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FACTORIZATIONS OF NONNEGATIVE SYMMETRIC OPERATORS

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Dedicated to F. S. Rofe-Beketov on the occasion of his 80-th birthday

ABSTRACT. We prove that each closed densely defined and nonnegative symmetric operator $\dot{A}$ having disjoint nonnegative self-adjoint extensions admits infinitely many factorizations of the form $\dot{A} = \mathcal{L}\mathcal{L}_0$, where $\mathcal{L}_0$ is a closed nonnegative symmetric operator and $\mathcal{L}$ its nonnegative self-adjoint extension. The same factorizations are also established for a non-densely defined nonnegative closed symmetric operator with infinite deficiency indices while for operator with finite deficiency indices we prove impossibility of such a kind factorization. A construction of pairs $\mathcal{L}_0 \subset \mathcal{L}$ ($\mathcal{L}_0$ is closed and densely defined, $\mathcal{L} = \mathcal{L}^* \geq 0$) having the property $\operatorname{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}$ (and, in particular, $\operatorname{dom}(\mathcal{L}_0^2) = \{0\}$) is given.

1. Introduction

Notations.

We use the symbols $\operatorname{dom}(T)$, $\operatorname{ran}(T)$, $\ker(T)$ for the domain, the range, and the null-subspace of a linear operator T. The closures of $\operatorname{dom}(T)$, $\operatorname{ran}(T)$ are denoted by $\overline{\operatorname{dom}}(T)$, $\overline{\operatorname{ran}}(T)$, respectively. The identity operator in a Hilbert space $\mathfrak{H}$ is denoted by I and sometimes by $I_{\mathfrak{H}}$. If $\mathfrak{L}$ is a subspace, i.e., a closed linear subset of $\mathfrak{H}$, the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{L}$ is denoted by $P_{\mathfrak{L}}$. The notation $T{\upharpoonright}\mathcal{N}$ means the restriction of a linear operator T to the set $\mathcal{N} \subset \operatorname{dom}(T)$. The linear space of bounded operators acting between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ is denoted by $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$ and the Banach algebra $\mathbf{L}(\mathfrak{H}, \mathfrak{H})$ by $\mathbf{L}(\mathfrak{H})$. A linear operator $\mathcal{A}$ in a Hilbert space is called nonnegative if $(\mathcal{A}f, f) \geq 0$ for all $f \in \operatorname{dom}(\mathcal{A})$. If M_1 and M_2 are linear operators acting from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ and from $\mathfrak{H}_2$ into $\mathfrak{H}_3$, respectively, then the product M_2M_1 we understand as follows:

$$\begin{aligned}\operatorname{dom}(M_2M_1) &= \{\varphi \in \operatorname{dom}(M_1) : M_1\varphi \in \operatorname{dom}(M_2)\},\\ (M_2M_1)\varphi &:= M_2(M_1\varphi), \quad \varphi \in \operatorname{dom}(M_2M_1).\end{aligned}$$

Let L_0 and L_1 be closed linear operators in a Hilbert space H taking values in a Hilbert space $\mathfrak{H}$ and possessing the condition

$$L_0 \subset L_1. \tag{1.1}$$

The operators $L_0^*L_0$ and $L_1^*L_1$ are self-adjoint and nonnegative in H. Since $L_1^* \subset L_0^*$, the following relations are valid:

$$\operatorname{dom}(L_1^*L_0) = \operatorname{dom}(L_0^*L_0) \cap \operatorname{dom}(L_1^*L_1) = \operatorname{dom}(L_0) \cap \operatorname{dom}(L_1^*L_1).$$

If

$$\operatorname{dom}(L_0^*L_0) \cap \operatorname{dom}(L_1^*L_1) \neq \{0\}, \tag{1.2}$$

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then the operator $\dot{A}$ defined as follows

$$\operatorname{dom}(\dot{A}) := \operatorname{dom}(L_1^* L_0), \quad \dot{A}f := L_1^* L_0 f, \quad f \in \operatorname{dom}(\dot{A}) \tag{1.3}$$

is closed, symmetric. Since $(\dot{A}f, f) = ||L_0 f||^2 \geq 0$ for all $f \in \operatorname{dom}(\dot{A})$, the operator $\dot{A}$ is nonnegative. Such kind of operators $\dot{A}$ we call *operators in divergence form.*

Observe that both operators $L_0^* L_0$ and $L_1^* L_1$ are nonnegative self-adjoint extensions of $\dot{A}$. In accordance with the first representation theorem [20] they are associated with the closed sesquilinear forms

$$\begin{aligned} \tau_0[\varphi, \psi] &= (L_0\varphi, L_0\psi)_{\mathfrak{H}}, \quad \varphi, \psi \in \operatorname{dom}(L_0), \\ \tau_1[u, v] &= (L_1 u, L_1 v)_{\mathfrak{H}}, \quad u, v \in \operatorname{dom}(L_1), \end{aligned}$$

respectively, and due to (1.1) the form τ_0 is a closed restriction of the form τ_1.

It is well known that if a linear manifold $\mathcal{D}$ is dense in a Banach space $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ is a subspace of $\mathcal{B}$ with finite co-dimension, then the linear manifold $\mathcal{D} \cap \widetilde{\mathcal{B}}$ is dense in $\widetilde{\mathcal{B}}$. Hence, if the condition

$$\dim\left(\operatorname{dom}(L_1)/\operatorname{dom}(L_0)\right) < \infty$$

is fulfilled, then (1.2) holds. Moreover, [7], [12], since $\operatorname{dom}(L_1^* L_0)$ is dense in $\operatorname{dom}(L_0)$ w.r.t. the graph norm in $\operatorname{dom}(L_0)$ we obtain that

(1) the operator $\dot{A} = L_1^* L_0$ has dense domain,
(2) the operator $L_0^* L_0$ is the *Friedrichs extension* of $\dot{A}$.

Recall that a densely defined nonnegative symmetric operator has at least one nonnegative self-adjoint extensions, the Friedrichs extension. M. G. Kreĭn established [21], [22] that the set of all nonnegative self-adjoint extensions of $\dot{A}$ forms the operator interval $[A_K, A_F]$ in the sense of quadratic forms [20], where the "minimal" operator A_K is discovered by Kreĭn. The operator A_K is called the *Kreĭn-von Neumann extension* (it is often called the *Kreĭn extension*).

Recall also that two self-adjoint extensions A_0 and A_1 of a closed densely defined symmetric operator $\dot{A}$ with equal deficiency indices are called *disjoint* (or relatively prime) if

$$\operatorname{dom}(A_0) \cap \operatorname{dom}(A_1) = \operatorname{dom}(\dot{A})$$

and *transversal* if, in addition,

$$\operatorname{dom}(A_0) + \operatorname{dom}(A_1) = \operatorname{dom}(\dot{A}^*).$$

Due to (1.3) the operators $A_0 = L_0^* L_0$ and $A_1 = L_1^* L_1$ are disjoint nonnegative self-adjoint extensions of $\dot{A}$. Now observe that

if a closed operator $\dot{A}$ is given by (1.3)*, where L_0 and L_1 satisfy* (1.1)*, then $\dot{A}$ admits the factorization*

$$\dot{A} = \mathcal{L}\mathcal{L}_0,$$

where $\mathcal{L}_0$ is a closed densely defined symmetric and nonnegative operator in H and $\mathcal{L}$ is its nonnegative self-adjoint extension.

Actually, define

$$\begin{aligned} \operatorname{dom}(\mathcal{L}) &:= \operatorname{dom}(L_1), \quad \mathcal{L}u := (L_1^* L_1)^{1/2} u, \quad u \in \operatorname{dom}(L_1), \\ \operatorname{dom}(\mathcal{L}_0) &:= \operatorname{dom}(L_0), \quad \mathcal{L}_0\varphi := (L_1^* L_1)^{1/2}\varphi, \quad \varphi \in \operatorname{dom}(L_0). \end{aligned}$$

One of the aim of this paper is to prove the following statement.

