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SPECTRAL SINGULARITIES OF DIFFERENTIAL OPERATOR WITH TRIANGULAR MATRIX COEFFICIENTS

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I dedicate this work to my dear Teacher Fedor Semenovich Rofe-Beketov with deep gratitude and respect in honour of his glorious anniversary

Abstract. For a non-selfadjoint Sturm-Liouville operator with a triangular matrix potential growing at infinity, we construct an example of such an operator having spectral singularities.

In the study of the connection between spectral and oscillating properties of non-selfadjoint differential operators with block-triangular matrix coefficients growing at infinity (see [3]), there arises the question on the structure of the spectrum of such operators. For an operator with a triangular matrix potential decaying at infinity which first moment is bounded, due to the inverse scattering problem, the spectral structure was established in [2], [1], [8]. In [4], there are presented sufficient conditions where a non-selfadjoint operator with a block-triangular matrix potential growing at infinity has no spectral singularities, and its spectrum is real and discrete. The points at which the resolvent of a non-selfadjoint operator has a pole but which are not eigenvalues of the operator, are said to be *spectral singularities*. A special role of these points was found first by M. A. Naimark in [6]. The notion "spectral singularity" was introduced later due to J. Schwartz [9] (see also M. A. Naimark's monograph [7] and Supplement I of [5] due to V. E. Ljance).

In this paper we construct an example where a non-selfadjoint differential operator with a triangular matrix potential has spectral singularities.

Consider an equation with a block-triangular matrix potential,

$$l[\bar{y}] = -\bar{y}'' + V(x)\bar{y} = \lambda\bar{y}, \quad 0 \leqslant x < \infty, \tag{1}$$

where

$$V(x) = w(x)\cdot I_m + U(x), \quad U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \dots & U_{1r}(x) \\ 0 & U_{22}(x) & \dots & U_{2r}(x) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & U_{rr} \end{pmatrix}, \tag{2}$$

$w(x)$ is a real-valued function, $0 < w(x) \to \infty$ monotonically as $x \to \infty$, and has monotone absolutely continuous derivative. The diagonal blocks U_{kk}, $k = \overline{1,r}$, are Hermitian matrices of order $m_k \geqslant 1$ (in particular, for $m_k = 1$ they are real scalar functions). Let $\sum_{k=1}^{r} m_k = m$, and let I_m be a unit matrix of order m. Denote by H_m an m-dimensional Hilbert space.

In the case of

$$w(x) \geqslant Cx^{2\alpha}, \quad C > 0, \quad \alpha > 1, \tag{3}$$

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we suppose that the coefficients of the equation (1) satisfy the relations

$$\int_0^\infty |U(t)| \cdot w^{-\frac{1}{2}}\, dt < \infty, \tag{4}$$

$$\int_0^\infty w'^2(t) \cdot w^{-\frac{5}{2}}(t)\, dt < \infty, \qquad \int_0^\infty w''(t) \cdot w^{-\frac{3}{2}}(t)\, dt < \infty. \tag{5}$$

Denote

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4w(x)}} \cdot \exp\left(-\int_0^x \sqrt{w(u)}\, du \right),$$

$$\gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4w(x)}} \cdot \exp\left(\int_0^x \sqrt{w(u)}\, du \right).$$

In the case where $w(x) = x^{2\alpha}$, $0 < \alpha \leqslant 1$, suppose that the coefficients of the equation (1) satisfy the relation

$$\int_a^\infty |U(t)| \cdot t^{-\alpha}\, dt < \infty, \quad a > 0, \tag{6}$$

and then put

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(-\int_a^x \sqrt{u^{2\alpha} - \lambda}\, du \right),$$

$$\gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha} - \lambda)}} \cdot \exp\left(\int_a^x \sqrt{u^{2\alpha} - \lambda}\, du \right).$$

In [4], there was established the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ as $x \to \infty$. With a use of it, the Theorem as well as its Corollary below was proved.

