

## FACTORIZATIONS OF NONNEGATIVE SYMMETRIC OPERATORS

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*Dedicated to F. S. Rofe-Beketov on the occasion of his 80-th birthday*

ABSTRACT. We prove that each closed densely defined and nonnegative symmetric operator  $\dot{A}$  having disjoint nonnegative self-adjoint extensions admits infinitely many factorizations of the form  $\dot{A} = \mathcal{L}\mathcal{L}_0$ , where  $\mathcal{L}_0$  is a closed nonnegative symmetric operator and  $\mathcal{L}$  its nonnegative self-adjoint extension. The same factorizations are also established for a non-densely defined nonnegative closed symmetric operator with infinite deficiency indices while for operator with finite deficiency indices we prove impossibility of such a kind factorization. A construction of pairs  $\mathcal{L}_0 \subset \mathcal{L}$  ( $\mathcal{L}_0$  is closed and densely defined,  $\mathcal{L} = \mathcal{L}^* \geq 0$ ) having the property  $\text{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}$  (and, in particular,  $\text{dom}(\mathcal{L}_0^2) = \{0\}$ ) is given.

### 1. INTRODUCTION

#### Notations.

We use the symbols  $\text{dom}(T)$ ,  $\text{ran}(T)$ ,  $\ker(T)$  for the domain, the range, and the null-subspace of a linear operator  $T$ . The closures of  $\text{dom}(T)$ ,  $\text{ran}(T)$  are denoted by  $\overline{\text{dom}(T)}$ ,  $\overline{\text{ran}(T)}$ , respectively. The identity operator in a Hilbert space  $\mathfrak{H}$  is denoted by  $I$  and sometimes by  $I_{\mathfrak{H}}$ . If  $\mathfrak{L}$  is a subspace, i.e., a closed linear subset of  $\mathfrak{H}$ , the orthogonal projection in  $\mathfrak{H}$  onto  $\mathfrak{L}$  is denoted by  $P_{\mathfrak{L}}$ . The notation  $T \upharpoonright \mathcal{N}$  means the restriction of a linear operator  $T$  to the set  $\mathcal{N} \subset \text{dom}(T)$ . The linear space of bounded operators acting between Hilbert spaces  $\mathfrak{H}$  and  $\mathfrak{K}$  is denoted by  $\mathbf{L}(\mathfrak{H}, \mathfrak{K})$  and the Banach algebra  $\mathbf{L}(\mathfrak{H}, \mathfrak{H})$  by  $\mathbf{L}(\mathfrak{H})$ . A linear operator  $\mathcal{A}$  in a Hilbert space is called nonnegative if  $(\mathcal{A}f, f) \geq 0$  for all  $f \in \text{dom}(\mathcal{A})$ . If  $M_1$  and  $M_2$  are linear operators acting from  $\mathfrak{H}_1$  into  $\mathfrak{H}_2$  and from  $\mathfrak{H}_2$  into  $\mathfrak{H}_3$ , respectively, then the product  $M_2M_1$  we understand as follows:

$$\begin{aligned} \text{dom}(M_2M_1) &= \{\varphi \in \text{dom}(M_1) : M_1\varphi \in \text{dom}(M_2)\}, \\ (M_2M_1)\varphi &:= M_2(M_1\varphi), \quad \varphi \in \text{dom}(M_2M_1). \end{aligned}$$

Let  $L_0$  and  $L_1$  be closed linear operators in a Hilbert space  $H$  taking values in a Hilbert space  $\mathfrak{H}$  and possessing the condition

$$(1.1) \quad L_0 \subset L_1.$$

The operators  $L_0^*L_0$  and  $L_1^*L_1$  are self-adjoint and nonnegative in  $H$ . Since  $L_1^* \subset L_0^*$ , the following relations are valid:

$$\text{dom}(L_1^*L_0) = \text{dom}(L_0^*L_0) \cap \text{dom}(L_1^*L_1) = \text{dom}(L_0) \cap \text{dom}(L_1^*L_1).$$

If

$$(1.2) \quad \text{dom}(L_0^*L_0) \cap \text{dom}(L_1^*L_1) \neq \{0\},$$

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then the operator  $\dot{A}$  defined as follows

$$(1.3) \quad \text{dom}(\dot{A}) := \text{dom}(L_1^*L_0), \quad \dot{A}f := L_1^*L_0f, \quad f \in \text{dom}(\dot{A})$$

is closed, symmetric. Since  $(\dot{A}f, f) = \|L_0f\|^2 \geq 0$  for all  $f \in \text{dom}(\dot{A})$ , the operator  $\dot{A}$  is nonnegative. Such kind of operators  $\dot{A}$  we call *operators in divergence form*.

Observe that both operators  $L_0^*L_0$  and  $L_1^*L_1$  are nonnegative self-adjoint extensions of  $\dot{A}$ . In accordance with the first representation theorem [20] they are associated with the closed sesquilinear forms

$$\begin{aligned} \tau_0[\varphi, \psi] &= (L_0\varphi, L_0\psi)_{\mathfrak{H}}, & \varphi, \psi &\in \text{dom}(L_0), \\ \tau_1[u, v] &= (L_1u, L_1v)_{\mathfrak{H}}, & u, v &\in \text{dom}(L_1), \end{aligned}$$

respectively, and due to (1.1) the form  $\tau_0$  is a closed restriction of the form  $\tau_1$ .

It is well known that if a linear manifold  $\mathcal{D}$  is dense in a Banach space  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  is a subspace of  $\mathcal{B}$  with finite co-dimension, then the linear manifold  $\mathcal{D} \cap \tilde{\mathcal{B}}$  is dense in  $\tilde{\mathcal{B}}$ . Hence, if the condition

$$\dim(\text{dom}(L_1)/\text{dom}(L_0)) < \infty$$

is fulfilled, then (1.2) holds. Moreover, [7], [12], since  $\text{dom}(L_1^*L_0)$  is dense in  $\text{dom}(L_0)$  w.r.t. the graph norm in  $\text{dom}(L_0)$  we obtain that

- (1) the operator  $\dot{A} = L_1^*L_0$  has dense domain,
- (2) the operator  $L_0^*L_0$  is the *Friedrichs extension* of  $\dot{A}$ .

Recall that a densely defined nonnegative symmetric operator has at least one nonnegative self-adjoint extensions, the Friedrichs extension. M. G. Kreĭn established [21], [22] that the set of all nonnegative self-adjoint extensions of  $\dot{A}$  forms the operator interval  $[A_K, A_F]$  in the sense of quadratic forms [20], where the "minimal" operator  $A_K$  is discovered by Kreĭn. The operator  $A_K$  is called the *Kreĭn-von Neumann extension* (it is often called the *Kreĭn extension*).

Recall also that two self-adjoint extensions  $A_0$  and  $A_1$  of a closed densely defined symmetric operator  $\dot{A}$  with equal deficiency indices are called *disjoint* (or relatively prime) if

$$\text{dom}(A_0) \cap \text{dom}(A_1) = \text{dom}(\dot{A})$$

and *transversal* if, in addition,

$$\text{dom}(A_0) + \text{dom}(A_1) = \text{dom}(\dot{A}^*).$$

Due to (1.3) the operators  $A_0 = L_0^*L_0$  and  $A_1 = L_1^*L_1$  are disjoint nonnegative self-adjoint extensions of  $\dot{A}$ . Now observe that

*if a closed operator  $\dot{A}$  is given by (1.3), where  $L_0$  and  $L_1$  satisfy (1.1), then  $\dot{A}$  admits the factorization*

$$\dot{A} = \mathcal{L}\mathcal{L}_0,$$

*where  $\mathcal{L}_0$  is a closed densely defined symmetric and nonnegative operator in  $H$  and  $\mathcal{L}$  is its nonnegative self-adjoint extension.*

Actually, define

$$\begin{aligned} \text{dom}(\mathcal{L}) &:= \text{dom}(L_1), & \mathcal{L}u &:= (L_1^*L_1)^{1/2}u, & u &\in \text{dom}(L_1), \\ \text{dom}(\mathcal{L}_0) &:= \text{dom}(L_0), & \mathcal{L}_0\varphi &:= (L_1^*L_1)^{1/2}\varphi, & \varphi &\in \text{dom}(L_0). \end{aligned}$$

One of the aim of this paper is to prove the following statement.

