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ON EXIT SPACE EXTENSIONS OF SYMMETRIC OPERATORS WITH APPLICATIONS TO FIRST ORDER SYMMETRIC SYSTEMS

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Dedicated with respect to F. S. Rofe-Beketov on the occasion of his anniversary

Abstract. Let A be a symmetric linear relation with arbitrary deficiency indices. By using the concept of the boundary triplet we describe exit space self-adjoint extensions $\widetilde{A}^\tau$ of A in terms of a boundary parameter τ. We characterize certain geometrical properties of $\widetilde{A}^\tau$ and describe all $\widetilde{A}^\tau$ with $\operatorname{mul}\widetilde{A}^\tau=\{0\}$. Applying these results to general (possibly non- Hamiltonian) symmetric systems $Jy'-B(t)y=\Delta(t)y,\ t\in[a,b\rangle$, we describe all matrix spectral functions of the minimally possible dimension such that the Parseval equality holds for any function $f\in L^2_\Delta([a,b\rangle)$.

1. Introduction

Assume that $\mathfrak{H}$ is a Hilbert space, A is a not necessarily densely defined symmetric operator in $\mathfrak{H}$ with deficiency indices $n_\pm(A)$ and A^* is the adjoint linear relation of A. Let also $[\mathfrak{H}_1,\mathfrak{H}_2]$ be the set of all bounded operators between $\mathfrak{H}_1$ and $\mathfrak{H}_2$ and let $[\mathfrak{H}]=[\mathfrak{H},\mathfrak{H}]$.

As is known the exit space self-adjoint extension of A is a linear relation $\widetilde{A}=\widetilde{A}^*\supset A$ in a Hilbert space $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$. Denote by $\mathcal{S}_A$ the set of all such extensions $\widetilde{A}$ and let $\mathcal{S}_A^0$ be the set of all $\widetilde{A}\in\mathcal{S}_A$ with $\operatorname{mul}\widetilde{A}=\{0\}$ (that is $\widetilde{A}=\widetilde{A}^*$ is an operator in $\widetilde{\mathfrak{H}}$). It is known that $\mathcal{S}_A\neq\emptyset$ for any A and $\mathcal{S}_A=\mathcal{S}_A^0$ if and only if A is densely defined.

Each extension $\widetilde{A}\in\mathcal{S}_A$ generates a generalized resolvent

$$R(\lambda)=P_{\mathfrak{H}}(\widetilde{A}-\lambda)^{-1}\upharpoonright\mathfrak{H},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}\tag{1.1}$$

of A and the (minimal) extension $\widetilde{A}\in\mathcal{S}_A$ is defined by $R(\lambda)$ uniquely up to the unitary equivalence. In the particular case $\widetilde{\mathfrak{H}}=\mathfrak{H}$ the extension $\widetilde{A}\in\mathcal{S}_A$ is canonical and $R(\lambda)$ is a canonical resolvent of A (the later is possible if and only if $n_+(A)=n_-(A)$).

A description of the classes $\mathcal{S}_A$ and $\mathcal{S}_A^0$ for a given A is an important problem in the extension theory of symmetric operators. In the paper by A. V. Štraus [31] the class $\mathcal{S}_A^0$ is parametrized by means of a contractive holomorphic parameter $F(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathfrak{N}_{\lambda_0},\mathfrak{N}_{\overline{\lambda}_0}]$ with a certain limit property at ∞ (here $\lambda_0\in\mathbb{C}_+$ and $\mathfrak{N}_\lambda$ is a defect subspace of A). In the case $n_+(A)=n_-(A)$ another description of the sets $\mathcal{S}_A$ and $\mathcal{S}_A^0$ is given by the Krein-Naimark formula for generalized resolvents [16, 17, 20]

$$R_\tau(\lambda)=P_{\mathfrak{H}}(\widetilde{A}^\tau-\lambda)^{-1}\upharpoonright\mathfrak{H}=(A_0-\lambda)^{-1}-\gamma(\lambda)(\tau(\lambda)+M(\lambda))^{-1}\gamma^*(\overline{\lambda}),\quad \lambda\in\mathbb{C}\setminus\mathbb{R},\tag{1.2}$$

where $A_0=A_0^*$ is a fixed extension of A, $\gamma(\lambda)$ is the so called γ-field and $M(\lambda)$ is the Weyl function (Q-function) of the pair (A,A_0). Formula (1.2) gives a bijective correspondence between all extensions $\widetilde{A}=\widetilde{A}^\tau\in\mathcal{S}_A$ and all Nevanlinna families of linear relations

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$\tau = \tau(\lambda)$ in the auxiliary Hilbert space $\mathcal{H}$. Moreover, H. Langer and B. Textorious showed in [20] that $\widetilde{A}^\tau \in \mathcal{S}_A^0$ if and only if

$$s - \lim_{y\to\infty} \tfrac{1}{y}[M(iy) - (M(iy) - M^*(z_0))(M(iy) + \tau(iy))^{-1}(M(iy) - M(z_0))] = 0. \tag{1.3}$$

Note also that formula for generalized resolvents of an operator A with arbitrary (possibly unequal) deficiency indices $n_\pm(A)$ was obtained in [19, 2]. This formula is more complicated than (1.2); it contains as a parameter a contractive holomorphic operator-function $F(\cdot)$ from the Štraus' paper [31].

During the last three decades an approach to the extension theory based on the concept of a boundary triplet has been extensively developed (see [4, 6, 8, 9, 12, 14, 22, 25, 27] and references therein). This approach goes back to the pioneering paper by J. W. Calkin [5], where all self-adjoint extensions of symmetric operators with arbitrary deficiency indices were described in terms of hyper-maximal symmetric subspaces of some auxiliary Hilbert space (see also review [27]). Later on similar methods were applied to various classes of boundary value problems in [3, 11, 29, 32]. It should be especially singled out the paper by F. S. Rofe-Beketov [29], in which for the first time self-adjoint boundary conditions for ordinary differential operators with operator valued coefficients were described in terms of self-adjoint linear relations. These papers influenced the appearance of the concept of a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* in [4, 14]. Such a triplet consists of an auxiliary Hilbert space $\mathcal{H}$ and two linear mappings $\Gamma_0, \Gamma_1 : A^* \to \mathcal{H}$ such that the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective and the following abstract Green identity holds:

