N}_\mu^m(\lambda)$ takes the form for every $\mu \in \mathfrak{h}_m^+$

$$\mathsf{N}_\mu^m(\lambda) = \begin{bmatrix} r(\lambda) & \frac{1}{q(\lambda)}\Lambda & \frac{r(\lambda)}{q^\#(\lambda)} m_0(\lambda)\Lambda \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & B_{pq} \\ 0 & B_{pq}^* & 0 \end{bmatrix} \begin{bmatrix} r^\#(\overline{\mu}) \mathsf{N}_\mu^{m_0}(\lambda) \\ \frac{1}{q^\#(\overline{\mu})} M^* \\ \frac{r^\#(\overline{\mu}) m_0(\overline{\mu})}{q(\overline{\mu})} M^* \end{bmatrix}.$$

Clearly, $\mathcal{H}$ is a Pontryagin space with negative index κ. Moreover, $\mathcal{H}$ is RKPS with the kernel $\mathsf{N}_\mu^m(\lambda)$, since for every function f of the form (3.6) one gets

$$\begin{aligned} \big\langle f(\cdot), \mathsf{N}_\mu^m(\cdot) \big\rangle_{\mathcal{H}} &= \big(f_0(\cdot), \mathsf{N}_\mu^{m_0}(\cdot) r^\#(\overline{\mu})\big)_{\mathcal{H}(m_0)} + \begin{bmatrix} \frac{1}{q(\mu)} M & m_0^*(\overline{\mu}) \frac{r(\mu)}{q^\#(\mu)} M \end{bmatrix} \mathcal{B}^* \mathcal{B}^{-1} \mathcal{F} \\ &= r(\mu) f_0(\mu) + \frac{1}{q(\mu)} \varphi_1(\mu) + m_0(\mu) \frac{r(\mu)}{q^\#(\mu)} \varphi_2(\mu), \end{aligned}$$

where $\varphi_j(\mu) = M f^{(j)}$ $(j = 1, 2)$. This proves the reproducing kernel property for the space $\mathcal{H}$. Therefore, $\mathcal{H}$ coincides with $\mathcal{H}(m)$. $\square$

3.1. Some properties of the space $\mathcal{H}(m)$.

Lemma 3.4. *Let $j \in \mathbb{Z}_+$ then a function $m \in N_\kappa$ satisfies the condition*

$$m(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \frac{s_2}{\lambda^3} - \cdots - \frac{s_{j-1}}{\lambda^j} + O\Big(\frac{1}{\lambda^{j+1}}\Big) \quad (\lambda\widehat{\to}\infty) \tag{3.8}$$

if and only if

$$\lambda^j m(\lambda) + s_0\lambda^{j-1} + s_1\lambda^{j-2} + \cdots + s_{j-1} \in \mathcal{H}(m). \tag{3.9}$$

Proof. A proof of Lemma 3.4 is in Appendix. □

Corollary 3.5. *Let a function $m \in N_\kappa$ satisfies $m(\lambda) = O(1/\lambda)$ for $\lambda\widehat{\to}\infty$. Define a kernel $\mathsf{K}_\mu^m(\lambda)$ by the formula*

$$\mathsf{K}_\mu^m(\lambda) = \frac{\lambda m(\lambda) - \overline{\mu} m(\overline{\mu})}{\lambda - \overline{\mu}} \quad (\lambda, \mu \in \mathfrak{h}_m). \tag{3.10}$$

Then

$$\mathsf{K}_\mu^m(\lambda) \in \mathcal{H}(m). \tag{3.11}$$

Proof. The relation (3.11) is implied by the identity

$$\mathsf{K}_\mu^m(\lambda) = m(\lambda) + \overline{\mu}\mathsf{N}_\mu^m(\lambda). \tag{3.12}$$

The inclusion $m \in \mathcal{H}(m)$ is proved in Lemma 3.4 and the inclusion $\mathsf{N}_\mu^m \in \mathcal{H}(m)$ follows from the definition of RKPS. □

4. Problem AIP_κ associated with $MP_\kappa(\mathbf{s})$

Let $\mathcal{X} = \mathbb{C}[x]$ be the space of polynomials $h(x) = \sum_{j=0}^n h_j x^j (n = 0, 1, 2, \ldots)$. Let a sequence $\mathbf{s} = \{s_j\}_0^\infty \in H_\kappa$ be given. Define a sesquilinear form $K(\cdot, \cdot)$ by the formula (1.5).

Define the operators B_1, B_2 and C_1, C_2 by the equalities (cf. [19], [10])

$$\begin{aligned} &B_1, B_2 : \mathcal{X} \to \mathcal{X}, \qquad B_1 h = \frac{h(x) - h(0)}{x}, \qquad B_2 h = h, \\ &C_1, C_2 : \mathcal{X} \to \mathbb{C}, \qquad C_1 h = \sum_{j=1}^n s_{j-1} h_j, \qquad C_2 h = -h(0). \end{aligned} \tag{4.1}$$

We will show that the data set (B_1, B_2, C_2, C_2, K) satisfies the assumptions (A1)–(A4) but first we state some useful properties.

Proposition 4.1. *The definition* (4.1) *of C_1 can be rewritten as*

$$C_1 h = \widetilde{h}(0), \tag{4.2}$$

where the adjacent polynomial $\widetilde{h}$ is defined by

$$\widetilde{h}(\lambda) = K\left(\frac{h(x) - h(\lambda)}{x - \lambda}, \mathbf{1}\right). \tag{4.3}$$

Proof. Indeed, since

$$\frac{h(x) - h(0)}{x} = \sum_{j=1}^n h_j x^{j-1},$$

then

$$\widetilde{h}(0) = K\Big(\sum_{j=1}^n h_j x^{j-1}, \mathbf{1}\Big) = \sum_{j=1}^n s_{j-1} h_j = C_1 h.$$

□

Proposition 4.2. *Let operators B_1, C_1, C_2 be defined by* (4.1). *Then the following relations hold:*

$$(I-\lambda B_1)^{-1}h = \frac{xh(x)-\lambda h(\lambda)}{x-\lambda} \quad (h \in \mathcal{X}), \tag{4.4}$$

$$C_1(I-\lambda B_1)^{-1}h = \widetilde{h}(\lambda), \quad C_2(I-\lambda B_1)^{-1}h = -h(\lambda), \tag{4.5}$$

where the adjacent polynomial $\widetilde{h}$ is defined by (4.3).

Proof. Let $f \in \mathcal{X}$. Then

$$(I-\lambda B_1)f(x) = f(x) - \lambda\frac{f(x)-f(0)}{x} =: h(x). \tag{4.6}$$

Substituting $x=\lambda$ in equation (4.6), one obtains $f(0)=h(\lambda)$. It follows from (4.6) that

$$f(x) = \frac{xh(x)-\lambda h(\lambda)}{x-\lambda}. \tag{4.7}$$

This proves the equation (4.4).