**Theorem 1.1.** *Let  $\dot{A}$  be a densely defined closed nonnegative symmetric operator in a Hilbert space  $H$  having disjoint nonnegative self-adjoint extensions. Then  $\dot{A}$  admits infinitely many factorizations of the form*

$$\dot{A} = \mathcal{L}\mathcal{L}_0,$$

where  $\mathcal{L}_0$  is a densely defined closed nonnegative symmetric operator in  $H$  and  $\mathcal{L}$  is nonnegative self-adjoint extension of  $\mathcal{L}_0$ . Moreover,

- (1) if the deficiency indices of  $\dot{A}$  are finite, then it is necessary that the operator  $\mathcal{L}_0^*\mathcal{L}_0$  coincides with the Friedrichs extension  $A_F$  of  $\dot{A}$ ;
- (2) if the deficiency indices of  $\dot{A}$  are infinite, then the operator  $\mathcal{L}_0$  can be chosen such that  $\mathcal{L}_0^*\mathcal{L}_0$  coincides or does not coincide with the Friedrichs extension of  $\dot{A}$ ;
- (3) if  $\dot{A}$  admits transversal nonnegative self-adjoint extensions and if  $\mathcal{L}^2$  is transversal to  $A_F$  (in particular, if  $\mathcal{L}^2$  coincides with the Kreĭn-von Neumann extensions of  $\dot{A}$ ), then it is necessary that  $\mathcal{L}_0^*\mathcal{L}_0$  is the Friedrichs extension of  $\dot{A}$ .

If a closed symmetric operator  $\dot{A}$  is non-densely defined, then its adjoint  $\dot{A}^* = \{ \langle x, x' \rangle \}$  is a linear relation (a subspace in  $H \oplus H$ ) defined as follows:

$$\langle \dot{A}\varphi, x \rangle = \langle \varphi, x' \rangle \quad \text{for all } \varphi \in \text{dom}(\dot{A}).$$

The Friedrichs extension of a non-densely defined closed nonnegative operator is not a linear operator. It is a linear relation [30], [32]. But it is possible that the minimal extension (the Kreĭn-von Neumann extension) is an operator [3]. For a non-densely defined case we prove the following analog of Theorem 1.1.

**Theorem 1.2.** 1) A non-densely defined closed nonnegative symmetric operator with finite deficiency indices does not admit representation in divergence form.

2) A non-densely defined closed nonnegative symmetric operator  $\dot{A}$  with infinite deficiency indices and having disjoint nonnegative self-adjoint extensions (operators) admits infinitely many factorizations

$$\dot{A} = \mathcal{L}\mathcal{L}_0,$$

where  $\mathcal{L}_0$  is a densely defined closed nonnegative symmetric operator and  $\mathcal{L}$  is nonnegative self-adjoint extension of  $\mathcal{L}_0$ .

In the proves of Theorem 1.1 and Theorem 1.2 we essentially use M. Kreĭn’s approach [21], [22], [23] completed by Ando and Nishio [3] in the theory of nonnegative self-adjoint extensions of nonnegative symmetric operator. Notice that the inclusion  $\dot{A} \subseteq L_1^*L_0$  for some special  $L_0$  and  $L_1$  provided conditions (1.1) and  $A_F = L_0^*L_0$ ,  $A_K = L_1^*L_1$  are established for densely defined nonnegative  $\dot{A}$  in [29], [34], [11] and for nonnegative linear relations  $\dot{A}$  in [18]. In [7] and [12] some properties of extensions of the operators in divergence form are established and applications to boundary value problems are given.

M. A. Naĭmark in [26], [27] found an example of a densely defined closed symmetric operator  $T$  whose square  $T^2$  is zero defined, i.e.,  $\text{dom}(T^2) = \{0\}$ . A more concrete nonnegative symmetric operator with the same property is constructed in [14]. The results related to the powers of symmetric operators are obtained in [33]. In particular it is established [33, Theorem 5.2] that for each unbounded self-adjoint operator  $T$  there exist closed symmetric restrictions  $T_1$  and  $T_2$  of  $T$  such that

$$\text{dom}(T_1) \cap \text{dom}(T_2) = \{0\} \quad \text{and} \quad \text{dom}(T_1^2) = \text{dom}(T_2^2) = \{0\}.$$

In [13] it is shown that the above result remains true for a closed symmetric non-self-adjoint  $T$ .

In the present paper we give an example of a densely defined closed nonnegative symmetric operator  $\mathcal{L}_0$  and its nonnegative self-adjoint extension  $\mathcal{L}$  such that  $\text{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}$ . In particular,  $\text{dom}(\mathcal{L}_0^2) = \{0\}$ .

For this purpose we construct two nonnegative unbounded self-adjoint operators  $A_0$  and  $A$  in  $H$  such that

$$\begin{aligned} \text{dom}(A_0) \cap \text{dom}(A) &= \{0\}, \\ \text{dom}(A_0^{1/2}) \subset \text{dom}(A^{1/2}), \quad \|A_0^{1/2}\varphi\| &= \|A^{1/2}\varphi\|, \quad \varphi \in \text{dom}(A_0^{1/2}). \end{aligned}$$

Our construction is also based on the results in [21] related to the special kind of operators which are called nowadays the Kreĭn shorted operators.

It turns out that in our example the product  $\mathcal{L}_0\mathcal{L}$  is densely defined.

## 2. THE KREĬN SHORTED OPERATOR

For every nonnegative bounded operator  $B$  in the Hilbert space  $\mathcal{H}$  and every subspace  $\mathcal{K} \subset \mathcal{H}$  M. G. Kreĭn [21] defined the operator  $B_{\mathcal{K}}$  by the relation

$$B_{\mathcal{K}} = \max \{ Z \in \mathbf{L}(\mathcal{H}) : 0 \leq Z \leq B, \text{ran}(Z) \subseteq \mathcal{K} \}.$$

The equivalent definition is

$$(2.1) \quad (B_{\mathcal{K}}f, f) = \inf_{\varphi \in \mathcal{K}^{\perp}} \{(B(f + \varphi), f + \varphi)\}, \quad f \in \mathcal{H}.$$

Here  $\mathcal{K}^{\perp} := \mathcal{H} \ominus \mathcal{K}$ . The operator  $B_{\mathcal{K}}$  is called the *shorted operator* (see [1, 2]). Let the subspace  $\Omega$  be defined as follows:

$$\Omega = \{ f \in \overline{\text{ran}}(B) : B^{1/2}f \in \mathcal{K} \} = \overline{\text{ran}}(B) \ominus B^{1/2}\mathcal{K}^{\perp}.$$

It is proved in [21] that  $B_{\mathcal{K}}$  takes the form  $B_{\mathcal{K}} = B^{1/2}P_{\Omega}B^{1/2}$ . Hence, (see [21])

$$(2.2) \quad \text{ran}(B_{\mathcal{K}}^{1/2}) = \text{ran}(B^{1/2}) \cap \mathcal{K}.$$

It follows that

$$(2.3) \quad B_{\mathcal{K}} = 0 \iff \text{ran}(B^{1/2}) \cap \mathcal{K} = \{0\}.$$

Let a bounded self-adjoint operator  $B$  is given by the block operator matrix

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} : \begin{array}{c} \mathcal{K} \\ \mathcal{K}^{\perp} \end{array} \rightarrow \begin{array}{c} \mathcal{K} \\ \mathcal{K}^{\perp} \end{array},$$

where  $B_{11} \in \mathbf{L}(\mathcal{K})$ ,  $B_{22} \in \mathbf{L}(\mathcal{K}^{\perp})$ ,  $B_{12} \in \mathbf{L}(\mathcal{K}^{\perp}, \mathcal{K})$ . It is well known (see [23]) that *the operator  $B$  is nonnegative if and only if*

$$B_{22} \geq 0, \quad \text{ran}(B_{12}^*) \subset \text{ran}(B_{22}^{1/2}), \quad B_{11} \geq \left( B_{22}^{[-1/2]} B_{12}^* \right)^* \left( B_{22}^{[-1/2]} B_{12}^* \right)$$

*and the operator  $B_{\mathcal{K}}$  is given by the block matrix*

$$B_{\mathcal{K}} = \begin{pmatrix} B_{11} - \left( B_{22}^{[-1/2]} B_{12}^* \right)^* \left( B_{22}^{[-1/2]} B_{12}^* \right) & 0 \\ 0 & 0 \end{pmatrix},$$

*where  $B^{[-1/2]}$  is the Moore-Penrose pseudo-inverse.*

## 3. NONNEGATIVE SELF-ADJOINT EXTENSIONS OF NONNEGATIVE SYMMETRIC OPERATOR

Let  $H$  be a separable Hilbert space and let  $\dot{A}$  be a densely defined closed, symmetric, and nonnegative operator, i.e.,  $(\dot{A}f, f) \geq 0$  for all  $f \in \text{dom}(\dot{A})$ . The Friedrichs extension  $A_F$  of  $\dot{A}$  is defined as follows [20]. Denote by  $\dot{A}[\cdot, \cdot]$  the closure of the sesquilinear form

$$\dot{A}[f, g] = (\dot{A}f, g), \quad f, g \in \text{dom}(\dot{A}),$$

and let  $\mathcal{D}[\dot{A}]$  be the domain of this closure. According to the first representation theorem [20] there exists a nonnegative self-adjoint operator  $A_F$  associated with  $\dot{A}[\cdot, \cdot]$ , i.e.,

$$(A_F h, \psi) = \dot{A}[h, \psi], \quad \psi \in \mathcal{D}[\dot{A}], \quad h \in \text{dom}(A_F).$$

