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SCHRÖDINGER OPERATORS WITH NONLOCAL POTENTIALS

SERGIO ALBEVERIO AND LEONID NIZHNIK

Dedicated to Professor F. S. Rofe-Beketov on the occasion of his 80th birthday

Abstract. We describe selfadjoint nonlocal boundary-value conditions for new exact solvable models of Schrödinger operators with nonlocal potentials. We also solve the direct and the inverse spectral problems on a bounded line segment, as well as the scattering problem on the whole axis for first order operators with a nonlocal potential.

1. Introduction

Exact solvable models have been used for modeling complex problems in quantum mechanics for a long time, with a wide use of point interaction models [1, 2]. Such models can be applied in the case of short-range potentials, replacing them with zero-range potentials. In the simplest case, such a potential has support in a single point x_0, and is regarded as a distribution with point support, for example, a multiple of Dirac's δ-function at the point x_0. Another approach to Schrödinger operators with point interaction goes back to [6] and is based on selfadjoint extensions of a minimal symmetric operator $L_{\min}$ defined on functions that are zero in neighborhoods of the point interactions. Point interaction models are exactly solvable [1, 2].

Solvable models also include Schrödinger operators of the form $L\psi \equiv -\Delta\psi + \int K(x,s)\psi(s)\,ds$ that describe nonlocal interactions, where the Hermitian integral operator K is finite dimensional. In particular, the operator

$$L\psi = -\frac{d^2\psi}{dx^2} + v_1(x)(\psi, v_2) + v_2(x)(\psi, v_1)$$

is of such a kind. Setting $v_1(x) = v(x) \in L_2$ and $v_2(x) = \delta(x - x_0)$ we get Schrödinger operators with nonlocal potentials [5].

Operators with nonlocal potentials are self-adjoint extensions of a symmetric operator A_{sym} whose domain is not dense and the operator is a restriction of the operator $L_{\min}$ to functions that are orthogonal to the nonlocal potentials. Hence, the class of Schrödinger operators with nonlocal potentials is an extension of the class of Schrödinger operators with point interactions. Such models, together with numerical parameters (coupling constants), also contain functional parameters (nonlocal potentials). Examples of one-dimensional and three dimensional Schrödinger operators with nonlocal potentials were considered in [5]. Exact solvable models for operators with nonlocal potentials permitted to obtain a number of exact results in [5], in particular, an explicit representation of the resolvent and the scattering operator was obtained in terms of the nonlocal potentials for a one-dimensional Schrödinger operator. A solution of the direct and the inverse problems on a bounded line segment for a one-dimensional Schrödinger operator with nonlocal potential is given in [3]. Let us also remark that the algorithm for finding the nonlocal potentials from the set of all eigenvalues, when solving the inverse problem,

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does not contain the Gelfand–Levitan–Marchenko integral equation. The characteristic function $\chi(\lambda)$ can be explicitly constructed from the set of all eigenvalues as an infinite product. Its values $\chi(\lambda)$ in the points of the unperturbed spectrum (if the nonlocal potential is absent) directly give the Fourier coefficients of the nonlocal potential when the potential is expanded in eigenfunctions of the accompanying selfadjoint operator with point interaction. The results obtained in [3] are generalized to encompass other cases in [12, 13, 14, 15].

Sections 2, 3, 4 describe nonlocal selfadjoint boundary-value conditions for first order differential one-dimensional Schrödinger operators and Dirac operators with nonlocal potentials.

A solution of the direct and the inverse problems on bounded line segments is given in Section 5 for first order differential operators with various nonlocal interactions.

In Section 6, we give an explicit expression for the scattering operator for first order differential operators on the whole axis in terms of nonlocal potentials; the inverse scattering problem is also discussed.

The scattering problem on the half-axis for Schrödinger equation with nonlocal potentials is studied in Section 7.

Section 8 discusses a possibility to use nonlocal potentials for modeling the usual short-range potentials.

2. First order differential operators

On the Hilbert space $L_2(\mathbb{R}^1)$, consider selfadjoint operators A generated by the differential expression $i\frac{d}{dx}$ with nonlocal potentials $v_1, v_2 \in L_2(\mathbb{R}^1)$. The selfadjoint operator A will be called an operator with nonlocal potentials v_1, v_2 corresponding to a point $x = x_0$ if the operator A is defined on all the functions $\varphi \in W_2^1(\mathbb{R}^1 \setminus \{x_0\})$ that are zero in $x = x_0$, if $A\varphi = i\varphi'$ for all functions φ that are orthogonal to v_1 and v_2. We denote such a restriction by $A_{\mathrm{sym}} = \left(i\frac{d}{dx}\right)_{\mathrm{sym}}$. Hence, A is a selfadjoint extension of A_{sym}.

Let A_{v_1,v_2} be a two-dimensional selfadjoint operator on $L_2(\mathbb{R}^1)$, taking values in the subspace $\mathrm{span}\{v_1, v_2\}$. Such an operator can be written as $A_{v_1,v_2}\psi = \sum_{k,n=1}^{2} a_{k,n} v_k(\psi, v_n)$, where the number matrix $a = ||a_{k,n}||_{k,n=1}^2$ is selfadjoint. It is clear that if the selfadjoint operator A is regarded with nonlocal potentials v_1, v_2, then $A + A_{v_1,v_2}$ will also be selfadjoint with the same nonlocal potentials, since $(A + A_{v_1,v_2})_{\mathrm{sym}} = A_{\mathrm{sym}}$. Hence, we can construct selfadjoint operators A with nonlocal potentials up to the operator of the type A_{v_1,v_2}.

Instead of considering the operators A as selfadjoint extensions of the operator $\left(i\frac{d}{dx}\right)_{\mathrm{sym}}$ on $L_2(\mathbb{R}^1)$, one can consider selfadjoint restrictions of the maximal operator $A_{\max}$, which is defined on $L_2(\mathbb{R}^1)$, to the set of functions from the Sobolev space $W_2^1(\mathbb{R}^1 \setminus \{x_0\})$ via the identity

$$A_{\max}\psi(x) = i\frac{d\psi}{dx} + v_1(x)\psi(x_0 - 0) + v_2(x)\psi(x_0 + 0) + iA_{v_1,v_2}, \quad x \neq x_0 \tag{2.1}$$

for one special choice of the rank-two selfadjoint operator A_{v_1,v_2}.