Theorem 1. *Suppose that, for equation* (1)*, either conditions* (3)*,* (4)*,* (5) *for* $\alpha > 1$*, or condition* (6) *for* $0 < \alpha \leqslant 1$*, hold. Then equation* (1) *has a unique matrix solution* $\Phi(x,\lambda)$ *decaying at infinity and satisfying the relation*

$$\lim_{x\to\infty} \frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I_m$$

such that

$$\lim_{x\to\infty} \frac{\Phi'(x,\lambda)}{\gamma_0'(x,\lambda)} = I_m.$$

Also, this equation has a matrix solution $\Psi(x,\lambda)$ *growing at infinity and satisfying the relation*

$$\lim_{x\to\infty} \frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)} = I_m$$

such that

$$\lim_{x\to\infty} \frac{\Psi'(x,\lambda)}{\gamma_\infty'(x,\lambda)} = I_m.$$

Corollary 1. *If* $\alpha = 1$*, i.e., the coefficient* $w(x) = x^2$*, then, under condition* (6)*, Theorem 1 holds true for the functions* $\gamma_0(x,\lambda) = x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)$*,* $\gamma_\infty(x,\lambda) = x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right)$*. If* $\alpha = \frac{1}{2}$*, i.e., the coefficient* $w(x) = x$*, and condition* (6) *holds, then* $\gamma_0(x,\lambda) = x^{-\frac{1}{4}} \cdot \exp\left(-\frac{2}{3}x^{\frac{3}{2}} + \lambda x^{\frac{1}{2}}\right)$*,* $\gamma_\infty(x,\lambda) = x^{-\frac{1}{4}} \cdot \exp\left(\frac{2}{3}x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}}\right)$*.*

Remark 1. *In monograph* [10]*, for the scalar equation*

$$-\varphi'' + x^2 \cdot \varphi = \lambda\varphi \tag{7}$$

it was shown that, for $\lambda = 2n+1$, this equation has the solution

$$\varphi_n(x) = H_n(x) \cdot \exp\left(-\frac{x^2}{2}\right),$$

where $H_n(x)$ is the Chebyshev–Hermite polynomial. The differential and recursion formulas for the polynomial are also provided there. Note that the Chebyshev–Hermite polynomial has the following asymptotics as $x \to \infty$: $H_n(x) = (2x)^n(1+o(1))$. Hence the solution $\varphi_n(x)$ of the equation (7) for $\lambda = 2n+1$ will have the following asymptotics at infinity:

$$\varphi_n(x) = (2x)^n \cdot \exp\left(-\frac{x^2}{2}\right) \cdot (1+o(1)).$$

In the case of $U(x) = 0$ and $w(x) = x^2$ in (2), the matrix equation (1) is splitting into m scalar equations of the form (7). The matrix solution $\Phi(x,\lambda)$ will be diagonal in this case. Denote by $\varphi(x,\lambda)$ the diagonal elements of the matrix $\Phi(x,\lambda)$. Then, by Corollary 1, the solution $\varphi(x,\lambda)$ will have the following asymptotics at infinity:

$$\varphi(x,\lambda) = x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)(1+o(1)).$$

In particular, for $\lambda = 2n+1$, this yields the solution proportional to $\varphi_n(x)$.

Let the following boundary condition be given at $x = 0$:

$$\cos A \cdot \bar{y}'(0) - \sin A \cdot \bar{y}(0) = 0, \tag{8}$$

where A is a block-triangular matrix of a similar structure as the coefficients of the differential equation (1).

Together with problem (1), (8), we consider a separate system,

$$l_k[\bar{y}_k] = -\bar{y}_k'' + (w(x)I_{m_k} + U_{kk}(x))\,\bar{y}_k = \lambda\bar{y}_k, \quad k = \overline{1,r},$$

with the boundary conditions

$$\cos A_{kk} \cdot \bar{y}_k'(0) - \sin A_{kk} \cdot \bar{y}_k(0) = 0, \quad k = \overline{1,r}, \tag{9}$$

where A_{kk} are diagonal elements of the matrix A, A_{kk}, $k = \overline{1,r}$, are Hermitian matrices of order $m_k \geqslant 1$, $\sum_{k=1}^{r} m_k = m$.