Clearly  $\dot{A} \subset A_F \subset \dot{A}^*$ , where  $\dot{A}^*$  is adjoint to  $\dot{A}$ . It follows that

$$\text{dom}(A_F) = \mathcal{D}[\dot{A}] \cap \text{dom}(\dot{A}^*).$$

By the second representation theorem the equalities

$$\mathcal{D}[\dot{A}] = \text{dom}(A_F^{1/2}) \quad \text{and} \quad \dot{A}[\phi, \psi] = (A_F^{1/2}\phi, A_F^{1/2}\psi), \quad \phi, \psi \in \mathcal{D}[\dot{A}]$$

hold. If  $A$  is a nonnegative self-adjoint operator, then

$$\mathcal{D}[A] = \text{dom}(A^{1/2}), \quad A[u, v] = (A^{1/2}u, A^{1/2}v), \quad A[u] = \|A^{1/2}u\|^2.$$

If  $A$  is a linear relation, then

$$\mathcal{D}[A] = \text{dom}(A_{\text{op}}^{1/2}), \quad A[u, v] = (A_{\text{op}}^{1/2}u, A_{\text{op}}^{1/2}v), \quad A[u] = \|A_{\text{op}}^{1/2}u\|^2,$$

where  $A_{\text{op}}$  is the operator part of  $A$  [31].

In his fundamental paper [21] M. Kreĭn reduced the problem of finding all nonnegative self-adjoint extensions for a nonnegative symmetric operator to the problem of self-adjoint contractive extensions (*sc*-extensions) for a given non-densely defined Hermitian contraction. He used the fact that the Cayley transform

$$S = (I - A)(I + A)^{-1}, \quad A = (I - S)(I + S)^{-1}$$

gives a one-to-one correspondence between closed densely defined nonnegative symmetric operators  $A$  in a Hilbert space  $H$  and non-densely defined closed symmetric contractions  $S$  such that  $\ker(S + I) = \{0\}$ . Moreover, the operator  $S$  is a self-adjoint if and only if  $A$  is self-adjoint.

Let  $\dot{S}$  be a closed non-densely defined symmetric contraction in  $H$ . M. Kreĭn proved that the set of all *sc*-extensions of  $\dot{S}$  forms an operator interval  $[S_\mu, S_M]$ . If

$$\dot{S} = (I - \dot{A})(I + \dot{A})^{-1}, \quad \text{dom}(\dot{S}) = \text{ran}(I + \dot{A}),$$

where  $\dot{A}$  is a densely defined closed and nonnegative symmetric (non-self-adjoint) operator, then, as it is shown by M. Kreĭn, the Cayley transform

$$(3.1) \quad A_F = (I - S_\mu)(I + S_\mu)^{-1}$$

of the extremal extension (the "rigid" extension of  $\dot{A}$  in M. Kreĭn terminology) coincides with the Friedrichs extension of  $\dot{A}$ . Another extremal nonnegative self-adjoint extension

$$(3.2) \quad A_K = (I - S_M)(I + S_M)^{-1}$$

was called by M. Kreĭn the "soft" extension of  $\dot{A}$ . It was proved in [21] that a nonnegative self-adjoint operator  $A$  is an extension of  $\dot{A}$  if and only if for some  $a > 0$  (then for all  $a > 0$ ) hold the inequalities

$$(A_F + aI)^{-1} \leq (A + aI)^{-1} \leq (A_K + aI)^{-1}$$

or equivalently  $A_K \leq A \leq A_F$  in the sense of corresponding quadratic forms [20], [21], i.e.,

$$(3.3) \quad \begin{aligned} \mathcal{D}[\dot{A}] &\subset \mathcal{D}[A] \subseteq \mathcal{D}[A_K], \\ A[\varphi] &= \dot{A}[\varphi] \quad \text{for all } \varphi \in \mathcal{D}[\dot{A}], \\ A[u] &\geq A_K[u] \quad \text{for all } u \in \mathcal{D}[A]. \end{aligned}$$

When  $\dot{A}$  is positive definite, i.e., the lower bound of  $\dot{A}$  is a positive number, it is shown in [21], [22] that

$$\text{dom}(A_K) = \text{dom}(\dot{A}) \dot{+} \ker(\dot{A}^*).$$

Thus, in that case the Kreĭn-von Neumann extension  $A_K$  coincides with self-adjoint extension constructed by J. von Neumann. Let  $\mathfrak{N}_z := H \ominus \text{ran}(\dot{A} - zI)$  be the defect subspace of  $\dot{A}$ . For densely defined  $\dot{A}$  one has

$$\mathfrak{N}_z = \ker(\dot{A}^* - zI).$$

Let  $A$  be a nonnegative self-adjoint extension of densely defined  $\dot{A}$ . It is established by M.G. Kreĭn [21] that the domain  $\mathcal{D}[A]$  admits the decomposition

$$(3.4) \quad \mathcal{D}[A] = \mathcal{D}[\dot{A}] \dot{+} (\mathcal{D}[A] \cap \mathfrak{N}_{-a})$$

for arbitrary  $a > 0$ . The operator  $\dot{A}$  has unique nonnegative self-adjoint extension (see [21] for densely defined  $\dot{A}$  and [5], [17] when  $\dot{A}$  is a linear relation) if and only if for some  $a > 0$  (and then for all  $a > 0$ ) holds the condition

$$\sup_{f \in \text{dom}(\dot{A})} \frac{|(f, \varphi_{-a})|^2}{(\dot{A}f, f)} = \infty \quad \text{for every } \varphi_{-a} \in \mathfrak{N}_{-a} \setminus \{0\}.$$

This condition is equivalent to  $\text{ran}(A_F^{1/2}) \cap \mathfrak{N}_{-a} = \{0\}$ .

Let  $S$  be any  $sc$ -extension of Hermitian contraction  $\dot{S}$  and let  $\mathfrak{N} = H \ominus \text{dom}(\dot{S})$ . The subspace  $\mathfrak{N}$  coincides with defect subspace  $\mathfrak{N}_{-1}$  of the operator  $\dot{A}$ . The operators  $S_\mu$  and  $S_M$  can be defined by the relations [21]

$$(3.5) \quad S_\mu = S - (I + S)_{\mathfrak{N}}, \quad S_M = S + (I - S)_{\mathfrak{N}}.$$

Thus, extremal  $sc$ -extensions  $S_\mu$  and  $S_M$  of  $\dot{S}$  possess the properties

$$(I_{\mathfrak{N}} + S_\mu)_{\mathfrak{N}} = (I_{\mathfrak{N}} - S_M)_{\mathfrak{N}} = 0.$$

The operator interval  $[S_\mu, S_M]$  can be parametrized as follows (see [23])

$$(3.6) \quad [S_\mu, S_M] \ni S \iff S = S_\mu + (S_M - S_\mu)^{1/2} X (S_M - S_\mu)^{1/2},$$

where  $X$  is a nonnegative self-adjoint contraction in the subspace  $\overline{\text{ran}}(S_M - S_\mu) (\subseteq \mathfrak{N})$ .

Basic Kreĭn's results remain true for non-densely defined closed nonnegative symmetric operators, for nonnegative linear relations, and for general case of sectorial operators and linear relations [3], [5], [6], [15], [17]. As it has been mentioned above, the Friedrichs extension of a non-densely defined nonnegative operator  $\dot{A}$  is the linear relation [31]. It takes the form

$$(3.7) \quad A_F = Gr((P_{H_0} \dot{A})_F) \oplus \langle 0, \mathfrak{B} \rangle,$$

where  $H_0 = \overline{\text{dom}}(\dot{A})$ ,  $\mathfrak{B} = H \ominus H_0$ , and the operator  $(P_{H_0} \dot{A})_F$  is the Friedrichs extension of the operator  $P_{H_0} \dot{A}$  in the Hilbert space  $H_0$ . The linear relation  $A_F$  is connected with the minimal  $sc$ -extension  $S_\mu$  of the contraction  $\dot{S}$  by the Cayley transform (3.1)

$$A_F = \{ \langle (I + S_\mu)h, (I - S_\mu)h \rangle, h \in H \}.$$

If  $\dot{A}$  is bounded and non-densely defined with  $\text{dom}(\dot{A}) = H_0 \subset H$ , then it admits bounded nonnegative self-adjoint extensions if and only if (see [35])

$$\sup_{\varphi \in H_0} \frac{\|\dot{A}\varphi\|^2}{(\dot{A}\varphi, \varphi)} < \infty.$$