Formulas (4.5) and (4.1) yield

$$C_2(1-\lambda B_1)^{-1}h = C_2 f = -f(0) = -h(\lambda).$$

Similarly, formulas (4.5) and (4.2) yield

$$\begin{aligned} C_1(1-\lambda B_1)^{-1}h &= C_1 f = \widetilde{f}(0) \\ &= K\Big(\frac{f(x)-f(0)}{x}, \mathbf{1}\Big) = K\Big(\frac{h(x)-h(\lambda)}{x-\lambda}, \mathbf{1}\Big) = \widetilde{h}(\lambda). \end{aligned}$$

□

Proposition 4.3. *The data set (B_1, B_2, C_1, C_2, K) satisfies the assumptions (A1)–(A4).*

Proof. The assumption `(A1)` is checked by straightforward calculations.

`(A2)`. We will prove (A2) by contradiction. Assume that $\ker K \neq 0$. Then there is a polynomial $h(x)$ of degree n such that $K(h,u)=0$ for any polynomial $u \in \mathbb{C}[x]$. Hence the Hankel matrix $D_n := \{s_{j+k}\}_{j,k=0}^n$ is degenerate ($\det D_n = 0$). Since the problem MP_κ (1.3) is solvable, then by [13, Theorem 1.3] the solution to this problem is unique. But this contradicts to the assumption (I). So $\ker K = \{0\}$.

`(A3)`. Let $\mathcal{H}$ be the completion of the space $\mathcal{X}$ endowed with the inner product $K(\cdot,\cdot)$. Let M_0 be a multiplication operator in $\mathcal{X}$ and let the operator M be the closure of M_0 in $\mathcal{H}$. As follows from [22, Proposition 1.1] the operator M is an entire π-symmetric operator in $\mathcal{H}$ with the scale $\mathcal{L}=\mathbb{C}$.

Let $\widetilde{B}_1$ be the closure of the graph of B_1

$$\widetilde{B}_1^{-1} = \{\{h, Mh+u\} : h \in \operatorname{dom} M,\ u \in \mathbb{C}\}.$$

Since the operator M is entire with the scale $\mathcal{L}=\mathbb{C}$, then $0 \in \rho(M,\mathbb{C})$ and

$$\operatorname{ran}\widetilde{B}_1^{-1} = \operatorname{ran} M \dot{+} \mathbb{C} = \mathcal{H}, \quad \ker \widetilde{B}_1^{-1} = \operatorname{ran} M \cap \mathbb{C} = \{0\}.$$

Hence $\widetilde{B}_1$ is the graph of a bounded operator in $\mathcal{H}$.

The boundedness of the operator C_1 follows from the already proved boundedness of the operator B_1 and Proposition 4.1. Indeed

$$C_1 h = K(\frac{h(x)-h(0)}{x}, \mathbf{1}) = \langle B_1 h, \mathbf{1}\rangle_{\mathcal{H}}.$$

The operator C_2 is bounded since $0 \in \rho(M,\mathbb{C})$ and

$$C_2 h = -\mathcal{P}_{M,\mathbb{C}}(0)h,$$

where $\mathcal{P}_{M,\mathbb{C}}(0)$ is a skew projection on the space $\mathbb{C}$ in the decomposition $\mathcal{H} = \operatorname{ran} M \dot{+} \mathbb{C}$.

(A4). Now we show that the data set (B_1, B_2, C_1, C_2) satisfies the condition (A4). It is sufficient to show that the next formula holds for different points $\lambda_j \in \mathbb{C}_+$ $(j = 1, \dots, \kappa)$

$$\operatorname{ran} \begin{bmatrix} C_2 \\ C_2(1 - \overline{\lambda}_1 B_1)^{-1} \\ \vdots \\ C_2(1 - \overline{\lambda}_\kappa B_1)^{-1} \end{bmatrix} = \mathbb{C}^{\kappa+1}. \tag{4.8}$$

As follows from the relation (4.5)

$$\begin{bmatrix} C_2 \\ C_2(1 - \overline{\lambda}_1 B_1)^{-1} \\ \vdots \\ C_2(1 - \overline{\lambda}_\kappa B_1)^{-1} \end{bmatrix} f = \begin{bmatrix} -f(0) \\ -f(\overline{\lambda}_1) \\ \vdots \\ -f(\overline{\lambda}_\kappa) \end{bmatrix}.$$

Since the polynomial $f(x)$ is arbitrary we have gotten condition (4.8). □

The abstract interpolation problem AIP_κ associated with $MP_\kappa(\mathbf{s})$ can be formulated as follows.

Problem $AIP_\kappa(B_1, B_2, C_1, C_2, K)$. Let a sequence of real numbers $\mathbf{s} = \{s_j\}_{j=0}^\infty$ belong to H_κ and let the property (I) hold. Let the data set B_1, B_2, C_1, C_2, K be defined by (4.1) and let F be defined by (2.5). Find an N_κ function $m(\lambda)$, such that

(C1) $Fh \in \mathcal{H}(m)$ for any $h \in \mathcal{X}$;
(C2) $\langle Fh, Fh\rangle_{\mathcal{H}(m)} \le K(h, h)$ for any $h \in \mathcal{X}$.

Proposition 4.4. *Let $\mathbf{s} = \{s_j\}_{j=0}^\infty \in H_\kappa$ and let the data set (B_1, B_2, C_1, C_2, K) be defined by (4.1). Then the function m defined by (2.4) belongs to N_κ.*

Proof. Let $\varphi \in \widetilde{N} = N \cup \{\infty\}$ and let $m \in N_{\kappa'}$ ($\kappa' \le \kappa$, the case $\kappa' > \kappa$ is impossible) be defined by (2.4). By [25, Theorems 4.13, 4.14] there is a selfadjoint extension $\widetilde{A}$ of the linear relation A (in (2.7)), such that

$$(Fh)(\lambda) = P_{\mathcal{L}}(\widetilde{A} - \lambda)^{-1}h \in \mathcal{H}(m) \quad (h \in \mathcal{X})$$

and the mapping $F : \mathcal{X} \to \mathcal{H}(m)$ satisfies the identity $(C1)$. On the other hand by Remark 2.2 the mapping F takes the form (2.5), where $G(\lambda)$ is defined by (2.6)

$$(Fh)(\lambda) = \begin{bmatrix} \mathbf{1} & -m(\lambda) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (\mathbf{1} - \lambda B_1)^{-1}h \quad (h \in \mathcal{X}).$$

It follows from (4.1) and (4.4) that for $h = x^j$

$$G(\lambda)h = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (x^j + x^{j-1}\lambda + \dots + \lambda^j) = \begin{bmatrix} s_0\lambda^{j-1} + s_1\lambda^{j-2} + \dots + s_{j-1} \\ -\lambda^j \end{bmatrix},$$

therefore

$$(Fh)(\lambda) = \lambda^j m(\lambda) + s_0\lambda^{j-1} + s_1\lambda^{j-2} + \dots + s_{j-1} \in \mathcal{H}(m) \quad (h = x^j). \tag{4.9}$$