Theorem 2.1. *For the operator $A_{\max}$ of form (2.1) to have a selfadjoint restriction A on $L_2(\mathbb{R}^1)$, it is necessary and sufficient that the operator $A_{\max}$ should have the form*

$$A_{\max}\psi(x) = i\frac{d\psi(x)}{dx} + v_1(x)\Big[\psi(x_0 - 0) + \frac{i}{2}(\psi, v_1)] + v_2(x)[\psi(x_0 + 0) - \frac{i}{2}(\psi, v_2)\Big], \tag{2.2}$$

and its domain should consist of all the functions $\psi \in W_2^1(\mathbb{R}^1 \setminus \{x_0\})$ that satisfy the nonlocal boundary-value condition

$$\psi(x_0 + 0) - i(\psi, v_2) = e^{i\theta}[\psi(x_0 - 0) + i(\psi, v_1)] \tag{2.3}$$

for some real θ. Also, the selfadjoint operator A corresponds to the nonlocal potentials v_1, v_2 in the point $x = x_0$.

Proof. Operators of the form (2.2) satisfy the Lagrange formula

$$(A_{\max}\psi(x), \varphi) - (\psi(x), A_{\max}\varphi) = i[\Gamma_1\psi \cdot \overline{\Gamma_2\varphi} - \Gamma_2\psi \cdot \overline{\Gamma_1\varphi}], \tag{2.4}$$

where $\Gamma_1\psi = \psi(x_0 - 0) + i(\psi, v_1)$ and $\Gamma_2\psi = \psi(x_0 + 0) - i(\psi, v_2)$ are nonlocal boundary data functions ψ that span the Euclidean space E^2. Hence, (2.3) gives a general form of selfadjoint nonlocal boundary-value conditions [9]. The operator A, with the domain defined by condition (2.3), is a selfadjoint restriction of the operator $A_{\max}$ of form (2.2). The operator of the form (2.2) itself can be obtained from (2.1) by setting $A_{v_1,v_2}\psi = \frac{1}{2}v_1(\psi, v_1) - \frac{1}{2}v_2(\psi, v_2)$. Any other operator A_{v_1,v_2} will give a skew-symmetric perturbation of A and can not yield a selfadjoint restriction of an operator of type (2.1). □

An important particular case of selfadjoint boundary-value conditions (2.3) is where $v_1 = v_2 \equiv \frac{1}{2}v$, and $\theta = 0$, as well as $v_2 = -v_1 \equiv v$ and $\theta = \pi$. In these cases, the operator with nonlocal potential v has the form

$$A\psi \equiv i\psi'(x) + v(x)\psi_r(x_0), \quad \psi_s(x_0) - i(\psi, v) = 0, \tag{2.5}$$

$$A\psi \equiv i\psi'(x) + v(x)\psi_s(x_0), \quad \psi_r(x_0) - i(\psi, v) = 0, \tag{2.6}$$

where $\psi_r(x_0) = \frac{1}{2}[\psi(x_0 + 0) + \psi(x_0 - 0)]$, $\psi_s(x_0) = \psi(x_0 + 0) - \psi(x_0 - 0)$.

Remark 2.1. *To every selfadjoint operator A of the form (2.1)–(2.3) there corresponds an unperturbed operator $\tilde{A}_\theta$ related to the case $v_1 \equiv v_2 \equiv 0$. The operator $\tilde{A}_\theta$ defines a point interaction in the point $x = x_0$ for the operator $i\frac{d}{dx}$ with the boundary-value conditions $\psi(x_0+0) = e^{i\theta}\psi(x_0-0)$. For an operator A that corresponds to the problems (2.5) and (2.6) there are also accompanying selfadjoint operators $\hat{A}$ with point interaction in the point $x = x_0$; this corresponds to the conditions $\psi_r(x_0) = 0$ (for problem (2.5)) or $\psi_s(x_0) = 0$ (for problem (2.6)).*

3. One–dimensional Schrödinger operator

Selfadjoint boundary-value conditions for one–dimensional Schrödinger operators with nonlocal potentials $v_1, v_2 \in L_2(\mathbb{R}^1)$ are given in [5]. Similarly to Section 2 one can also consider a wider class of nonlocal potentials $v(x) = (v_1(x), v_2(x), v_3(x), v_4(x))$ with $v_k(x) \in L_2(\mathbb{R}^1)$ for $k = 1, \ldots, 4$. The maximal operator $A_{\max}$ on the space $L_2(\mathbb{R}^1)$, having nonlocal potentials $v(x)$, is defined on all functions from the Sobolev space $W_2^1(\mathbb{R}^1 \setminus \{0\})$ by

$$A_{\max}\psi(x) = -\frac{d^2\psi(x)}{dx^2} + v(x)\vec{\psi}(0) + v(x)a(\psi, v^+), \quad x \neq 0, \tag{3.1}$$

where $\vec{\psi}(0) = \operatorname{col}(\psi(-0), \psi'(-0), \psi(+0), \psi'(+0))$, $v^+(x) = \operatorname{col}(v_1, v_2, v_3, v_4)$, $(\psi, v^+) = \operatorname{col}((\psi, v_1), (\psi, v_2), \ldots, (\psi, v_4))$, and a is a numeric skew-symmetric 4×4-matrix.

Theorem 3.1. *For the operator $A_{\max}$ of form (3.1) to admit a selfadjoint restriction A on $L_2(\mathbb{R}^1)$, it is necessary and sufficient that the operator $A_{\max}$ would have the form*

$$\begin{aligned} A_{\max}\psi(x) = &-\frac{d^2\psi(x)}{dx^2} + v_1(x)\Big[\psi(-0) - \frac{1}{2}(\psi, v_2)\Big] + v_2(x)\Big[\psi'(-0) + \frac{1}{2}(\psi, v_1)\Big] \\ &+ v_3(x)\Big[\psi(+0) + \frac{1}{2}(\psi, v_4)\Big] + v_4(x)\Big[\psi'(+0) - \frac{1}{2}(\psi, v_3)\Big], \end{aligned} \tag{3.2}$$

and its domain $\mathcal{D}(A)$ would consist of all the functions $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ satisfying the nonlocal selfadjoint boundary-value conditions

$$\Gamma_1\psi + i\Gamma_2\psi = U(\Gamma_1\psi - i\Gamma_2\psi), \tag{3.3}$$

where the two-dimensional vectors $\Gamma_1\psi$ and $\Gamma_2\psi$ from the Euclidean space E^2 define the nonlocal boundary data for the function ψ

$$\begin{aligned}\Gamma_1\psi &= \mathrm{col}\,(\psi'(+0) - (\psi, v_3), -\psi'(-0) - (\psi, v_1)),\\ \Gamma_2\psi &= \mathrm{col}\,(\psi(+0) + (\psi, v_4), \psi(-0) - (\psi, v_2))\end{aligned} \tag{3.4}$$

and U is a unitary matrix on E^2.

