Methods of Functional Analysis and Topology
Vol. 19 (2013), no. 2, pp. 191–196

DECOMPOSITION OF A UNITARY SCALAR OPERATOR INTO A PRODUCT OF ROOTS OF THE IDENTITY

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Abstract. We prove that for all $m_1, m_2, m_3 \in \mathbb{N}$, $\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3} \le 1$, every unitary scalar operator γI on a complex infinite-dimensional Hilbert space is a product $\gamma I = U_1U_2U_3$ where U_i is a unitary operator such that $U_i^{m_i} = I$.

1. Introduction

Let H be a complex Hilbert space, for $i = \overline{1, n}$, let A_i be a self-adjoint operator with finite spectrum $\sigma(A_i)$. Let I denote the identity operator on H. Consider the following equation:

$$A_1 + A_2 + \cdots + A_n = \lambda I, \quad \lambda \in \mathbb{C}. \tag{1}$$

In [6], [5] and related works the following problems were studied.

1) Describe the set of all possible values of λ if $\sigma(A_i)$ are given.

2) Classify unitary nonequivalent tuples of operators $(A_i)_{i=1}^n$ that satisfy equation (1) if λ and $\sigma(A_i)$ are given.

In this work we continue to study the multiplicative analog of the mentioned problems. It was known that every unitary operator on an infinite-dimensional Hilbert space H is a product of four symmetries (see [2]), that is,

$$\forall U \in Uni(H)\ \exists U_i \in Uni(H) :\ U = U_1U_2U_3U_4, \quad U_i^2 = I,$$

(here $Uni(H)$ denotes the set of all unitary operators on H), and every $U \in Uni(H)$ is a product of three n-th roots of the identity if $n \ge 3$ (see [3]), that is,

$$\forall n \ge 3\ \forall U \in Uni(H)\ \exists U_i \in Uni(H) :\ U = U_1U_2U_3, \quad U_i^n = I.$$

In recent papers we have proved that if $\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3} \le 1$ and two numbers from $\{m_1, m_2, m_3\}$ are even then every unitary scalar operator is a product of three m_i-th roots of I (see [9]), moreover, every unitary operator is a product of three m_i-th roots (see [10]). In the present paper using a technique different from [9] and [10] we prove the existence of decomposition of a scalar unitary operator without the condition on parity of m_i (see Theorem 1).

2. Statements and proofs

The main result of this work is the following theorem.

Theorem 1. *For all $m_1, m_2, m_3 \in \mathbb{N}$, $\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3} \le 1$, every unitary scalar operator γI on a complex infinite-dimensional Hilbert space is a product $\gamma I = U_1U_2U_3$ where U_i is a unitary operator such that $U_i^{m_i} = I$.*

2010 *Mathematics Subject Classification.* Primary 47A62; Secondary 20F55, 20F10.

Key words and phrases. Hilbert space, unitary operator, group representation, string rewriting.

Proof. From now on we suppose that $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \leq 1$ so we will omit this condition in the statements.

Consider the following central extension of an ordinary triangle group

$$E = \langle x, y, z, q | x^{m_1} = y^{m_2} = z^{m_3} = 1, xyz = q, xq = qx, yq = qy, zq = qz \rangle.$$

This group is infinite since an ordinary triangle group is infinite [7].

It is clear that if $\gamma I = U_1U_2U_3$, $U_i^{m_i} = I$ on H then $\pi : E \to H$ with $\pi(q) = \gamma I$, $\pi(x) = U_1$, $\pi(y) = U_2$, $\pi(z) = U_3$ is a unitary representation of the group E. Also, every irreducible unitary representation of E gives us a solution to $\gamma I = U_1U_2U_3$, $U_i^{m_i} = I$.

If every γI is a product of three m_i-th roots of I then the element q from E is an element of infinite order. Our proof of Theorem 1 consists of two steps: to prove that the element q is an element of infinite order in E and to deduce Theorem 1 from it. Let us start with the easiest one.