It follows from Lemma 3.4 that the function m satisfies the decomposition (1.3). So from [22, Proposition 1.3] one obtains, that $m \in N_{\kappa'}$ ($\kappa' \ge \kappa$). Therefore, m belongs to the class N_κ. □

5. Equivalence of problems AIP_κ and $MP_\kappa(\mathbf{s})$

Theorem 5.1. *Let $\{s_j\} \in H_\kappa$, let the data set (B_1, B_2, C_1, C_2, K) be defined by (4.1). Then a function m is a solution of the problem $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ if and only if the function m is a solution of the problem $MP_\kappa(\mathbf{s})$.*

Proof. Necessity. Let $m \in N_\kappa$ be a solution of $AIP_\kappa(B_1, B_2, C_1, C_2, K)$. It follows from Remark 2.2 that the mapping F corresponding to the function m is unique. This mapping takes the form (2.5) and satisfies (4.9). It follows from Lemma 3.4 that the function m has the asymptotic expansion (3.8) for every $j \in \mathbb{Z}_+$ and hence the function m is a solution of $MP_\kappa(\mathbf{s})$.

Sufficiency. Let $m \in N_\kappa$ be a solution of $MP_\kappa(\mathbf{s})$. Define a mapping F by the formula (2.5).

Step 1. *Let us show that F satisfies* (C1). One gets from (4.9) and Lemma 3.4 that

$$Fh \in \mathcal{H}(m) \quad (h = x^j, \quad j \in \mathbb{Z}_+),$$

hence $Fh \in \mathcal{H}(m)$ for every $h \in \mathcal{X}$.

Step 2. *Let us show that the F satisfies* (C2). Let $\widetilde{A}$ be a selfadjoint operator in $\mathcal{H}(m)$ (see [24])

$$\widetilde{A} = \left\{\{f,\, f'\} \in \mathcal{H}(\varphi, \psi)^2 : f'(\lambda) - \lambda f(\lambda) \equiv \text{const} \in \mathbb{C}\right\}. \tag{5.1}$$

Define the functions

$$f_j(\lambda) := \lambda^j m(\lambda) + \lambda^{j-1} s_0 + \lambda^{j-2} s_1 + \cdots + s_{j-1} \quad (j = 0, 1, \ldots).$$

If follows from Lemma 3.4 that $f_j \in \mathcal{H}(m)$ $(j = 0, 1, \ldots)$. Also the functions f_j for $j = 0, 1, \ldots$ satisfy the relation

$$f_{j+1}(\lambda) - \lambda f_j(\lambda) \equiv s_j = \text{const}.$$

So $\{f_j, f_{j+1}\} \in \widetilde{A}$ that is

$$\widetilde{A} f_j = f_{j+1}. \tag{5.2}$$

It follows from definition of RKPS $\mathcal{H}(m)$ and Corollary 3.5 that

$$\mathsf{N}^m_{\bar\mu}(\cdot) \in \mathcal{H}(m) \quad \text{and} \quad \mathsf{K}^m_{\bar\mu}(\cdot) \in \mathcal{H}(m).$$

Moreover, it follows from the identity (3.12) that

$$\left\{\mathsf{N}^m_{\bar\mu}(\cdot),\, \mathsf{K}^m_{\bar\mu}(\cdot)\right\} = \left\{\frac{m(\lambda) - m(\mu)}{\lambda - \mu}, \lambda \frac{m(\lambda) - m(\mu)}{\lambda - \mu} + m(\mu)\right\} \in \widetilde{A}$$

and hence

$$\left\{\mathsf{N}^m_{\bar\mu}(\cdot),\, m(\cdot)\right\} = \left\{\mathsf{N}^m_{\bar\mu}(\cdot), \mathsf{K}^m_{\bar\mu}(\cdot) - \mu \mathsf{N}^m_{\bar\mu}(\cdot)\right\} \in \widetilde{A} - \mu.$$

This implies

$$(\widetilde{A} - \mu)^{-1} m(\cdot) = (\widetilde{A} - \mu)^{-1} f_0(\cdot) = \mathsf{N}^m_{\bar\mu}(\cdot).$$

Then by the reproducing kernel property in $\mathcal{H}(m)$

$$\left\langle (\widetilde{A} - \mu)^{-1} f_0(\cdot), f_0(\cdot)\right\rangle_{\mathcal{H}(m)} = \left\langle \mathsf{N}^m_{\bar\mu}(\cdot), m(\cdot)\right\rangle_{\mathcal{H}(m)} = m(\mu). \tag{5.3}$$

Since m admits the asymptotic expansion (1.3) then $f_0 \in \operatorname{dom}(\widetilde{A}^j)$ for every $j \in \mathbb{N}$ (see [20, Satz 1.10], see also [12]) and

$$\langle \widetilde{A}^j f_0, f_0\rangle_{\mathcal{H}(m)} = s_j \quad (j = 0, 1, \ldots). \tag{5.4}$$

Since $\widetilde{A}^j f_0 = f_j$ then

$$\langle f_j, f_i\rangle_{\mathcal{H}(m)} = \langle \widetilde{A}^{i+j} f_0, f_0\rangle_{\mathcal{H}(m)} = s_{i+j} \quad (i, j = 0, 1, 2, \ldots). \tag{5.5}$$

Hence one obtains for every $h=\sum_{j=0}^n h_j x^j \in \mathcal{X}$ that $Fh=\sum_{j=0}^n h_j f_j$ and by (5.5)

$$\begin{aligned}\langle Fh, Fh\rangle_{\mathcal{H}(m)} &= \Big\langle \sum_{j=0}^n h_j f_j, \sum_{j=0}^n h_j f_j \Big\rangle_{\mathcal{H}(m)} \\ &= \sum_{i,j=0}^n \langle h_i f_i, h_j f_j\rangle_{\mathcal{H}(m)} = \sum_{i,j=0}^n h_i \overline{h}_j s_{i+j} = K(h,h).\end{aligned}$$

□

Remark 5.2. Note that in the course of the proof of Theorem 5.1 we have shown the Parseval equality

$$\langle Fh,\, Fh\rangle_{\mathcal{H}(m)} = K(h,h) \quad (\forall h \in \mathcal{X}). \tag{5.6}$$

In the case $\kappa=0$ the equality (5.6) was proved in [19].

