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ℓ^1-MUNN IDEAL AMENABILITY OF CERTAIN SEMIGROUP ALGEBRAS

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Abstract. In this paper we investigate ideal amenability of $\ell^1(G_p)$, where G_p is a maximal subgroup of inverse semigroup S with uniformly locally finite idempotent. Also we find some conditions for ideal amenability of Rees matrix semigroup.

1. Introduction

The notion of amenability for ℓ^1-Munn Banach algebra was studied by G. H. Esslamzadeh in [4]. In [6] he characterized various types of ideals in ℓ^1-Munn Banach algebras.

A Banach algebra $\mathcal{A}$ is ideally amenable if for every closed ideal I of $\mathcal{A}$, the first cohomology group of $\mathcal{A}$ with coefficients in I^* is trivial. In [9] M. E. Gorji and T. Yazdanpanah have characterized some properties of ideally amenable Banach algebras.

In this paper we study ideal amenability in the category of ℓ^1-Munn Banach algebra. We show that for an inverse semigroup S with uniformly locally finite idempotent, the ideal amenability of $\ell^1(S)$ implies ideal amenability of $\ell^1(G_p)$ where G_p is a maximal subgroup of S. In a particular case, we characterized ideal amenability of $\ell^1(S)$ for a Rees matrix semigroup S.

2. Preliminaries

Throughout we use notations of [4]. Let $\mathcal{A}$ be a Banach algebra. I and J be arbitrary index sets and P be $J\times I$ matrix over $\mathcal{A}$ such that $\|P\|_\infty = \sup\{\|P_{ji}\| : j\in J, i\in I\}\le 1$. The set $\mathcal{LM}(\mathcal{A},P)$ of all $I\times J$ matrices A over $\mathcal{A}$ such that $\|A\|_1=\sum_{i\in I, j\in J}\|A_{ij}\|<\infty$ with ℓ^1-norm and product $A\circ B = APB, A, B\in\mathcal{LM}(\mathcal{A},P)$ is a Banach algebra that we call the ℓ^1-Munn $I\times J$ matrix algebra over $\mathcal{A}$ with sandwich matrix P or briefly ℓ^1-Munn algebra. When $\mathcal{A}$ is unital and P is the identity $J\times J$ matrix over $\mathcal{A}, \mathcal{LM}(\mathcal{A},P)$ is the algebra $\mathcal{M}_J(\mathcal{A})$.

Let G be a group, I and J be arbitrary nonempty sets and $G^0 = G\cup\{0\}$ be the group with zero arising from G by adjunction of a zero element 0. An $I\times J$ matrix A over G^0 that has at most one nonzero entry $a = A(i,j)$ is called a Rees $I\times J$ matrix over G^0 and is denoted by $a\epsilon_{ij}$. Let P be a $J\times I$ matrix over G. The set $S = G\times I\times J$ with the composition $(a,i,j)\circ(b,l,k) = (aP_{jl}b,i,k), (a,i,j),(b,k,l)\in S$ is a semigroup that we denote by $\mathcal{M}(G,P)$. Similarly if P is a $J\times I$ matrix over G^0, then $S = G\times I\times J\cup\{0\}$ is a semigroup under the following composition operation which is denoted by $\mathcal{M}^0(G,P)$:

$$(a,i,j)\circ(b,l,k)=\begin{cases}(aP_{jl}b,i,k), & P_{jl}\neq 0\\ 0, & P_{jl}=0\end{cases},$$

$$(a,i,j)\circ 0 = 0\circ(a,i,j) = 0\circ 0 = 0.$$

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$\mathcal{M}^0(G,P)$ is isomorphic to the semigroup of all Rees $I \times J$ matrices over G^0 with binary operation $A \circ B = APB$. $\mathcal{M}^0(G,P)$ [resp. $\mathcal{M}(G,P)$] is called the Rees $I \times J$ matrix semigroup over G^0 [resp. G] with the sandwich matrix P.