If  $\dot{A}$  is non-densely defined, then in general the Kreĭn-von Neumann nonnegative self-adjoint extension  $A_K$  is a linear relation. The relationship between  $A_K$  and  $S_M$  is given by the Cayley transform (3.2).  $A_K$  is the operator if and only if  $\dot{A}$  is positively closable [3], i.e.,

$$\text{if } \{ \varphi_n \} \subset \text{dom}(\dot{A}) \text{ and } \lim_{n \rightarrow \infty} \dot{A}\varphi_n = g, \quad \lim_{n \rightarrow \infty} (\dot{A}\varphi_n, \varphi_n) = 0, \quad \text{then } g = 0.$$

The domain  $\mathcal{D}[A_K]$  can be characterized as follows [3]:

$$\mathcal{D}[A_K] = \left\{ u \in H : \sup_{\varphi \in \text{dom}(\dot{A})} \frac{|(\dot{A}\varphi, u)|^2}{(\dot{A}\varphi, \varphi)} < \infty \right\},$$

$$A_K[u] = \sup_{\varphi \in \text{dom}(\dot{A})} \frac{|(\dot{A}\varphi, u)|^2}{(\dot{A}\varphi, \varphi)}, \quad u \in \mathcal{D}[A_K].$$

For a non-densely defined  $\dot{A}$  the decomposition (3.4) remains true for any arbitrary nonnegative self-adjoint extension  $A$  (possibly a linear relation) [5].

We will need the following proposition (see [9], [10]).

**Proposition 3.1.** (1) *Let  $B$  be a non-negative self-adjoint operator and let*

$$S = (I - B)(I + B)^{-1}$$

*be its Cayley transform. Then*

$$\mathcal{D}[B] = \text{ran}((I + S)^{1/2}),$$

$$B[u, v] = -(u, v) + 2 \left( (I + S)^{-1/2}u, (I + S)^{-1/2}v \right), \quad u, v \in \mathcal{D}[B].$$

(2) *Let  $\dot{A}$  be a closed non-negative symmetric operator and let  $A$  be its non-negative self-adjoint extension (a linear relation, in general). If  $\dot{S} = (I - \dot{A})(I + \dot{A})^{-1}$ ,  $S = (I - A)(I + A)^{-1}$ , then*

$$\mathcal{D}[A] = \mathcal{D}[\dot{A}] \dot{+} \text{ran}((S - S_\mu)^{1/2}).$$

#### 4. DISJOINTNESS AND TRANSVERSALITY OF NON-NEGATIVE SELF-ADJOINT EXTENSIONS

The disjointness of self-adjoint extensions  $A_0$  and  $A_1$  of a symmetric linear relation  $\dot{A}$  means that  $A_0 \cap A_1 = \dot{A}$ , while  $A_0$  and  $A_1$  are transversal if the equality  $A_0 + A_1 = \dot{A}^*$  is valid. Clearly,  $A_0$  and  $A_1$  are transversal implies  $A_0$  and  $A_1$  are disjoint. The following equivalences for two self-adjoint extensions  $A_1$  and  $A_0$  of  $\dot{A}$  holds true :

$$(4.1) \quad \begin{aligned} A_1, A_0 \text{ are disjoint} &\iff \overline{\text{ran}} \left( (A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1} \right) = \mathfrak{N}_\lambda, \\ A_1, A_0 \text{ are transversal} &\iff \text{ran} \left( (A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1} \right) = \mathfrak{N}_\lambda \end{aligned}$$

for at least one (then for all)  $\lambda \in \rho(A_1) \cap \rho(A_0)$ . If the deficiency indices of  $\dot{A}$  are finite (and equal), then two self-adjoint extensions of  $\dot{A}$  are transversal if and only they are disjoint. The equivalences of statements in the next proposition can be found in [5], [6], [8], [18], [25].

**Proposition 4.1.** *Let  $\dot{A}$  be a non-negative closed symmetric relation.*

- (1) *The following statements are equivalent:*
  - (a)  $\dot{A}$  has two disjoint nonnegative self-adjoint extensions,
  - (b) the Friedrichs and Kreĭn - von Neumann extensions  $A_F$  and  $A_K$  are disjoint,
  - (c)  $\mathfrak{N}_z \cap \mathcal{D}[A_K]$  is dense in  $\mathfrak{N}_z$  at least for one (then for all)  $z \in \mathbb{C} \setminus [0, \infty)$ ,
  - (d)  $\ker(S_M - S_\mu) = \text{dom}(\dot{S}) (= \text{ran}(\dot{A} + I))$ ,
  - (e) from  $\lim_{n \rightarrow \infty} (I + \dot{A}\dot{A}^*)^{-1/2} \dot{A}\varphi_n = g$  and  $\lim_{n \rightarrow \infty} (\dot{A}\varphi_n, \varphi_n) = 0$  follows  $g = 0$  (for densely defined  $\dot{A}$ ).
- (2) *The conditions*
  - (a)  $\dot{A}$  has two transversal nonnegative self-adjoint extensions,
  - (b) the Friedrichs and Kreĭn extensions  $A_F$  and  $A_K$  are transversal,
  - (c)  $\text{dom}(A^*) \subset \mathcal{D}[A_K]$ ,
  - (d)  $\mathfrak{N}_z \subset \mathcal{D}[A_K]$  at least for one (then for all)  $z \in \mathbb{C} \setminus [0, \infty)$ ,

- (e)  $\text{ran}(S_M - S_\mu) = \mathfrak{N} (= \mathfrak{N}_{-1})$ ,  
 (f)  $\sup_{f \in \text{dom}(A)} \frac{\|(I + \dot{A}\dot{A}^*)^{-1/2}\dot{A}f\|^2}{(\dot{A}f, f)} < \infty$  (for densely defined  $\dot{A}$ )  
 are equivalent.

Now we get the following statement.

**Proposition 4.2.** ([18]). *If a non-densely defined nonnegative symmetric operator  $\dot{A}$  admits disjoint nonnegative self-adjoint extensions, then the Kreĭn-von Neumann extension  $A_K$  of  $\dot{A}$  is an operator.*

5. NONNEGATIVE SYMMETRIC OPERATOR  $\mathcal{L}_0$  AND ITS NONNEGATIVE SELF-ADJOINT EXTENSION  $\mathcal{L}$  SUCH THAT  $\text{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}$ .

Let  $H$  be a separable infinite-dimensional complex Hilbert space and let  $\mathfrak{M}$  be an infinite-dimensional subspace of  $H$  with infinite-dimensional orthogonal complement  $\mathfrak{M}^\perp = H \ominus \mathfrak{M}$ . It is well known, see e.g. [28], [19], that there exist unbounded self-adjoint operators  $B_1$  and  $B_2$  on  $\mathfrak{M}$  such that

$$\overline{\text{dom}}(B_1) = \overline{\text{dom}}(B_2) = \mathfrak{M}, \quad \text{dom}(B_1) \cap \text{dom}(B_2) = \{0\}.$$

Let  $D_k = (B_k^* B_k)^{1/2}$ ,  $k = 0, 1$ . Since  $\text{dom}(D_k) = \text{dom}(B_k)$ ,  $k = 1, 2$ , we get that  $\text{dom}(D_1) \cap \text{dom}(D_2) = \{0\}$ . Consequently, the operators

$$F = (I_{\mathfrak{M}} + D_1)^{-1}, \quad V = (I_{\mathfrak{M}} + D_2)^{-1}$$

possess the properties

$$\begin{aligned} \overline{\text{ran}}(F) = \overline{\text{ran}}(V) = \mathfrak{M}, \quad \text{ran}(F) \cap \text{ran}(V) = \{0\}, \\ 0 \leq F \leq I_{\mathfrak{M}}, \quad \ker(F) = \{0\}, \quad 0 \leq V \leq I_{\mathfrak{M}}, \quad \ker(V) = \{0\}. \end{aligned}$$

Replace  $V$  with  $U = V\Phi$ , where  $\Phi$  is a unitary operator from  $\mathfrak{M}^\perp$  onto  $\mathfrak{M}$ . Let a self-adjoint bounded operator  $G$  in  $H$  be given by the operator matrix

$$G = \begin{bmatrix} I_{\mathfrak{M}} & U \\ U^* & U^*U \end{bmatrix} : \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array} \rightarrow \begin{array}{c} \mathfrak{M} \\ \oplus \\ \mathfrak{M}^\perp \end{array}.$$

Clearly

$$\ker(G) = \left\{ \begin{bmatrix} -Uh \\ h \end{bmatrix} : h \in \mathfrak{M} \right\}.$$

Define

$$X = \begin{bmatrix} F & 0 \\ 0 & I_{\mathfrak{M}^\perp} \end{bmatrix} G \begin{bmatrix} F & 0 \\ 0 & I_{\mathfrak{M}^\perp} \end{bmatrix} = \begin{bmatrix} F^2 & FU \\ U^*F & U^*U \end{bmatrix}.$$