6. Description of solutions of MP_κ

We need the notion of biorthogonal bases to determine the matrix function $\Theta(\lambda)\in\mathbb{C}^{2\times 2}$ defined by (2.3). The bases $\{f_k\}_{k=0}^\infty$ and $\{g_k\}_{k=0}^\infty$ in $\mathbb{C}[x]$ are called *biorthogonal* with respect to the form $K(\cdot,\cdot)$, if

$$K(f_j, g_k) = \delta_{jk} \quad (j,k=0,1,\dots),$$

where $\delta_{jk}=0$ for $k\neq j$ and $\delta_{jj}=1$ $(k,j=0,1,\dots)$. If the form $K(\cdot,\cdot)$ is nonnegative then any orthonormal basis of polynomials is biorthogonal to itself. Construction of biorthogonal bases in general will be given below.

Proposition 6.1. *Let the bases $\{f_k\}_{k=0}^\infty$ and $\{g_k\}_{k=0}^\infty$ be biorthogonal with respect to the form $K(\cdot,\cdot)$. Then the following relations hold for $u\in\mathbb{C}$:*

$$C_1^* u = \sum_{k=0}^\infty f_k(\cdot)\widetilde{g}_k(0)^* u, \quad C_2^* u = -\sum_{k=0}^\infty f_k(\cdot) g_k(0)^* u. \tag{6.1}$$

Proof. Indeed, if follows from the expansion with basis $\{f_k\}$ of functions C_1^* and C_2^* and the formulas (4.2), (4.1) that

$$\begin{aligned}(C_1^*1)(\lambda) &= \sum_{k=0}^\infty f_k(\lambda)\big\langle C_1^*1, g_k(\cdot)\big\rangle_{\mathcal{H}} = \sum_{k=0}^\infty f_k(\lambda)\big(1, C_1 g_k(\cdot)\big)_{\mathbb{C}} = \sum_{k=0}^\infty f_k(\lambda)\widetilde{g}_k(0)^*,\\ (C_2^*1)(\lambda) &= \sum_{k=0}^\infty f_k(\lambda)\big\langle C_2^*1, g_k(\cdot)\big\rangle_{\mathcal{H}} = \sum_{k=0}^\infty f_k(\lambda)\big(1, C_2 g_k(\cdot)\big)_{\mathbb{C}} = -\sum_{k=0}^\infty f_k(\lambda) g_k(0)^*.\end{aligned}$$

□

6.1. **Proof of Theorem 1.1.** Let the matrix valued function $\Theta(\lambda)$ be defined by (2.3). It follows from (4.5) and (6.1) that

$$\begin{aligned}\theta_{11}(\lambda) &= 1+\lambda C_1(I-\lambda B_1)^{-1}\sum_{k=0}^\infty f_k(\lambda)g_k(0)^* = 1+\lambda\sum_{k=0}^\infty \widetilde{f}_k(\lambda)g_k(0)^*,\\ \theta_{12}(\lambda) &= \lambda C_1(I-\lambda B_1)^{-1}\sum_{k=0}^\infty f_k(\lambda)\widetilde{g}_k(0)^* = \lambda\sum_{k=0}^\infty \widetilde{f}_k(\lambda)\widetilde{g}_k(0)^*,\\ \theta_{21}(\lambda) &= \lambda C_2(I-\lambda B_1)^{-1}\sum_{k=0}^\infty f_k(\cdot)g_k(0)^* = -\lambda\sum_{k=0}^\infty f_k(\lambda)g_k(0)^*,\\ \theta_{22}(\lambda) &= 1+\lambda C_2(I-\lambda B_1)^{-1}\sum_{k=0}^\infty f_k(\lambda)\widetilde{g}_k(0)^* = 1-\lambda\sum_{k=0}^\infty f_k(\lambda)\widetilde{g}_k(0)^*,\end{aligned} \tag{6.2}$$

and hence $\Theta(\lambda)$ coincides with the matrix valued function in (1.8).

Let m be a solution of $MP_\kappa(\mathbf{s})$. Then by Theorem 5.1 the function m is a solution of $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ and hence it admits the representation (2.4) with $\varphi \in \widetilde{N}$.

Conversely let $\varphi \in \widetilde{N}$. Then the function m defined by (2.4) belongs to N_κ by Proposition 4.4. Hence, by Theorem 2.1 the function m is a solution of $AIP_\kappa(B_1, B_2, C_1, C_2, K)$ and by Theorem 5.1 the function m is also a solution of $MP_\kappa(\mathbf{s})$. □

In the case of $\kappa = 0$ the formulas (6.2) are identical to those in [1].

6.2. **Description of solutions of $MP_\kappa(\mathbf{s})$ in the form of Krein-Langer.** The set of polynomials $\{g_k\}_{k=0}^\infty$ is called an *almost-orthogonal system* ([21, §7.1], see also [22, §3.1]) with respect to the form $K(\cdot,\cdot)$, if for each g_k there exists $g_{k'}$ with the properties

(i) $K(g_k, g_j) = 0$ for all j ($j \neq k'$);
(ii) $K(g_k, g_{k'}) = \pm 1$.

Put $\varepsilon_k := K(g_k, g_{k'})$ for $k = 0, 1, \dots$

It follows from [21, Behauptung 7.1]) that there exists an almost-orthogonal system $\{g_k\}_{k=0}^\infty$ to the form $K(\cdot,\cdot)$ such that $g_0 \equiv 1$ and g_k is a real polynomial of degree k. Therefore the basis $\{g_k\}_{k=0}^\infty$ and $\{\varepsilon_{k'} g_{k'}\}_{k=0}^\infty$ are biorthogonal with respect to the form $K(\cdot,\cdot)$. So one obtains from the formula (1.8) that

$$\theta_{11}(\lambda) = 1 + \lambda \sum_{k=0}^\infty \varepsilon_k \widetilde{g}_k(\lambda) g_{k'}(0), \qquad \theta_{12}(\lambda) = \lambda \sum_{k=0}^\infty \varepsilon_k \widetilde{g}_k(\lambda) \widetilde{g}_{k'}(0)$$
$$\theta_{21}(\lambda) = -\lambda \sum_{k=0}^\infty \varepsilon_k g_k(\lambda) g_{k'}(0), \qquad \theta_{22}(\lambda) = 1 - \lambda \sum_{k=0}^\infty \varepsilon_k g_k(\lambda) \widetilde{g}_{k'}(0),$$

where $\{\widetilde{g}_k(\lambda)\}_{k=0}^\infty$ is the system of adjacent polynomial defined by the formula (1.7).

This description of solution of the problem $MP_\kappa(\mathbf{s})$ coincides with [22, Theorem 1.4] (see also [21, Satz 7.5]).

6.3. **Description of solutions of $MP_\kappa(\mathbf{s})$ in the form of Derevyagin-Derkach.** Another description of solutions of the problem $MP_\kappa(\mathbf{s})$ was given in [8]. This description also can be derived from Theorem 1.1 at the expense of a choice of another biorthogonal system $\{f_k\}_{k=0}^\infty$ and $\{g_k\}_{k=0}^\infty$.