Theorem 1.1. *Let $\dot{A}$ be a densely defined closed nonnegative symmetric operator in a Hilbert space H having disjoint nonnegative self-adjoint extensions. Then $\dot{A}$ admits infinitely many factorizations of the form*

$$\dot{A} = \mathcal{L}\mathcal{L}_0,$$

where $\mathcal{L}_0$ is a densely defined closed nonnegative symmetric operator in H and $\mathcal{L}$ is nonnegative self-adjoint extension of $\mathcal{L}_0$. Moreover,

(1) *if the deficiency indices of $\dot{A}$ are finite, then it is necessary that the operator $\mathcal{L}_0^*\mathcal{L}_0$ coincides with the Friedrichs extension A_F of $\dot{A}$;*
(2) *if the deficiency indices of $\dot{A}$ are infinite, then the operator $\mathcal{L}_0$ can be chosen such that $\mathcal{L}_0^*\mathcal{L}_0$ coincides or does not coincide with the Friedrichs extension of $\dot{A}$;*
(3) *if $\dot{A}$ admits transversal nonnegative self-adjoint extensions and if $\mathcal{L}^2$ is transversal to A_F (in particular, if $\mathcal{L}^2$ coincides with the Kreĭn-von Neumann extensions of $\dot{A}$), then it is necessary that $\mathcal{L}_0^*\mathcal{L}_0$ is the Friedrichs extension of $\dot{A}$.*

If a closed symmetric operator $\dot{A}$ is non-densely defined, then its adjoint $\dot{A}^* = \{\langle x, x'\rangle\}$ is a linear relation (a subspace in $H \oplus H$) defined as follows:

$$(\dot{A}\varphi, x) = (\varphi, x') \quad \text{for all} \quad \varphi \in \operatorname{dom}(\dot{A}).$$

The Friedrichs extension of a non-densely defined closed nonnegative operator is not a linear operator. It is a linear relation [30], [32]. But it is possible that the minimal extension (the Kreĭn-von Neumann extension) is an operator [3]. For a non-densely defined case we prove the following analog of Theorem 1.1.

Theorem 1.2. *1) A non-densely defined closed nonnegative symmetric operator with finite deficiency indices does not admit representation in divergence form.*

2) A non-densely defined closed nonnegative symmetric operator $\dot{A}$ with infinite deficiency indices and having disjoint nonnegative self-adjoint extensions (operators) admits infinitely many factorizations

$$\dot{A} = \mathcal{L}\mathcal{L}_0,$$

where $\mathcal{L}_0$ is a densely defined closed nonnegative symmetric operator and $\mathcal{L}$ is nonnegative self-adjoint extension of $\mathcal{L}_0$.

In the proves of Theorem 1.1 and Theorem 1.2 we essentially use M. Kreĭn's approach [21], [22], [23] completed by Ando and Nishio [3] in the theory of nonnegative self-adjoint extensions of nonnegative symmetric operator. Notice that the inclusion $\dot{A} \subseteq L_1^*L_0$ for some special L_0 and L_1 provided conditions (1.1) and $A_F = L_0^*L_0$, $A_K = L_1^*L_1$ are established for densely defined nonnegative $\dot{A}$ in [29], [34], [11] and for nonnegative linear relations $\dot{A}$ in [18]. In [7] and [12] some properties of extensions of the operators in divergence form are established and applications to boundary value problems are given.

M. A. Naĭmark in [26], [27] found an example of a densely defined closed symmetric operator T whose square T^2 is zero defined, i.e., $\operatorname{dom}(T^2) = \{0\}$. A more concrete nonnegative symmetric operator with the same property is constructed in [14]. The results related to the powers of symmetric operators are obtained in [33]. In particular it is established [33, Theorem 5.2] that for each unbounded self-adjoint operator T there exist closed symmetric restrictions T_1 and T_2 of T such that

$$\operatorname{dom}(T_1) \cap \operatorname{dom}(T_2) = \{0\} \quad \text{and} \quad \operatorname{dom}(T_1^2) = \operatorname{dom}(T_2^2) = \{0\}.$$

In [13] it is shown that the above result remains true for a closed symmetric non-self-adjoint T.

In the present paper we give
an example of a densely defined closed nonnegative symmetric operator $\mathcal{L}_0$ and its nonnegative self-adjoint extension $\mathcal{L}$ such that $\operatorname{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}$. *In particular,* $\operatorname{dom}(\mathcal{L}_0^2) = \{0\}$.

For this purpose we construct two nonnegative unbounded self-adjoint operators A_0 and A in H such that

$$\operatorname{dom}(A_0) \cap \operatorname{dom}(A) = \{0\},$$
$$\operatorname{dom}(A_0^{1/2}) \subset \operatorname{dom}(A^{1/2}), \quad ||A_0^{1/2}\varphi|| = ||A^{1/2}\varphi||, \quad \varphi \in \operatorname{dom}(A_0^{1/2}).$$

Our construction is also based on the results in [21] related to the special kind of operators which are called nowadays the Kreĭn shorted operators.

It turns out that in our example the product $\mathcal{L}_0\mathcal{L}$ is densely defined.

2. The Kreĭn shorted operator

For every nonnegative bounded operator B in the Hilbert space $\mathcal{H}$ and every subspace $\mathcal{K} \subset \mathcal{H}$ M. G. Kreĭn [21] defined the operator $B_{\mathcal{K}}$ by the relation

$$B_{\mathcal{K}} = \max\{\, Z \in \mathbf{L}(\mathcal{H}) : 0 \le Z \le B,\ \mathrm{ran}\,(Z) \subseteq \mathcal{K}\,\}.$$

The equivalent definition is

$$(B_{\mathcal{K}} f, f) = \inf_{\varphi \in \mathcal{K}^\perp} \{(B(f+\varphi), f+\varphi)\}, \quad f \in \mathcal{H}. \tag{2.1}$$

Here $\mathcal{K}^\perp := \mathcal{H} \ominus \mathcal{K}$. The operator $B_{\mathcal{K}}$ is called the *shorted operator* (see [1, 2]). Let the subspace Ω be defined as follows:

$$\Omega = \{\, f \in \overline{\mathrm{ran}}\,(B) : B^{1/2} f \in \mathcal{K}\,\} = \overline{\mathrm{ran}}\,(B) \ominus B^{1/2}\mathcal{K}^\perp.$$

It is proved in [21] that $B_{\mathcal{K}}$ takes the form $B_{\mathcal{K}} = B^{1/2} P_\Omega B^{1/2}$. Hence, (see [21])

$$\mathrm{ran}\,(B_{\mathcal{K}}^{1/2}) = \mathrm{ran}\,(B^{1/2}) \cap \mathcal{K}. \tag{2.2}$$

It follows that

$$B_{\mathcal{K}} = 0 \iff \mathrm{ran}\,(B^{1/2}) \cap \mathcal{K} = \{0\}. \tag{2.3}$$

Let a bounded self-adjoint operator B is given by the block operator matrix

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} : \begin{array}{c} \mathcal{K} \\ \oplus \\ \mathcal{K}^\perp \end{array} \to \begin{array}{c} \mathcal{K} \\ \oplus \\ \mathcal{K}^\perp \end{array},$$

where $B_{11} \in \mathbf{L}(\mathcal{K})$, $B_{22} \in \mathbf{L}(\mathcal{K}^\perp)$, $B_{12} \in \mathbf{L}(\mathcal{K}^\perp, \mathcal{K})$. It is well known (see [23]) that *the operator B is nonnegative if and only if*

$$B_{22} \ge 0, \quad \mathrm{ran}\,(B_{12}^*) \subset \mathrm{ran}\,(B_{22}^{1/2}), \quad B_{11} \ge \left(B_{22}^{[-1/2]} B_{12}^*\right)^* \left(B_{22}^{[-1/2]} B_{12}^*\right)$$

and the operator $B_{\mathcal{K}}$ is given by the block matrix

$$B_{\mathcal{K}} = \begin{pmatrix} B_{11} - \left(B_{22}^{[-1/2]} B_{12}^*\right)^* \left(B_{22}^{[-1/2]} B_{12}^*\right) & 0 \\ 0 & 0 \end{pmatrix},$$

where $B^{-[1/2]}$ is the Moore-Penrose pseudo-inverse.