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g}) - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g}), \quad \widehat{f} = \{f, f'\},\ \widehat{g} = \{g, g'\} \in A^*. \tag{1.4}$$

A connection between the Krein-Naimark formula (1.2) and a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* has been discovered in [8] for a densely defined operator and in [9, 22] for a nondensely defined operator A. Namely, it was shown in [8, 9, 22] that each boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ gives rise to the formula (1.2) with $A_0 = \ker \Gamma_0$, $\gamma(\lambda) = (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1}$, $\tau(\lambda) = \Gamma(R^{-1}(\lambda) + \lambda I)$ and the Weyl function $M(\lambda)(\in [\mathcal{H}])$ defined by

$$\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda = M(\lambda)\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.5}$$

These results together with a coupling method made it possible to describe in [6, 7] the class $\mathcal{S}_A^0$ in terms of $M(\cdot)$ and $\tau(\cdot)$ as follows: $\widetilde{A}^\tau \in \mathcal{S}_A^0$ if and only if

$$s - \lim_{y\to\infty} \tfrac{1}{y}(\tau(iy) + M(iy))^{-1} = 0 \quad \text{and} \quad s - \lim_{y\to\infty} \tfrac{1}{y}(\tau^{-1}(iy) + M^{-1}(iy))^{-1} = 0. \tag{1.6}$$

Moreover, in [6, 9, 22] several other criteria for $\widetilde{A}^\tau \in \mathcal{S}_A^0$ were found. In particular, it was firstly shown in [22] that in the case $\operatorname{mul} A_0 = \{0\}$ the following equivalence holds:

$$\widetilde{A}^\tau \in \mathcal{S}_A^0 \iff s - \lim_{y\to\infty} \tfrac{1}{y}(\tau(iy) + M(iy))^{-1} = 0. \tag{1.7}$$

In [9] these results were applied to truncated power moment problem.

Since a triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* satisfies $n_+(A) = n_-(A) = \dim \mathcal{H}$, the above results on boundary triplets are applicable only to operators A with equal deficiency indices. To cover the case $n_+(A) \neq n_-(A)$ we generalized in [25] definition of a boundary triplet as follows. Assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, P_j is the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$ and $\Gamma_j : A^* \to \mathcal{H}_j$, $j \in \{0, 1\}$, are linear mappings. A collection $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* if the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective and the identity (1.4) holds with a certain additional term in the right hand side (see (3.1)). Associated with a triplet Π_+ is the Weyl function $M_+(\lambda)(\in [\mathcal{H}_0, \mathcal{H}_1])$ and the Nevanlinna operator function $M(\lambda)$ defined by (cf. (1.5))

$$\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda = M_+(\lambda)\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda, \quad M(\lambda) = M_+(\lambda) \upharpoonright \mathcal{H}_1, \quad \lambda \in \mathbb{C}_+. \tag{1.8}$$

It turns out that a boundary triplet Π_+ exists for any A with $n_-(A) \leq n_+(A)$. Moreover, it is shown in [25] that each boundary triplet $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* gives a parametrization of all extensions $\widetilde{A}=\widetilde{A}^{\tau}\in\mathcal{S}_A$ by means of the formula for generalized resolvents

$$R_\tau(\lambda)=P_{\mathfrak{H}}(\widetilde{A}^{\tau}-\lambda)^{-1}\restriction\mathfrak{H}=(A_0-\lambda)^{-1}-\gamma_+(\lambda)(\tau_+(\lambda)+M_+(\lambda))^{-1}\gamma_-^*(\overline{\lambda}), \tag{1.9}$$

which holds for $\lambda\in\mathbb{C}_+$. In this formula $A_0=\ker\Gamma_0$ is a maximal symmetric extension of A and $\gamma_\pm(\lambda)$ are γ-fields of the triplet Π_+. The role of a boundary parameter in (1.9) is played by holomorphic families of linear relations $\tau_+(\lambda),\lambda\in\mathbb{C}_+$, belonging to the special Nevanlinna type class $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ (see Subsection 2.2 below).

In the present paper we first develop the known results on exit space extensions and then apply them to symmetric systems of differential equations.

Let A be a symmetric operator in $\mathfrak{H}$ with $n_-(A)\le n_+(A)$, let $\Pi_+=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^* , let $M_+(\cdot)$ be the Weyl function of the triplet Π_+ and let $M(\lambda)$ be given by the second equality in (1.8). We prove that A is densely defined if and only if $s-\lim_{y\to+\infty}\frac{1}{y}M(iy)=0$ and

$$\lim_{y\to+\infty}y\left(\operatorname{Im}(M_+(iy)h_0,h_0)_{\mathcal{H}_0}+\tfrac{1}{2}||P_2h_0||^2\right)=+\infty,\quad h_0\in\mathcal{H}_0,\quad h_0\ne0.$$

This is a generalization of the known results obtained in [18, 20, 9] for the case $n_+(A)=n_-(A)$. Next we show that $\widetilde{A}^{\tau}\in\mathcal{S}_A^0$ if and only if the following two conditions are satisfied

$$s-\lim_{y\to+\infty}\tfrac{1}{iy}P_1(\tau_+(iy)+M_+(iy))^{-1}=0,\qquad s-\lim_{y\to+\infty}\tfrac{1}{iy}P_1(\widehat{\tau}_+(iy)+\widehat{M}_+(iy))^{-1}=0, \tag{1.10}$$

where $\widehat{\tau}_+(\lambda)$ and $\widehat{M}_+(\lambda)$ are constructed in terms of $\tau_+(\lambda)$ and $M_+(\lambda)$ (see (4.14) and (4.17)). Moreover, we show that this criterion for $\widetilde{A}^{\tau}\in\mathcal{S}_A^0$ is a consequence of the following equivalences:

$$\operatorname{mul}\widetilde{A}^{\tau}\subset\operatorname{mul}A_0\oplus\mathfrak{H}_1\iff s-\lim_{y\to+\infty}\tfrac{1}{iy}P_1(\tau_+(iy)+M_+(iy))^{-1}=0, \tag{1.11}$$

$$\operatorname{mul}\widetilde{A}^{\tau}\subset\operatorname{mul}A_1\oplus\mathfrak{H}_1\iff s-\lim_{y\to+\infty}\tfrac{1}{iy}P_1(\widehat{\tau}_+(iy)+\widehat{M}_+(iy))^{-1}=0, \tag{1.12}$$

where $A_1=\ker\Gamma_1\cap\ker P_2\Gamma_0$, $\operatorname{mul}\widetilde{A}^{\tau}$ and $\operatorname{mul}A_j$ are the multivalued parts of $\widetilde{A}^{\tau}$ and $A_j,\ j\in\{0,1\}$, respectively and $\mathfrak{H}_1=\widetilde{\mathfrak{H}}\ominus\mathfrak{H}_0$. Note that equivalences (1.11) and (1.12) clarify the geometrical sense of each of the conditions in (1.10).