It follows from $\{s_j\}_0^\infty \in H_\kappa$ that there exists a number M such that $\kappa = \nu_-(D_M) = \nu_-(D_{M+1}) = \nu_-(D_{M+2}) = \cdots$, where $\nu_-(D_n)$ is a number of negative eigenvalues of the Hankel matrix $D_n := \{s_{j+k}\}_{j,k=0}^{n-1}$. An index n is called *normal*, if $\det D_n \neq 0$. Let $n_0 = 0$ and $n_1 < n_2 < \cdots$ be a sequence of all normal indices of $\{s_j\}_{j=0}^\infty$. Let $k_j = n_{j+1} - n_j$ ($j \in \mathbb{Z}_+$). Define the polynomials P_{n_j} by

$$P_{n_j}(\lambda) = c_j \det \begin{bmatrix} s_0 & s_1 & \dots & s_{n_j} \\ s_1 & s_2 & \dots & s_{n_j - 1} \\ \vdots & & \ddots & \vdots \\ 1 & \lambda & \dots & \lambda^{n_j} \end{bmatrix} \quad (j \in \mathbb{Z}_+), \tag{6.3}$$

where normalizing coefficients c_j are determined by the conditions

$$K\big(P_{n_j}(\lambda), \lambda^{k_j - 1} P_{n_j}(\lambda)\big) = \varepsilon_j, \quad |\varepsilon_j| = 1 \quad (j \in \mathbb{Z}_+).$$

Next define missing polynomials $P_n(\lambda)$ ($n \neq n_j$) by

$$P_{n_j + k}(\lambda) = \lambda^k P_{n_j}(\lambda), \quad k = 1, 2, \dots, k_j - 1 \quad (j \in \mathbb{Z}_+).$$

Then the Gram matrix

$$G = (G_{jk})_{j,k=0}^\infty, \quad G_{jk} = K(P_j, P_k)$$

of the system $\{P_j\}_{j=0}^{\infty}$ takes the block matrix from (see [7])

$$G = \operatorname{diag}(G^{(0)}, G^{(1)}, \ldots),$$

where G^j are an $n_{j-1} \times n_{j-1}$ nondegenerate matrices and $G_j = 1$ for $j \geq M$. As follows from [7] the space $\mathcal{H}$ is isometrically isomorphic to the space $\big(\mathbf{l}_2, \langle G\cdot, \cdot\rangle_{\mathbf{l}_2}\big)$ via the mapping

$$V : P_j \mapsto e_j \quad (j \in \mathbb{N} \cup \{0\}),$$

where $\{e_j\}_{j=1}^{\infty}$ is the standard basis in space $\mathbf{l}_2$.

Let $\pi(\lambda) = \big(P_0(\lambda), P_1(\lambda), \ldots\big)$. It follows from condition (I) that $\pi(\lambda) \in \mathbf{l}_2$. Let

$$f_k(\lambda) = P_k(\lambda), \quad g_k(\lambda) = \big(G^{-1}\pi(\lambda), e_k\big)_{\mathbf{l}_2} \quad (k \in \mathbb{Z}_+).$$

The system $\{g_k\}_{k=0}^{\infty}$ is biorthogonal to $\{f_k\}_{k=0}^{\infty}$ with respect to the form $K(\cdot,\cdot)$ since

$$K(f_j, g_k) = (GVf_j, Vg_k)_{\mathbf{l}_2} = (Ge_j, G^{-1}e_k)_{\mathbf{l}_2} = (e_j, e_k)_{\mathbf{l}_2} = \delta_{jk}.$$

Proposition 6.2. *Let a polynomial P_{n_j} $(j \in \mathbb{Z}_+)$ be defined by* (6.3) *and let Q_{n_j} be the adjacent polynomial*

$$Q_{n_j}(\lambda) := \widetilde{P}_{n_j}(\lambda) = K\Big(\frac{P_{n_j}(\lambda) - P_{n_j}(t)}{\lambda - t}, 1\Big).$$

Let

$$Q_{n_j+k}(\lambda) := \lambda^k P_{n_j}(\lambda) \quad (k = 1, 2, \ldots, k_j - 1).$$

Then

$$Q_{n_j+k}(\lambda) = \widetilde{P}_{n_j+k}(\lambda) \quad (k = 1, 2, \ldots, k_j - 1). \tag{6.4}$$

Proof. For every $j \in \mathbb{N}$, $k = 1, 2, \ldots, k_j - 1$ one obtains

$$\begin{aligned} \widetilde{P}_{n_j+k}(\lambda) &= K\Big(\frac{\lambda^k P_{n_j}(\lambda) - t^k P_{n_j}(t)}{\lambda - t}, 1\Big) \\ &= \lambda^k K\Big(\frac{P_{n_j}(\lambda) - P_{n_j}(t)}{\lambda - t}, 1\Big) + K\Big(P_{n_j}(t), \frac{\overline{\lambda}^{\,k} - t^k}{\overline{\lambda} - t}\Big). \end{aligned} \tag{6.5}$$

Now (6.4) follows from (6.5) since the latter term is equal to 0. $\square$

So the formula (1.8) coincides with the result [8, Corollary 3.17] for this choice of biorthogonal systems $\{f_k\}_{k=0}^{\infty}$ and $\{g_k\}_{k=0}^{\infty}$.

Appendix A. Proof of Lemma 3.4

We will need an auxiliary result proved in [10, Lemma 2.12] in a more general case.

Lemma A.1. ([10]). *Let $m_0 \in N$, then*

(i) *$f(\lambda) = O(1)$ $(\lambda \widehat{\to} \infty)$ for all $f \in \mathcal{H}(m_0)$;*
(ii) *If, additionally, $m_0 \in N^0$ then $f(\lambda) = O(\frac{1}{\lambda})$ $(\lambda \widehat{\to} \infty)$ for all $f \in \mathcal{H}(m_0)$.*

A.1. **Proof of Lemma 3.4.** ***Sufficiency.*** Assume that the function $m \in N_\kappa$ satisfies (3.9). Let us show that the condition (3.8) holds. Let polynomials p, q and a function $m_0 \in N$ be determined by the factorization (3.2) for the function $m \in N_\kappa$.