Let us show that

$$(5.1) \quad \ker\{X\} = \{0\}, \quad X_{\mathfrak{M}} = 0, \quad X_{\mathfrak{M}^\perp} = 0.$$

Set  $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ , where  $f_1 \in \mathfrak{M}$ ,  $f_2 \in \mathfrak{M}^\perp$ . Then

$$(5.2) \quad (Xf, f) = \|Ff_1 + Uf_2\|^2.$$

It follows that

$$Xf = 0 \iff Ff_1 + Uf_2 = 0.$$

Since  $\text{ran}(F) \cap \text{ran}(U) = \{0\}$ ,  $\ker(F) = \{0\}$ ,  $\ker(U) = \{0\}$ , we get  $f_1 = 0$ ,  $f_2 = 0$ . From (5.2) and relations  $\overline{\text{ran}}(F) = \overline{\text{ran}}(U) = \mathfrak{M}$  we get the equalities

$$\inf_{\varphi \in \mathfrak{M}^\perp} (X(f - \varphi), f - \varphi) = 0, \quad \inf_{\psi \in \mathfrak{M}} (X(f - \psi), f - \psi) = 0.$$



Equality (2.1) now implies  $X_{\mathfrak{M}} = 0$  and  $X_{\mathfrak{M}^\perp} = 0$ . Applying (2.3) we obtain

$$(5.3) \quad \mathfrak{M} \cap \text{ran}(X^{1/2}) = \{0\}, \quad \mathfrak{M}^\perp \cap \text{ran}(X^{1/2}) = \{0\}.$$

Define in  $H = \mathfrak{M} \oplus \mathfrak{M}^\perp$  the operator

$$(5.4) \quad X_0 = X^{1/2}P_{\mathfrak{M}}X^{1/2}.$$

Then

$$X - X_0 = X^{1/2}P_{\mathfrak{M}^\perp}X^{1/2}.$$

Equalities (5.1) yield

$$\ker(X_0) = \{0\}, \quad \ker(X - X_0) = \{0\}.$$

Notice that

$$(5.5) \quad \text{ran}(X) \cap \text{ran}(X_0) = \{0\}.$$

Actually, if  $Xf = X_0h$ , then  $X^{1/2}f = P_{\mathfrak{M}}X^{1/2}h$  and (5.3) yields  $f = h = 0$ .

Set

$$A = X^{-1}, \quad A_0 = X_0^{-1}.$$

The operators  $A_0$  and  $A$  are nonnegative and self-adjoint in  $H$ . Relation (5.5) implies

$$\text{dom}(A_0) \cap \text{dom}(A) = \{0\}.$$

In addition

$$\text{dom}(A_0^{1/2}) = \text{dom}(X_0^{-1/2}), \quad \text{dom}(A^{1/2}) = \text{dom}(X^{-1/2}).$$

From (5.4) we get  $\text{ran}(X_0^{1/2}) \subset \text{ran}(X^{1/2})$  and

$$X_0^{1/2} = X^{1/2}W_0,$$

where  $W_0$  is unitary operator from  $H$  onto  $\mathfrak{M}$ . It follows that

$$X^{-1/2}g = W_0X_0^{-1/2}g, \quad g \in \text{ran}(X_0^{1/2}).$$

Hence, the pair  $\langle A_0, A \rangle$  possess the property

$$\text{dom}(A_0^{1/2}) \subset \text{dom}(A^{1/2}) \quad \text{and} \quad \|A_0^{1/2}\varphi\| = \|A^{1/2}\varphi\| \quad \text{for all } \varphi \in \text{dom}(A_0^{1/2}).$$

Now define

$$\text{dom}(\mathcal{L}) = \text{dom}(A^{1/2}), \quad \mathcal{L}h = A^{1/2}h, \quad h \in \text{dom}(\mathcal{L}),$$

$$\text{dom}(\mathcal{L}_0) = \text{dom}(A_0^{1/2}), \quad \mathcal{L}_0g = A_0^{1/2}g, \quad g \in \text{dom}(\mathcal{L}_0).$$

The operator  $\mathcal{L}$  is self-adjoint and nonnegative, the operator  $\mathcal{L}_0$  is densely defined, symmetric and nonnegative, and is a restriction of  $\mathcal{L}$ , i.e.,  $\mathcal{L}_0 \subset \mathcal{L}$ . The sesquilinear form

$$\tau_0[\varphi, \psi] = (\mathcal{L}_0\varphi, \mathcal{L}_0\psi) = (A^{1/2}\varphi, A^{1/2}\psi) = (A_0^{1/2}\varphi, A_0^{1/2}\psi), \quad \varphi, \psi \in \text{dom}(A_0^{1/2})$$

is closed. This implies that  $\mathcal{L}_0$  is closed operator and the operator  $\mathcal{L}_0^*\mathcal{L}_0 = A_0$  is associated with the form  $\tau_0$ . In addition  $\mathcal{L}^2 = A$ . Since  $\text{dom}(\mathcal{L}_0^*\mathcal{L}_0) \cap \text{dom}(\mathcal{L}^2) = \{0\}$ , we get that

$$\text{dom}(\mathcal{L}\mathcal{L}_0) = \{0\}.$$

In particular,  $\text{dom}(\mathcal{L}_0^2) = \{0\}$ .

**Remark 5.1.** The operators  $\mathcal{L}$  and  $\mathcal{L}_0$  are positive definite. It follows that

$$\text{dom}(\mathcal{L}_0\mathcal{L}) = \mathcal{L}^{-1}\text{dom}(\mathcal{L}_0), \quad (\mathcal{L}_0\mathcal{L})(\mathcal{L}^{-1}\varphi) = \mathcal{L}_0\varphi, \quad \varphi \in \text{dom}(\mathcal{L}_0).$$

This yields, that  $\text{dom}(\mathcal{L}_0\mathcal{L})$  is dense in  $\text{dom}(\mathcal{L})$  w.r.t. the graph norm in  $\text{dom}(\mathcal{L})$ . Hence, the operator  $\mathcal{L}_0\mathcal{L}$  is densely defined in  $H$  and, moreover,  $(\mathcal{L}_0\mathcal{L})_F = \mathcal{L}^2$ .

Clearly, the equality  $\ker((\mathcal{L}_0\mathcal{L})^*) = \ker(\mathcal{L}_0^*)$  holds true. Therefore, relations

$$\text{dom}(\mathcal{L}_0^*) = \text{dom}(\mathcal{L}) \dot{+} \ker(\mathcal{L}_0^*), \quad \text{dom}((\mathcal{L}_0\mathcal{L})^*) = \text{dom}(\mathcal{L}^2) \dot{+} \ker((\mathcal{L}_0\mathcal{L})^*)$$

lead to the equality  $(\mathcal{L}_0\mathcal{L})^* = \mathcal{L}\mathcal{L}_0^*$ . Since  $\text{ran}(\mathcal{L}) = H$ , we get  $(\mathcal{L}_0\mathcal{L})_K = \mathcal{L}_0\mathcal{L}_0^*$  (see [12, Theorem 3.1]).

**Remark 5.2.** We have constructed an example of two unbounded nonnegative self-adjoint operators  $A_0$  and  $A$  such that

- (1)  $\text{dom}(A_0) \cap \text{dom}(A) = \{0\}$ ,
- (2)  $A_0 \geq A$  and the form  $A_0[\cdot, \cdot]$  is a closed restriction of the form  $A[\cdot, \cdot]$ .

**Remark 5.3.** It is proved in [33] that for any closed unbounded densely defined operator  $\mathcal{B}$  in  $H$  there exists a subspace  $\mathfrak{L}$  such that

$$\mathfrak{L} \cap \text{dom}(\mathcal{B}) = \mathfrak{L}^\perp \cap \text{dom}(\mathcal{B}) = \{0\}.$$

Hence it follows that for any bounded nonnegative self-adjoint operator  $\mathcal{F}$  with dense range  $\text{ran}(\mathcal{F})$  in  $H$  there exists a subspace  $\mathfrak{L}$  such that

$$\mathfrak{L} \cap \text{ran}(\mathcal{F}^{1/2}) = \mathfrak{L}^\perp \cap \text{ran}(\mathcal{F}^{1/2}) = \{0\}.$$

For any subspace  $\mathfrak{M}$ , with  $\dim(\mathfrak{M}) = \dim(\mathfrak{M}^\perp) = \infty$ , we have constructed above a bounded nonnegative self-adjoint operator  $X$  with dense range such that  $\mathfrak{M} \cap \text{ran}(X^{1/2}) = \mathfrak{M}^\perp \cap \text{ran}(X^{1/2}) = \{0\}$ .