Proof. It is similar to the proof of Theorem 2.1. We prove only sufficiency. Let us show that a restriction of the operator $A_{\max}$ of form (3.2) to functions $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$ satisfying the boundary-value conditions (3.3) for any unitary matrix U defines a selfadjoint operator $A_{U,v}$ on the space $L_2(\mathbb{R}^1)$, which are Schrödinger operators with nonlocal potentials v. Indeed, the operator $A_{\max}$ satisfies the Lagrange formula. For any $\psi, \varphi \in W_2^2(\mathbb{R}^1 \setminus \{x_0\})$

$$(A_{\max}\psi, \varphi) - (\psi, A_{\max}\varphi) = \omega(\Gamma\psi, \Gamma\varphi),$$

where the bilinear form ω is defined on the Euclidean space E^4 of boundary-value data for the functions ψ and φ

$$\omega(\Gamma\psi, \Gamma\varphi) =< \Gamma_1\psi, \Gamma_2\varphi >_{E^2} - < \Gamma_2\psi, \Gamma_1\varphi >_{E^2} . \tag{3.5}$$

Since the boundary-value data $\Gamma\psi = \{\Gamma_1\psi, \Gamma_2\psi\}$ span the whole Euclidean space E^4 for $\psi \in W_2^2(\mathbb{R}^1 \setminus \{0\})$, see [5, Lemma 1], selfadjoint restrictions of the operator $A_{\max}$ correspond to selfadjoint boundary-value conditions that are uniquely determined by the form ω with Lagrangian planes in E^4, which, in turn, are parameterized with unitary matrices U on the space E^2. Hence, (3.3) defines general nonlocal selfadjoint boundary-value conditions for a Schrödinger operator of the form (3.2). □

Let us look closely at some interesting particular cases of Schrödinger operators of the form (3.2)–(3.3). If the matrix U in the boundary-value conditions (3.3) is diagonal, the nonlocal potentials $v_3(x)$ and $v_4(x)$ are zero for $x < 0$, and $v_1(x)$, $v_2(x)$ are zero for $x > 0$, then the operator $A_{U,v}$ admits a representation as a direct sum of operators that are selfadjoint on $L_2(0,\infty)$ and $L_2(-\infty, 0)$, $A_{U,v} = A^+_{U,v} \oplus A^-_{U,v}$.We also have that, if $v_3(x) \equiv v(x)\sin\theta$ and $v_4(x) \equiv v(x)\cos\theta$, then the operator $A^+_{U,v}$ on the space $L_2(0,\infty)$ acts as

$$A^+_{U,v}\psi(x) = -\frac{d^2\psi(x)}{dx^2} + v(x)[\sin\theta\psi(0) + \cos\theta\psi'(0)], \tag{3.6}$$

and its domain consists of all functions from the space $W_2^2(0,\infty)$ that satisfy the boundary-value conditions

$$\psi'(0)\sin\theta - \psi(0)\cos\theta = (\psi, v)_{L_2(0,\infty)}. \tag{3.7}$$

Representations (3.2)–(3.3) also give the following particular cases:

the operators:	**selfadjoint boundary conditions:**
(3.8) $-\psi''(x) + v_1(x)\psi_s(0) + v_2(x)\psi_s'(0);$	$\psi_r' - (\psi, v_1) = 0, \quad \psi_r + (\psi, v_2) = 0.$
(3.9) $-\psi''(x) + v_1(x)\psi_r(0) + v_2(x)\psi_r'(0);$	$\psi_s' - (\psi, v_1) = 0, \quad \psi_s + (\psi, v_2) = 0.$
(3.10) $-\psi''(x) + v_1(x)\psi(+0) + v_2(x)\psi(-0);$	$\psi'(+0) - (\psi, v_1) = 0, \quad \psi'(-0) + (\psi, v_2) = 0.$

4. Dirac operator

A one-dimensional Dirac operator A with a matrix-valued potential $v(x)$ is defined on the space $L_2(\mathbb{R}^1; E^n)$, which is the space of square integrable vector-valued functions $\psi(x) = \operatorname{col}(\psi_1(x), \psi_2(x), \ldots, \psi_n(x))$, $\psi_k(x) \in L_2(\mathbb{R}^1)$, $k = 1, 2, \ldots, n$, by the differential expression

$$A\psi = B\frac{d\psi}{dx} + v(x)\psi, \tag{4.1}$$

where B is a constant skew-symmetric matrix on the Euclidean space E^n. Most important are the cases $n = 4$, where B is a Dirac matrix, and the case $n = 2$, where B is a Pauli matrix. If $n = 2$, one usually takes $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $B = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

An analog of the Dirac operator with nonlocal matrix-valued potentials $v_1(x)$, $v_2(x)$, with the components being in the space $L_2(\mathbb{R}^1)$, is given by the following theorem.

Theorem 4.1. *Let the operator*

$$A\psi = B\frac{d\psi}{dx} + v_1(x)\Big[\psi(-0) - \frac{1}{2}(B\psi, v_2)\Big] + v_2\Big[\psi(+0) + \frac{1}{2}(B\psi, v_1)\Big] \tag{4.2}$$

be defined on all vector-valued functions $\psi(x) = \operatorname{col}(\psi_1(x), \psi_2(x), \ldots, \psi_n(x))$, with the component lying in the space $\psi_k(x) \in W_2^1(\mathbb{R}^1 \setminus \{0\})$. Let the functions in the domain of the operator A satisfy the following nonlocal boundary-value conditions in the point $x = 0$:

$$\psi(+0) + (B\psi, v_1) = U[\psi(-0) - (B\psi, v_2)], \tag{4.3}$$

where U is a nondegenerate matrix on E^n such that $U^*BU = B$. Then the operator A on the space $L_2(\mathbb{R}^1; E^n)$ is selfadjoint.