Proposition 1. *If q is an element of infinite order in E then for every $\gamma \in \mathbb{C}, |\gamma| = 1$, the operator γI on the infinite-dimensional Hilbert space H is the product $\gamma I = U_1U_2U_3$ where $U_iU_i^* = I$, $U_i^{m_i} = I$.*

Proof. Since q is an element of infinite order in E, the normal subgroup $Q = \langle q \rangle$ of E is an infinite cyclic group and E/Q is an ordinary triangle group. Let $\gamma \in \mathbb{C}, |\gamma| = 1$ and $\pi(q) = \gamma$ be a 1-dimensional unitary representation of Q. Consider the induced unitary representation $\tau = Ind_Q^E \pi$ that acts on an infinite-dimensional Hilbert space H since E/Q is infinite. The element q belongs to the center of E, therefore $\tau(q) = \gamma I$. We have $\tau(q) = \tau(x)\tau(y)\tau(z)$ hence $\gamma I = \tau(x)\tau(y)\tau(z)$ gives the decomposition we are looking for. $\square$

Proposition 2. *The element q is an element of infinite order in E.*

Proof. To prove this fact we need the notion of a finite complete rewriting system [1], [4]. Let $A = \{a_1, \dots, a_n\}$ be a finite set. The set A is called an alphabet and the elements of A are called letters. The ordered sets of letters from A are called words. The empty word is denoted by 1. The set of all words including the empty word with the concatenation operation form the free monoid A^*. Given a word $W \in A^*$, we will denote its length by $|W|$, defined as the numbers of letters in W.

A rewriting system R over A is a set of rules $U \to V$, $U, V \in A^*$, that is, $R \subset A^* \times A^*$. A word $W_1 \in A^*$ is said to be rewritten to another word $W_2 \in A^*$ by a one-step reduction induced by R, if $W_1 = Z_1XZ_2, W_2 = Z_1YZ_2$ for some rule $X \to Y$ in R. In this situation we write $W_1 \to_R W_2$. The reflexive transitive closure and the reflexive symmetric transitive closure of $\to_R$ are denoted by $\to_R^*$ and $\leftrightarrow_R^*$, respectively. The relation $\leftrightarrow_R^*$ is defined to be a congruence on A^* generated by R.

Let $Left(R) = \{X \in A^* : X \to Y \in R\}$ and $Irr(R) = A^* \setminus A^*Left(R)A^*$. That is, $Irr(R)$ is the set of all words from A^* that can not be reduced by any rule from R. A word $W \in A^*$ is called an irreducible word if $W \in Irr(R)$. From now on we suppose that $1 \notin Left(R)$ hence $1 \in Irr(R)$.

We say R is Noetherian if there is no infinite reduction sequence

$$W_1 \to_R W_2 \to_R W_3 \to_R \cdots .$$

System R is said to be confluent if whenever $U \to_R^* V$ and $U \to_R^* W$, then there is an $X \in A^*$ such that $V \to_R^* X$ and $W \to_R^* X$. If R is both Noetherian and confluent, we say that R is a complete rewriting system. R is a finite complete rewriting system if additionally R is a finite set.

The following fact is well known.

Proposition 3. *Suppose R is a complete rewriting system for A. Then for each $W \in A^*$ there is a unique $W' \in Irr(R)$ such that $W \to_R^* W'$. The word W' is denoted by $irr(W)$.*

It is clear that if R is a complete rewriting system and $W \in A^*$, then to find $irr(W)$ we just need to apply rules from R to W in an arbitrary order till we stop. If R is a finite system then this algorithm is computable and it computes $irr(W)$ in a finite number of steps.

Finite complete rewriting systems make a useful tool in solving word problem for groups. Suppose we have a finitely presented group

$$G \;=\; \langle A = \{a_1, \dots, a_n\} \mid S = \{s_1, \dots, s_m\} \subset A^{\pm}\rangle,$$

here A is a set of generators, $A^{\pm}$ is the set of all words from the alphabet $\{a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}\}$ and S is a set of relations, that is $S \subset A^{\pm}$. Words from S determine relations $s_i = 1$ in G.

For a word $W \in A^{\pm}$ we denote by W^{-1} the word constructed from W by reversing the order of the letters and changing the sign of every letter to the opposite. The word W^k denotes the concatenation of k words W. Note that words from $A^{\pm}$ with the concatenation and the inverse operation form a free group F_n.