Denote by L_0 the minimal differential operator generated by the differential expression $l[\bar{y}]$ and the boundary condition (8), and by L_k, $k = \overline{1,r}$, the minimal symmetric operators on $L_2(H_{m_k}, (0,\infty))$ generated by the differential expressions $l_k[\bar{y}_k]$ and the boundary conditions (9). Taking into account the conditions on the coefficients, we conclude that, for every symmetric operator L_k, $k = \overline{1,r}$, there is a limit point at infinity. Hence their self-adjoint extensions $\widetilde{L_k}$ are the closures of the operators L_k, respectively. The operators $\widetilde{L_k}$ are semi-bounded, and their spectra are discrete.

Denote by L the extension of the operator L_0 generated by the requirement on functions from the domain of the operator L to belong to $L_2(H_m, (0,\infty))$.

It can be shown (see Lemma 2 in [3]) that the discrete spectrum of the operator L is real and belongs to the union of the spectra of the self-adjoint operators $\widetilde{L_k}$,

$$\sigma_d(L) \subseteq \bigcup_{k=1}^{r} \sigma\left(\widetilde{L_k}\right).$$

If the perturbation $U(x)$ of the equation (1) is subordinated to the growth of the function $w(x)$, then this assertion can be refined. The following theorem is proved in [4].

Theorem 2. *Suppose that, for equation* (1), *either conditions* (3), (4), (5) *for* $\alpha > 1$, *or condition* (6) *for* $0 < \alpha \leqslant 1$, *hold. Then the spectrum of the operator* L *is real and coincides with the union of spectra of the self-adjoint operators* $\widetilde{L_k}$, $k = \overline{1, r}$, *i.e.,*

$$\sigma(L) = \bigcup_{k=1}^{r} \sigma\left(\widetilde{L_k}\right). \tag{10}$$

Remark 2. *If the perturbation* $U(x)$ *in the equation* (1) *does not satisfy either conditions* (4), (5) *or condition* (6), *then, as the following example shows, Theorem 2 is no longer true.*

Example 1. Consider the equation:

$$l\,[\bar{y}] = -\bar{y}'' + \begin{pmatrix} x^2 & q(x) \\ 0 & \pi^2 x^2 \end{pmatrix} \bar{y} = \lambda \bar{y}, \quad 0 \leqslant x < \infty, \quad \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{11}$$

with the boundary condition

$$\bar{y}(0) = 0. \tag{12}$$

Together with the problem (11), (12), consider a separate system,

$$l_1[y_1] = -y_1'' + x^2 y_1 = \lambda y_1, \tag{13}$$

$$l_2[y_2] = -y_2'' + \pi^2 x^2 y_2 = \lambda y_2 \tag{14}$$

with the boundary conditions

$$y_1(0) = 0, \tag{15}$$

$$y_2(0) = 0. \tag{16}$$

As above, denote by L_0 the differential operator generated by the differential expression $l\,[\bar{y}]$ (11) and the boundary condition (12), and by L_1, L_2 denote the minimal symmetric operators on $L_2(0;\infty)$ generated by the differential expressions $l_1\,[y_1]$, $l_2\,[y_2]$ and the boundary conditions (15), (16), respectively. Their self-adjoint extensions $\widetilde{L_1}$, $\widetilde{L_2}$ are the closures of the operators L_1, L_2 respectively. The operators $\widetilde{L_1}$, $\widetilde{L_2}$ are semi-bounded; let us denote their spectra by $\sigma_1 = \sigma(\widetilde{L_1})$, $\sigma_2 = \sigma(\widetilde{L_2})$.

The equation (13) (cf. (7)) has the solution $y_{1,n}(x) = H_n(x)\cdot\exp\left(-\frac{x^2}{2}\right)$ for $\lambda = 2n+1$. Since $H_{2n+1}(0) = 0$, the eigenvalues of the operator $\widetilde{L_1}$ are $\lambda_n = 4n + 3$.

The sets σ_1 and σ_2 do not intersect.

Denote by L the extension of the operator L_0 generated by the requirement on the functions from the domain of the operator L to belong to $L_2\,(H_2, (0;\infty))$, and by $\sigma(L)$ its spectrum.