Case 1. Let $\kappa = n = \deg q > \deg p = k$. Then $r(\lambda) = O(1/\lambda)$ as $\lambda \widehat{\to} \infty$. Since $m_0 \in N$, then

$$m_0(\lambda) = O(\lambda) \quad (\lambda \widehat{\to} \infty). \tag{A.1}$$

It follows from Lemma A.1 that

$$f_0(\lambda) = O(1) \quad (\lambda \widehat{\to} \infty) \tag{A.2}$$

for all $f_0 \in \mathcal{H}(m_0)$. One obtains from the formula (3.5) that for all $f \in \mathcal{H}(m)$

$$f(\lambda) = O(1/\lambda) \quad (\lambda \widehat{\to} \infty). \tag{A.3}$$

So it follows from (3.9) that

$$m(\lambda) + \frac{s_0}{\lambda} + \frac{s_1}{\lambda^2} + \cdots + \frac{s_{j-1}}{\lambda^j} = O\Big(\frac{1}{\lambda^{j+1}}\Big) \quad (\lambda \widehat{\to} \infty).$$

Case 2. Let $\kappa = \deg q = \deg p$. Then $r(\lambda) = O(1/\lambda)$ as $\lambda \widehat{\to} \infty$. The rate of growth at infinity of $m_0(\lambda)$ is given by (A.1) and (A.2). Then it follows from (3.5) for all $f \in \mathcal{H}(m)$ that

$$f(\lambda) = O(1) \quad (\lambda \widehat{\to} \infty). \tag{A.4}$$

One obtains from (3.9) that

$$m(\lambda) = O(1/\lambda) \quad (\lambda \widehat{\to} \infty).$$

Therefore, it follows from the assumption $\deg q = \deg p$ and the factorization (3.2) that

$$m_0(\lambda) = O(1/\lambda) \quad (\lambda \widehat{\to} \infty). \tag{A.5}$$

One obtains from Lemma A.1

$$f_0(\lambda) = O(1/\lambda) \quad (\lambda \widehat{\to} \infty) \tag{A.6}$$

for all $f_0 \in \mathcal{H}(m_0)$. Now (3.5), (A.5) and (A.6) yields (A.3). The end of the proof is similar to that in *Case 1.*

Case 3. Let $n = \deg q < \deg p = k = \kappa$. Then $r(\lambda) = O(\lambda^{k-n})$ as $\lambda \widehat{\to} \infty$. It follows from (3.5) and relations (A.1), (A.2) that for all $f \in \mathcal{H}(m)$

$$f(\lambda) = O(\lambda^{2k-2n}) \quad (\lambda \widehat{\to} \infty). \tag{A.7}$$

One obtains from (3.9) that

$$m(\lambda) = O(\lambda^{2k-2n-1}) \quad (\lambda \widehat{\to} \infty).$$

Therefore the relation (A.5) follows from the factorization (3.2). By Lemma A.1 one obtains the relation (A.6). Again the relations (3.5), (A.5) and (A.6) yield

$$f(\lambda) = O(\lambda^{2k-2n-2}) \quad (\lambda \widehat{\to} \infty).$$

Now one obtains from (3.9) that

$$m(\lambda) = O(\lambda^{2k-2n-2}) \quad (\lambda \widehat{\to} \infty).$$

So it follows from the factorization (3.2) that

$$m_0(\lambda) = O(\lambda^{-2}) \quad (\lambda \widehat{\to} \infty). \tag{A.8}$$

The relation (A.8) contradicts the condition $m_0 \in N$. Therefore, the inequality $\deg q < \deg p$ never occurs for function m, which satisfies the condition (3.9). $\square$

A.2. **Some auxiliary statements.** We start with some algebraic statements concerning formal power series

$$p(\lambda) = p_0\lambda^n + p_1\lambda^{n-1} + \cdots \tag{A.9}$$

and the corresponding matrices $T_k(p) \in \mathbb{C}^{k\times k}$ of their k leading coefficients

$$T_k(p) := \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & p_{k-1} \\ 0 & p_0 & p_1 & \cdots & p_{k-2} \\ 0 & 0 & p_0 & \cdots & p_{k-3} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_0 \end{bmatrix}. \tag{A.10}$$

Proposition A.2. *Let p and q be formal series with a leading degree k and n, respectively*

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + \cdots, \quad q(\lambda) = q_0\lambda^n + q_1\lambda^{n-1} + \cdots \quad (p_0, q_0 \neq 0).$$

Let $T_j(p)$ and $T_j(q)$ be matrices of j leading coefficients of series p and q. Then

$$T_j(pq) = T_j(p)T_j(q). \tag{A.11}$$

Proof. The product $T_j(p)T_j(q)$ is equal to

(A.12)

$$T_j(p)T_j(q) = \begin{bmatrix} p_0q_0 & p_0q_1 + p_1q_0 & p_0q_2 + p_1q_1 + p_2q_0 & \cdots & p_0q_{j-1} + p_1q_{j-2} + \cdots + p_{j-1}q_0 \\ 0 & p_0q_0 & p_0q_1 + p_1q_0 & \cdots & p_0q_{j-2} + p_1q_{j-3} + \cdots + p_{j-2}q_0 \\ 0 & 0 & p_0q_0 & \cdots & p_0q_{j-3} + p_1q_{j-4} + \cdots + p_{j-3}q_0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_0q_0 \end{bmatrix}.$$

The product of series $p(\lambda)q(\lambda)$ takes the form

$$\begin{aligned} p(\lambda)q(\lambda) &= p_0q_0\lambda^{k+n} + (p_0q_1 + p_1q_0)\lambda^{k+n-1} + (p_0q_2 + p_1q_1 + p_2q_0)\lambda^{k+n-2} \\ &\quad + \cdots + (p_0q_{j-1} + p_1q_{j-2} + p_{j-2}q_1 + p_{j-1}q_0)\lambda^{k+n-j+1} + o(\lambda^{k+n-j+1}). \end{aligned} \tag{A.13}$$

Therefore, the formula (A.11) holds. □

Remark A.3. Since the products $p(\lambda)q(\lambda)$ and $q(\lambda)p(\lambda)$ of formal series coincide, then the matrices $T_j(p)$ and $T_j(q)$ commute

$$T_j(p)T_j(q) = T_j(q)T_j(p).$$

Corollary A.4. *Let $q(\lambda) = q_0\lambda^n + q_1\lambda^{n-1} + \cdots$ be a formal series ($q_0 \neq 0$). Then j leading coefficients of series*

$$\frac{1}{q(\lambda)} = \frac{q_0'}{\lambda^n} + \frac{q_1'}{\lambda^{n+1}} + \frac{q_2'}{\lambda^{n+2}} + \cdots$$

generate a matrix $T_j(1/q)$, which is connected with $T_j(q)$ by $T_j\big(1/q\big) = \big(T_j(q)\big)^{-1}$.