Similarly to Π_+ we introduce in the paper a boundary triplet Π_- for A^* and extend the above results to such a triplet. This enables us to treat the case $n_+(A)\le n_-(A)$.

Observe that our results seem to be simpler and more convenient for applications than those of [19, 2] (see, for instance, Section 5 below).

In the case of equal deficiency indices $n_+(A)=n_-(A)$ and an ordinary boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* one has $\widehat{\tau}_+(\lambda)=-\tau^{-1}(\lambda)$, $\widehat{M}_+(\lambda)=-M^{-1}(\lambda)$ and the equalities (1.10) take the form (1.6); moreover, in this case equivalences (1.11) and (1.12) can be written as

$$\operatorname{mul}\widetilde{A}^{\tau}\subset\operatorname{mul}A_0\oplus\mathfrak{H}_1\iff s-\lim_{y\uparrow\infty}\tfrac{1}{iy}(\tau(iy)+M(iy))^{-1}=0, \tag{1.13}$$

$$\operatorname{mul}\widetilde{A}^{\tau}\subset\operatorname{mul}A_1\oplus\mathfrak{H}_1\iff s-\lim_{y\uparrow\infty}\tfrac{1}{iy}(\tau^{-1}(iy)+M^{-1}(iy))^{-1}=0. \tag{1.14}$$

Note that equivalences (1.13) and (1.14) are not contained in [6, 7]; in fact they can be derived from [7, Theorem 5.14 and Proposition 3.17(i)]. If $\operatorname{mul}A_0=\{0\}$, then the left hand side of (1.13) takes the form $\operatorname{mul}\widetilde{A}^{\tau}=\{0\}$. Hence the equivalence (1.7) is an elementary consequence of (1.13). Observe also that criterion (1.6) was proved in [7] with the aid of a rather complicated construction of a (possibly multivalued) boundary relation $\Gamma:A^*\to\mathcal{H}^2$, while our approach enables one to remain in the framework of an ordinary boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^*.

Next assume that H and $\widehat{H}$ are finite dimensional Hilbert spaces, $\mathbb{H} := H \oplus \widehat{H} \oplus H$ and let $J \in [\mathbb{H}]$ be the operator given by

$$J = \begin{pmatrix} 0 & 0 & -I_H \\ 0 & iI_{\widehat{H}} & 0 \\ I_H & 0 & 0 \end{pmatrix} : H \oplus \widehat{H} \oplus H \to H \oplus \widehat{H} \oplus H. \tag{1.15}$$

A first order symmetric system on an interval $\mathcal{I} = [a, b\rangle, -\infty < a < b \leq \infty$, (with the regular endpoint a) is of the form

$$Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in \mathcal{I}, \tag{1.16}$$

where $B(t) = B^*(t)$, $\Delta(t) \geq 0$ and $B(t), \Delta(t) \in [\mathbb{H}]$, $t \in \mathcal{I}$. Investigations of systems (1.16) is motivated by the fact that a formally self-adjoint differential equation of an arbitrary (even or odd) order with matrix coefficients is reduced to a system of the form (1.16) with the operator J given by (1.15) (see [15]).

As is known [13, 21, 28] system (1.16) generates minimal and maximal linear relations $T_{\min}$ and $T_{\max}$ in $\mathfrak{H} := L^2_\Delta(\mathcal{I})$. Moreover, $T_{\min}$ is a closed symmetric relation with finite not necessarily equal deficiency indices $n_\pm(T_{\min})$ and $T_{\max} = T^*_{\min}$.

In [1] systems (1.16) are studied in the framework of a boundary triplets approach under the assumptions $n_-(T_{\min}) \leq n_+(T_{\min})$. This enables the authors to describe boundary problems for system (1.16) with λ-depending (in particular, self-adjoint) boundary conditions, which generate eigenfunction expansions with the matrix spectral function $\Sigma_\tau(\cdot)$ of the minimally possible dimension (for more details see Subsection 5.1 below). Moreover, in the case $n_+(T_{\min}) = n_-(T_{\min})$ the class SF of all such spectral functions $\Sigma_\tau(\cdot)$ as well as its most interesting subclass SF_0 are parametrized in [1] by means of the formula similar to the formula for resolvents (1.2). In the present paper we extend this result to the case of possibly unequal deficiency indices $n_-(T_{\min}) \leq n_+(T_{\min})$ (see Theorem 5.5). For this purpose we use the mentioned above criterion (1.10).

2. Preliminaries

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on the Hilbert space $\mathcal{H}_1$ with values in the Hilbert space $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $A \upharpoonright \mathcal{L}$ is the restriction of an operator A onto the linear manifold $\mathcal{L}$; $P_\mathcal{L}$ is the orthogonal projector in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$; $\mathbb{C}_+$ ($\mathbb{C}_-$) is the upper (lower) half-plane of the complex plane.

Recall that a closed linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ is a closed linear subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\widetilde{\mathcal{C}}(\mathcal{H})$). A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\operatorname{gr} T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\operatorname{dom} T$, $\operatorname{ran} T$, $kerT$ and $\operatorname{mul} T$ the domain, range, kernel and the multivalued part of T respectively. Recall also that the inverse and adjoint linear relations of T are the relations $T^{-1} \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ and $T^* \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ defined by

$$T^{-1} = \{\{h_1, h_0\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : \{h_0, h_1\} \in T\},$$
$$T^* = \{\{k_1, k_0\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : (k_0, h_0) - (k_1, h_1) = 0, \ \{h_0, h_1\} \in T\}.$$

In the case $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we write $0 \in \rho(T)$ if $\ker T = \{0\}$ and $\operatorname{ran} T = \mathcal{H}_1$, or equivalently if $T^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$; $0 \in \widehat{\rho}(T)$ if $\ker T = \{0\}$ and $\operatorname{ran} T$ is a closed subspace in $\mathcal{H}_1$. For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T) := \{\lambda \in \mathbb{C} : \ 0 \in \rho(T - \lambda)\}$ and $\widehat{\rho}(T) = \{\lambda \in \mathbb{C} : \ 0 \in \widehat{\rho}(T - \lambda)\}$ the resolvent set and the set of regular type points of T respectively.