Denote by $Y(x,\lambda) = \begin{pmatrix} y_{11}(x,\lambda) & y_{12}(x,\lambda) \\ 0 & y_{22}(x,\lambda) \end{pmatrix}$ the matrix solution of the equation (1) satisfying the initial conditions $Y(0,\lambda) = 0$, $Y'(0,\lambda) = I$.

If some $\lambda_0 \in \sigma(\widetilde{L_1})$ and $y(x,\lambda_0)$ is the corresponding eigenfunction of the operator $\widetilde{L_1}$, then the vector function $\bar{y}(x,\lambda_0) = \begin{pmatrix} y(x,\lambda_0) \\ 0 \end{pmatrix}$ is the eigenfunction of the operator L corresponding to the eigenvalue λ_0, i.e., $\lambda_0 \in \sigma(L)$. Moreover, $\lambda_0 \in \sigma(\widetilde{L_2})$ is the eigenvalue of the operator L if and only if the solution $y_{12}(x,\lambda_0)$ of the equation

$$-y_{12}'' + x^2 y_{12} + q(x) y_{22} = \lambda_0 y_{12} \tag{17}$$

satisfying the initial conditions $y_{12}(0,\lambda) = y_{12}'(0,\lambda) = 0$ belongs to $L_2(0;\infty)$. Let $u(x,\lambda)$ and $v(x,\lambda)$ be the solutions of the equation (13) satisfying the initial conditions

$$u(0,\lambda) = 0, \quad u'(0,\lambda) = 1, \quad v(0,\lambda) = -1, \quad v'(0,\lambda) = 0,$$

and let $C(x,t,\lambda) = u(x,\lambda)v(t,\lambda) - v(x)u(t,\lambda)$ be the Cauchy function of the equation (13). Then the solution $y_{12}(x,\lambda_0)$ is given by

$$y_{12}(x,\lambda_0) = \int_0^x q(t)\cdot C(x,t,\lambda_0)\cdot y_{22}(t,\lambda_0)\,dt.$$

Choose the coefficient $q(x) = y_{22}(x,\lambda_0)e^{x^\mu}$, where $\mu > 2$ (for instance, $\mu = 4$), and show that the integral $\int_0^\infty y_{12}^2(x,\lambda_0)\,dx$ diverges and, consequently, $\lambda_0 \notin \sigma(L)$. Indeed, since the solution $y_{22}(x,\lambda_0)$ has finitely many zeros, we conclude that, for any $x \geqslant N_1 > 0$,

$$y_{22}(x,\lambda_0) \geqslant c_1 e^{-\alpha x^2}, \quad \alpha > 0, \tag{18}$$

and the Cauchy function decays no faster than $e^{-(x-t)^2}$. Hence, if $|x-t| > N_2$, we have

$$C(x,t,\lambda_0) \geqslant c_2 e^{-(x-t)^2}. \tag{19}$$

In the case of $\frac{x}{4} \leqslant t \leqslant \frac{x}{2}$ and $x \geqslant \max(4N_1, 2N_2)$, the inequalities (18) and (19) are fulfilled simultaneously, therefore,

$$y_{12}\,(x,\lambda_0) > c_3 \int_{\frac{x}{4}}^{\frac{x}{2}} e^{\,t^4}\cdot e^{-2\alpha\,t^2}\cdot e^{-\,(x-t)^2}\,dt.$$

Since $e^{-(x-t)^2} \geqslant e^{-\frac{x^2}{4}}$ for $t \leqslant \frac{x}{2}$, we get $y_{12}(x,\lambda_0) > c_3 e^{-\frac{x^2}{4}} \int_{\frac{x}{4}}^{\frac{x}{2}} e^{t^4}\cdot e^{-2\alpha t^2}\,dt$.

If x is sufficiently large and $t \in \left[\frac{x}{4},\frac{x}{2}\right]$, we have $e^{t^4 - 2\alpha t^2} > e^{\frac{1}{2}t^4} \geqslant e^{\frac{x^4}{32}}$, hence

$$y_{12}(x,\lambda_0) > c_3 \frac{x}{4} e^{-\frac{x^2}{4} + \frac{x^4}{32}} \to \infty \quad \text{for} \quad x \to \infty.$$

It follows that $y_{12}(x,\lambda_0) \notin L_2(0;\infty)$ and $\lambda_0 \notin \sigma(L)$.