Proof. It is similar to the proof of Theorems 2.1 and 3.1. □

5. Spectral problem

Consider the following spectral problem with nonlocal potentials $v \in L_2(-\pi, \pi)$ for the equation

$$l\psi \equiv i\psi'(x) + v(x)\psi_s(0) = \lambda\psi(x), \quad -\pi < x < \pi, \quad \psi_s(0) = \psi(+0) - \psi(-0) \tag{5.1}$$

with the periodic boundary-value conditions

$$\psi(-\pi) = \psi(\pi) \tag{5.2}$$

and the nonlocal conditions in the point $x = 0$

$$\psi_r(0) \equiv \frac{1}{2}[\psi(+0) + \psi(-0)] = i\int_{-\pi}^{\pi} \psi(x)\overline{v(x)}\, dx. \tag{5.3}$$

The eigenvalue problem (5.1)–(5.3) is selfadjoint on the space $L_2(-\pi, \pi)$. The domain of the corresponding operator A on the space $L_2(-\pi, \pi)$ consists of the functions $\psi \in W_2^1((-\pi, \pi) \setminus \{0\})$ satisfying the conditions (5.2)–(5.3), and the operator is given by the left-hand side of equation (5.1)

$$A\psi(x) = i\psi'(x) + v(x)\psi_s(0), \quad x \neq 0. \tag{5.4}$$

Theorem 5.1. *The operator A is selfadjoint on the space $L_2(-\pi, \pi)$ and has purely discrete spectrum. All eigenvalues of the operator A that are not integer are simple. The number $n \in Z$ is an eigenvalue if and only if*

$$\int_{-\pi}^{\pi} v(x)e^{inx}\, dx = i. \tag{5.5}$$

Proof. We can apply the Lagrange formula to functions $\psi, \varphi \in W_2^1((-\pi,\pi)\setminus 0) = W_2^1((-\pi,\pi)\setminus\{0\}) \cap \{\psi : \psi(-\pi) = \psi(\pi)\}$ in the space $L_2(-\pi,\pi)$

$$(l\psi, \varphi) - (\psi, l\varphi) = \omega(\Gamma\psi, \Gamma\varphi),$$

where the form $\omega(\xi,\eta) = i[(\xi_1,\eta_1) - (\xi_2,\eta_2)]$ is defined on two-dimensional vectors $\xi = \Gamma\psi$ $\eta = \Gamma\varphi$, which make boundary data $\Gamma\psi = \operatorname{col}(\psi(-0) + \frac{i}{2}(\psi, v), \psi(+0) - \frac{i}{2}(\psi, v))$ for the functions ψ and φ. Since the boundary-value conditions (5.3) are selfadjoint, see (2.6)), the operator A is selfadjoint on the space $L_2(-\pi,\pi)$. It is easy to see that the selfadjoint operator A_0 defined by the expression $i\frac{d}{dx}$ on functions from $W_2^1(-\pi,\pi)$ satisfying periodic boundary-value conditions has eigenvalues $\lambda_n = n$ with the corresponding eigenfunctions $\psi_n = e^{-inx}$. The operators A_0 and A are distinct selfadjoint extensions of the symmetric operator A_{sym} defined as $i\frac{d}{dx}$ on functions from the space $W_2^1(-\pi,\pi)$ which are orthogonal to the function v and satisfying the boundary-value conditions (5.2) and $\psi(0) = 0$. Since the deficiency indices of the operator A_{sym} are finite and equal to $(2,2)$, the spectrum of the operator A is discrete and the eigenvalues satisfy $|\lambda_n| \to \infty$ as $|n| \to \infty$. If the selfadjoint operator A would have a multiple eigenvalue $\tilde{\lambda} \notin Z$, then there would be at least two linearly independent eigenfunctions $\psi_1(x;\lambda)$ and $\psi_2(x;\lambda)$ corresponding to $\tilde{\lambda}$, and then there would exist their nontrivial linear combination $\psi = \alpha\psi_1 + \beta\psi_2 \neq 0$ such that $\psi_s(0) = 0$. But then (5.1)–(5.2) would yield that $\tilde{\lambda}$ is an integer, which contradicts the assumption.

Finally, a direct verification shows that if (5.5) holds then e^{-inx} is an eigenfunction of the operator A with the eigenvalue $\lambda = n$. If $\lambda = n$ is an eigenvalue of the problem (5.1)–(5.3), then e^{-inx} is an eigenfunction with condition $\psi_s(0) = 0$ and (5.5) follows from (5.3). If $\psi_s(0) \neq 0$ then (5.5) follows from (5.1)–(5.2). □

Expand the nonlocal potential $v \in L_2$ in a Fourier series

$$v(x) = \sum_{n=-\infty}^{+\infty} v_n e^{-inx}, \quad v_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x)e^{inx}\,dx. \tag{5.6}$$

To give an exact description of the eigenvalues λ_n of problem (5.1)–(5.3) it becomes useful to consider the following special solutions of problem (5.1), (5.2):

$$\varphi(x;\lambda) = \varphi_0(x;\lambda) + 2i\sin\lambda\pi \sum_{n=-\infty}^{+\infty} \frac{v_n e^{-inx}}{\lambda - n}, \tag{5.7}$$

$$\varphi_0(x;\lambda) = e^{-i\lambda(x-\pi\,\mathrm{sign}\,x)} = \begin{cases} e^{-i\lambda(x+\pi)}, & x<0, \\ e^{-i\lambda(x-\pi)}, & x>0. \end{cases} \tag{5.8}$$

By substituting the special solution $\varphi(x;\lambda)$ into the boundary-value condition (5.3), we obtain the following characteristic function $\chi(\lambda) \equiv \varphi_r(0;\lambda) - i(\varphi, v)$:

$$\chi(\lambda) = \cos\lambda\pi + 4\pi\sin\lambda\pi \sum_{n=-\infty}^{+\infty} \frac{\alpha_n}{\lambda - n}, \tag{5.9}$$

where

$$\alpha_n = |v_n - \frac{i}{2\pi}|^2 - \frac{1}{4\pi^2} \tag{5.10}$$

and v_n is a Fourier coefficient of the nonlocal potential $v(x)$ in decomposition (5.6).

Theorem 5.2. *A number λ is an eigenvalue of problem (5.1)–(5.3) if and only if λ is a zero of the characteristic function $\chi(\lambda)$. Multiplicity of the eigenvalue λ coincides with its multiplicity as a zero of the characteristic function.*

Proof. It is similar to the proof of Theorem 2 in [14].