For a Banach algebra $\mathcal{A}$ let X be a Banach $\mathcal{A}$-bimodule. A derivation from $\mathcal{A}$ into X is a continuous linear operator D such that

$$D(ab) = a.D(b) + D(a).b \quad (a, b \in \mathcal{A}).$$

We define a derivation $\delta_x(a) = a.x - x.a$ for each $x \in X$ and $a \in \mathcal{A}$ from $\mathcal{A}$ into X, which is called an inner derivation. Let J be a closed ideal of $\mathcal{A}$, then $\mathcal{A}$ is said to be J-weakly amenable if every derivation from $\mathcal{A}$ into J^* is inner. We call $\mathcal{A}$ ideally amenable if $\mathcal{A}$ is J-weakly amenable for every closed ideal J of $\mathcal{A}$.

Let I be a non-empty set. We denote by $M_I(\mathcal{A})$, the set $I \times I$ matrices (a_{ij}) with entries in $\mathcal{A}$ such that

$$\|(a_{ij})\| = \sum_{i.j \in I} \|a_{ij}\| < \infty.$$

Then $M_I(\mathcal{A})$ with the usual matrix multiplication is a Banach algebra that belongs to the class of ℓ^1-Munn algebras. It is an easy verification that the map $\theta : M_I(\mathcal{A}) \to M_I(\mathbb{C})\hat{\otimes}\mathcal{A}$ defined by

$$\theta((a_{ij})) = \sum_{i,j \in I} E_{ij} \otimes a_{ij} \quad ((a_{ij})) \in M_I(\mathcal{A}))$$

is an isometric isomorphism of Banach algebras, where (E_{ij}) are the matrix units in $M_I(\mathbb{C})$.

Let $\{\mathcal{A}_\alpha : \alpha \in I\}$ be a collection of Banach algebras. Then the ℓ^1-direct sum of $\{\mathcal{A}_\alpha\}$ is denoted by

$$\ell^1 - \oplus\{\mathcal{A}_\alpha : \alpha \in I\},$$

which is a Banach algebra with componentwise operations.

Now we give some definitions and properties of semigroups for further details, see [10]. Let S be a semigroup and $E(S) = \{p \in S : p^2 = p\}$. We say that S is a semilattice if S is commutative and $E(S) = S$. The canonical partial order on $E(S)$ is given by

$$s \leq t \ \Leftrightarrow \ s = st = ts \quad (s, t \in E(S)). \tag{2.1}$$

The semigroup S is an inverse semigroup if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. By [10, Proposition 5.2.1], for any inverse semigroup S, there is a partial order on S defined by

$$s \leq t \ \Leftrightarrow \ s = ss^*t \quad (s, t \in S). \tag{2.2}$$

It is easily verified that the partial order given in (2.2) coincides with that given in (2.1) on $E(S)$.

If $(S, \leq)$ is a partially order set, we set $(x] = \{y \in S : y \leq x\}$. The partially ordered set $(S, \leq)$ is called locally finite if $(x]$ is finite for every $x \in S$ and is called uniformly locally finite if $\sup\{|(x]| : x \in S\} < \infty$.

Notation: Let S be an inverse semigroup and let $p \in E(S)$. The maximal subgroup of S at p is denoted by G_p. It is easily checked that $G_p = \{s \in S : ss^* = s^*s = p\}$.

3. ℓ^1-Munn ideal amenability of certain semigroup algebras

In this section, we investigate ideal amenability of Banach algebra $\ell^1(S)$ for an inverse semigroup S with uniformly locally finite idempotent set. We find some conditions for ideal amenability of $\ell^1(G)$ in Brandt semigroup algebra.

Lemma 3.1. *Let A be a Banach algebra and I be a non-empty set. If $\mathcal{A} = \mathcal{M}_I(A)$ is ideally amenable, then A is ideally amenable.*

Proof. Suppose B is a closed ideal of A and $d : A \to B^*$ be a continuous derivation. Define $D : \mathcal{A} \to \mathcal{B}^*$ by setting $D(a)_{ij} = (d(a_{ji}))$, where $\mathcal{B} = M_I(B)$. By [1, Theorem 2.7], D is a continuous derivation. Since $\mathcal{A}$ is ideally amenable, there exists $\Lambda = (\lambda_{ij}) \in \mathcal{B}^*$ such that

$$D(a) = a.\Lambda - \Lambda.a \quad (a \in \mathcal{A}).$$

Take $a \in A$ and identify a with the matrix that has a in the (1,1)-th position and 0 elsewhere. Then $\lambda_{1,1} \in B^*$ and

$$d(a) = D(a)_{1,1} = (a.\lambda - \lambda.a)_{1,1} = a.\lambda_{1,1} - \lambda_{1,1}.a \quad (a \in A).$$