Recall also the following definition.

Definition 2.1. A holomorphic operator function $\Phi(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathcal{H}]$ is called a Nevanlinna function if $\operatorname{Im}\lambda\cdot\operatorname{Im}\Phi(\lambda)\geq 0$ and $\Phi^*(\lambda)=\Phi(\overline{\lambda}),\ \lambda\in\mathbb{C}\setminus\mathbb{R}$.

The class of all $[\mathcal{H}]$-valued Nevanlinna functions will be denoted by $R[\mathcal{H}]$.

As is known for each function $\Phi\in R[\mathcal{H}]$ there exists the strong limit

$$\mathcal{B}=\mathcal{B}_\Phi:=s-\lim_{y\to\infty}\tfrac{1}{iy}\Phi(iy); \tag{2.1}$$

moreover, $\mathcal{B}_\Phi=\mathcal{B}_\Phi^*$ and $\mathcal{B}_\Phi\geq 0$.

The following proposition will be useful in the sequel.

Proposition 2.2. *Let $\mathcal{H}'$ and $\mathcal{H}''$ be Hilbert spaces and let*

$$\Phi(\lambda)=\begin{pmatrix}\Phi_{11}(\lambda) & \Phi_{12}(\lambda)\\ \Phi_{21}(\lambda) & \Phi_{22}(\lambda)\end{pmatrix}:\mathcal{H}'\oplus\mathcal{H}''\to\mathcal{H}'\oplus\mathcal{H}'',\quad \lambda\in\mathbb{C}\setminus\mathbb{R}$$

be the block matrix representation of a function $\Phi(\cdot)\in R[\mathcal{H}'\oplus\mathcal{H}'']$. Then: (i) $\Phi_{11}(\cdot)\in R[\mathcal{H}']$ *and* $\Phi_{22}(\cdot)\in R[\mathcal{H}'']$*;* (ii) $\mathcal{B}_\Phi=0$ *if and only if* $\mathcal{B}_{\Phi_{11}}=0$ *and* $\mathcal{B}_{\Phi_{22}}=0$.

The statement of the proposition follows from the relation $\mathcal{B}_\Phi=\begin{pmatrix}\mathcal{B}_{\Phi_{11}} & C\\ C^* & \mathcal{B}_{\Phi_{22}}\end{pmatrix}\geq 0$.

2.2. **Holomorphic operator pairs.** Let Λ be an open set in $\mathbb{C}$ and let $\mathcal{K},\mathcal{H}_0,\mathcal{H}_1$ be Hilbert spaces. A pair of holomorphic operator functions (in short a holomorphic pair) $C_j(\cdot):\Lambda\to[\mathcal{H}_j,\mathcal{K}],\ j\in\{0,1\}$, is called admissible if, for each $\lambda\in\Lambda$, the range of the operator

$$(C_0(\lambda),\,C_1(\lambda)):\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{K} \tag{2.2}$$

coincides with $\mathcal{K}$. Below, unless otherwise stated, all the pairs (2.2) are admissible .

Two holomorphic pairs $(C_0^{(j)}(\cdot),\,C_1^{(j)}(\cdot)):\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{K}_j,\ j\in\{1,2\}$, are said to be equivalent if there exists a holomorphic operator function $\varphi(\cdot):\Lambda\to[\mathcal{K}_1,\mathcal{K}_2]$ such that $0\in\rho(\varphi(\lambda))$ and $C_j^{(2)}(\lambda)=\varphi(\lambda)C_j^{(1)}(\lambda),\ \lambda\in\Lambda,\ j\in\{1,2\}$. Clearly, the set of all holomorphic pairs splits into disjoint equivalence classes; moreover, the equality

$$\tau(\lambda)=\{(C_0(\lambda),C_1(\lambda));\mathcal{K}\}:=\{\{h_0,h_1\}\in\mathcal{H}_0\oplus\mathcal{H}_1:C_0(\lambda)h_0+C_1(\lambda)h_1=0\} \tag{2.3}$$

allows us to identify such a class with the $\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$-valued function $\tau(\lambda),\ \lambda\in\Lambda$. In the case $\Lambda=\overline{\mathbb{C}}$ one has $C_j(\lambda)\equiv C_j\in[\mathcal{H}_j,\mathcal{K}]$ and the equality (2.3) defines the relation

$$\theta=\{(C_0,C_1);\mathcal{K}\}:=\{\{h_0,h_1\}\in\mathcal{H}_0\oplus\mathcal{H}_1:C_0h_0+C_1h_1=0\},\quad \theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1). \tag{2.4}$$

Conversely, each $\theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ can be represented in the form (2.4).

In what follows, unless otherwise stated, $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$, $\mathcal{H}_2:=\mathcal{H}_0\ominus\mathcal{H}_1$ and P_j is the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j,\ j\in\{1,2\}$.

Let $\alpha\in\{-1,+1\}$. With each linear relation $\theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ we associate the $\times$-adjoint linear relation $\theta_\alpha^\times\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ given by

$$\theta_\alpha^\times=\{\{k_0,k_1\}\in\mathcal{H}_0\oplus\mathcal{H}_1:(k_1,h_0)-(k_0,h_1)+i\alpha(P_2k_0,P_2h_0)=0\ \text{ for all }\ \{h_0,h_1\}\in\theta\}.$$

Samples of calculating of $\times$-adjoint linear relations can be fond in [24, Proposition 3.1].

For a linear relation $\theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ we let

$$S_{\theta,\alpha}(\widehat{h})=2\operatorname{Im}(h_1,h_0)+\alpha||P_2h_0||^2,\quad \widehat{h}=\{h_0,h_1\}\in\theta.$$

Since $\mathcal{H}_1\subset\mathcal{H}_0$, one may consider relations $\theta+\lambda I_{\mathcal{H}_0}\in\widetilde{\mathcal{C}}(\mathcal{H}_0)$ and $\theta+\lambda P_1\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$.