Since the characteristic function $\chi(\lambda)$ of the form (5.9) is an entire analytic function, it is uniquely defined by a two-sided sequence of its zeros $\lambda_n = n - \frac{1}{2}\operatorname{sign}(n) + \varepsilon_n,\ n \in Z \setminus \{0\},\ \sum_n |\varepsilon|^2 < +\infty$ as an infinite product

$$\chi(z) = \prod_{k=1}^{\infty} \Big(k - \frac{1}{2}\Big)^{-2} (\lambda_k - z)(z - \lambda_{-k}). \tag{5.11}$$

On the other hand, it follows from (5.9) that

$$\chi(n) = (-1)^n |2\pi v_n - i|^2. \tag{5.12}$$

Formulas (5.11)–(5.12) give an algorithm for solving the inverse spectral problem of recovering the nonlocal potential in problem (5.1)–(5.3) from all of its eigenvalues, is similar to the algorithms of [3, 12, 13, 14, 15]. One can as well describe all isospectral nonlocal potentials.

The algorithm for solving the inverse spectral problem consists of the following three steps.

Step 1. Use the spectrum $\{\lambda_n\}$ to construct the characteristic function $\chi(\lambda)$ as the infinite product (5.11).

Step 2. Calculating the values $\chi(n)$, $n \in Z$, by solving the equation (5.12), we find the Fourier coefficients v_n of the nonlocal potential $v(x)$.

Step 3. The nonlocal potential is defined by its Fourier series (5.6). □

Let us briefly consider a spectral problem close to the one in (5.1)–(5.3), namely, to find λ such that the equation

$$i\psi'(x) + v(x)\psi_r(0) = \lambda\psi(x), \quad -\pi < x < \pi, \quad v \in L_2(-\pi, \pi) \tag{5.13}$$

has a nontrivial solution $\psi(x)$ satisfying the boundary-value condition

$$\psi(-\pi) = \psi(\pi) \tag{5.14}$$

and the nonlocal boundary-value conditions

$$\psi_s(0) - i(\psi, v) = 0. \tag{5.15}$$

Replace the expansion of the nonlocal potential into a Fourier series (5.6) for problem (5.1)–(5.3) with the expansion in eigenfunctions of the following accompanying selfadjoint problem:

$$i\psi'(x) = \lambda\psi(x), \quad \psi(-\pi) = \psi(\pi), \quad \psi_r(0) = 0. \tag{5.16}$$

The problem (5.16) is selfadjoint in the space $L_2(-\pi, \pi)$. Its eigenvalues are the numbers $\mu_n = \frac{1}{2} + n$, $n \in Z$, and the corresponding eigenfunctions $e_n = \operatorname{sign} x e^{-i\mu_n x}$ make a complete orthogonal system in $L_2(-\pi, \pi)$. Hence,

$$v(x) = \sum_{n=-\infty}^{+\infty} v_n e_n(x), \quad v_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(x)\overline{e_n(x)}\, dx. \tag{5.17}$$

Since $e_n(+0) + e_n(-0) = 0$, a special solution of problem (5.13)–(5.15) can be represented as

$$\varphi(x, \lambda) = e^{-i\lambda(x - \pi \operatorname{sign} x)} + \cos\lambda\pi \sum_{n=1}^{\infty} \frac{v_n e_n(x)}{\lambda - \mu_n}, \tag{5.18}$$

where v_n are the Fourier coefficients (5.17) of the nonlocal potential. By substituting the solution $\varphi(x, \lambda)$ into the boundary-value condition (5.15) we get the characteristic function $\chi(\lambda) = \frac{1}{2i}[\varphi_s(0;\lambda) - (\varphi, v)]$

$$\chi(\lambda) = \sin\lambda\pi + \cos\lambda\pi \sum_{n\in\mathbb{Z}} \frac{\alpha_n}{\lambda - \mu_n}, \tag{5.19}$$

where $\alpha_n = \pi|v_n|^2 + i(v_n - \overline{v_n})$.

Zeros of the characteristic function $\chi(\lambda)$ determine spectrum of problem (5.13)–(5.15). Here, the value of the characteristic function in the point $\mu_n = \frac{1}{2} + n$ is explicitly connected with values of the Fourier coefficients v_n in the expansion (5.17)

$$\chi(\mu_n) = (-1)^n |1 + i\pi v_n|^2. \tag{5.20}$$

Formulas (5.19)–(5.20) give a description of spectrum of problem (5.13)–(5.15) and a solution of the inverse spectral problem similar to the ones above for the problem (5.1)–(5.3) and close problems in [14].

Remark 5.1. *The approach to the problems (5.1)–(5.3) and (5.13)–(5.15) can be explained using the abstract perturbation theory for operators on a Hilbert space* [4], [7], [8], [9], [10], [11], [16]. *Let A_0 and A_1 be two selfadjoint operators on a Hilbert space H. Let $A_{\min} = A_0 \wedge A_1$ be the maximal common part of the operators A_0 and A_1. Let $A_{\min}$ be a densely defined symmetric operator with deficiency indices $(1,1)$. The operators A_0 and A_1 are distinct selfadjoint extensions of the operator $A_{\min}$. If in H there is a element $v \in H$, then the operator $A_{\min}$ can be restricted to elements orthogonal to v. Then one can consider the operator $A_{\mathrm{sym}} = A_{\min} \restriction_{\mathcal{D}}$, $\mathcal{D} \equiv \mathcal{D}(A_{\min})^h = \{x : x \in \mathcal{D}(A_{\min}),\, x \perp h\}$, which is not densely defined. The element v will be called a nonlocal potential. Selfadjoint extensions A of the operator A_{sym} will be called operators with nonlocal potentials.*

In the abstract theory, the meanings of the nonperturbed operator A_0 and the accompanying operator A_1 are given, together with defining the element v. The selfadjoint extension A of the operator A_{sym} is subject to the condition $\mathcal{D}(A) \cap \mathcal{D}(A_1) = \mathcal{D}(A \wedge A_1)$, $\mathcal{D}(A) \cap \mathcal{D}(A_0) = \mathcal{D}(A_0)^h$, where $\mathcal{D}(A_0)^h = \{\psi : \psi \in \mathcal{D}(A_0), \psi \perp h\}$. These conditions uniquely define the selfadjoint operator A in terms of A_0, A_1, and h. This construction of the operator A with nonlocal potential, regarded as a selfadjoint extension of a Hermitian operator A_{sym} that is not densely defined, as well as the form of the resolvent of the operator A follow from [4].