Proposition A.5. *Let $c(\lambda) = \lambda^j + c_1\lambda^{j-1} + cdots + c_j$ be a polynomial of a degree j. Let a function $m_0 \in N^0$ have an integral representation* (2.10) *and admit the expansion in a series* (2.11)*. Then the function*

$$d(\lambda) = \int_{-\infty}^{\infty} \frac{c(t) - c(\lambda)}{t - \lambda}\, d\sigma(t)$$

is a polynomial in λ of degree $j-1$ and

$$T_j(d) = T_j(c)T_j(s^0), \tag{A.14}$$

where $T_j(c)$ and $T_j(s^0)$ are matrices of j leading coefficients of polynomials $c(\lambda)$ and

$$s^0(\lambda) = s_0^0\lambda^{j-1} + s_1^0\lambda^{j-2} + \cdots + s_j^0.$$

Proof. Denote $c_0 = 1$. Then the function $d(\lambda)$ take the form

(A.15)

$$\begin{aligned} d(\lambda) &= \int_{-\infty}^{\infty} \Big(\frac{t^j - \lambda^j}{t - \lambda}c_0 + \frac{t^{j-1} - \lambda^{j-1}}{t - \lambda}c_1 + \cdots + \frac{t - \lambda}{t - \lambda}c_{j-1}\Big)\, d\sigma(t) \\ &= \int_{-\infty}^{\infty} \big(c_0(t^{j-1} + t^{j-2}\lambda + \cdots + \lambda^{j-1}) + c_1(t^{j-2} + \cdots + \lambda^{j-2}) + \cdots + c_{j-1}\big) d\sigma(t). \end{aligned}$$

It follows from equations (2.12) and (A.15) that

$$\begin{aligned} d(\lambda) &= c_0(s^0_{j-1} + s^0_{j-2}\lambda + \cdots + \lambda^{j-1}) + c_1(s^0_{j-2} + s^0_{j-3}\lambda + \cdots + \lambda^{j-2}) + \cdots + c_{j-1}s^0_0 \\ &= c_0 s_0 \lambda^{j-1} + (c_0 s^0_1 + c_1 s^0_0)\lambda^{j-2} + \cdots + (c_0 s^0_{j-1} + c_1 s^0_{j-2} + \cdots + c_{j-1}s^0_0). \end{aligned}$$

In view of (A.12) this proves (A.14). □

A.3. **Proof of Lemma 3.4.** ***Necessity.*** Let a function $m \in N_\kappa$ satisfy the condition (3.8). Define a polynomial $s(\lambda)$ of a formal degree $j-1$ by the formula

$$s(\lambda) := s_0\lambda^{j-1} + s_1\lambda^{j-2} + \cdots + s_{j-1}. \tag{A.16}$$

Let p and q be monic polynomials of degrees k and n, respectively

$$\begin{aligned} p(\lambda) &= \lambda^k + p_1\lambda^{k-1} + \cdots + p_{k-1}\lambda + p_k, \\ q(\lambda) &= \lambda^n + q_1\lambda^{n-1} + \cdots + q_{n-1}\lambda + q_n, \end{aligned} \tag{A.17}$$

and let a function $m_0 \in N$ be defined by the factorization (3.2).

Define by $\mathcal{P}_n$ the set of polynomials of a formal degree n.

Case 1: $\kappa = k = \deg p = \deg q = n$**.**

It follows from the condition (3.8) that $m_0 = O(1/\lambda)$, moreover the function m_0 belongs to N^0 and admits the integral representation (2.10).

Let c be some monic polynomial of degree j ($c_i \in \mathbb{C}$ for $i = 1, 2, \ldots, j$)

$$c(\lambda) := \lambda^j + c_1\lambda^{j-1} + c_2\lambda^{j-2} + \cdots + c_j. \tag{A.18}$$

Consider the next decomposition of $\lambda^j m(\lambda) + s(\lambda)$

$$\lambda^j m(\lambda) + s(\lambda) = \frac{p(\lambda)}{q(\lambda)} f(\lambda) + \frac{p(\lambda)}{q(\lambda)q^\#(\lambda)} m_0(\lambda)\varphi_2(\lambda) + \frac{1}{q(\lambda)}\varphi_1(\lambda), \tag{A.19}$$

where

$$f(\lambda) := \int_{-\infty}^{\infty} \frac{c(t)}{t-\lambda}\, d\sigma(t), \tag{A.20}$$

$$\varphi_1(\lambda) := q(\lambda)s(\lambda) + p(\lambda)m_0(\lambda)c(\lambda) - p(\lambda)\int_{-\infty}^{\infty} \frac{c(t)}{t-\lambda}\, d\sigma(t), \tag{A.21}$$

$$\varphi_2(\lambda) := p^\#(\lambda)\lambda^j - q^\#(\lambda)c(\lambda). \tag{A.22}$$

We will show that there is a choice of c such that

$$f(\cdot) \in \mathcal{H}(m_0), \quad \varphi_1(\cdot), \varphi_2(\cdot) \in \mathcal{P}_{\kappa-1}. \tag{A.23}$$

Then it will imply by Theorem 3.3 that the relation (3.9) holds.

The inclusion $f(\cdot) \in \mathcal{H}(m_0)$ follows from Remark 2.4 and Theorem 3.1.

Let $P_i = T_i(p)$, $Q_i = T_i(q)$, $S_i = T_i(s)$, $C_i = T_i(c)$, $S^0_i = T_i(s^0)$ be matrices of i leading coefficients of the polynomials $p(\lambda)$, $q(\lambda)$, $s(\lambda)$, $c(\lambda)$, $s^0(\lambda)$, respectively, and let $\overline{P}_i$, $\overline{Q}_i$ be matrices with complex adjoint elements of the matrices P_i and Q_i. Note that $\overline{P}_i = T_i(p^\#)$, $\overline{Q}_i = T_i(q^\#)$.

The function φ_2 is a polynomial of formal degree $\kappa + j$. Define a polynomial $c(\lambda)$ so that $\varphi_2(\cdot) \in \mathcal{P}_{\kappa-1}$. This condition is equivalent to

$$\overline{P}_i - \overline{Q}_i C_i = 0 \quad (i = 1, 2, \ldots, j+1)$$

so

$$C_i = (\overline{Q}_i)^{-1}\overline{P}_i = \overline{P}_i(\overline{Q}_i)^{-1} \quad (i = 1, 2, \ldots, j+1). \tag{A.24}$$

Note that the matrix $\overline{Q}_{j+1}$ is invertible and the coefficients $c_1, c_2, \ldots, c_j$ are uniquely defined.

Next the inclusion $\varphi_1(\cdot) \in \mathcal{P}_{\kappa-1}$ is showed. It follows from the factorization (3.2) that

$$-S_j = P_j \overline{P}_j Q_j^{-1} (\overline{Q}_j)^{-1} (-S_j^0). \tag{A.25}$$

By the integral representation (2.10)

$$\varphi_1(\lambda) = q(\lambda)s(\lambda) - p(\lambda) \int_{-\infty}^{\infty} \frac{c(t) - c(\lambda)}{t - \lambda} \, d\sigma(t).$$

It follows from Proposition A.5 that $\varphi_1(\lambda)$ is a polynomial of formal degree $\kappa + j - 1$ and

$$T_j(\varphi_1) = Q_j S_j - P_j C_j S_j^0. \tag{A.26}$$

Using the formula (A.24) for $i = j$ and the relation (A.25) one obtains

$$T_j(\varphi_1) = Q_j P_j \overline{P}_j Q_j^{-1} (\overline{Q}_j)^{-1} S_j^0 - P_j (\overline{Q}_j)^{-1} \overline{P}_j S_j^0 = 0.$$

So the relation (A.23) is held.