Definition 2.3. A linear relation $\theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ belongs to the class:

(1) $\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if $S_{\theta,\alpha}(\widehat{h})\geq 0,\ \widehat{h}\in\theta$, and there exists $\lambda\in\mathbb{C}_+$ such that

$$0\in\rho(\theta+\lambda I_{\mathcal{H}_0})\ \text{ in the case }\ \alpha=+1\ \text{ and }\ 0\in\rho(\theta+\lambda P_1)\ \text{ in the case }\ \alpha=-1; \tag{2.5}$$

(2) $\mathrm{Ac}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if $S_{\theta,\alpha}(\widehat{h})\le 0$, $\widehat{h}\in\theta$, and there exists $\lambda\in\mathbb{C}_-$ such that

$$0\in\rho(\theta+\lambda P_1) \text{ in the case } \alpha=+1 \text{ and } 0\in\rho(\theta+\lambda I_{\mathcal{H}_0}) \text{ in the case } \alpha=-1; \tag{2.6}$$

(3) $\mathrm{Sym}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if $\theta\in\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)\cup\mathrm{Ac}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ and $S_{\theta,\alpha}(\widehat{h})=0$ for all $\widehat{h}\in\theta$;
(4) $\mathrm{Self}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$, if $\theta=\theta_\alpha^\times$.

A description of the classes Dis_α, Ac_α, Sym_α and Self_α in terms of operator pairs is given in the following proposition.

Proposition 2.4. *Let a relation $\theta\in\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ be given by* (2.4) *with $C_0=(C_{01},C_{02}):\mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{K}$ and $C_1\in[\mathcal{H}_1,\mathcal{K}]$. Moreover, let*

$$\widetilde{S}_{\theta,\alpha}:=2\mathrm{Im}(C_1C_{01}^*)-\alpha C_{02}C_{02}^*,\quad \widetilde{S}_{\theta,\alpha}\in[\mathcal{K}].$$

Then: (1) *$\theta\in\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if and only if $\widetilde{S}_{\theta,\alpha}\ge 0$ and there exists $\lambda\in\mathbb{C}_+$ such that*

$$0\in\rho(C_{01}-\lambda C_1) \text{ if } \alpha=+1 \text{ and } 0\in\rho(C_0-\lambda C_1P_1) \text{ if } \alpha=-1; \tag{2.7}$$

(2) *$\theta\in\mathrm{Ac}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if and only if $\widetilde{S}_{\theta,\alpha}\le 0$ and there exists $\lambda\in\mathbb{C}_-$ such that*

$$0\in\rho(C_0-\lambda C_1P_1) \text{ if } \alpha=+1 \text{ and } 0\in\rho(C_{01}-\lambda C_1) \text{ if } \alpha=-1; \tag{2.8}$$

(3) *$\theta\in\mathrm{Sym}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ ($\theta\in\mathrm{Self}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$) if and only if $\widetilde{S}_{\theta,\alpha}=0$ and at least one of the conditions (respectively both the conditions)* (2.7), (2.8) *is fulfilled. Therefore $\theta\in\mathrm{Self}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if and only if $\theta\in\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)\cap\mathrm{Ac}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$.*

Moreover, if $\theta\in\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ ($\theta\in\mathrm{Ac}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$), then the relations (2.5) *and* (2.7) *(resp.* (2.6) *and* (2.8)*) hold for all $\lambda\in\mathbb{C}_+$ (resp. $\lambda\in\mathbb{C}_-$).*

Remark 2.5. (1) In the case $\alpha=+1$ the classes Dis_α, Ac_α, Sym_α and Self_α coincide with those introduced (without index α) in [24]. Moreover, Proposition 2.4 for $\alpha=+1$ was also proved in [24]. The passage to the case $\alpha=-1$ can be realized by means of the equivalence $\theta\in\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)\iff -\theta\in Ac_{-\alpha}(\mathcal{H}_0,\mathcal{H}_1)$.

(2) In the case $\mathcal{H}_0=\mathcal{H}_1=:\mathcal{H}$ one has $\theta^\times=\theta^*$ and the classes Dis_α, Ac_α, Sym_α and Self_α coincide with the well known classes of all maximal dissipative, maximal accumulative, maximal symmetric and self-adjoint linear relations in $\mathcal{H}$ respectively.

Let as before $\alpha\in\{-1,+1\}$ and let $\tau=\{\tau_+,\tau_-\}$ be a collection of functions $\tau_+(\cdot):\mathbb{C}_+\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and $\tau_-(\cdot):\mathbb{C}_-\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$.

Definition 2.6. A collection $\tau=\{\tau_+,\tau_-\}$ belongs to the class $\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if
(1) $-\tau_+(\lambda)\in\mathrm{Ac}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$, $\lambda\in\mathbb{C}_+$, and $-\tau_-(\lambda)\in\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$, $\lambda\in\mathbb{C}_-$;
(2) $(-\tau_+(\lambda))_\alpha^\times=-\tau_-(\overline{\lambda})$, $\lambda\in\mathbb{C}_+$;
(3) The operator function $(\tau_+(\lambda)+iP_1)^{-1}(\in[\mathcal{H}_1,\mathcal{H}_0])$ in the case $\alpha=+1$ $((\tau_+(\lambda)+iI_{\mathcal{H}_0})^{-1}(\in[\mathcal{H}_0])$ in the case $\alpha=-1)$ is holomorphic in $\mathbb{C}_+$.

A collection $\tau=\{\tau_+,\tau_-\}$ belongs to the class $\widetilde{R}^0_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if $-\tau_\pm(\lambda)\equiv\theta$, $\lambda\in\mathbb{C}_\pm$, with some $\theta\in\mathrm{Self}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ (this implies that $\widetilde{R}^0_\alpha(\mathcal{H}_0,\mathcal{H}_1)\subset\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$).

In the following we write $\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ (resp. $\widetilde{R}_-(\mathcal{H}_0,\mathcal{H}_1)$) in place of $\widetilde{R}_{+1}(\mathcal{H}_0,\mathcal{H}_1)$ (resp. $\widetilde{R}_{-1}(\mathcal{H}_0,\mathcal{H}_1)$).