There arises the question on the nature of such values λ.

Consider the equation with a triangular matrix potential:

$$l\,[\bar{y}] = -\bar{y}'' + \begin{pmatrix} p(x) & q(x) \\ 0 & r(x) \end{pmatrix}\bar{y} = \lambda\bar{y}, \quad 0 \leqslant x < \infty, \quad \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{20}$$

where $p(x)$, $q(x)$, $r(x)$ are scalar functions, $p(x)$, $r(x)$ are real functions and $p(x)$, $r(x) \to \infty$ monotonically as $x \to \infty$.

Let the boundary condition is given at $x = 0$

$$\cos A\cdot\bar{y}'(0) - \sin A\cdot\bar{y}(0) = 0, \tag{21}$$

where A is a triangular matrix, $\cos A = \begin{pmatrix} \cos\alpha_{11} & \cos\alpha_{12} \\ 0 & \cos\alpha_{22} \end{pmatrix}$.

Consider the separated system

$$l_1[y_1] = -y_1'' + p(x)y_1 = \lambda y_1, \tag{22}$$

$$l_2[y_2] = -y_2'' + r(x)y_2 = \lambda y_2 \tag{23}$$

with the boundary conditions

$$\cos\alpha_{11}y_1'(0) - \sin\alpha_{11}y_1(0) = 0, \tag{24}$$

$$\cos\alpha_{22}y_2'(0) - \sin\alpha_{22}y_2(0) = 0. \tag{25}$$

Let L_0 be the differential operator generated by the differential expression $l\,[\bar{y}]$ (20) and the boundary condition (21), and let L_1, L_2 be minimal symmetric operators on $L_2(0,\infty)$ generated by the differential expressions $l_1[y_1]$, $l_2[y_2]$ and the boundary conditions (24), (25) respectively. Denote by $\widetilde{L_1}$, $\widetilde{L_2}$ the self-adjoint extensions of the operators L_1, L_2 respectively. The operators $\widetilde{L_1}$, $\widetilde{L_2}$ are semi-bounded; let us denote their spectra by σ_1 and σ_2 respectively.

Denote by L the extension of the operator L_0 and by $\sigma(L)$ its spectrum.

Let $u(x,\lambda)$ and $v(x,\lambda)$ be the solutions of the equation (22) with the boundary conditions

$$u(0,\lambda)=0, \qquad u'(0,\lambda)=1,$$
$$v(0,\lambda)=-1, \quad v'(0,\lambda)=0.$$

The general solution of the equation (20) has the form $\varphi(x,\lambda)=u(x,\lambda)+lv(x,\lambda)$ up to a constant. Choose an l such that the condition $\varphi(b,\lambda)=0$ holds true. This equality is valid for $l=l(b,\lambda)=-\frac{u(b,\lambda)}{v(b,\lambda)}$ (the solution $v(x,\lambda)$ has finitely many zeros for a fixed λ, hence $v(b,\lambda)\neq 0$ whenever b is sufficiently large). Put $\varphi_{11}^{(b)}(x,\lambda)=u(x,\lambda)+l(b,\lambda)v(x,\lambda)$. Since for the operator L_1 there is the case of a limit point, then, as is known, $l(b,\lambda)$ has a unique limit $m(\lambda)$ as $b\to\infty$, and the solution of the equation (22) satisfies $\varphi_{11}(x,\lambda)=u(x,\lambda)+m(\lambda)v(x,\lambda)\in L_2(0,\infty)$. Similarly we obtain that the solution of the equation (23) satisfies $\varphi_{22}(x,\lambda)\in L_2(0,\infty)$.