6. Scattering problem for first order differential operators

On the whole axis $-\infty < x < +\infty$, consider the scattering problem for first order equation with nonlocal potential

$$i\psi'(x) + v(x)\psi_s(0) = \lambda\psi(x), \quad x \in \mathbb{R}^1 \setminus \{0\}, \tag{6.1}$$

where the nonlocal potential satisfies $v \in L_2(\mathbb{R}^1) \cap L_1(\mathbb{R}^1)$, $\psi_s(0) = \psi(+0) - \psi(-0)$ is the jump of the function $\psi(x)$ in the point $x = 0$. A solution of equation (6.1) satisfies the nonlocal boundary-value conditions (2.6) in the point $x = 0$,

$$\psi_r(0) = i\int_{-\infty}^{+\infty} \psi(x)\overline{v(x)}\,dx. \tag{6.2}$$

If λ is real, then there exists a nontrivial bounded solution of problem (6.1)–(6.2), it is unique, and has the following asymptotic expansion as $x \to \pm\infty$:

$$\psi(x;\lambda) = A_\pm e^{-i\lambda x} + o(1), \quad x \to \pm\infty. \tag{6.3}$$

The coefficients A_+ and A_- are amplitudes of the incoming and the scattered waves, and their ratio gives the scattering operator

$$A_- = -S(\lambda)A_+. \tag{6.4}$$

The minus sign in formula (6.4) is chosen so that the scattering operator would become identity if $v \equiv 0$ and the solution is $\psi(x;\lambda) = Ae^{-i\lambda x}\operatorname{sign} x$.

Theorem 6.1. *Problem (6.1)–(6.2) has a bounded solution $\psi(x;\lambda)$, for real λ,*

$$\psi(x;\lambda) = [A + \operatorname{sign} x]e^{-i\lambda x} + i\int_{-\infty}^{x} e^{-i\lambda(x-s)}v(s)\,ds - i\int_{x}^{+\infty} e^{-i\lambda(x-s)}v(s)\,ds \tag{6.5}$$

and the constant A is uniquely defined by the nonlocal potential v as follows:

$$A = i\beta(\lambda)[1 - i\tilde{v}^*(\lambda)]^{-1}, \tag{6.6}$$

where

$$\beta(\lambda) \equiv J\alpha = \frac{1}{\pi}\int{}' \frac{\alpha(p)}{\lambda - p}\,dp, \tag{6.7}$$

and the integral is understood in the principal value sense, $\alpha(\lambda) = |1 + i\tilde{v}(\lambda)|^2 - 1$, $\tilde{v}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}v(x)\,dx$ is the Fourier transform of the nonlocal potential v, $\tilde{v}^*(\lambda)$ is the comlex conjugate function of $\tilde{v}(\lambda)$. Here, the scattering operator $S(\lambda)$ is uniquely defined by the function $\alpha(\lambda)$ as

$$S(\lambda) = \frac{1 + \alpha(\lambda) - i(J\alpha)(\lambda)}{1 + \alpha(\lambda) + i(J\alpha)(\lambda)}. \tag{6.8}$$

Proof. The function $\psi(x;\lambda)$ in (6.5) satisfies equation (6.1). This function can be represented in the form:

$$\psi(x;\lambda) = [A + \operatorname{sign} x]e^{-i\lambda x} + \frac{1}{\pi}\int{}' \frac{e^{-ipx}\tilde{v}(p)}{\lambda - p}\,dp. \tag{6.9}$$

Substituting $\psi(x;\lambda)$ (6.9) into the boundary-value condition (6.2) we get identity (6.5). The solution $\psi(x;\lambda)$ has the following asymptotics as $x \to \pm\infty$:

$$\psi(x;\lambda) = [A \pm 1 \mp i\tilde{v}]e^{-i\lambda x} + o(1), \;\; x \to \pm\infty.$$

Hence, by definition (6.4) of the scattering operator we get

$$S(\lambda) = \frac{1 + i\tilde{v} - A}{1 + i\tilde{v} + A}. \tag{6.10}$$

Using relation (6.6), we get an explicit formula for the scattering operator in terms of the potential (6.8). □

Remark 6.1. *Since, by (6.8), the scattering operator $S(\lambda)$ can be explicitly expressed in terms of $|1 + \tilde{v}(\lambda)|^2$, two different nonlocal potentials v_1 and v_2 such that*

$$|1 + \tilde{v_1}(\lambda)|^2 = |1 + \tilde{v_2}(\lambda)|^2 \tag{6.11}$$

give the same scattering operator. This shows that the inverse scattering problem for recovering the nonlocal potential from a known scattering operator has a non-unique solution in the class of L_2-potentials. However, one can single out a sufficiently broad classes $(\mathcal{P})$ of potentials for which the identity (6.11) leads to $v_1 \equiv v_2$ and, consequently, the inverse scattering problem would have a unique solution in the class $(\mathcal{P})$.

7. Scattering problem for Schrödinger operators on the half-axis

Consider a selfadjoint problem of the form (3.6)–(3.7) for Schrödinger operators with a nonlocal potential $v \in L_2(0,\infty)$,

$$-\psi''(x) + v(x)\psi(0) = \lambda^2\psi(x), \quad 0 < x < +\infty, \quad \psi'(0) = (\psi, v)_{L_2(0,\infty)}. \tag{7.1}$$

For real λ, a bounded solution of problem (7.1), the Jost solution, has the form

$$\psi(x;\lambda) = e^{-i\lambda x} + S(\lambda)e^{i\lambda x} - \psi(0)\int_{x}^{+\infty} \frac{\sin\lambda(x-s)}{\lambda}v(s)\,ds, \tag{7.2}$$

where $S(\lambda)$ is the reflection coefficient. The value $\psi(0)$ is determined by (7.2), assuming $x = 0$

$$\psi(0) = (1 + S)(1 - \frac{1}{\lambda}\tilde{v_s})^{-1}, \tag{7.3}$$

where $\tilde{v_s}(\lambda) = \int_0^{+\infty} \sin \lambda s \, v(s) \, ds$ is the sin-Fourier transform of the nonlocal potential. Substituting (7.2) into (7.1) and a use of (7.3) leads to an explicit expression for $S(\lambda)$ in terms of the nonlocal potential. If v is a real-valued nonlocal potential, then .

$$S(\lambda) = [\lambda - \tilde{v_s} - i\beta] \cdot [\lambda - \tilde{v_s} + i\beta]^{-1}, \tag{7.4}$$

where $\beta = \frac{1}{2\pi} \int_0^{+\infty}{}' \frac{\alpha(p)}{p^2-\lambda^2} \, dp$, and the integral is understood in the principal value sense, and $\alpha(p) = |\tilde{v_s}(p) - p|^2 - p^2$, can be explicitly found from the potential v.