## 6. PROVES OF THEOREMS 1.1 AND 1.2

**6.1. Auxiliary statements.** We start with two propositions.

**Proposition 6.1.** *For nonnegative self-adjoint unbounded operators  $A_0$  and  $A_1$  and their Cayley transformations  $S_k = (I - A_k)(I + A_k)^{-1}$ ,  $k = 0, 1$  the following statements are equivalent:*

(i) *the pair  $\langle A_0, A_1 \rangle$  possess the property*

$$(6.1) \quad \begin{aligned} &\text{dom}(A_0^{1/2}) \subset \text{dom}(A_1^{1/2}) \quad \text{and} \\ &\|A_0^{1/2}\varphi\| = \|A_1^{1/2}\varphi\| \quad \text{for all } \varphi \in \text{dom}(A_0^{1/2}); \end{aligned}$$

(ii) *the pair  $\langle S_0, S_1 \rangle$  possess the property*

$$(6.2) \quad \begin{aligned} &S_1 \geq S_0 \quad \text{and} \\ &\text{ran}((I + S_0)^{1/2}) \cap \text{ran}((S_1 - S_0)^{1/2}) = \{0\}; \end{aligned}$$

(iii) *the pair  $\langle S_0, S_1 \rangle$  possess the property*

$$(6.3) \quad I + S_0 = (I + S_1)^{1/2}P(I + S_1)^{1/2},$$

where  $P$  is an orthogonal projection in  $H$ .

If in addition  $A_0$  and  $A_1$  both are extensions of a densely defined closed symmetric non-negative operator  $\dot{A}$ , then each of the conditions (i), (ii), and (iii) is equivalent to the condition

$$(6.4) \quad \begin{aligned} &S_1 \geq S_0 \quad \text{and} \\ &\text{ran}((S_0 - S_\mu)^{1/2}) \cap \text{ran}((S_1 - S_0)^{1/2}) = \{0\}, \end{aligned}$$

where  $S_\mu = (I - A_F)(I + A_F)^{-1}$  and  $A_F$  is the Friedrichs extension of  $\dot{A}$ .

*Proof.* (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). From Proposition 3.1 follows that

$$\|(I + S_1)^{-1/2}\varphi\| = \|(I + S_0)^{-1/2}\varphi\| \quad \text{for all } \varphi \in \text{ran}((I + S_0)^{1/2}).$$

Hence

$$(I + S_1)^{-1/2}\varphi = \mathcal{V}(I + S_0)^{-1/2}\varphi, \quad \varphi \in \text{ran}((I + S_0)^{1/2}),$$

where  $\mathcal{V}$  is a isometry in  $H$ ,  $\text{ran}(\mathcal{V}) = (I + S_1)^{-1/2}\text{ran}((I + S_0)^{1/2})$ . Then

$$(I + S_0)^{1/2} = (I + S_1)^{1/2}\mathcal{V}$$

and

$$I + S_0 = (I + S_1)^{1/2} \mathcal{V} \mathcal{V}^* (I + S_1)^{1/2} = (I + S_1)^{1/2} P_{\text{ran}(\mathcal{V})} (I + S_1)^{1/2},$$

where  $P_{\text{ran}(\mathcal{V})}$  is the orthogonal projection in  $H$  onto  $\text{ran}(\mathcal{V})$ , i.e., (6.3) holds. It follows that

$$S_1 - S_0 = (I + S_1) - (I + S_0) = (I + S_1)^{1/2} (I - P_{\text{ran}(\mathcal{V})}) (I + S_1)^{1/2}.$$

We recall that if bounded self-adjoint nonnegative operators  $X$  and  $Y$  are connected by the relation  $X = Y^{1/2} \mathcal{Z} Y^{1/2}$ , where  $\mathcal{Z} \in \mathbf{L}(\overline{\text{ran}}(Y))$  is a nonnegative operator, then (see [16])

$$\text{ran}(X^{1/2}) = Y^{1/2} \text{ran}(\mathcal{Z}^{1/2}).$$

Therefore,  $\text{ran}((S_1 - S_0)^{1/2}) = (I + S_1)^{1/2} (H \ominus \text{ran}(\mathcal{V}))$  and (6.2) holds.

Clearly, (iii)  $\Rightarrow$  (ii).

Let us show (ii)  $\Rightarrow$  (iii) and (ii)  $\Rightarrow$  (i). Since  $I + S_1 \geq I + S_0$ , the equality

$$I + S_0 = (I + S_1)^{1/2} P (I + S_1)^{1/2}$$

is valid with  $0 \leq P \leq I$ . The equality  $S_1 - S_0 = (I + S_1) - (I + S_0)$  yields

$$S_1 - S_0 = (I + S_1)^{1/2} (I - P) (I + S_1)^{1/2}.$$

Due to (6.2) we get now

$$\text{ran} \left( (I - P)^{1/2} \right) \cap \text{ran} (P^{1/2}) = \{0\}.$$

Finally, since

$$\text{ran} \left( (I - P)^{1/2} \right) \cap \text{ran} (P^{1/2}) = \text{ran} \left( (P - P^2)^{1/2} \right),$$

we get that  $P^2 = P$ , i.e.,  $P$  is an orthogonal projection in  $H$ . Thus, (6.3) holds. From (6.3) we obtain

$$(I + S_0)^{1/2} h = (I + S_1)^{1/2} \mathcal{U} h, \quad h \in H,$$

where  $\mathcal{U}$  is unitary operator from  $H$  onto  $\text{ran}(P)$ . Hence

$$(I + S_1)^{-1/2} g = \mathcal{U} (I + S_0)^{-1/2} g \quad \text{for all } g \in \text{ran} \left( (I + S_0)^{1/2} \right).$$

Thus

$$(6.5) \quad \|(I + S_1)^{-1/2} g\|^2 = \|(I + S_0)^{-1/2} g\|^2, \quad g \in \text{ran} \left( (I + S_0)^{1/2} \right).$$

Now (6.1) follows from Proposition 3.1 and (6.5).

Suppose  $A_0$  and  $A_1$  both are extensions of a densely defined closed symmetric nonnegative operator  $\dot{A}$ . Let  $\dot{S} = (I - \dot{A})(I + \dot{A})^{-1}$  and let  $\mathfrak{N} = H \ominus \text{dom}(\dot{S})$ . Applying (3.5) and (2.2) we get

$$\mathfrak{N} \cap \text{ran} \left( (I + S_0)^{1/2} \right) = \text{ran} \left( (S_0 - S_\mu)^{1/2} \right).$$

This yields the equivalence of (6.4) and (6.2). □

**Remark 6.2.** Relation (6.4) is equivalent to the statement:  $S_0$  is an extremal point of the operator interval  $[S_\mu, S_1]$  (see [10] and references therein).

Let  $\dot{S}$  be a non-densely defined closed symmetric contraction. For simplicity we denote

$$C := S_M - S_\mu.$$

Using (3.6), one can get that if  $S_k = S_\mu + C^{1/2}X_kC^{1/2}$ ,  $k = 0, 1$  are two *sc*-extensions of  $\dot{S}$ , where  $X_k$ ,  $k = 0, 1$  are nonnegative self-adjoint contractions in  $\overline{\text{ran}}(C)$ , then

$$(6.6) \quad \begin{aligned} S_0 \leq S_1 &\iff X_0 \leq X_1, \\ \text{ran} \left( (S_0 - S_\mu)^{1/2} \right) \cap \text{ran} \left( (S_1 - S_0)^{1/2} \right) &= \{0\} \\ &\iff \text{ran} (X_0^{1/2}) \cap \text{ran} ((X_1 - X_0)^{1/2}) = \{0\} \\ &\iff X_0 = X_1^{1/2} P X_1^{1/2}, \end{aligned}$$

where  $P$  is an orthogonal projection in  $\overline{\text{ran}}(C)$ .

**Proposition 6.3.** *Let  $\dot{A}$  be a densely defined closed symmetric nonnegative operator in  $H$  having disjoint nonnegative self-adjoint extensions. Then there is a one-to-one correspondence between all factorizations of  $\dot{A}$  in the form  $\dot{A} = \mathcal{L}\mathcal{L}_0$ , where  $\mathcal{L}_0$  is a nonnegative densely defined closed symmetric operator in  $H$  and  $\mathcal{L}$  its nonnegative self-adjoint extension, and all pairs  $\langle A_0, A_1 \rangle$  of disjoint nonnegative self-adjoint extensions of  $\dot{A}$ , satisfying condition (6.1). This correspondence is given by the relations*

$$(6.7) \quad \begin{aligned} \text{dom}(\mathcal{L}) &= \text{dom}(A_1^{1/2}), \quad \mathcal{L}u = A_1^{1/2}u, \quad u \in \text{dom}(\mathcal{L}), \\ \text{dom}(\mathcal{L}_0) &= \text{dom}(A_0^{1/2}), \quad \mathcal{L}_0\varphi = A_1^{1/2}\varphi, \quad \varphi \in \text{dom}(\mathcal{L}_0). \end{aligned}$$

*Proof.* Let  $\dot{A} = \mathcal{L}\mathcal{L}_0$  be a factorization of  $\dot{A}$ , where  $\mathcal{L}_0$  is a nonnegative densely defined closed symmetric operator in  $H$  and  $\mathcal{L}$  its nonnegative self-adjoint extension. Then the operators  $A_0 = \mathcal{L}_0^*\mathcal{L}_0$ ,  $A_1 = \mathcal{L}^2$  are disjoint nonnegative self-adjoint extensions of  $\dot{A}$  and (6.1) holds. Therefore, (6.7) is valid.