Denote by $\Phi_b(x,\lambda)=\begin{pmatrix}\varphi_{11}^{(b)}(x,\lambda) & \varphi_{12}^{(b)}(x,\lambda)\\ 0 & \varphi_{22}^{(b)}(x,\lambda)\end{pmatrix}$ the matrix solution of the equation (20) satisfying the initial conditions $\Phi_b(b,\lambda)=0$, ${\Phi_b}'(b,\lambda)=I$. We have $\varphi_{11}^{(b)}(x,\lambda)\to\varphi_{11}(x,\lambda)\in\in L_2(0,\infty)$, $\varphi_{22}^{(b)}(x,\lambda)\to\varphi_{22}(x,\lambda)\in L_2(0,\infty)$ as $b\to\infty$. The solution $\varphi_{12}^{(b)}(x,\lambda)$ is given by

$$\varphi_{12}^{(b)}(x,\lambda)=\int_0^x q(t)\cdot C(x,t,\lambda)\cdot\varphi_{22}^{(b)}(t,\lambda)\,dt,$$

where $C(x,t,\lambda)=u(x,\lambda)\,v(t,\lambda)-v(x,\lambda)\,u(t,\lambda)$ is the Cauchy function of the equation (22).

Further, we have $\varphi_{12}^{(b)}(x,\lambda)\to\int_0^x q(t)\cdot C(x,t,\lambda)\cdot\varphi_{22}(t,\lambda)\,dt:=\varphi_{12}(x,\lambda)$ as $b\to\infty$. Put

$$\Phi(x,\lambda)=\begin{pmatrix}\varphi_{11}(x,\lambda) & \varphi_{12}(x,\lambda)\\ 0 & \varphi_{22}(x,\lambda)\end{pmatrix}.$$

Together with the equation (20), we consider the left equation

$$\widetilde{l}\,[\widetilde{y}]=-\widetilde{y}''+\widetilde{y}V(x)=\lambda\widetilde{y},\quad \widetilde{y}=(y_1,y_2). \tag{26}$$

The matrix solutions of the equation (26) will be denoted by $\tilde{\Phi}_b(x,\lambda)$ and $\tilde{\Phi}(x,\lambda)$.

Denote by $Y(x,\lambda)$ and $\widetilde{Y}(x,\lambda)$ the solutions of the equations (20) and (26) respectively satisfying the initial conditions

$$Y(0,\lambda)=\cos A,\quad Y'(0,\lambda)=\sin A,\quad \widetilde{Y}(0,\lambda)=\cos A,\quad \widetilde{Y}'(0,\lambda)=\sin A,\quad \lambda\in\mathbb{C}. \tag{27}$$

Put

$$G_b(x,t,\lambda)=\begin{cases} Y(x,\lambda)\left(W\{\tilde{\Phi}_b,Y\}\right)^{-1}\tilde{\Phi}_b(t,\lambda), & 0\leqslant x\leqslant t\\ -\Phi_b(x,\lambda)\left(W\{\tilde{Y},\Phi_b\}\right)^{-1}\tilde{Y}(t,\lambda), & t\leqslant x\leqslant b\end{cases}. \tag{28}$$

The function $G_b(x,t,\lambda)$ is the Green function of the operator L_b^0 generated by the problem (20), (21), $y(b)=0$, which spectrum coincides (see Lemma 1 from [3]) with the union of spectra of the operators $L_{b,1}^0$, $L_{b,2}^0$ generated by the problems (22), (24), $y_1(b)=0$ and (23), (25), $y_2(b)=0$ respectively. Eigenvalues of the operators $L_{b,1}^0$ and $L_{b,2}^0$ tend to ones of the operators $\widetilde{L_1}$ and $\widetilde{L_2}$ respectively as $b\to\infty$, $\Phi_b(x,\lambda)\to\Phi(x,\lambda)$,

$\widetilde{\Phi}_b(x,\lambda) \to \widetilde{\Phi}(x,\lambda)$, and

$$\begin{aligned} W\{\widetilde{Y}, \Phi_b\} &= \cos A \cdot {\Phi_b}'(0,\lambda) - \sin A \cdot \Phi_b(0,\lambda) \\ &\quad \to \cos A \cdot \Phi'(0,\lambda) - \sin A \cdot \Phi(0,\lambda) = W\{\widetilde{Y}, \Phi\}, \\ W\{\tilde{\Phi}_b, Y\} &\to W\{\tilde{\Phi}, Y\}, \end{aligned}$$

$$G_b(x,t,\lambda) \to G(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W\{\tilde{\Phi}, Y\}\right)^{-1} \tilde{\Phi}(t,\lambda), & 0 \leqslant x \leqslant t \\ -\Phi(x,\lambda)\left(W\{\tilde{Y}, \Phi\}\right)^{-1} \tilde{Y}(t,\lambda), & t \leqslant x \end{cases}. \tag{29}$$