Case 2: $k = \deg p < \deg q = n = \kappa$**.** Denote $d := n - k (> 0)$. It follows from the decomposition (2.8) for the function $m_0 \in N$ that

$$m_0(\lambda) = a\lambda + b + m_{00}(\lambda). \tag{A.27}$$

One obtains from the factorization (3.2) for $m \in N_\kappa$ and the conditions (1.3) that $m_{00}(\lambda) = O(1/\lambda)$ so $m_{00} \in N^0$. Therefore the function m_{00} admits the integral representation similar (2.10).

Let the coefficient a in (A.27) not equal 0 so $m_0(\lambda) = O(\lambda)$ for $\lambda \widehat{\to} \infty$ (cases when $m_0(\lambda) = O(1)$ and $m_0(\lambda) = m_{00}(\lambda) = O(1/\lambda)$ for $\lambda \widehat{\to} \infty$ researched by analogical). It follows from the factorization (3.2) that $m(\lambda) = O(\frac{1}{\lambda^{2d-1}})$ for $\lambda \widehat{\to} \infty$ and

$$s_0 = s_1 = \cdots = s_{2d-3} = 0, \quad s_{2d-2} \neq 0 \tag{A.28}$$

($s_0 = \cdots = s_{2d-2} = 0$, $s_{2d-1} \neq 0$ for $a = 0$; $s_0 = \cdots = s_{2d-1} = 0$, $s_{2d} \neq 0$ for $a = b = 0$). Moreover the polynomial $s(\lambda)$ defined by (A.16) has the degree $j - 2d + 1$

$$s(\lambda) = s_{2d-2}\lambda^{j-2d+1} + s_{2d-1}\lambda^{j-2d} + \cdots + s_{j-1}.$$

Let $j \geq 2d - 1$. Consider the decomposition (A.19) of the function $\lambda^j m(\lambda) + s(\lambda)$. The difference from the considered case $d = 0$ is next: the function $m_0 \in N$ admits the representation (A.27); the measure $d\sigma$ is defined from the integral representation (2.10) for the function m_{00}; the polynomial c has a degree $j - d$

$$c(\lambda) := \lambda^{j-d} + c_1\lambda^{j-d-1} + c_2\lambda^{j-d-2} + \cdots + c_{j-d}. \tag{A.29}$$

We will show that conditions (A.23) are right too.

It follows from Remark 2.4 and Theorem 3.1 that the including $f(\cdot) \in \mathcal{H}(m_0)$. The condition $\varphi_2(\cdot) \in \mathcal{P}_{\kappa-1}$ is equivalent to (A.24) for $i = j - d + 1$.

Next the inclusion $\varphi_1(\cdot) \in \mathcal{P}_{\kappa-1}$ is showed. It follows from the representations (A.27) and (2.10) that

$$\varphi_1(\lambda) = q(\lambda)s(\lambda) + p(\lambda)c(\lambda)(a\lambda + b) - p(\lambda) \int_{-\infty}^{\infty} \frac{c(t) - c(\lambda)}{t - \lambda} \, d\sigma(t). \tag{A.30}$$

Let

$$A_{j-2d+2} = T_{j-2d+2}(a\lambda + b) := \begin{bmatrix} a & b & 0 & \cdots & 0 \\ 0 & a & b & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix}$$

(respectively $A = bI$ where I is the identity matrix for $a = 0$, $b \neq 0$ and $A = 0$ for $a = b = 0$). Define the polynomial s^0 with formal degree $\deg s^0 = j - d + 2$

$$s^0(\lambda) := 0 \cdot \lambda^{j-d+2} + 0 \cdot \lambda^{j-d+1} + s_0^0\lambda^{j-d} + s_1^0\lambda^{j-d-1} + \cdots + s_{j-d}^0,$$

where $\{s_i^0\}_{i=0}^{j-d}$ is coefficients from decomposition (2.11) for $m_{00} \in N^0$. We have added to polynomial s^0 zero-summands for the degrees of summands of φ_1 are coincidence. Let

$$S^0_{j-2d+2} = T_{j-2d+2}(s^0) := \begin{bmatrix} 0 & 0 & s_0^0 & s_1^0 & \cdots & s^0_{j-2d-1} \\ 0 & 0 & 0 & s_0^0 & \cdots & s^0_{j-2d-2} \\ 0 & 0 & 0 & 0 & \cdots & s^0_{j-2d-3} \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It follows from Proposition A.5 that φ_1 is a polynomial of formal degree $j+n-2d+1$ and

$$T_{j-2d+2}(\varphi_1) = Q_{j-2d+2}S_{j-2d+2} + P_{j-2d+2}C_{j-2d+2}A_{j-2d+2} - P_{j-2d+2}C_{j-2d+2}S^0_{j-2d+2}.$$

It follows from the factorization (3.2) and the representation (A.27) that

$$-S = P\overline{P}Q^{-1}(\overline{Q})^{-1}(A - S^0), \tag{A.31}$$

where all matrices have the size $(j-2d+2)\times(j-2d+2)$.

One obtains from the formula (A.24) for $i = j-2d+2$ and the relation (A.30) that

$$T(\varphi_1) = QP\overline{P}Q^{-1}(\overline{Q})^{-1}(-A+S^0) + P\overline{P}(\overline{Q})^{-1}A - P\overline{P}(\overline{Q})^{-1}S^0 = 0.$$

So $\varphi_1 \in \mathcal{P}_{n-1}$ and the relation (A.23) is held.

Let $d \le j < 2d-1$. One obtains from the conditions (A.28) that $s(\lambda) \equiv 0$. Consider the decomposition kind of (A.19) for the function $\lambda^j m(\lambda)$ where the polynomial c is defined by formula (A.29).

The inclusions $f(\cdot) \in \mathcal{H}(m_0)$ and $\varphi_2(\cdot) \in \mathcal{P}_{\kappa-1}$ are proved similarly. Consider the function

$$\varphi_1(\lambda) = p(\lambda)c(\lambda)(a\lambda+b) - p(\lambda)\int_{-\infty}^{\infty} \frac{c(t)-c(\lambda)}{t-\lambda}\,d\sigma(t).$$

One obtains from Proposition A.5 and condition $j < 2d-1$ that φ_1 is polynomial and

$$\deg \varphi_1 = j+k-d+1 < d+k = n = \kappa.$$

So $\varphi_1 \in \mathcal{P}_{n-1}$ and the relation (A.23) holds.

Let $j < d$. One obtains from the conditions (A.28) that $s(\lambda) \equiv 0$. Consider the degree of the polynomial $p(\lambda)\lambda^j$

$$\deg\bigl(p(\lambda)\lambda^j\bigr) = k+j < k+d = n = \kappa.$$

It follows from Theorem 3.3 that $\lambda^j m(\lambda) \in \mathcal{H}(m)$.

Case 3: $\kappa = \deg p > \deg q$ is impossible from the condition (3.8). □

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