To the problem (7.1), one can connect a selfadjoint operator A on $L_2(0, \infty)$. Its action is defined by the left-hand side of identity (7.1), $A\psi = -\psi''(x) + v(x)\psi(0)$. The domain $\mathcal{D}(A)$ consists of all functions from the Sobolev space $W_2^2(0, \infty)$ satisfying the nonlocal boundary-value condition $\psi(0) = (\psi, v)$. The positive half-axis $[0, \infty)$ is the continuous part of spectrum of the operator A. The operator A can also have eigenvalues. This is the case for the potential $v(x) = -2e^{-x}$. Here, the number $\lambda^2 = -3$ is an eigenvalue of A with the eigenfunction e^{-x}.

If $\psi(0)$ and $\psi'(0)$ are interchanged in problem (7.1), that is the problem becomes

$$\begin{gathered} -\psi''(x) + v(x)\psi'(0) = \lambda^2\psi(x), \quad 0 < x < +\infty, \quad \psi(0) + (\psi, v)_{L_2(0,\infty)} = 0, \\ \psi(x) = e^{-i\lambda x} - S(\lambda)e^{i\lambda x} + o(1), \quad x \to \infty, \end{gathered} \tag{7.5}$$

then the scattering operator S can be expressed in terms of $\tilde{v_c}(\lambda) = \int_0^{+\infty} \cos \lambda s \, v(s) \, ds$, which is the cos-Fourier transform, similarly to (7.4) as

$$S(\lambda) = [1 + \tilde{v_c} + i\beta] \cdot [1 + \tilde{v_c} - i\beta]^{-1}, \tag{7.6}$$

where $\beta(\lambda) = \frac{\lambda}{2\pi} \int_0^{+\infty}{}' \frac{\alpha(p)}{p^2-\lambda^2} \, dp$, and the integral is taken as the principal value, where $\alpha(p) = |\tilde{v_c}(p) - 1|^2 - 1$.

8. Nonlocal potentials as models for short-range potentials

Consider the one-dimensional Schrödinger equation

$$-\psi''(x) + v(x)\psi(x) = \lambda^2\psi(x), \tag{8.1}$$

where the potential v has support in a small neighborhood of the point $x = 0$, that is, $\operatorname{supp} v(x) \subset (-\varepsilon, \varepsilon)$. The product $v(x)\psi(x)$, for $x < 0$, can be approximated with $v_1(x)\psi(-0) + v_2(x)\psi'(-0)$ and, for $x > 0$, with $v_3(x)\psi(+0) + v_4(x)\psi'(+0)$, where $v_1(x) = \theta(-x)v(x)$, $v_2(x) = x\theta(-x)v(x)$, $v_3(x) = \theta(x)v(x)$, $v_4(x) = x\theta(x)v(x)$, and $\theta(x)$ is the Heaviside step function. Then we have a Schrödinger operator with nonlocal potential of the form (3.2)–(3.3), which can be regarded as a solvable model for the initial operator.

If λ is real and $v \in L_2(\mathbb{R}^1) \cap L_1(\mathbb{R}^1)$, problem (8.5) has a unique Jost solution, bounded on the whole axis, with the asymptotics

$$\psi(x; \lambda) = e^{i\lambda x} + o(1), \quad x \to -\infty; \quad \psi(x; \lambda) = ae^{i\lambda x} + be^{-i\lambda x} + o(1), \quad x \to +\infty. \tag{8.2}$$

The coefficients $a(\lambda)$, $b(\lambda)$ in (8.2) have important physical meanings. The quantity a^{-1} gives the coefficient of transmission from the right, and ba^{-1} is the coefficient of reflection from the right. We have $|a|^2 - |b|^2 = 1$. These quantities are used to construct the unitary 2×2- scattering matrix $S(\lambda) = a^{-1} \begin{pmatrix} 1 & -\bar{b} \\ b & 1 \end{pmatrix}$.

Let us also give expressions for the coefficients a and b in the Jost solution (8.2) for a Schrödinger operator with point interaction in the point $x = 0$ and subject to the boundary-value conditions

$$\begin{pmatrix} \psi_s' \\ \psi_s \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \gamma & -\beta \end{pmatrix} \begin{pmatrix} \psi_r \\ -\psi_r' \end{pmatrix} \tag{8.3}$$

for real α, β, γ. In this case

$$\begin{aligned} b &= \Big[\gamma + \frac{i}{2\lambda}(\alpha + \lambda^2\beta)\Big] \cdot \Big[1 - \frac{1}{4}(\gamma^2 + \alpha\beta)\Big]^{-1}, \\ a &= \Big[1 + \frac{1}{4}(\gamma^2 + \alpha\beta) - \frac{i}{2\lambda}(\alpha - \lambda^2\beta)\Big] \cdot \Big[1 - \frac{1}{4}(\gamma^2 + \alpha\beta)\Big]^{-1}. \end{aligned} \tag{8.4}$$

Consider the simplest case of Schrödinger operator with nonlocal potential of the form (3.2)–(3.3),where $v_1(x) = \theta(-x)v(x)$, $v_3(x) = \theta(x)v(x)$, and $v_2(x) = v_4(x) = 0$. For the sake of simplicity, we will assume that $v(x)$ is an even function, $v(x) = v(-x)$. Then we consider the problem on the whole axis

$$\begin{aligned} -\psi''(x) + v(x)\psi_r + \hat{v}(x)\psi_s = \lambda^2\psi(x), \quad \hat{v}(x) = \frac{1}{2}\operatorname{sign} x \cdot v(x), \\ \psi_s = \hat{\beta}[\psi_r' - (\psi, \hat{v})], \quad \psi_s' = \hat{\alpha}\psi_r + (\psi, v) \end{aligned} \tag{8.5}$$

for real $\hat{\alpha}, \hat{\beta}$ in the boundary conditions.