Poles of the Green function $G(x,t,\lambda)$ of the operator L coincide with the zero set of the determinant $\Delta(\lambda) := \det \Omega(\lambda)$, where

$$\Omega(\lambda) = W\{\widetilde{Y}, \Phi\}\Big|_{x=0} = \cos A \cdot \Phi'(0,\lambda) - \sin A \cdot \Phi(0,\lambda).$$

Since the matrices $\cos A$, $\sin A$, $\Phi(0,\lambda)$, $\Phi'(0,\lambda)$ are triangle, we have $\Delta(\lambda) = \Delta_1(\lambda) \cdot \Delta_2(\lambda)$, where $\Delta_k(\lambda) = \cos\alpha_{kk} \cdot \varphi'_{kk}(0,\lambda) - \sin\alpha_{kk} \cdot \varphi_{kk}(0,\lambda)$, $k = 1,2$. On the other hand, zeros of the function $\Delta_k(\lambda)$ are eigenvalues of the self-adjoint operator $\widetilde{L}_k$. Hence the poles of the Green function $G(x,t,\lambda)$ of the operator L are situated on the real axis, and their set coincides with the union of spectra of the operators $\widetilde{L_1}$ and $\widetilde{L_2}$.

Consider the operator $R_{\lambda,b}$ defined on $L_2(H_2,(0;b))$ by

$$\begin{aligned} \left(R_{\lambda,b}\bar{f}\right)(x) &= \int_0^b G_b(x,t,\lambda)\,\bar{f}(t)\,dt \\ &= -\int_0^x \Phi_b(x,\lambda)\left(W\{\tilde{Y}, \Phi_b\}\right)^{-1} \tilde{Y}(t,\lambda)\,\bar{f}(t)\,dt \\ &\quad + \int_x^b Y(x,\lambda)\left(W\{\tilde{\Phi}_b, Y\}\right)^{-1} \tilde{\Phi}(t,\lambda)\,\bar{f}(t)\,dt. \end{aligned} \tag{30}$$

One can directly verify that the operator $R_{\lambda,b}$ is the resolvent of the operator L_b^0.

Let $\bar{f}(x)$ be an arbitrary vector function square integrable on $[0,\infty)$. Choose a sequence of finite continuous vector functions $\left\{\bar{f}_n(x)\right\}$ $(n = 1,2,\ldots)$ converging in mean square to $\bar{f}(x)$. Substituting $\bar{f}_n$ for $\bar{f}$ in (30) and letting first $b \to \infty$ and then $n \to \infty$, we obtain the following formula for the resolvent R_λ of the operator L:

$$\left(R_\lambda \bar{f}\right)(x) = \int_0^\infty G(x,t,\lambda)\bar{f}(t)dt,$$

where the Green function of the operator L is defined by the formula (29).

Theorem 3. *The operator R_λ is the resolvent of the operator L. The poles of the resolvent coincide with the union of the spectra of the self-adjoint operators $\widetilde{L}_1$ and $\widetilde{L}_2$.*

Remark 3. *As in Example 1, if $\lambda_0 \in \sigma(\widetilde{L}_2)$ and $\varphi_{12}(x,\lambda_0) \notin L_2(0,\infty)$, then λ_0 is the pole of the resolvent R_λ of the operator L but it is not the eigenvalue of this operator, i.e., λ_0 is the point of the spectral singularity of the operator L.*

Theorem 2 implies that, if the rate of the coefficient's growth $q(x)$ of the equation (20) is subordinated to one of $p(x)$ and $r(x)$, then the operator L has no spectral singularities, and its spectrum is real and coincides with the union of the spectra of the operators $\widetilde{L}_1$ and $\widetilde{L}_2$.

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