Conversely, if a pair  $\langle A_0, A_1 \rangle$  of disjoint nonnegative self-adjoint extensions of  $\dot{A}$ , satisfying condition (6.1), is given, then define the pair  $\langle \mathcal{L}_0, \mathcal{L} \rangle$  by (6.7). Clearly,  $\mathcal{L}^2 = A_1$  and from (6.1) follows that  $\mathcal{L}_0^*\mathcal{L}_0 = A_0$ . In addition due to  $\text{dom}(A_0) \cap \text{dom}(A_1) = \text{dom}(\dot{A})$ , we get  $\dot{A} = \mathcal{L}\mathcal{L}_0$ .  $\square$

**6.2. Proof of Theorem 1.1.** Let

$$\begin{aligned} \dot{S} &= (I - \dot{A})(I + \dot{A})^{-1}, \\ S_\mu &= (I - A_F)(I + A_F)^{-1}, \quad S_M = (I - A_K)(I + A_K)^{-1} \end{aligned}$$

be the Cayley transforms of  $\dot{A}$ ,  $A_F$  and  $A_K$ , respectively. Since  $A_F$ , and  $A_K$  are disjoint, we have  $\ker(C) = \text{dom}(\dot{S})$ .

**Defect of  $\dot{A}$  is finite.** Then  $n := \dim(\mathfrak{N}) < \infty$  and  $\text{ran}(C) = \mathfrak{N}$ . Suppose  $\dot{A}$  is factorized as  $\dot{A} = \mathcal{L}\mathcal{L}_0$ , where  $\mathcal{L}_0$  is closed densely defined nonnegative symmetric operator and  $\mathcal{L}$  is its self-adjoint extension. Since  $A_0 = \mathcal{L}_0^*\mathcal{L}_0$  and  $A_1 = \mathcal{L}^2$  are nonnegative self-adjoint extensions of  $\dot{A}$  and

$$\text{dom}(\dot{A}) = \text{dom}(A_0) \cap \text{dom}(A_1),$$

from (4.1), (3.4), and the relations

$$\text{dom}(\mathcal{L}_0) = \text{dom}(A_0^{1/2}) \subset \text{dom}(A_1^{1/2}) = \text{dom}(\mathcal{L})$$

it follows that the deficiency indices of  $\mathcal{L}_0$  are  $\langle n, n \rangle$  and  $A_0 = \mathcal{L}_0^*\mathcal{L}_0$  is the Friedrichs extension of  $\dot{A}$  [7].

In order to construct a factorization let take an arbitrary  $A_1$  transversal to  $A_F$  and let  $A_0 = A_F$ . Then due to (3.3) the sesquilinear form  $A_F[\cdot, \cdot] = \dot{A}[\cdot, \cdot]$  is a closed restriction of the form  $A_1[\cdot, \cdot]$ . Further we use (6.7).

**Defect of  $\dot{A}$  is infinite.** In this case  $\dim(\mathfrak{N}) = \infty$ . By Proposition 4.1 we have  $\overline{\text{ran}}(C) = \mathfrak{N}$ . Due to Propositions 6.1 and 6.3 we need to describe *all pairs*  $\langle S_0, S_1 \rangle$  of *sc*-extensions of  $\dot{S}$ , satisfying (6.4) and such that  $\ker(S_1 - S_0) = \text{dom}(\dot{S})$ .

Let  $S_k = S_\mu + C^{1/2}X_kC^{1/2}$ ,  $k = 0, 1$ , and  $0 \leq X_0 \leq X_1 \leq I_{\mathfrak{N}}$ . According to (6.6) the operator  $X_0$  takes the form

$$X_0 = X_1^{1/2}PX_1^{1/2},$$

where  $P$  is an orthogonal projection with  $\text{ran}(P) \subset \mathfrak{N}$ . We need to find such  $P$  that  $\ker(S_1 - S_0) = \text{dom}(\dot{S})$ . We have

$$\begin{aligned} S_1 - S_0 &= C^{1/2}(X_1 - X_0)C^{1/2} = C^{1/2}X_1^{1/2}(I_{\mathfrak{N}} - P)X_1^{1/2}C^{1/2}, \\ \|(S_1 - S_0)^{1/2}h\|^2 &= \|(I_{\mathfrak{N}} - P)X_1^{1/2}C^{1/2}h\|^2, \quad h \in H \end{aligned}$$

and

$$\mathfrak{N} \ni h \in \ker(S_1 - S_0) \iff X_1^{1/2}C^{1/2}h \in \text{ran}(P).$$

Therefore

$$(6.8) \quad \ker(S_1 - S_0) = \text{dom}(\dot{S}) \iff \begin{cases} \ker(X_1) \cap \text{ran}(C^{1/2}) = \{0\}, \\ \text{ran}(X_1^{1/2}C^{1/2}) \cap \text{ran}(P) = \{0\}. \end{cases}$$

The choice of  $X_1$  depends on the case:  $\text{ran}(C) = \mathfrak{N}$  or  $\text{ran}(C) \neq \mathfrak{N}$ . Recall that  $\overline{\text{ran}}(C) = \mathfrak{N}$ .

*In the case  $\text{ran}(C) = \mathfrak{N}$  ( $\iff A_F$  and  $A_K$  are transversal) there is an equivalence*

$$\ker(X_1) \cap \text{ran}(C^{1/2}) = \{0\} \iff \ker(X_1) = \{0\}.$$

If  $\text{ran}(X_1) = \mathfrak{N}$ , then it is only one possibility to satisfy conditions

$$\text{ran}(P) \cap \text{ran}(X_1^{1/2}) = \{0\}$$

is to choose  $P = 0$ . This means that  $X_0 = 0$ , i.e.,  $S_0 = S_\mu$  and  $A_0 = A_F$ . In particular,

$$X_1 = I_{\mathfrak{N}} \iff S_1 = S_M \iff A_1 = A_K \Rightarrow A_0 = A_F.$$

If  $\ker(X_1) = \{0\}$  and  $\text{ran}(X_1) \neq \mathfrak{N}$ , then it is possible to choose a nontrivial subspace  $\mathfrak{M}$  in  $\mathfrak{N}$  such that

$$\mathfrak{M} \cap \text{ran}(X_1^{1/2}) = \{0\}$$

and  $X_0 = X_1^{1/2}P_{\mathfrak{M}}X_1^{1/2}$ . If we take  $\mathfrak{M} = \{0\}$ , then we get  $A_0 = A_F$ .

*In the case  $\text{ran}(C) \neq \mathfrak{N}$  it is also possible to choose  $X_1$  satisfying conditions in (6.8).* For example, one can take  $X_1 \in [0, I_{\mathfrak{N}}]$  with  $\ker(X_1) = \{0\}$  and then take  $\mathfrak{M} \subset \mathfrak{N}$  such that  $\mathfrak{M} \cap (X_1^{1/2}\text{ran}(C^{1/2})) = \{0\}$ . In particular,

$$X_1 = I_{\mathfrak{N}} \iff S_1 = S_M \iff A_1 = A_K \Rightarrow S_0 = S_\mu + C^{1/2}P_{\mathfrak{M}}C^{1/2},$$

where  $\mathfrak{M}$  is a subspace in  $\mathfrak{N}$  and  $\mathfrak{M} \cap \text{ran}(C^{1/2}) = \{0\}$ . The proof is complete.

Let us make a few remarks.

1) As it follows from the proof, the operator  $\mathcal{L}_0 = \mathcal{L} \upharpoonright \text{dom}(A_0^{1/2})$  depends on the choice of

- disjoint to  $A_F$  a nonnegative self-adjoint extension operator  $A_1 (= \mathcal{L}^2)$ ,
- nonnegative self-adjoint extension  $A_0$ , which is disjoint with  $A_1$  and possess property (6.1).

The minimal domain  $\text{dom}(\mathcal{L}_0)$  of symmetric operators  $\mathcal{L}_0$  coincides with  $\mathcal{D}[\dot{A}] = \text{dom}(A_F^{1/2})$ .