Proposition 4. *Let R be a finite complete rewriting system on $A^{\pm}$ and suppose that*
1) $1 \in Irr(R)$;
2) rules $a_i a_i^{-1} \to 1, a_i^{-1} a_i \to 1$ belong to R, $i = \overline{1,n}$;
3) if $U \to V \in R$ then $U = V$ in group G;
4) if $U \in S$ then $U \to_R^* 1$, $U^{-1} \to_R^* 1$.
Then $W = 1$ in G if and only if $W \to_R^* 1$ (that is $irr(W) = 1$).

Such a system R is called a finite complete rewriting system for a group G. To check if $W = 1$ in G we just need to find $irr(W)$ using R.

Let's go back to our group E. Note that E has equal presentation

$$E \;=\; \langle x, y, q | x^{m_1} = y^{m_2} = 1, (xy)^{m_3} = q^{m_3}, xq = qx, yq = qy\rangle,$$

which can be obtained using Tits transformations.

For simplicity we construct a finite complete rewriting system for the subgroup E' generated by $\langle x, y, q^{m_3}\rangle$ and which has the presentation

$$E' \;=\; \langle x, y, q | x^{m_1} = y^{m_2} = 1, (xy)^{m_3} = q, xq = qx, yq = qy\rangle.$$

It is clear that $|q| = \infty$ in E if and only if $|q| = \infty$ in E'.

Our finite complete rewriting system R for E' depends on parity of the numbers m_1, m_2, m_3.

For simplicity we denote the letter x^{-1} as X and the letter y^{-1} as Y.

Finite complete rewriting system for E' (alphabet $\{x, y, q, X, Y, q^{-1}\}$):

Case 1: $(m_1, m_2, m_3) \equiv (0, 0, 0) \bmod 2$
Rules:
$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1,$
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY,$
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y,$
$x^{\frac{m_1}{2}+1} \to X^{\frac{m_1}{2}-1},\ X^{\frac{m_1}{2}} \to x^{\frac{m_1}{2}},$
$y^{\frac{m_2}{2}+1} \to Y^{\frac{m_2}{2}-1},\ Y^{\frac{m_1}{2}} \to y^{\frac{m_1}{2}},$
$(xy)^{\frac{m_3}{2}} x \to q(YX)^{\frac{m_3}{2}-1}Y,\ (yx)^{\frac{m_3}{2}} y \to q(XY)^{\frac{m_3}{2}-1}X,$
$(YX)^{\frac{m_3}{2}} \to q^{-1}(xy)^{\frac{m_3}{2}},\ (XY)^{\frac{m_3}{2}} \to q^{-1}(yx)^{\frac{m_3}{2}},$
$(YX)^{\frac{m_3}{2}-1} Y x^{\frac{m_1}{2}} \to q^{-1}(xy)^{\frac{m_3}{2}} X^{\frac{m_1}{2}-1},$
$X^{\frac{m_1}{2}-1}(yx)^{\frac{m_3}{2}} \to q^{-1}x^{\frac{m_1}{2}}(YX)^{\frac{m_3}{2}-1}Y,$

$(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2}{2}} \to q^{-1}(yx)^{\frac{m_3}{2}}Y^{\frac{m_2}{2}-1}$,
$Y^{\frac{m_1}{2}-1}(xy)^{\frac{m_3}{2}} \to qy^{\frac{m_2}{2}}(XY)^{\frac{m_3}{2}-1}X\}$.