3. Nonnegative self-adjoint extensions of nonnegative symmetric operator

Let H be a separable Hilbert space and let $\dot{A}$ be a densely defined closed, symmetric, and nonnegative operator, i.e., $(\dot{A}f, f) \ge 0$ for all $f \in \mathrm{dom}\,(\dot{A})$. The Friedrichs extension A_F of $\dot{A}$ is defined as follows [20]. Denote by $\dot{A}[\cdot,\cdot]$ the closure of the sesquilinear for

$$\dot{A}[f,g] = (\dot{A}f, g), \quad f, g \in \mathrm{dom}\,(\dot{A}),$$

and let $\mathcal{D}[\dot{A}]$ be the domain of this closure. According to the first representation theorem [20] there exists a nonnegative self-adjoint operator A_F associated with $\dot{A}[\cdot,\cdot]$, i.e.,

$$(A_F h, \psi) = \dot{A}[h, \psi], \quad \psi \in \mathcal{D}[\dot{A}], \quad h \in \mathrm{dom}\,(A_F).$$

Clearly $\dot{A} \subset A_F \subset \dot{A}^*$, where $\dot{A}^*$ is adjoint to $\dot{A}$. It follows that

$$\mathrm{dom}\,(A_F) = \mathcal{D}[\dot{A}] \cap \mathrm{dom}\,(\dot{A}^*).$$

By the second representation theorem the equalities

$$\mathcal{D}[\dot{A}] = \operatorname{dom}(A_F^{1/2}) \quad \text{and} \quad \dot{A}[\phi, \psi] = (A_F^{1/2}\phi, A_F^{1/2}\psi), \quad \phi,\, \psi \in \mathcal{D}[\dot{A}]$$

hold. If A is a nonnegative self-adjoint operator, then

$$\mathcal{D}[A] = \operatorname{dom}(A^{1/2}), \quad A[u,v] = (A^{1/2}u, A^{1/2}v), \quad A[u] = ||A^{1/2}u||^2.$$

If A is a linear relation, then

$$\mathcal{D}[A] = \operatorname{dom}(A_{\mathrm{op}}^{1/2}), \quad A[u,v] = (A_{\mathrm{op}}^{1/2}u, A_{\mathrm{op}}^{1/2}v), \quad A[u] = ||A_{\mathrm{op}}^{1/2}u||^2,$$

where A_{op} is the operator part of A [31].

In his fundamental paper [21] M. Kreĭn reduced the problem of finding all nonnegative self-adjoint extensions for a nonnegative symmetric operator to the problem of self-adjoint contractive extensions (*sc*-extensions) for a given non-densely defined Hermitian contraction. He used the fact that the Cayley transform

$$S = (I - A)(I + A)^{-1}, \quad A = (I - S)(I + S)^{-1}$$

gives a one-to-one correspondence between closed densely defined nonnegative symmetric operators A in a Hilbert space H and non-densely defined closed symmetric contractions S such that $\ker(S + I) = \{0\}$. Moreover, the operator S is a self-adjoint if and only if A is self-adjoint.

Let $\dot{S}$ be a closed non-densely defined symmetric contraction in H. M. Kreĭn proved that the set of all *sc*-extensions of $\dot{S}$ forms an operator interval $[S_\mu, S_M]$. If

$$\dot{S} = (I - \dot{A})(I + \dot{A})^{-1}, \quad \operatorname{dom}(\dot{S}) = \operatorname{ran}(I + \dot{A}),$$

where $\dot{A}$ is a densely defined closed and nonnegative symmetric (non-self-adjoint) operator, then, as it is shown by M. Kreĭn, the Cayley transform

$$A_F = (I - S_\mu)(I + S_\mu)^{-1} \tag{3.1}$$

of the extremal extension (the "rigid" extension of $\dot{A}$ in M. Kreĭn terminology) coincides with the Friedrichs extension of $\dot{A}$. Another extremal nonnegative self-adjoint extension

$$A_K = (I - S_M)(I + S_M)^{-1} \tag{3.2}$$

was called by M. Kreĭn the "soft" extension of $\dot{A}$. It was proved in [21] that a nonnegative self-adjoint operator A is an extension of $\dot{A}$ if and only if for some $a > 0$ (then for all $a > 0$) hold the inequalities

$$(A_F + aI)^{-1} \le (A + aI)^{-1} \le (A_K + aI)^{-1}$$

or equivalently $A_K \le A \le A_F$ in the sense of corresponding quadratic forms [20], [21], i.e.,

$$\begin{aligned} &\mathcal{D}[\dot{A}] \subset \mathcal{D}[A] \subseteq \mathcal{D}[A_K], \\ &A[\varphi] = \dot{A}[\varphi] \quad \text{for all} \quad \varphi \in \mathcal{D}[\dot{A}], \\ &A[u] \ge A_K[u] \quad \text{for all} \quad u \in \mathcal{D}[A]. \end{aligned} \tag{3.3}$$

When $\dot{A}$ is positive definite, i.e., the lower bound of $\dot{A}$ is a positive number, it is shown in [21], [22] that

$$\operatorname{dom}(A_K) = \operatorname{dom}(\dot{A}) \dot{+} \ker(\dot{A}^*).$$

Thus, in that case the Kreĭn-von Neumann extension A_K coincides with self-adjoint extension constructed by J. von Neumann. Let $\mathfrak{N}_z := H \ominus \operatorname{ran}(\dot{A} - \bar{z}I)$ be the defect subspace of $\dot{A}$. For densely defined $\dot{A}$ one has

$$\mathfrak{N}_z = \ker(\dot{A}^* - zI).$$

Let A be a nonnegative self-adjoint extension of densely defined $\dot{A}$. It is established by M.G. Kreĭn [21] that the domain $\mathcal{D}[A]$ admits the decomposition

$$\mathcal{D}[A] = \mathcal{D}[\dot{A}]\dot{+}(\mathcal{D}[A] \cap \mathfrak{N}_{-a}) \tag{3.4}$$

for arbitrary $a > 0$. The operator $\dot{A}$ has unique nonnegative self-adjoint extension (see [21] for densely defined $\dot{A}$ and [5], [17] when $\dot{A}$ is a linear relation) if and only if for some $a > 0$ (and then for all $a > 0$) holds the condition

$$\sup_{f \in \operatorname{dom}(\dot{A})} \frac{|(f, \varphi_{-a})|^2}{(\dot{A}f, f)} = \infty \quad \text{for every} \quad \varphi_{-a} \in \mathfrak{N}_{-a} \setminus \{0\}.$$

This condition is equivalent to $\operatorname{ran}(A_F^{1/2}) \cap \mathfrak{N}_{-a} = \{0\}$.

Let S be any *sc*-extension of Hermitian contraction $\dot{S}$ and let $\mathfrak{N} = H \ominus \operatorname{dom}(\dot{S})$. The subspace $\mathfrak{N}$ coincides with defect subspace $\mathfrak{N}_{-1}$ of the operator $\dot{A}$. The operators S_μ and S_M can be defined by the relations [21]

$$S_\mu = S - (I + S)_{\mathfrak{N}}, \quad S_M = S + (I - S)_{\mathfrak{N}}. \tag{3.5}$$

Thus, extremal *sc*-extensions S_μ and S_M of $\dot{S}$ possess the properties

$$(I_{\mathfrak{H}} + S_\mu)_{\mathfrak{N}} = (I_{\mathfrak{H}} - S_M)_{\mathfrak{N}} = 0.$$

The operator interval $[S_\mu, S_M]$ can be parametrized as follows (see [23])

$$[S_\mu, S_M] \ni S \iff S = S_\mu + (S_M - S_\mu)^{1/2} X (S_M - S_\mu)^{1/2}, \tag{3.6}$$

where X is a nonnegative self-adjoint contraction in the subspace $\overline{\operatorname{ran}}\,(S_M - S_\mu)(\subseteq \mathfrak{N})$.

Basic Kreĭn's results remain true for non-densely defined closed nonnegative symmetric operators, for nonnegative linear relations, and for general case of sectorial operators and linear relations [3], [5], [6], [15], [17]. As it has been mentioned above, the Friedrichs extension of a non-densely defined nonnegative operator $\dot{A}$ is the linear relation [31]. It takes the form

$$A_F = Gr((P_{H_0}\dot{A})_F) \oplus \langle 0, \mathfrak{B} \rangle, \tag{3.7}$$

where $H_0 = \overline{\operatorname{dom}}\,(\dot{A})$, $\mathfrak{B} = H \ominus H_0$, and the operator $(P_{H_0}\dot{A})_F$ is the Friedrichs extension of the operator $P_{H_0}\dot{A}$ in the Hilbert space H_0. The linear relation A_F is connected with the minimal *sc*-extension S_μ of the contraction $\dot{S}$ by the Cayley transform (3.1)

$$A_F = \{\langle (I + S_\mu)h, (I - S_\mu)h \rangle, \ h \in H\}.$$

If $\dot{A}$ is bounded and non-densely defined with $\operatorname{dom}(\dot{A}) = H_0 \subset H$, then it admits bounded nonnegative self-adjoint extensions if and only if (see [35])

$$\sup_{\varphi \in H_0} \frac{||\dot{A}\varphi||^2}{(\dot{A}\varphi, \varphi)} < \infty.$$

If $\dot{A}$ is non-densely defined, then in general the Kreĭn-von Neumann nonnegative self-adjoint extension A_K is a linear relation. The relationship between A_K and S_M is given by the Cayley transform (3.2). A_K is the operator if and only if $\dot{A}$ is positively closable [3], i.e.,

$$\text{if} \quad \{\varphi_n\} \subset \operatorname{dom}(\dot{A}) \quad \text{and} \quad \lim_{n\to\infty} \dot{A}\varphi_n = g, \quad \lim_{n\to\infty} (\dot{A}\varphi_n, \varphi_n) = 0, \quad \text{then} \quad g = 0.$$

The domain $\mathcal{D}[A_K]$ can be characterized as follows [3]:

$$\mathcal{D}[A_K] = \left\{ u \in H : \sup_{\varphi \in \mathrm{dom}\,(\dot{A})} \frac{|(\dot{A}\varphi, u)|^2}{(\dot{A}\varphi, \varphi)} < \infty \right\},$$

$$A_K[u] = \sup_{\varphi \in \mathrm{dom}\,(\dot{A})} \frac{|(\dot{A}\varphi, u)|^2}{(\dot{A}\varphi, \varphi)}, \quad u \in \mathcal{D}[A_K].$$

For a non-densely defined $\dot{A}$ the decomposition (3.4) remains true for any arbitrary nonnegative self-adjoint extension A (possibly a linear relation) [5].