Next assume that $\mathcal{K}_+$ and $\mathcal{K}_-$ are auxiliary Hilbert spaces and

$$\begin{aligned}\tau_+(\lambda)&=\{(C_0(\lambda),C_1(\lambda));\mathcal{K}_+\},\quad \lambda\in\mathbb{C}_+,\\ \tau_-(\lambda)&=\{(D_0(\lambda),D_1(\lambda));\mathcal{K}_-\},\quad \lambda\in\mathbb{C}_-\end{aligned} \tag{2.9}$$

are equivalence classes of holomorphic operator pairs (cf. (2.3))

$$\begin{aligned}(C_0(\lambda),C_1(\lambda))&:\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{K}_+,\quad \lambda\in\mathbb{C}_+,\\ (D_0(\lambda),D_1(\lambda))&:\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{K}_-,\quad \lambda\in\mathbb{C}_-.\end{aligned}$$

Assume also that

$$C_0(\lambda)=(C_{01}(\lambda),C_{02}(\lambda)):\mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{K}_+, \tag{2.10}$$

$$D_0(\lambda)=(D_{01}(\lambda),D_{02}(\lambda)):\mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{K}_- \tag{2.11}$$

are the block representations of $C_0(\lambda)$ and $D_0(\lambda)$.

In the following proposition we describe the class $\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ in terms of holomorphic operator pairs.

Proposition 2.7. *Let $\tau=\{\tau_+,\tau_-\}$ be a collection of functions $\tau_\pm(\cdot):\mathbb{C}_\pm\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ given by* (2.9)*. Then $\tau\in\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if and only if the following relations are satisfied:*

$$2\,\mathrm{Im}(C_1(\lambda)C_{01}^*(\lambda))+\alpha C_{02}(\lambda)C_{02}^*(\lambda)\ge 0,\quad \lambda\in\mathbb{C}_+; \tag{2.12}$$

$$2\,\mathrm{Im}(D_1(\lambda)D_{01}^*(\lambda))+\alpha D_{02}(\lambda)D_{02}^*(\lambda)\le 0,\quad \lambda\in\mathbb{C}_-; \tag{2.13}$$

$$C_1(\lambda)D_{01}^*(\overline{\lambda})-C_{01}(\lambda)D_1^*(\overline{\lambda})+i\alpha C_{02}(\lambda)D_{02}^*(\overline{\lambda})=0,\quad \lambda\in\mathbb{C}_+; \tag{2.14}$$

$$\text{if } \alpha=+1, \text{ then } 0\in\rho(C_0(\lambda)-iC_1(\lambda)P_1) \text{ and } 0\in\rho(D_{01}(\lambda)+iD_1(\lambda)); \tag{2.15}$$

$$\text{if } \alpha=-1, \text{ then } 0\in\rho(C_{01}(\lambda)-iC_1(\lambda)) \text{ and } 0\in\rho(D_0(\lambda)+iD_1(\lambda)P_1). \tag{2.16}$$

Moreover, if $\tau\in\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ (so that (2.12)*–*(2.16) *hold), then $\tau\in\widetilde{R}^0_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if and only if for some (and hence for any) $\lambda\in\mathbb{C}_+$ the inequality in* (2.12) *turns into the equality and $0\in\rho(C_{01}(\lambda)+iC_1(\lambda))$ in the case $\alpha=+1$ ($0\in\rho(C_0(\lambda)+iC_1(\lambda)P_1)$ in the case $\alpha=-1$).*

Conversely, each collection $\tau=\{\tau_+,\tau_-\}\in\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ admits the representation in the form of holomorphic pairs (2.9)*. In particular, each collection $\tau=\{\tau_+,\tau_-\}\in\widetilde{R}^0_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ admits the constant-valued representation*

$$\tau_\pm(\lambda)\equiv\{(C_0,C_1);\mathcal{K}\}=-\theta,\quad \lambda\in\mathbb{C}_\pm,$$

where $C_j\in[\mathcal{H}_j,\mathcal{K}]$, $j\in\{0,1\}$, and $\theta\in\mathrm{Self}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$.

In view of Proposition 2.7 we identify in the sequel a collection of functions $\tau=\{\tau_+,\tau_-\}\in\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ with a collection of two holomorphic pairs (2.9) satisfying (2.12)–(2.16) (more precisely, with a collection of two equivalence classes of holomorphic pairs). Moreover, in view of (2.15) and (2.16) we may assume in what follows that $\mathcal{K}_+$ and $\mathcal{K}_-$ in (2.9) are: $\mathcal{K}_+=\mathcal{H}_0$ and $\mathcal{K}_-=\mathcal{H}_1$ if $\dim\mathcal{H}_1<\infty$ and $\alpha=+1$; $\mathcal{K}_+=\mathcal{H}_1$ and $\mathcal{K}_-=\mathcal{H}_0$ if $\dim\mathcal{H}_1<\infty$ and $\alpha=-1$; $\mathcal{K}_+=\mathcal{K}_-=\mathcal{H}_1$ if $\dim\mathcal{H}_1=\infty(=\dim\mathcal{H}_0)$.

Remark 2.8. (1) In the case $\alpha=+1$ the class $\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)=\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ coincides with the class $\widetilde{R}(\mathcal{H}_0,\mathcal{H}_1)$ introduced in [24]; moreover, Proposition 2.7 for this class follows from [24, Proposition 4.3]. The case $\alpha=-1$ can be treated by means of the following assertion: if $\tau_j=\{\tau_{+,j},\tau_{-,j}\}$ are collections of functions $\tau_{\pm,j}:\mathbb{C}_\pm\to\widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$, $j\in\{1,2\}$, such that $\tau_{\pm,2}(\lambda)=-\tau_{\mp,1}(-\lambda)$, $\lambda\in\mathbb{C}_\pm$, then $\tau_2\in\widetilde{R}_-(\mathcal{H}_0,\mathcal{H}_1)\iff\tau_1\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$.

(2) The set $\widetilde{R}^0_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ is not empty if and only if $\dim\mathcal{H}_0=\dim\mathcal{H}_1$. Therefore in the case $\dim\mathcal{H}_1<\infty$ the set $\widetilde{R}^0_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ is not empty if and only if $\mathcal{H}_0=\mathcal{H}_1=:\mathcal{H}$.