Case 2: $(m_1, m_2, m_3) \equiv (0, 0, 1) \bmod 2$

Rules:

$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1,$
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY,$
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y,$
$x^{\frac{m_1}{2}+1} \to X^{\frac{m_1}{2}-1},\ X^{\frac{m_1}{2}} \to x^{\frac{m_1}{2}},$
$y^{\frac{m_2}{2}+1} \to Y^{\frac{m_2}{2}-1},\ Y^{\frac{m_1}{2}} \to y^{\frac{m_1}{2}},$
$(xy)^{\frac{m_3+1}{2}} \to q(YX)^{\frac{m_3-1}{2}},\ (yx)^{\frac{m_3+1}{2}} \to q(XY)^{\frac{m_3-1}{2}},$
$(YX)^{\frac{m_3-1}{2}}Y \to q^{-1}(xy)^{\frac{m_3-1}{2}}x,\ (XY)^{\frac{m_3-1}{2}}X \to q^{-1}(yx)^{\frac{m_3-1}{2}}y,$
$X^{\frac{m_1}{2}-1}y(xy)^{\frac{m_3-1}{2}} \to q^{-1}x^{\frac{m_1}{2}}(YX)^{\frac{m_3-1}{2}},$
$(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1}{2}} \to q^{-1}(yx)^{\frac{m_3-1}{2}}yX^{\frac{m_1}{2}-1},$
$Y^{\frac{m_1}{2}-1}x(yx)^{\frac{m_3-1}{2}} \to qy^{\frac{m_2}{2}}(XY)^{\frac{m_3-1}{2}},$
$(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2}{2}} \to q^{-1}(xy)^{\frac{m_3-1}{2}}xY^{\frac{m_2}{2}-1}\}.$

Case 3: $(m_1, m_2, m_3) \equiv (1, 1, 0) \bmod 2$

Rules:

$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1,$
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY,$
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y,$
$x^{\frac{m_1+1}{2}} \to X^{\frac{m_1-1}{2}},\ X^{\frac{m_1+1}{2}} \to x^{\frac{m_1-1}{2}},$
$y^{\frac{m_2+1}{2}} \to Y^{\frac{m_2-1}{2}},\ Y^{\frac{m_1+1}{2}} \to y^{\frac{m_1-1}{2}},$
$(xy)^{\frac{m_3}{2}}x \to q(YX)^{\frac{m_3}{2}-1}Y,\ (yx)^{\frac{m_3}{2}}y \to q(XY)^{\frac{m_3}{2}-1}X,$
$(YX)^{\frac{m_3}{2}} \to q^{-1}(xy)^{\frac{m_3}{2}},\ (XY)^{\frac{m_3}{2}} \to q^{-1}(yx)^{\frac{m_3}{2}},$
$(xy)^{\frac{m_3}{2}}X^{\frac{m_1-1}{2}} \to q(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}},\ X^{\frac{m_1-1}{2}}(yx)^{\frac{m_3}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3}{2}-1}Y,$
$(yx)^{\frac{m_3}{2}}Y^{\frac{m_2-1}{2}} \to q(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}},\ Y^{\frac{m_2-1}{2}}(xy)^{\frac{m_3}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3}{2}-1}X,$
$X^{\frac{m_1-1}{2}}y(xy)^{\frac{m_3}{2}-1}X^{\frac{m_1-1}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}},$
$(xy)^{\frac{m_3}{2}}X^{\frac{m_1-3}{2}}(yx)^{\frac{m_3}{2}} \to q^2(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1},$
$(YX)^{\frac{m_3}{2}-1}Yx^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1}x^{\frac{m_1-1}{2}} \to q^{-2}(xy)^{\frac{m_3}{2}}X^{\frac{m_1-3}{2}}y(xy)^{\frac{m_3}{2}-1}X^{\frac{m_1-1}{2}},$
$X^{\frac{m_1-1}{2}}y(xy)^{\frac{m_3}{2}-1}X^{\frac{m_1-3}{2}}(yx)^{\frac{m_3}{2}} \to q^2x^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1}x^{\frac{m_1-1}{2}}Y(XY)^{\frac{m_3}{2}-1},$
$Y^{\frac{m_2-1}{2}}x(yx)^{\frac{m_3}{2}-1}Y^{\frac{m_2-1}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}},$
$(yx)^{\frac{m_3}{2}}Y^{\frac{m_2-3}{2}}(xy)^{\frac{m_3}{2}} \to q^2(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1},$
$(XY)^{\frac{m_3}{2}-1}Xy^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1}y^{\frac{m_2-1}{2}} \to q^{-2}(yx)^{\frac{m_3}{2}}Y^{\frac{m_2-3}{2}}x(yx)^{\frac{m_3}{2}-1}Y^{\frac{m_2-1}{2}},$
$Y^{\frac{m_2-1}{2}}x(yx)^{\frac{m_3}{2}-1}Y^{\frac{m_2-3}{2}}(xy)^{\frac{m_3}{2}} \to q^2y^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1}y^{\frac{m_2-1}{2}}X(YX)^{\frac{m_3}{2}-1}.$