We will need the following proposition (see [9], [10]).

Proposition 3.1. *(1) Let B be a non-negative self-adjoint operator and let*

$$S = (I - B)(I + B)^{-1}$$

be its Cayley transform. Then

$$\mathcal{D}[B] = \mathrm{ran}\,((I + S)^{1/2}),$$

$$B[u, v] = -(u, v) + 2\left((I + S)^{-1/2}u, (I + S)^{-1/2}v\right), \quad u, v \in \mathcal{D}[B].$$

(2) Let $\dot{A}$ be a closed non-negative symmetric operator and let A be its non-negative self-adjoint extension (a linear relation, in general). If $\dot{S} = (I - \dot{A})(I + \dot{A})^{-1}$, $S = (I - A)(I + A)^{-1}$, then

$$\mathcal{D}[A] = \mathcal{D}[\dot{A}] \dotplus \mathrm{ran}\,((S - S_\mu)^{1/2}).$$

4. Disjointness and transversality of non-negative self-adjoint extensions

The disjointness of self-adjoint extensions A_0 and A_1 of a symmetric linear relation $\dot{A}$ means that $A_0 \cap A_1 = \dot{A}$, while A_0 and A_1 are transversal if the equality $A_0 + A_1 = \dot{A}^*$ is valid. Clearly, A_0 and A_1 are transversal implies A_0 and A_1 are disjoint. The following equivalences for two self-adjoint extensions A_1 and A_0 of $\dot{A}$ holds true :

$$\begin{aligned} A_1, A_0 \text{ are disjoint } &\iff \overline{\mathrm{ran}}\,\left((A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1}\right) = \mathfrak{N}_\lambda, \\ A_1, A_0 \text{ are transversal } &\iff \mathrm{ran}\,\left((A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1}\right) = \mathfrak{N}_\lambda \end{aligned} \tag{4.1}$$

for at least one (then for all) $\lambda \in \rho(A_1) \cap \rho(A_0)$. If the deficiency indices of $\dot{A}$ are finite (and equal), then two self-adjoint extensions of $\dot{A}$ are transversal if and only they are disjoint. The equivalences of statements in the next proposition can be found in [5], [6], [8], [18], [25].

Proposition 4.1. *Let $\dot{A}$ be a non-negative closed symmetric relation.*

(1) *The following statements are equivalent:*
   (a) *$\dot{A}$ has two disjoint nonnegative self-adjoint extensions,*
   (b) *the Friedrichs and Kreĭn - von Neumann extensions A_F and A_K are disjoint,*
   (c) *$\mathfrak{N}_z \cap \mathcal{D}[A_K]$ is dense in $\mathfrak{N}_z$ at least for one (then for all) $z \in \mathbb{C} \setminus [0, \infty)$,*
   (d) $\ker(S_M - S_\mu) = \mathrm{dom}\,(\dot{S})(= \mathrm{ran}\,(\dot{A} + I))$,
   (e) *from $\lim_{n\to\infty}(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}\varphi_n = g$ and $\lim_{n\to\infty}(\dot{A}\varphi_n, \varphi_n) = 0$ follows $g = 0$ (for densely defined $\dot{A}$).*

(2) *The conditions*
   (a) *$\dot{A}$ has two transversal nonnegative self-adjoint extensions,*
   (b) *the Friedrichs and Kreĭn extensions A_F and A_K are transversal,*
   (c) $\mathrm{dom}\,(A^*) \subset \mathcal{D}[A_K]$,
   (d) *$\mathfrak{N}_z \subset \mathcal{D}[A_K]$ at least for one (then for all) $z \in \mathbb{C} \setminus [0, \infty)$,*

(e) $\operatorname{ran}(S_M - S_\mu) = \mathfrak{N}(= \mathfrak{N}_{-1})$,

(f) $\sup\limits_{f\in\operatorname{dom}(\dot A)} \dfrac{||(I+\dot A\dot A^*)^{-1/2}\dot Af||^2}{(\dot Af,f)} < \infty$ *(for densely defined* $\dot A$*)*

are equivalent.

Now we get the following statement.

Proposition 4.2. ([18]). *If a non-densely defined nonnegative symmetric operator $\dot A$ admits disjoint nonnegative self-adjoint extensions, then the Kreĭn-von Neumann extension A_K of $\dot A$ is an operator.*

5. Nonnegative symmetric operator $\mathcal{L}_0$ and its nonnegative self-adjoint extension $\mathcal{L}$ such that $\operatorname{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}$.

Let H be a separable infinite-dimensional complex Hilbert space and let $\mathfrak{M}$ be an infinite-dimensional subspace of H with infinite-dimensional orthogonal complement $\mathfrak{M}^\perp = H \ominus \mathfrak{M}$. It is well known, see e.g. [28], [19], that there exist unbounded self-adjoint operators B_1 and B_2 on $\mathfrak{M}$ such that

$$\overline{\operatorname{dom}}(B_1) = \overline{\operatorname{dom}}(B_2) = \mathfrak{M}, \quad \operatorname{dom}(B_1)\cap\operatorname{dom}(B_2) = \{0\}.$$

Let $D_k = (B_k^*B_k)^{1/2}$, $k = 0,1$. Since $\operatorname{dom}(D_k) = \operatorname{dom}(B_k)$, $k = 1,2$, we get that $\operatorname{dom}(D_1)\cap\operatorname{dom}(D_2) = \{0\}$. Consequently, the operators

$$F = (I_\mathfrak{M} + D_1)^{-1}, \quad V = (I_\mathfrak{M} + D_2)^{-1}$$

possess the properties

$$\overline{\operatorname{ran}}(F) = \overline{\operatorname{ran}}(V) = \mathfrak{M}, \quad \operatorname{ran}(F)\cap\operatorname{ran}(V) = \{0\},$$
$$0 \le F \le I_\mathfrak{M}, \quad \ker(F) = \{0\}, \quad 0 \le V \le I_\mathfrak{M}, \quad \ker(V) = \{0\}.$$

Replace V with $U = V\Phi$, where Φ is a unitary operator from $\mathfrak{M}^\perp$ onto $\mathfrak{M}$. Let a self-adjoint bounded operator G in H be given by the operator matrix

$$G = \begin{bmatrix} I_\mathfrak{M} & U \\ U^* & U^*U \end{bmatrix} : \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}^\perp\end{array} \to \begin{array}{c}\mathfrak{M}\\ \oplus \\ \mathfrak{M}^\perp\end{array}.$$

Clearly

$$\ker(G) = \left\{ \begin{bmatrix} -Uh \\ h \end{bmatrix} : h \in \mathfrak{M} \right\}.$$

Define

$$X = \begin{bmatrix} F & 0 \\ 0 & I_{\mathfrak{M}^\perp} \end{bmatrix} G \begin{bmatrix} F & 0 \\ 0 & I_{\mathfrak{M}^\perp} \end{bmatrix} = \begin{bmatrix} F^2 & FU \\ U^*F & U^*U \end{bmatrix}.$$

Let us show that

$$\ker\{X\} = \{0\}, \quad X_\mathfrak{M} = 0, \quad X_{\mathfrak{M}^\perp} = 0. \tag{5.1}$$