Explicit expressions for the scattering matrix for problem (8.5) with $\hat{\alpha} = 0$ and $\hat{\beta} = 0$ in terms of nonlocal potentials are obtained in [5]. If the function $v(x)$ is real and even, we have exact explicit expressions for the coefficients a and b in the Jost solution (8.2) in terms of nonlocal potential v and the numbers $\hat{\alpha}$, $\hat{\beta}$ from the boundary conditions

$$\begin{aligned} a = \frac{1}{2}\Big[\psi_+ E^{-1} - \frac{\psi_+' + \psi_+\tilde{v}_{+,c}}{i\lambda}\Big], \quad b = \frac{1}{2}\Big[\psi_+ E^{-1} + \frac{\psi_+' + \psi_+\tilde{v}_{+,c}}{i\lambda}\Big], \\ \psi_+ = \Big[\Big(1 + \frac{AB}{4}\Big)E + B(i\lambda + E\tilde{v}_{+,c})\Big]\Big(1 - \frac{1}{4}AB\Big)^{-1}, \\ \psi_+' = \Big[\Big(1 + \frac{AB}{4}\Big)(i\lambda + E\tilde{v}_{+,c}) + AE\Big]\Big(1 - \frac{1}{4}AB\Big)^{-1}, \\ \tilde{v}_{+,c} = \int_0^\infty \cos\lambda x v(x)\,dx, \quad \tilde{v}_{+,s} = \int_0^\infty \sin\lambda x v(x)\,dx, \\ k = \int_0^\infty v(x)\Big[\int_0^\infty \frac{\sin\lambda(x-s)}{\lambda} v(s)\,ds\Big]\,dx, \\ E = \Big(1 - \frac{1}{\lambda}\tilde{v}_{+,s}\Big)^{-1}, \quad A = (\hat{\alpha} + 2\tilde{v}_{+,c} + 2k)E, \quad B = \hat{\beta}E^{-1}\Big(1 + \frac{\hat{\beta}}{2}(\tilde{v}_{+,c} + k)\Big)^{-1}. \end{aligned} \tag{8.6}$$

Let us compare these values with those obtained for a potential well, where $v = v_{h,\varepsilon}(x) = -h$ for $-\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2}$ and $= v_{h,\varepsilon}(x) = 0$ for $|x| > \frac{\varepsilon}{2}$. In this case, the Schrödinger equation $-\psi''(x) + v_{h,\varepsilon}(x)\psi(x) = \lambda^2\psi(x)$ admits an explicit solution of the form (8.2) given by

$$\begin{aligned} a &= e^{-i\lambda\varepsilon}\Big[\cos\varepsilon\sqrt{h+\lambda^2} + \Big(\frac{-ih}{2\lambda} + i\lambda\Big)\frac{\sin\varepsilon\sqrt{h+\lambda^2}}{\varepsilon\sqrt{h+\lambda^2}}\Big] \approx 1 - \frac{h^2\varepsilon^4}{24} + \frac{ih\varepsilon}{2\lambda}\Big(1 - \frac{\varepsilon^2 h}{6}\Big), \\ b &= \frac{-ih\varepsilon}{2\lambda} \cdot \frac{\sin\varepsilon\sqrt{h+\lambda^2}}{\varepsilon\sqrt{h+\lambda^2}} \approx -\frac{ih\varepsilon}{2\lambda}\Big(1 - \frac{\varepsilon^2(h+\lambda^2)}{6}\Big). \end{aligned} \tag{8.7}$$

Formulas (8.6), for $v = v_{h,\varepsilon}(x)$ and $\hat{\alpha} = h\varepsilon$, $\hat{\beta} = \frac{1}{6}h\varepsilon^3$ give

$$a = 1 - \frac{h^2\varepsilon^4}{24} + \frac{ih\varepsilon}{2\lambda}\Big(1 - \frac{\varepsilon^2 h}{6}\Big), \quad b = -\frac{h^2\varepsilon^4}{24} - \frac{ih\varepsilon}{2\lambda}\Big(1 - \frac{\varepsilon^2(h+\lambda^2)}{6}\Big). \tag{8.8}$$

Comparing (8.4)–(8.7)–(8.8) we see that if the potential well has small width and large depth, the scattering problem is better modeled with a Schrödinger operator with nonlocal potential than with a point interaction.

References

1. S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer Verlag, Berlin, 1988; 2nd ed. with an Appendix by P. Exner, Chelsea, Amer. Math. Soc., Providence, RI, 2005.
2. S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators. Solvable Schrödinger Type Operators*, Cambridge University Press, Cambridge, 2000.
3. S. Albeverio, R. Hryniv, and L. Nizhnik, *Inverse spectral problems for nonlocal Sturm-Liouville operators*, Inverse Problems **23** (2007), 523–535.
4. S. Albeverio, S. Kuzhel, and L. Nizhnik, *On the perturbation theory of self-adjoint operators*, Tokyo Journal of Mathematics **31** (2008), no. 2, 273–292.
5. S. Albeverio, L. Nizhnik, *Schrödinger operators with nonlocal point interactions*, J. Math. Anal. Appl. **332** (2007), no. 2, 884–895.
6. F. A. Berezin and L. D. Faddeev, *Remarks on Schrödinger equation*, Soviet. Math. Doklady **137** (1961), 1011–1014.
7. V. A. Derkach and M. M. Malamud, *Generalized resolvents and boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), 1–95.
8. V. A. Derkach and M. M. Malamud, *Characteristic functions of almost solwable extensions of Hermitian operators*, Ukrainian Math. J. **44** (1992), 379–401.
9. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991.
10. M. Krasnosel'skii, *On self-adjoint extensions of Hermitian operators*, Ukrain. Mat. Zh. **1** (1949), 21–28.
11. A. N. Kochubei, *On extensions of nondensely defined symmetric operator*, *Sib. Math. J.* **18** (1977), no. 2, 314–320.
12. L. P. Nizhnik, *Inverse eigenvalue problems for nonlocal Sturm-Liouville operators*, Methods Funct. Anal. Topology **15** (2009), no. 1, 41–47.
13. L. P. Nizhnik, *Inverse nonlocal Sturm-Liouville problem*, Inverse Problems **26** (2010), 125006 (9 pp.).
14. L. Nizhnik, *Inverse spectral nonlocal problem for the first order ordinary differential equation*, Tamkang Journal of Mathematics **42** (2011), no. 3, 385–394.
15. L. P. Nizhnik, *Inverse eigenvalue problems for nonlocal Sturm-Liouville operators on a star graph*, *Methods Funct. Anal. Topology* **18** (2012), no. 1, 68–78.
16. F. Rofe-Beketov and A. Kholkin, *Spectral Analysis of Differential Operators. Interplay Between Spectral and Oscillatory Properties*, World Scientific Monograph Series in Mathematics, vol. 7, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

Institut für Angewandte Mathematik, Universität Bonn, Endenicherallee 60, D-53 115 Bonn; HCM, SFB 611, Universität Bonn; BiBoS (Universities of Bielefeld and Bonn)

E-mail address: albeverio@uni-bonn.de

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: nizhnik@imath.kiev.ua

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