Case 4: $(m_1, m_2, m_3) \equiv (1, 1, 1) \bmod 2$

Rules:

$\{xX \to 1,\ Xx \to 1,\ yY \to 1,\ Yy \to 1,\ qq^{-1} \to 1,\ q^{-1}q \to 1,$
$xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY,$
$xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y,$
$x^{\frac{m_1+1}{2}} \to X^{\frac{m_1-1}{2}},\ X^{\frac{m_1+1}{2}} \to x^{\frac{m_1-1}{2}},$
$y^{\frac{m_2+1}{2}} \to Y^{\frac{m_2-1}{2}},\ Y^{\frac{m_1+1}{2}} \to y^{\frac{m_1-1}{2}},$
$(xy)^{\frac{m_3+1}{2}} \to q(YX)^{\frac{m_3-1}{2}},\ (yx)^{\frac{m_3+1}{2}} \to q(XY)^{\frac{m_3-1}{2}},$
$(YX)^{\frac{m_3-1}{2}}Y \to q^{-1}(xy)^{\frac{m_3-1}{2}}x,\ (XY)^{\frac{m_3-1}{2}}X \to q^{-1}(yx)^{\frac{m_3-1}{2}}y,$
$X^{\frac{m_1-1}{2}}y(xy)^{\frac{m_3-1}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}},\ (yx)^{\frac{m_3-1}{2}}yX^{\frac{m_1-1}{2}} \to q(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}},$
$Y^{\frac{m_2-1}{2}}x(yx)^{\frac{m_3-1}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}},\ (xy)^{\frac{m_3-1}{2}}xY^{\frac{m_2-1}{2}} \to q(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}},$
$X^{\frac{m_1-1}{2}}(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-1}{2}} \to qx^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}},$

$Y^{\frac{m_2-1}{2}}(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-1}{2}} \to qy^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}$,
$y(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-3}{2}}y(xy)^{\frac{m_3-1}{2}} \to q^2(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}$,
$x(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-3}{2}}x(yx)^{\frac{m_3-1}{2}} \to q^2(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}$,
$X^{\frac{m_1-1}{2}}(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-3}{2}}(xy)^{\frac{m_3-1}{2}}x \to q^2x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}$,
$(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}} \to q^{-2}(yx)^{\frac{m_3-1}{2}}yX^{\frac{m_1-3}{2}}(yx)^{\frac{m_3-1}{2}}Y^{\frac{m_2-1}{2}}$,
$Y^{\frac{m_2-1}{2}}(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-3}{2}}(yx)^{\frac{m_3-1}{2}}y \to q^2y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}}(YX)^{\frac{m_3-1}{2}}$,
$(YX)^{\frac{m_3-1}{2}}y^{\frac{m_2-1}{2}}(XY)^{\frac{m_3-1}{2}}x^{\frac{m_1-1}{2}} \to q^{-2}(xy)^{\frac{m_3-1}{2}}xY^{\frac{m_1-3}{2}}(xy)^{\frac{m_3-1}{2}}X^{\frac{m_1-1}{2}}\}$.

This system is constructed by a modification of the finite complete rewriting system R' for an ordinary triangle group $\langle x, y | x^{m_1} = y^{m_2} = 1, (xy)^{m_3} = 1\rangle$ (see [8]). If we substitute every letter q and q^{-1} in the rules of R by an empty word then we obtain R' exactly.

Proposition 5. *The constructed system R is a finite complete rewriting system for the group E'.*

Proof. It is suffice to prove the following:

a) the system R is Noetherian;

b) the system R is confluent;

c) the system R is corresponding to the group E', that is, the requirements of the proposition 4 are satisfied.