(3) In the case $\dim\mathcal{H}_0<\infty$ the statements of Proposition 2.7 can be reformulated in the "matrix" form. Namely, let $n_0:=\dim\mathcal{H}_0<\infty$, $n_1=\dim\mathcal{H}_1$ and let

$$n_\alpha=\begin{cases} n_0 & \text{if } \alpha=+1\\ n_1 & \text{if } \alpha=-1\end{cases},\quad m_\alpha=\begin{cases} n_1 & \text{if } \alpha=+1\\ n_0 & \text{if } \alpha=-1\end{cases}.$$

Assume also that a collection $\tau=\{\tau_+,\tau_-\}$ is given by (2.9) and let

$$C_0(\lambda)=(c_{kj,0}(\lambda))_{k=1}^{n_\alpha}{}_{j=1}^{n_0},\quad C_1(\lambda)=(c_{kj,1}(\lambda))_{k=1}^{n_\alpha}{}_{j=1}^{n_1}, \tag{2.17}$$

$$D_0(\lambda)=(d_{kj,0}(\lambda))_{k=1}^{m_\alpha}{}_{j=1}^{n_0},\quad D_1(\lambda)=(d_{kj,1}(\lambda))_{k=1}^{m_\alpha}{}_{j=1}^{n_1} \tag{2.18}$$

be the matrix representations of the operators $C_l(\lambda)$ and $D_l(\lambda)$, $l \in \{0,1\}$, in some orthonormal bases of $\mathcal{H}_0$ and $\mathcal{H}_1$. Then $\tau \in \widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ if and only if the matrices (2.17) and (2.18) satisfy (2.12)–(2.14) ant the matrices $(C_0(\lambda), C_1(\lambda))$ and $(D_0(\lambda), D_1(\lambda))$ have the maximally possible rank.

Remark 2.9. If $\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}$, then the class $\widetilde{R}(\mathcal{H}) := \widetilde{R}_\alpha(\mathcal{H},\mathcal{H})$ $(\alpha \in \{-1,+1\})$ coincides with the well-known class of Nevanlinna functions $\tau(\cdot)$ with values in $\widetilde{\mathcal{C}}(\mathcal{H})$ (see, for instance, [6]). In this case the collection (2.9) turns into the Nevanlinna pair

$$\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{2.19}$$

and $\tau(\cdot)$ belongs to the class $\widetilde{R}^0(\mathcal{H}) := \widetilde{R}^0_\alpha(\mathcal{H},\mathcal{H})$ if and only if

$$\tau(\lambda) \equiv \{(C_0, C_1); \mathcal{H}\} = \theta (= \theta^*), \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

with the operators $C_j \in [\mathcal{H}]$, $j \in \{0,1\}$, such that $\operatorname{Im}(C_1C_0^*) = 0$ and $0 \in \rho(C_0 \pm iC_1)$ (for more details see [1, Remark 2.5]).

3. Boundary triplets and exit space extensions

3.1. **Boundary triplets and Weyl functions.** Let A be a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$, let $\mathfrak{N}_\lambda(A) = \ker(A^* - \lambda)$ $(\lambda \in \widehat{\rho}(A))$ be a defect subspace of A, let $\widehat{\mathfrak{N}}_\lambda(A) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(A)\}$ and let $n_\pm(A) := \dim \mathfrak{N}_\lambda(A) \le \infty$, $\lambda \in \mathbb{C}_\pm$, be deficiency indices of A. Denote by Ext_A the set of all proper extensions of A, i.e., the set of all relations $\widetilde{A} \in \widetilde{\mathcal{C}}(\mathfrak{H})$ such that $A \subset \widetilde{A} \subset A^*$.

Next assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$ and $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$, so that $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$. Denote by P_j the orthoprojector in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1,2\}$.

Definition 3.1. Let $\alpha \in \{-1,+1\}$. A collection $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where $\Gamma_j : A^* \to \mathcal{H}_j$, $j \in \{0,1\}$ are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma : \widehat{f} \to \{\Gamma_0\widehat{f}, \Gamma_1\widehat{f}\}, \widehat{f} \in A^*$, from A^* into $\mathcal{H}_0 \oplus \mathcal{H}_1$ is surjective and the following Green's identity holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in A^*$:

$$(f', g) - (f, g') = (\Gamma_1\widehat{f}, \Gamma_0\widehat{g})_{\mathcal{H}_0} - (\Gamma_0\widehat{f}, \Gamma_1\widehat{g})_{\mathcal{H}_0} + i\alpha(P_2\Gamma_0\widehat{f}, P_2\Gamma_0\widehat{g})_{\mathcal{H}_2}. \tag{3.1}$$

In the sequel we will also use the notation Π_+ (resp. Π_-) instead of Π_{+1} (resp. Π_{-1}).

In the following propositions some properties of boundary triplets are specified.

Proposition 3.2. *Let $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then*

$$\dim \mathcal{H}_1 = n_-(A) \le n_+(A) = \dim \mathcal{H}_0, \quad \text{if } \alpha = +1; \tag{3.2}$$

$$\dim \mathcal{H}_1 = n_+(A) \le n_-(A) = \dim \mathcal{H}_0, \quad \text{if } \alpha = -1. \tag{3.3}$$

Conversely for any symmetric relation A with $n_-(A) \le n_+(A)$ (resp. $n_+(A) \le n_-(A)$) there exists a boundary triplet Π_+ (resp. Π_-) for A^*.

Proposition 3.3. *Let $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then*

(1) $\ker\Gamma_0 \cap \ker\Gamma_1 = A$ *and Γ_j is a bounded operator from A^* into $\mathcal{H}_j$, $j \in \{0,1\}$.*

(2) *The set of all proper extensions $\widetilde{A} \in Ext_A$ is parameterized by linear relations $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$. More precisely, the mapping*

$$\theta \to A_\theta := \{\hat{f} \in A^* : \{\Gamma_0\hat{f}, \Gamma_1\hat{f}\} \in \theta\}$$

establishes a bijective correspondence between the linear relations $\theta \in \widetilde{\mathcal{C}}(\mathcal{H}_0,\mathcal{H}_1)$ and the extensions $\widetilde{A} = A_\theta \in Ext_A$. If θ is given as an operator pair $\theta = \{(C_0, C_1); \mathcal{K}\}$ (see (2.4)), then A_θ can be represented in the form of an abstract boundary condition

$$A_\theta = \{\hat{f} \in A^* : C_0\Gamma_0\hat{f} + C_1\Gamma_1\hat{f} = 0\}. \tag{3.4}$$

Moreover, the equality $\widetilde{A} = A_\theta$ means that $\theta = \Gamma\widetilde{A} = \{\{\Gamma_0\widehat{f}, \Gamma_1\widehat{f}\} : \widehat{f} \in \widetilde{A}\}$.