Set $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where $f_1 \in \mathfrak{M}$, $f_2 \in \mathfrak{M}^\perp$. Then

$$(Xf,f) = ||Ff_1 + Uf_2||^2. \tag{5.2}$$

It follows that

$$Xf = 0 \iff Ff_1 + Uf_2 = 0.$$

Since $\operatorname{ran}(F)\cap\operatorname{ran}(U) = \{0\}$, $\ker(F) = \{0\}$, $\ker(U) = \{0\}$, we get $f_1 = 0$, $f_2 = 0$. From (5.2) and relations $\overline{\operatorname{ran}}(F) = \overline{\operatorname{ran}}(U) = \mathfrak{M}$ we get the equalities

$$\inf_{\varphi\in\mathfrak{M}^\perp}(X(f-\varphi), f-\varphi) = 0, \quad \inf_{\psi\in\mathfrak{M}}(X(f-\psi), f-\psi) = 0.$$

Equality (2.1) now implies $X_{\mathfrak{M}} = 0$ and $X_{\mathfrak{M}^\perp} = 0$. Applying (2.3) we obtain

$$\mathfrak{M} \cap \operatorname{ran}(X^{1/2}) = \{0\}, \quad \mathfrak{M}^\perp \cap \operatorname{ran}(X^{1/2}) = \{0\}. \tag{5.3}$$

Define in $H = \mathfrak{M} \oplus \mathfrak{M}^\perp$ the operator

$$X_0 = X^{1/2} P_{\mathfrak{M}} X^{1/2}. \tag{5.4}$$

Then

$$X - X_0 = X^{1/2} P_{\mathfrak{M}^\perp} X^{1/2}.$$

Equalities (5.1) yield

$$\ker(X_0) = \{0\}, \quad \ker(X - X_0) = \{0\}.$$

Notice that

$$\operatorname{ran}(X) \cap \operatorname{ran}(X_0) = \{0\}. \tag{5.5}$$

Actually, if $Xf = X_0 h$, then $X^{1/2} f = P_{\mathfrak{M}} X^{1/2} h$ and (5.3) yields $f = h = 0$.

Set

$$A = X^{-1}, \quad A_0 = X_0^{-1}.$$

The operators A_0 and A are nonnegative and self-adjoint in H. Relation (5.5) implies

$$\operatorname{dom}(A_0) \cap \operatorname{dom}(A) = \{0\}.$$

In addition

$$\operatorname{dom}(A_0^{1/2}) = \operatorname{dom}\left(X_0^{-1/2}\right), \quad \operatorname{dom}(A^{1/2}) = \operatorname{dom}\left(X^{-1/2}\right).$$

From (5.4) we get $\operatorname{ran}(X_0^{1/2}) \subset \operatorname{ran}(X^{1/2})$ and

$$X_0^{1/2} = X^{1/2} W_0,$$

where W_0 is unitary operator from H onto $\mathfrak{M}$. It follows that

$$X^{-1/2} g = W_0 X_0^{-1/2} g, \quad g \in \operatorname{ran}(X_0^{1/2}).$$

Hence, the pair $\langle A_0, A \rangle$ possess the property

$$\operatorname{dom}(A_0^{1/2}) \subset \operatorname{dom}(A^{1/2}) \quad \text{and} \quad ||A_0^{1/2}\varphi|| = ||A^{1/2}\varphi|| \quad \text{for all} \quad \varphi \in \operatorname{dom}(A_0^{1/2}).$$

Now define

$$\operatorname{dom}(\mathcal{L}) = \operatorname{dom}(A^{1/2}), \quad \mathcal{L}h = A^{1/2}h, \quad h \in \operatorname{dom}(\mathcal{L}),$$

$$\operatorname{dom}(\mathcal{L}_0) = \operatorname{dom}(A_0^{1/2}), \quad \mathcal{L}_0 g = A^{1/2} g, \quad g \in \operatorname{dom}(\mathcal{L}_0).$$

The operator $\mathcal{L}$ is self-adjoint and nonnegative, the operator $\mathcal{L}_0$ is densely defined, symmetric and nonnegative, and is a restriction of $\mathcal{L}$, i.e., $\mathcal{L}_0 \subset \mathcal{L}$. The sesquilinear form

$$\tau_0[\varphi, \psi] = (\mathcal{L}_0\varphi, \mathcal{L}_0\psi) = (A^{1/2}\varphi, A^{1/2}\psi) = (A_0^{1/2}\varphi, A_0^{1/2}\psi), \quad \varphi, \psi \in \operatorname{dom}(A_0^{1/2})$$

is closed. This implies that $\mathcal{L}_0$ is closed operator and the operator $\mathcal{L}_0^*\mathcal{L}_0 = A_0$ is associated with the form τ_0. In addition $\mathcal{L}^2 = A$. Since $\operatorname{dom}(\mathcal{L}_0^*\mathcal{L}_0) \cap \operatorname{dom}(\mathcal{L}^2) = \{0\}$, we get that

$$\operatorname{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}.$$

In particular, $\operatorname{dom}(\mathcal{L}_0^2) = \{0\}$.

Remark 5.1. The operators $\mathcal{L}$ and $\mathcal{L}_0$ are positive definite. It follows that

$$\operatorname{dom}(\mathcal{L}_0\mathcal{L}) = \mathcal{L}^{-1}\operatorname{dom}(\mathcal{L}_0), \quad (\mathcal{L}_0\mathcal{L})(\mathcal{L}^{-1}\varphi) = \mathcal{L}_0\varphi, \quad \varphi \in \operatorname{dom}(\mathcal{L}_0).$$

This yields, that $\operatorname{dom}(\mathcal{L}_0\mathcal{L})$ is dense in $\operatorname{dom}(\mathcal{L})$ w.r.t. the graph norm in $\operatorname{dom}(\mathcal{L})$. Hence, the operator $\mathcal{L}_0\mathcal{L}$ is densely defined in H and, moreover, $(\mathcal{L}_0\mathcal{L})_F = \mathcal{L}^2$.

Clearly, the equality $\ker((\mathcal{L}_0\mathcal{L})^*) = \ker(\mathcal{L}_0^*)$ holds true. Therefore, relations

$$\operatorname{dom}(\mathcal{L}_0^*) = \operatorname{dom}(\mathcal{L}) \dot{+} \ker(\mathcal{L}_0^*), \quad \operatorname{dom}((\mathcal{L}_0\mathcal{L})^*) = \operatorname{dom}(\mathcal{L}^2) \dot{+} \ker((\mathcal{L}_0\mathcal{L})^*)$$

lead to the equality $(\mathcal{L}_0\mathcal{L})^* = \mathcal{L}\mathcal{L}_0^*$. Since $\operatorname{ran}(\mathcal{L}) = H$, we get $(\mathcal{L}_0\mathcal{L})_K = \mathcal{L}_0\mathcal{L}_0^*$ (see [12, Theorem 3.1]).

Remark 5.2. We have constructed an example of two unbounded nonnegative self-adjoint operators A_0 and A such that

(1) $\operatorname{dom}(A_0) \cap \operatorname{dom}(A) = \{0\}$,
(2) $A_0 \geq A$ and the form $A_0[\cdot,\cdot]$ is a closed restriction of the form $A[\cdot,\cdot]$.

Remark 5.3. It is proved in [33] that for any closed unbounded densely defined operator $\mathcal{B}$ in H there exists a subspace $\mathfrak{L}$ such that

$$\mathfrak{L} \cap \operatorname{dom}(\mathcal{B}) = \mathfrak{L}^\perp \cap \operatorname{dom}(\mathcal{B}) = \{0\}.$$

Hence it follows that for any bounded nonnegative self-adjoint operator $\mathcal{F}$ with dense range $\operatorname{ran}(\mathcal{F})$ in H there exists a subspace $\mathfrak{L}$ such that

$$\mathfrak{L} \cap \operatorname{ran}(\mathcal{F}^{1/2}) = \mathfrak{L}^\perp \cap \operatorname{ran}(\mathcal{F}^{1/2}) = \{0\}.$$

For any subspace $\mathfrak{M}$, with $\dim(\mathfrak{M}) = \dim(\mathfrak{M}^\perp) = \infty$, we have constructed above a bounded nonnegative self-adjoint operator X with dense range such that $\mathfrak{M} \cap \operatorname{ran}(X^{1/2}) = \mathfrak{M}^\perp \cap \operatorname{ran}(X^{1/2}) = \{0\}$.

6. Proves of Theorems 1.1 and 1.2

6.1. **Auxiliary statements.** We start with two propositions.