Let us impose, on the words from $\{x, y, q\}^{\pm}$, the following partial order. Let $x < y < X < Y$. On the words from $\{x, y\}^{\pm}$ we set the shortlex order. That is, $W_1 < W_2$ if $|W_1| < |W_2|$ and $t_1W_1 < t_2W_2$ for any words W_1, W_2 with $|W_1| = |W_2|$ and any letters t_1, t_2 with $t_1 < t_2$. Let *eraseQ* is a function from $\{x, y, q\}^{\pm}$ to $\{x, y\}^{\pm}$ that erases all entries of the letters q and q^{-1} from the word. Set $W_1 < W_2$ if $eraseQ(W_1) < eraseQ(W_2)$. If $eraseQ(W_1) = eraseQ(W_2)$ we set $W_1 < W_2$ if W_2 can be obtained from W_1 using rules $xq \to qx,\ yq \to qy,\ Xq \to qX,\ Yq \to qY,\ xq^{-1} \to q^{-1}x,\ yq^{-1} \to q^{-1}y,\ Xq^{-1} \to q^{-1}X,\ Yq^{-1} \to q^{-1}Y,\ qq^{-1} \to 1,\ q^{-1}q \to 1$. This set of rules is equivalent to shifting the letters q, q^{-1} to the left and the erasing entries of $qq^{-1}, q^{-1}q$.

It is not hard to see that the constructed binary relation $<$ is actually a partial order on words from $\{x, y, q\}^{\pm}$ and there is no infinite descending chains with respect to this order.

It can be easily verified that if $(W_1 \to W_2) \in R$ then $W_2 < W_1$. Therefore if $W_1 \to_R W_2$ then $W_2 < W_1$. Hence R is Noetherian.

Checking the requirements of Proposition 4 is also a simple, straightforward task.

The hardest part is to prove the confluence of R. This can be done using critical pairs analysis as in the Knuth-Bendix algorithm. This analysis for R is essentially the same as for R'. There are different implementations of the Knuth-Bendix algorithm such as *kbmag* in the computer algebra system GAP which can be used to complete this task.

So R is a finite complete rewriting system for the group E'. □

It is not hard to see that $\forall k \in \mathbb{N}:\ q^k \in Irr(R)$, hence q has infinite order in E. End of proof of Proposition 2. Theorem 1 is a direct consequence of Propositions 1 and 2. □

The author is grateful to Yu. S. Samoilenko for setting the problem and useful discussions and remarks.

References

1. R. V. Book, F. Otto, *String-Rewriting Systems*, Springer-Verlag, New York, 1993.
2. P. R. Halmos, S. Kakutani, *Products of symmetries*, Bull. Amer. Math. Soc. **64** (1958), no. 3, Part 1, 77–78.
3. M. Hladnik, M. Omladic, and H. Radjavi, *Products of roots of the identity*, Proc. Amer. Math. Soc. **129** (2001), no. 2, 459–465.

4. M. Jantzen, *Confluent String Rewriting*, Birkhauser, 1988.
5. S. Kruglyak, S. Popovich, Yu. Samoilenko, *The spectral problem and *-representations of algebras associated with Dynkin graphs*, J. Algebra Appl., **4** (2005), no. 6, 761–776.
6. S. A. Kruglyak, V. I. Rabanovich, Yu. S. Samoilenko, *On sums of projections*, Funct. Anal. Appl., **36** (2002), no. 3, 182–195.
7. W. Magnus, *Noneuclidean Tesselations and Their Groups*, Academic Press, New York, 1974.
8. M. Pfeiffer, *Automata and Growth Functions for the Triangle Groups*, Diploma Thesis in Computer Science, Lehrstuhl D fur Mathematik RWTH Aachen, Aachen, 2008.
9. Yu. S. Samoilenko, D. Yu. Yakymenko, *Scalar operators equal to the product of unitary roots of the identity operator*, Ukrainian Math. J. **64** (2012), no. 6, 938–947.
10. D. Yu. Yakymenko, *On unitary operators which are product of roots of the identity*, Reports of NAS of Ukraine, to appear.

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Received 21/01/2013