So A is ideally amenable. □

Theorem 3.2. *Let S be an inverse semigroup such that $(E(S), \leq)$ is uniformly locally finite. If $\ell^1(S)$ is ideally amenable, then for each maximal subgroup G_p of S, $\ell^1(G_p)$ is ideally amenable.*

Proof. Suppose $\ell^1(S)$ is ideally amenable. Since $(E(S), \leq)$ is uniformly locally finite, by [13, Proposition 2.14], $(S, \leq)$ is such. Thus we have

$$\ell^1(S) \cong \ell^1 - \oplus\{M_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\}$$

as Banach algebras. Hence for each $\lambda \in \Lambda$, $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is a homomorphic image of $\ell^1(S)$. Now by [3, Theorem 4.1], $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is ideally amenable.

So by Lemma 3.1, $\ell^1(G_{p_\lambda})$ is ideally amenable. □

Let G be a group and let I be a non-empty set. Set

$$\mathcal{M}^0(G, I) = \{(g)_{ij} : g \in G, i, j \in I\} \cup \{0\},$$

where $(g)_{ij}$ denotes the $I \times I-$matrix with entry $g \in G$ in the (i, j) position and zero elsewhere. Then $\mathcal{M}^0(G, I)$ with the multiplication given by

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il}, & j = k \\ 0, & j \neq k \end{cases} \qquad (g, h, i, j, k, l \in I)$$

is an inverse semigroup with $(g)^*_{ij} = (g^{-1})_{ji}$, that is called the Brandt semigroup over G with index set I.

Corollary 3.3. *Let G be a group, I be a non-empty set and let $S = \mathcal{M}^0(G, I)$ be the Brandt semigroup over G with index set I. If $\ell^1(S)$ is ideally amenable, then $\ell^1(G)$ is ideally amenable.*

In the following we assume $\mathcal{LM}(\mathcal{A}, P)$ has a bounded approximate identity $\{E^\gamma\}_{\gamma\in\Gamma}$. Consequently $\mathcal{A}$ has a bounded approximate identity $\{e^\gamma\}_{\gamma\in\Gamma}$, the index sets I and J are finite and P is invertible [4, Lemma 3.7].

Theorem 3.4. *Let $\mathcal{A}$ be an ideally amenable Banach algebra. If I' is a closed ideal of $\mathcal{LM}(\mathcal{A}, P)$ with a bounded approximate identity, then $\mathcal{LM}(\mathcal{A}, P)$ is I'-weakly amenable.*

Proof. Since I' is a closed ideal of $\mathcal{LM}(\mathcal{A}, P)$ then by [6 ,Theorem 3.1 (i)], there is a closed ideal I of $\mathcal{A}$ such that $I' = \mathcal{LM}(I, P)$, also by [(6), Theorem 3.1 (ii)], I has a bounded approximate identity. On the other hand $\mathcal{A}$ is ideally amenable , so by [3, Theorem 2.6], I is weakly amenable and by [15, Theorem 2.3] I' is weakly amenable. Therefore by [3, Theorem 2.6] $\mathcal{LM}(\mathcal{A}, P)$ is I'-weakly amenable. □

Corollary 3.5. *Let S be a Rees matrix semigroup and I is a closed ideal of $\ell^1(S)$ with a bounded approximate identity. If $\ell^1(G)$ is ideally amenable, then $\ell^1(S)$ is I-weakly amenable.*

Proof. Suppose $S = \mathcal{M}^0(G, P)$. Then by [4, Proposition 5.6], $\dfrac{\ell^1(S)}{\ell^1(0)}$ is isomorphic to $\mathcal{LM}(\ell^1(G), P)$. Since $\ell^1(G)$ is ideally amenable then by Theorem 3.4, $\mathcal{LM}(\ell^1(G), P)$ is I-weakly amenable. So $\dfrac{\ell^1(S)}{\ell^1(0)}$ is I-weakly amenable, and hence by [3, Theorem 4.4], $\ell^1(S)$ is I-weakly amenable. □

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