Proposition 6.1. *For nonnegative self-adjoint unbounded operators A_0 and A_1 and their Cayley transformations $S_k = (I - A_k)(I + A_k)^{-1}$, $k = 0, 1$ the following statements are equivalent:*

(i) *the pair $\langle A_0, A_1\rangle$ possess the property*

$$\begin{aligned} &\operatorname{dom}(A_0^{1/2}) \subset \operatorname{dom}(A_1^{1/2}) \quad \textit{and} \\ &||A_0^{1/2}\varphi|| = ||A_1^{1/2}\varphi|| \quad \textit{for all} \quad \varphi \in \operatorname{dom}(A_0^{1/2}); \end{aligned} \tag{6.1}$$

(ii) *the pair $\langle S_0, S_1\rangle$ possess the property*

$$\begin{aligned} &S_1 \geq S_0 \quad \textit{and} \\ &\operatorname{ran}\left((I + S_0)^{1/2}\right) \cap \operatorname{ran}\left((S_1 - S_0)^{1/2}\right) = \{0\}; \end{aligned} \tag{6.2}$$

(iii) *the pair $\langle S_0, S_1\rangle$ possess the property*

$$I + S_0 = (I + S_1)^{1/2} P (I + S_1)^{1/2}, \tag{6.3}$$

where P is an orthogonal projection in H.

If in addition A_0 and A_1 both are extensions of a densely defined closed symmetric nonnegative operator $\dot{A}$, then each of the conditions (i), (ii), and (iii) is equivalent to the condition

$$\begin{aligned} &S_1 \geq S_0 \quad \textit{and} \\ &\operatorname{ran}\left((S_0 - S_\mu)^{1/2}\right) \cap \operatorname{ran}\left((S_1 - S_0)^{1/2}\right) = \{0\}, \end{aligned} \tag{6.4}$$

where $S_\mu = (I - A_F)(I + A_F)^{-1}$ and A_F is the Friedrichs extension of $\dot{A}$.

Proof. (i)⇒(ii) and (i)⇒(iii). From Proposition 3.1 follows that

$$||(I + S_1)^{-1/2}\varphi|| = ||(I + S_0)^{-1/2}\varphi|| \quad \text{for all} \quad \varphi \in \operatorname{ran}\left((I + S_0)^{1/2}\right).$$

Hence

$$(I + S_1)^{-1/2}\varphi = \mathcal{V}(I + S_0)^{-1/2}\varphi, \quad \varphi \in \operatorname{ran}\left((I + S_0)^{1/2}\right),$$

where $\mathcal{V}$ is a isometry in H, $\operatorname{ran}(\mathcal{V}) = (I + S_1)^{-1/2}\operatorname{ran}\left((I + S_0)^{1/2}\right)$. Then

$$(I + S_0)^{1/2} = (I + S_1)^{1/2}\mathcal{V}$$

and

$$I + S_0 = (I + S_1)^{1/2}\mathcal{V}\mathcal{V}^*(I + S_1)^{1/2} = (I + S_1)^{1/2}P_{\mathrm{ran}\,(\mathcal{V})}(I + S_1)^{1/2},$$

where $P_{\mathrm{ran}\,(\mathcal{V})}$ is the orthogonal projection in H onto $\mathrm{ran}\,(\mathcal{V})$, i.e., (6.3) holds. It follows that

$$S_1 - S_0 = (I + S_1) - (I + S_0) = (I + S_1)^{1/2}(I - P_{\mathrm{ran}\,(\mathcal{V})})(I + S_1)^{1/2}.$$

We recall that if bounded self-adjoint nonnegative operators X and Y are connected by the relation $X = Y^{1/2}\mathcal{Z}Y^{1/2}$, where $\mathcal{Z} \in \mathbf{L}(\overline{\mathrm{ran}}\,(Y))$ is a nonnegative operator, then (see [16])

$$\mathrm{ran}\,(X^{1/2}) = Y^{1/2}\mathrm{ran}\,(\mathcal{Z}^{1/2}).$$

Therefore, $\mathrm{ran}\,((S_1 - S_0)^{1/2}) = (I + S_1)^{1/2}(H \ominus \mathrm{ran}\,(\mathcal{V}))$ and (6.2) holds.

Clearly, (iii)⇒(ii).

Let us show (ii)⇒(iii) and (ii)⇒(i). Since $I + S_1 \geq I + S_0$, the equality

$$I + S_0 = (I + S_1)^{1/2}P(I + S_1)^{1/2}$$

is valid with $0 \leq P \leq I$. The equality $S_1 - S_0 = (I + S_1) - (I + S_0)$ yields

$$S_1 - S_0 = (I + S_1)^{1/2}(I - P)(I + S_1)^{1/2}.$$

Due to (6.2) we get now

$$\mathrm{ran}\,\left((I - P)^{1/2}\right) \cap \mathrm{ran}\,(P^{1/2}) = \{0\}.$$

Finally, since

$$\mathrm{ran}\,\left((I - P)^{1/2}\right) \cap \mathrm{ran}\,(P^{1/2}) = \mathrm{ran}\,\left((P - P^2)^{1/2}\right),$$

we get that $P^2 = P$, i.e., P is an orthogonal projection in H. Thus, (6.3) holds. From (6.3) we obtain

$$(I + S_0)^{1/2}h = (I + S_1)^{1/2}\mathcal{U}h, \quad h \in H,$$

where $\mathcal{U}$ is unitary operator from H onto $\mathrm{ran}\,(P)$. Hence

$$(I + S_1)^{-1/2}g = \mathcal{U}(I + S_0)^{-1/2}g \quad \text{for all} \quad g \in \mathrm{ran}\,\left((I + S_0)^{1/2}\right).$$

Thus

$$||(I + S_1)^{-1/2}g||^2 = ||(I + S_0)^{-1/2}g||^2, \quad g \in \mathrm{ran}\,\left((I + S_0)^{1/2}\right). \tag{6.5}$$

Now (6.1) follows from Proposition 3.1 and (6.5).

Suppose A_0 and A_1 both are extensions of a densely defined closed symmetric nonnegative operator $\dot{A}$. Let $\dot{S} = (I - \dot{A})(I + \dot{A})^{-1}$ and let $\mathfrak{N} = H \ominus \mathrm{dom}\,(\dot{S})$. Applying (3.5) and (2.2) we get

$$\mathfrak{N} \cap \mathrm{ran}\,\left((I + S_0)^{1/2}\right) = \mathrm{ran}\,\left((S_0 - S_\mu)^{1/2}\right).$$

This yields the equivalence of (6.4) and (6.2). □

Remark 6.2. Relation (6.4) is equivalent to the statement: S_0 is an extremal point of the operator interval $[S_\mu, S_1]$ (see [10] and references therein).

Let $\dot{S}$ be a non-densely defined closed symmetric contraction. For simplicity we denote

$$C := S_M - S_\mu.$$

Using (3.6), one can get that if $S_k = S_\mu + C^{1/2}X_kC^{1/2}$, $k = 0, 1$ are two *sc*-extensions of $\dot{S}$, where X_k, $k = 0, 1$ are nonnegative self-adjoint contractions in $\overline{\mathrm{ran}}\,(C)$, then

$$\begin{aligned} & S_0 \le S_1 \iff X_0 \le X_1, \\ & \mathrm{ran}\,\left((S_0 - S_\mu)^{1/2}\right) \cap \mathrm{ran}\,\left((S_1 - S_0)^{1/2}\right) = \{0\} \\ & \qquad \iff \mathrm{ran}\,(X_0^{1/2}) \cap \mathrm{ran}\,((X_1 - X_0)^{1/2}) = \{0\} \\ & \qquad\qquad \iff X_0 = X_1^{1/2} P X_1^{1/2}, \end{aligned} \tag{6.6}$$

where P is an orthogonal projection in $\overline{\mathrm{ran}}\,(C)$.

Proposition 6.3. *Let $\dot{A}$ be a densely defined closed symmetric nonnegative operator in H having disjoint nonnegative self-adjoint extensions. Then there is a one-to-one correspondence between all factorizations of $\dot{A}$ in the form $\dot{A} = \mathcal{L}\mathcal{L}_0$, where $\mathcal{L}_0$ is a nonnegative densely defined closed symmetric operator in H and $\mathcal{L}$ its nonnegative self-adjoint extension, and all pairs $\langle A_0, A_1\rangle$ of disjoint nonnegative self-adjoint extensions of $\dot{A}$, satisfying condition* (6.1). *This correspondence is given by the relations*

$$\begin{aligned} \mathrm{dom}\,(\mathcal{L}) = \mathrm{dom}\,(A_1^{1/2}), \quad & \mathcal{L}u = A_1^{1/2}u, \quad u \in \mathrm{dom}\,(\mathcal{L}), \\ \mathrm{dom}\,(\mathcal{L}_0) = \mathrm{dom}\,(A_0^{1/2}), \quad & \mathcal{L}_0\varphi = A_1^{1/2}\varphi, \quad \varphi \in \mathrm{dom}\,(\mathcal{L}_0). \end{aligned} \tag{6.7}$$

Proof. Let $\dot{A} = \mathcal{L}\mathcal{L}_0$ be a factorization of $\dot{A}$, where $\mathcal{L}_0$ is a nonnegative densely defined closed symmetric operator in H and $\mathcal{L}$ its nonnegative self-adjoint extension. Then the operators $A_0 = \mathcal{L}_0^*\mathcal{L}_0$, $A_1 = \mathcal{L}^2$ are disjoint nonnegative self-adjoint extensions of $\dot{A}$ and (6.1) holds. Therefore, (6.7) is valid.