2) Due to [12, Theorem 3.1] if  $\dot{A}^* = \mathcal{L}_0^*\mathcal{L}$ , then  $A_F = \mathcal{L}_0^*\mathcal{L}_0$ . In addition, in that case the Friedrichs and Kreĭn - von Neumann extensions of  $\dot{A}$  are transversal. Therefore, if the Friedrichs and Kreĭn - von Neumann extensions of  $\dot{A}$  are disjoint and not transversal, then for each representation  $\dot{A} = \mathcal{L}\mathcal{L}_0$  the adjoint operator  $\dot{A}^*$  is not equal to  $\mathcal{L}_0^*\mathcal{L}$ . On the other hand if the Friedrichs and Kreĭn - von Neumann extensions of  $\dot{A} = \mathcal{L}\mathcal{L}_0$  are transversal and  $A_F \neq \mathcal{L}_0^*\mathcal{L}_0$ , then also  $\dot{A}^* \neq \mathcal{L}_0^*\mathcal{L}$ .

3) A nonnegative self-adjoint extension  $\tilde{A}$  of  $\dot{A}$  is called *extremal* [4] if

$$\inf_{\varphi \in \text{dom}(\dot{A})} (\tilde{A}(f - \varphi), f - \varphi) = 0 \quad \text{for all } f \in \text{dom}(\tilde{A}).$$

Extensions  $A_F$  and  $A_K$  are extremal. Suppose  $A_F$  and  $A_K$  are not transversal (but disjoint). Then, if we select  $A_1 = A_K (= \mathcal{L}^2)$ , a nonnegative self-adjoint extension  $A_0 (= \mathcal{L}_0^* \mathcal{L}_0)$  should be taken such that it is extremal and disjoint with  $A_K$ . The Cayley transform  $S_0 = (I - A_0)(I + A_0)^{-1}$  is of the form

$$S_0 = S_\mu + C^{1/2} P_{\mathfrak{M}} C^{1/2},$$

where  $P_{\mathfrak{M}}$  is the orthogonal projection onto a subspace  $\mathfrak{M}$  in  $\mathfrak{N}$  and  $\mathfrak{M} \cap \text{ran}(C^{1/2}) = \{0\}$  (see the end of the proof of Theorem 1.1).

4) In [30, Corollary to Theorem X.25] it is stated without proof that if  $\mathcal{L}_0$  is a symmetric operator whose square  $\mathcal{L}_0^2$  is densely defined, then the Friedrichs extensions  $(\mathcal{L}_0^2)_F$  of  $\mathcal{L}_0^2$  is the operator  $\mathcal{L}_0^* \mathcal{L}_0$ . This result is true if one of the deficiency indices of  $\mathcal{L}_0$  is finite (this follows from [12, Proposition 3.3]). Another sufficient condition of the equality  $(\mathcal{L}_0^2)_F = \mathcal{L}_0^* \mathcal{L}_0$  (for densely defined  $\mathcal{L}_0^2$ ) is the relation  $(\mathcal{L}_0^2)^* = \mathcal{L}_0^{*2}$  (see [24]). On the other hand as it is follows from [33, Theorem 4.5] for any unbounded self-adjoint operator  $\mathcal{L}$  there exists a closed densely defined symmetric restriction  $\mathcal{L}_0$  such that  $\mathcal{L}_0^2$  is densely defined but  $\text{dom}(\mathcal{L}_0^2)$  is not dense in  $\text{dom}(\mathcal{L}_0)$  w.r.t. the graph norm, i.e.,  $(\mathcal{L}_0^2)_F \neq \mathcal{L}_0^* \mathcal{L}_0$ . Due to Theorem 1.1 if  $\dot{A} = \mathcal{L} \mathcal{L}_0$  and the Friedrichs extensions of  $\dot{A}$  does not coincide with  $\mathcal{L}_0^* \mathcal{L}_0$ , then from the assumption that  $\mathcal{L}_0^2$  is densely defined follows:  $\mathcal{L}_0^* \mathcal{L}_0$  does not coincide with the Friedrichs extension of  $\mathcal{L}_0^2$ .

**6.3. Proof of Theorem 1.2.** 1) Let  $\dot{A}$  has finite deficiency indices  $\langle n, n \rangle$ . Then for two nonnegative self-adjoint extensions  $A_0$  and  $A_1$  such that  $A_0 \geq A_1$  from (3.4) it follows

$$\dim(\mathcal{D}[A_1]/\mathcal{D}[A_0]) \leq n.$$

Suppose  $\dot{A} = L_1^* L_0$ , where  $L_0$  is closed and densely defined in  $H$  and  $L_1$  is a closed extension of  $L_0$  in  $H$ . Put  $A_0 = L_0^* L_0$ ,  $A_1 = L_1^* L_1$ . Then  $\dim(\text{dom}(L_1)/\text{dom}(L_0)) \leq n$ . This yields that  $\text{dom}(L_1^* L_1)$  is dense in  $H$ . Contradiction.

2) Let  $\dot{A}$  has infinite defect numbers. Since  $\dot{A}$  admits disjoint nonnegative self-adjoint extensions (operators), we get  $\ker(C) = \text{dom}(\dot{S})$  (see Proposition 4.2). Recall that  $C = S_M - S_\mu$ . Note that the Kreĭn-von Neumann extension  $A_K$  is the operator. This means that  $\ker(I + S_M) = \{0\}$ . Let

$$S_1 = S_\mu + C^{1/2} X_1 C^{1/2}, \quad 0 \leq X_1 \leq I_{\mathfrak{N}}$$

be *sc*-extension of  $\dot{S}$ . Using the equality  $I + S_1 = (I + S_\mu) + C^{1/2} X_1 C^{1/2}$  and (3.7) we get that

$$\ker(I + S_1) = \{0\} \iff \ker(X_1) \cap C^{1/2} \mathfrak{B} = \{0\},$$

where  $\mathfrak{B} = H \ominus \overline{\text{dom}(\dot{A})}$ . It follows that if, in particular,  $\ker(X_1) = \{0\}$ , then  $\ker(I + S_1) = \{0\}$ . Let  $P$  be an orthogonal projection in  $H$ ,  $\text{ran}(P) \subset \mathfrak{N}$ . Put  $X_0 = X_1^{1/2} P X_1^{1/2}$  and let

$$S_0 = S_\mu + C^{1/2} X_0 C^{1/2} = S_\mu + C^{1/2} X_1^{1/2} P X_1^{1/2} C^{1/2}.$$

We need to satisfy also the following conditions:

$$\ker(S_1 - S_0) = \{0\}, \quad \ker(I + S_0) = \{0\}.$$

Therefore, (see (6.8))

$$\begin{cases} \ker(X_1) \cap \text{ran}(C^{1/2}) = \{0\}, \\ \text{ran}(X_1^{1/2} C^{1/2}) \cap \text{ran}(P) = \{0\} \end{cases}$$

and

$$(\mathfrak{N} \ominus \operatorname{ran}(P)) \cap X_1^{1/2} C^{1/2} \mathfrak{B} = \{0\}.$$

So if we construct an operator  $X_1 \in [0, I_{\mathfrak{N}}]$  and a subspace  $\mathfrak{M} \subset \mathfrak{N}$  such that

$$\ker(X_1) = \{0\}, \quad \operatorname{ran}(X_1) \neq \mathfrak{N}, \quad \operatorname{ran}(X_1^{1/2}) \cap \mathfrak{M} = \operatorname{ran}(X_1^{1/2}) \cap (\mathfrak{N} \ominus \mathfrak{M}) = \{0\},$$

then we obtain nonnegative self-adjoint extensions

$$A_k = (I - S_k)(I + S_k)^{-1}, \quad k = 0, 1$$

of the operator  $\dot{A}$ , satisfying conditions in (6.1). For a construction of such  $X_1$  we can repeat the construction in Section 5 or to use the result in [33] (see Remark 5.3). The proof is complete.

In the case  $\operatorname{ran}(C) \neq \mathfrak{N}$  we can take  $X_1 = I_{\mathfrak{N}}$ , that is equivalent to the selection  $S_1 = S_M \iff A_1 = A_K$ . Then we can find (see Remark 5.3) a subspace  $\mathfrak{M} \subset \mathfrak{N}$  such that

$$\mathfrak{M} \cap \operatorname{ran}(C^{1/2}) = \{0\}, \quad (\mathfrak{N} \ominus \mathfrak{M}) \cap \operatorname{ran}(C^{1/2}) = \{0\}.$$

Hence,  $S_0 = S_\mu + C^{1/2} P_{\mathfrak{M}} C^{1/2}$  and  $A_0 = (I - S_0)(I + S_0)^{-1}$  is extremal extension of  $\dot{A}$ .

Notice that boundedness of  $\dot{A}$  is possible. So, a bounded  $\dot{A}$  having nonnegative self-adjoint operator extension admits factorization  $\dot{A} = \mathcal{L}\mathcal{L}_0$  with unbounded  $\mathcal{L}_0$  and  $\mathcal{L}$ .

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