Conversely, if a pair $\langle A_0, A_1\rangle$ of disjoint nonnegative self-adjoint extensions of $\dot{A}$, satisfying condition (6.1), is given, then define the pair $\langle \mathcal{L}_0, \mathcal{L}\rangle$ by (6.7). Clearly, $\mathcal{L}^2 = A_1$ and from (6.1) follows that $\mathcal{L}_0^*\mathcal{L}_0 = A_0$. In addition due to $\mathrm{dom}\,(A_0) \cap \mathrm{dom}\,(A_1) = \mathrm{dom}\,(\dot{A})$, we get $\dot{A} = \mathcal{L}\mathcal{L}_0$. □

6.2. **Proof of Theorem 1.1.** Let

$$\dot{S} = (I - \dot{A})(I + \dot{A})^{-1},$$
$$S_\mu = (I - A_F)(I + A_F)^{-1}, \quad S_M = (I - A_K)(I + A_K)^{-1}$$

be the Cayley transforms of $\dot{A}$, A_F and A_K, respectively. Since A_F, and A_K are disjoint, we have $\ker\,(C) = \mathrm{dom}\,(\dot{S})$.

Defect of $\dot{A}$ is finite. Then $n := \dim\,(\mathfrak{N}) < \infty$ and $\mathrm{ran}\,(C) = \mathfrak{N}$. Suppose $\dot{A}$ is factorized as $\dot{A} = \mathcal{L}\mathcal{L}_0$, where $\mathcal{L}_0$ is closed densely defined nonnegative symmetric operator and $\mathcal{L}$ is its self-adjoint extension. Since $A_0 = \mathcal{L}_0^*\mathcal{L}_0$ and $A_1 = \mathcal{L}^2$ are nonnegative self-adjoint extensions of $\dot{A}$ and

$$\mathrm{dom}\,(\dot{A}) = \mathrm{dom}\,(A_0) \cap \mathrm{dom}\,(A_1),$$

from (4.1), (3.4), and the relations

$$\mathrm{dom}\,(\mathcal{L}_0) = \mathrm{dom}\,(A_0^{1/2}) \subset \mathrm{dom}\,(A_1^{1/2}) = \mathrm{dom}\,(\mathcal{L})$$

it follows that the deficiency indices of $\mathcal{L}_0$ are $\langle n, n\rangle$ and $A_0 = \mathcal{L}_0^*\mathcal{L}_0$ is the Friedrichs extension of $\dot{A}$ [7].

In order to construct a factorization let take an arbitrary A_1 transversal to A_F and let $A_0 = A_F$. Then due to (3.3) the sesquilinear form $A_F[\cdot,\cdot] = \dot{A}[\cdot,\cdot]$ is a closed restriction of the form $A_1[\cdot,\cdot]$. Further we use (6.7).

Defect of $\dot{A}$ is infinite. In this case $\dim\,(\mathfrak{N}) = \infty$. By Proposition 4.1 we have $\overline{\mathrm{ran}}\,(C) = \mathfrak{N}$. Due to Propositions 6.1 and 6.3 we need to describe *all pairs $\langle S_0, S_1\rangle$ of sc-extensions of $\dot{S}$, satisfying* (6.4) *and such that* $\ker\,(S_1 - S_0) = \mathrm{dom}\,(\dot{S})$.

Let $S_k = S_\mu + C^{1/2}X_kC^{1/2}$, $k = 0, 1$, and $0 \le X_0 \le X_1 \le I_{\mathfrak{N}}$. According to (6.6) the operator X_0 takes the form

$$X_0 = X_1^{1/2} P X_1^{1/2},$$

where P is an orthogonal projection with $\operatorname{ran}(P) \subset \mathfrak{N}$. We need to find such P that $\ker(S_1 - S_0) = \operatorname{dom}(\dot{S})$. We have

$$S_1 - S_0 = C^{1/2}(X_1 - X_0)C^{1/2} = C^{1/2}X_1^{1/2}(I_{\mathfrak{N}} - P)X_1^{1/2}C^{1/2},$$

$$||(S_1 - S_0)^{1/2}h||^2 = ||(I_{\mathfrak{N}} - P)X_1^{1/2}C^{1/2}h||^2, \quad h \in H$$

and

$$\mathfrak{N} \ni h \in \ker(S_1 - S_0) \iff X_1^{1/2}C^{1/2}h \in \operatorname{ran}(P).$$

Therefore

$$\ker(S_1 - S_0) = \operatorname{dom}(\dot{S}) \iff \begin{cases} \ker(X_1) \cap \operatorname{ran}(C^{1/2}) = \{0\}, \\ \operatorname{ran}\left(X_1^{1/2}C^{1/2}\right) \cap \operatorname{ran}(P) = \{0\}. \end{cases} \tag{6.8}$$

The choice of X_1 depends on the case: $\operatorname{ran}(C) = \mathfrak{N}$ or $\operatorname{ran}(C) \ne \mathfrak{N}$. Recall that $\overline{\operatorname{ran}}(C) = \mathfrak{N}$.

In the case $\operatorname{ran}(C) = \mathfrak{N}$ *(* $\iff$ *A_F and A_K are transversal)* there is an equivalence

$$\ker(X_1) \cap \operatorname{ran}(C^{1/2}) = \{0\} \iff \ker(X_1) = \{0\}.$$

If $\operatorname{ran}(X_1) = \mathfrak{N}$, then it is only one possibility to satisfy conditions

$$\operatorname{ran}(P) \cap \operatorname{ran}(X_1^{1/2}) = \{0\}$$

is to choose $P = 0$. This means that $X_0 = 0$, i.e., $S_0 = S_\mu$ and $A_0 = A_F$. In particular,

$$X_1 = I_{\mathfrak{N}} \iff S_1 = S_M \iff A_1 = A_K \Rightarrow A_0 = A_F.$$

If $\ker(X_1) = \{0\}$ and $\operatorname{ran}(X_1) \ne \mathfrak{N}$, then it is possible to choose a nontrivial subspace $\mathfrak{M}$ in $\mathfrak{N}$ such that

$$\mathfrak{M} \cap \operatorname{ran}(X_1^{1/2}) = \{0\}$$

and $X_0 = X_1^{1/2}P_{\mathfrak{M}}X_1^{1/2}$. If we take $\mathfrak{M} = \{0\}$, then we get $A_0 = A_F$.

In the case $\operatorname{ran}(C) \ne \mathfrak{N}$ it is also possible to choose X_1 satisfying conditions in (6.8). For example, one can take $X_1 \in [0, I_{\mathfrak{N}}]$ with $\ker(X_1) = \{0\}$ and then take $\mathfrak{M} \subset \mathfrak{N}$ such that $\mathfrak{M} \cap \left(X^{1/2}\operatorname{ran}(C^{1/2})\right) = \{0\}$. In particular,

$$X_1 = I_{\mathfrak{N}} \iff S_1 = S_M \iff A_1 = A_K \Rightarrow S_0 = S_\mu + C^{1/2}P_{\mathfrak{M}}C^{1/2},$$

where $\mathfrak{M}$ is a subspace in $\mathfrak{N}$ and $\mathfrak{M} \cap \operatorname{ran}(C^{1/2}) = \{0\}$. The proof is complete.

Let us make a few remarks.

1) As it is follows from the proof, the operator $\mathcal{L}_0 = \mathcal{L}{\restriction}\operatorname{dom}(A_0^{1/2})$ depends on the choice of

- disjoint to A_F a nonnegative self-adjoint extension operator $A_1 (= \mathcal{L}^2)$,
- nonnegative self-adjoint extension A_0, which is disjoint with A_1 and possess property (6.1).

The minimal domain $\operatorname{dom}(\mathcal{L}_0)$ of symmetric operators $\mathcal{L}_0$ coincides with $\mathcal{D}[\dot{A}] = \operatorname{dom}(A_F^{1/2})$.

2) Due to [12, Theorem 3.1] if $\dot{A}^* = \mathcal{L}_0^*\mathcal{L}$, then $A_F = \mathcal{L}_0^*\mathcal{L}_0$. In addition, in that case the Friedrichs and Kreĭn - von Neumann extensions of $\dot{A}$ are transversal. Therefore, if the Friedrichs and Kreĭn - von Neumann extensions of $\dot{A}$ are disjoint and not transversal, then for each representation $\dot{A} = \mathcal{L}\mathcal{L}_0$ the adjoint operator $\dot{A}^*$ is not equal to $\mathcal{L}_0^*\mathcal{L}$. On the other hand if the Friedrichs and Kreĭn - von Neumann extensions of $\dot{A} = \mathcal{L}\mathcal{L}_0$ are transversal and $A_F \ne \mathcal{L}_0^*\mathcal{L}_0$, then also $\dot{A}^* \ne \mathcal{L}_0^*\mathcal{L}$.

3) A nonnegative self-adjoint extension $\widetilde{A}$ of $\dot{A}$ is called *extremal* [4] if

$$\inf_{\varphi\in\operatorname{dom}(\dot{A})} (\widetilde{A}(f-\varphi), f-\varphi) = 0 \quad \text{for all} \quad f \in \operatorname{dom}(\widetilde{A}).$$

Extensions A_F and A_K are extremal. Suppose A_F and A_K are not transversal (but disjoint). Then, if we select $A_1 = A_K (= \mathcal{L}^2)$, a nonnegative self-adjoint extension $A_0 (= \mathcal{L}_0^*\mathcal{L}_0)$ should be taken such that it is extremal and disjoint with A_K. The Cayley transform $S_0 = (I - A_0)(I + A_0)^{-1}$ is of the form

$$S_0 = S_\mu + C^{1/2} P_{\mathfrak{M}} C^{1/2},$$

where $P_{\mathfrak{M}}$ is the orthogonal projection onto a subspace $\mathfrak{M}$ in $\mathfrak{N}$ and $\mathfrak{M}\cap\operatorname{ran}(C^{1/2}) = \{0\}$ (see the end of the proof of Theorem 1.1).

4) In [30, Corollary to Theorem X.25] it is stated without proof that if $\mathcal{L}_0$ is a symmetric operator whose square $\mathcal{L}_0^2$ is densely defined, then the Friedrichs extensions $(\mathcal{L}_0^2)_F$ of $\mathcal{L}_0^2$ is the operator $\mathcal{L}_0^*\mathcal{L}_0$. This result is true if one of the deficiency indices of $\mathcal{L}_0$ is finite (this follows from [12, Proposition 3.3]). Another sufficient condition of the equality $(\mathcal{L}_0^2)_F = \mathcal{L}_0^*\mathcal{L}_0$ (for densely defined $\mathcal{L}_0^2$) is the relation $(\mathcal{L}_0^2)^* = \mathcal{L}_0^{*2}$ (see [24]). On the other hand as it is follows from [33, Theorem 4.5] for any unbounded self-adjoint operator $\mathcal{L}$ there exists a closed densely defined symmetric restriction $\mathcal{L}_0$ such that $\mathcal{L}_0^2$ is densely defined but $\operatorname{dom}(\mathcal{L}_0^2)$ is not dense in $\operatorname{dom}(\mathcal{L}_0)$ w.r.t. the graph norm, i.e., $(\mathcal{L}_0^2)_F \ne \mathcal{L}_0^*\mathcal{L}_0$. Due to Theorem 1.1 if $\dot{A} = \mathcal{L}\mathcal{L}_0$ and the Friedrichs extensions of $\dot{A}$ does not coincide with $\mathcal{L}_0^*\mathcal{L}_0$, then from the assumption that $\mathcal{L}_0^2$ is densely defined follows: $\mathcal{L}_0^*\mathcal{L}_0$ does not coincide with the Friedrichs extension of $\mathcal{L}_0^2$.

6.3. **Proof of Theorem 1.2.** 1) Let $\dot{A}$ has finite deficiency indices $\langle n, n\rangle$. Then for two nonnegative self-adjoint extensions A_0 and A_1 such that $A_0 \ge A_1$ from (3.4) it follows

$$\dim(\mathcal{D}[A_1]/\mathcal{D}[A_0]) \le n.$$

Suppose $\dot{A} = L_1^* L_0$, where L_0 is closed and densely defined in H and L_1 is a closed extension of L_0 in H. Put $A_0 = L_0^* L_0$, $A_1 = L_1^* L_1$. Then $\dim(\operatorname{dom}(L_1)/\operatorname{dom}(L_0)) \le n$. This yields that $\operatorname{dom}(L_1^* L_0)$ is dense in H. Contradiction.

2) Let $\dot{A}$ has infinite defect numbers. Since $\dot{A}$ admits disjoint nonnegative self-adjoint extensions (operators), we get $\ker(C) = \operatorname{dom}(\dot{S})$ (see Proposition 4.2). Recall that $C = S_M - S_\mu$. Note that the Kreĭn-von Neumann extension A_K is the operator. This means that $\ker(I + S_M) = \{0\}$. Let

$$S_1 = S_\mu + C^{1/2} X_1 C^{1/2}, \quad 0 \le X_1 \le I_{\mathfrak{N}}$$

be *sc*-extension of $\dot{S}$. Using the equality $I + S_1 = (I + S_\mu) + C^{1/2} X_1 C^{1/2}$ and (3.7) we get that

$$\ker(I + S_1) = \{0\} \iff \ker(X_1) \cap C^{1/2}\mathfrak{B} = \{0\},$$

where $\mathfrak{B} = H \ominus \overline{\operatorname{dom}}(\dot{A})$. It follows that if, in particular, $\ker(X_1) = \{0\}$, then $\ker(I + S_1) = \{0\}$. Let P be an orthogonal projection in H, $\operatorname{ran}(P) \subset \mathfrak{N}$. Put $X_0 = X_1^{1/2} P X_1^{1/2}$ and let

$$S_0 = S_\mu + C^{1/2} X_0 C^{1/2} = S_\mu + C^{1/2} X_1^{1/2} P X_1^{1/2} C^{1/2}.$$

We need to satisfy also the following conditions:

$$\ker(S_1 - S_0) = \{0\}, \quad \ker(I + S_0) = \{0\}.$$

Therefore, (see (6.8))

$$\begin{cases} \ker(X_1) \cap \operatorname{ran}(C^{1/2}) = \{0\}, \\ \operatorname{ran}\left(X_1^{1/2} C^{1/2}\right) \cap \operatorname{ran}(P) = \{0\} \end{cases}$$

and

$$(\mathfrak{N} \ominus \operatorname{ran}(P)) \cap X_1^{1/2} C^{1/2} \mathfrak{B} = \{0\}.$$

So if we construct an operator $X_1 \in [0, I_{\mathfrak{N}}]$ and a subspace $\mathfrak{M} \subset \mathfrak{N}$ such that

$$\ker(X_1) = \{0\}, \quad \operatorname{ran}(X_1) \neq \mathfrak{N}, \quad \operatorname{ran}(X_1^{1/2}) \cap \mathfrak{M} = \operatorname{ran}(X_1^{1/2}) \cap (\mathfrak{N} \ominus \mathfrak{M}) = \{0\},$$

then we obtain nonnegative self-adjoint extensions

$$A_k = (I - S_k)(I + S_k)^{-1}, \quad k = 0, 1$$

of the operator $\dot{A}$, satisfying conditions in (6.1). For a construction of such X_1 we can repeat the construction in Section 5 or to use the result in [33] (see Remark 5.3). The proof is complete.

In the case $\operatorname{ran}(C) \neq \mathfrak{N}$ we can take $X_1 = I_{\mathfrak{N}}$, that is equivalent to the selection $S_1 = S_M \iff A_1 = A_K$. Then we can find (see Remark 5.3) a subspace $\mathfrak{M} \subset \mathfrak{N}$ such that

$$\mathfrak{M} \cap \operatorname{ran}(C^{1/2}) = \{0\}, \quad (\mathfrak{N} \ominus \mathfrak{M}) \cap \operatorname{ran}(C^{1/2}) = \{0\}.$$

Hence, $S_0 = S_\mu + C^{1/2} P_{\mathfrak{M}} C^{1/2}$ and $A_0 = (I - S_0)(I + S_0)^{-1}$ is extremal extension of $\dot{A}$.

Notice that boundedness of $\dot{A}$ is possible. So, a bounded $\dot{A}$ having nonnegative self-adjoint operator extension admits factorization $\dot{A} = \mathcal{L}\mathcal{L}_0$ with unbounded $\mathcal{L}_0$ and $\mathcal{L}$.

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