(3) *The extension* A_θ *is maximal dissipative, maximal accumulative, maximal symmetric or self-adjoint if and only if* θ *belongs to the class* $\mathrm{Dis}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$, $\mathrm{Ac}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$, $\mathrm{Sym}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ *or* $\mathrm{Self}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$ *respectively.*

(4) *The equalities*

$$A_0 := \ker\Gamma_0 = \{\hat f \in A^* : \Gamma_0\hat f = 0\}, \quad A_1 := \{\hat f \in A^* : P_2\Gamma_0\widehat{f} = \Gamma_1\hat f = 0\} \tag{3.5}$$

define maximal symmetric extensions A_0 *and* A_1 *of* A *such that* $n_-(A_0) = n_-(A_1) = 0$ *in the case* $\alpha = +1$ *and* $n_+(A_0) = n_+(A_1) = 0$ *in the case* $\alpha = -1$. *Moreover, the equality* $A_1^* = \ker\Gamma_1 = \{\hat f \in A^* : \Gamma_1\hat f = 0\}$ *is valid.*

In the following two propositions we denote by π_1 the orthoprojector in $\mathfrak{H}\oplus\mathfrak{H}$ onto $\mathfrak{H}\oplus\{0\}$.

Proposition 3.4. *Let* $\Pi_+ = \{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ *be a boundary triplet for* A^* *(so that in view of* (3.2) $n_-(A)\le n_+(A)$*). Then*

(1) *The operators* $\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A)$, $\lambda\in\mathbb{C}_+$, *and* $P_1\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A)$, $\lambda\in\mathbb{C}_-$, *isomorphically map* $\widehat{\mathfrak{N}}_\lambda(A)$ *onto* $\mathcal{H}_0$ *and* $\widehat{\mathfrak{N}}_\lambda(A)$ *onto* $\mathcal{H}_\lambda$ *respectively. Therefore the equalities*

$$\gamma_+(\lambda) = \pi_1(\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A))^{-1},\ \lambda\in\mathbb{C}_+;\quad \gamma_-(\lambda) = \pi_1(P_1\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A))^{-1},\ \lambda\in\mathbb{C}_-, \tag{3.6}$$

$$\Gamma_1\restriction\widehat{\mathfrak{N}}_\lambda(A) = M_+(\lambda)\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\mathbb{C}_+, \tag{3.7}$$

$$(\Gamma_1 + iP_2\Gamma_0)\restriction\widehat{\mathfrak{N}}_\lambda(A) = M_-(\lambda)P_1\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\mathbb{C}_- \tag{3.8}$$

correctly define the operator functions $\gamma_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathfrak{H}]$, $\gamma_-(\cdot):\mathbb{C}_-\to[\mathcal{H}_1,\mathfrak{H}]$ *and* $M_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathcal{H}_1]$, $M_-(\cdot):\mathbb{C}_-\to[\mathcal{H}_1,\mathcal{H}_0]$, *which are holomorphic on their domains.*

(2) *The block matrix representations*

$$M_+(\lambda) = (M(\lambda), N_+(\lambda)) : \mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{H}_1,\quad \lambda\in\mathbb{C}_+, \tag{3.9}$$

$$M_-(\lambda) = (M(\lambda), N_-(\lambda))^\top : \mathcal{H}_1\to\mathcal{H}_1\oplus\mathcal{H}_2,\quad \lambda\in\mathbb{C}_- \tag{3.10}$$

define the operator function $M(\cdot)\in R[\mathcal{H}_1]$ *such that* $0\in\rho(\mathrm{Im}M(\lambda))$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$. *Moreover,* $M_-(\lambda) = M_+^*(\overline{\lambda})$, $\lambda\in\mathbb{C}_-$, *and, consequently,*

$$M(\lambda) = M^*(\overline{\lambda}),\quad \lambda\in\mathbb{C}\setminus\mathbb{R};\quad N_-(\lambda) = N_+^*(\overline{\lambda}),\quad \lambda\in\mathbb{C}_-. \tag{3.11}$$

Similar statements for the triplet Π_- are specified in the following proposition.

Proposition 3.5. *Let* $\Pi_- = \{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ *be a boundary triplet for* A^* *(so that in view of* (3.3) $n_+(A)\le n_-(A)$*). Then*

(1) *The equalities*

$$\gamma_+(\lambda) = \pi_1(P_1\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A))^{-1},\ \lambda\in\mathbb{C}_+;\quad \gamma_-(\lambda) = \pi_1(\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A))^{-1},\ \lambda\in\mathbb{C}_-, \tag{3.12}$$

$$(\Gamma_1 - iP_2\Gamma_0)\restriction\widehat{\mathfrak{N}}_\lambda(A) = M_+(\lambda)P_1\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\mathbb{C}_+, \tag{3.13}$$

$$\Gamma_1\restriction\widehat{\mathfrak{N}}_\lambda(A) = M_-(\lambda)\Gamma_0\restriction\widehat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\mathbb{C}_- \tag{3.14}$$

correctly define the holomorphic operator functions $\gamma_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_1,\mathfrak{H}]$, $\gamma_-(\cdot):\mathbb{C}_-\to[\mathcal{H}_0,\mathfrak{H}]$ *and* $M_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_1,\mathcal{H}_0]$, $M_-(\cdot):\mathbb{C}_-\to[\mathcal{H}_0,\mathcal{H}_1]$.

(2) *The block matrix representations*

$$M_+(\lambda) = (M(\lambda), N_+(\lambda))^\top : \mathcal{H}_1\to\mathcal{H}_1\oplus\mathcal{H}_2,\quad \lambda\in\mathbb{C}_+, \tag{3.15}$$

$$M_-(\lambda) = (M(\lambda), N_-(\lambda)) : \mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{H}_1,\quad \lambda\in\mathbb{C}_- \tag{3.16}$$

define the operator function $M(\cdot)\in R[\mathcal{H}_1]$ *such that* $0\in\rho(\mathrm{Im}M(\lambda))$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$. *Moreover,* $M_-(\lambda) = M_+^*(\overline{\lambda})$, $\lambda\in\mathbb{C}_-$, *so that the equalities* (3.11) *are valid.*

Definition 3.6. The operator functions $\gamma_\pm(\cdot)$ and $M_\pm(\cdot)$ defined in Propositions 3.4 and 3.5 are called the γ-fields and the Weyl functions, respectively, corresponding to the boundary triplet Π_α.

3.2. **Exit space extensions and generalized resolvents.**

Definition 3.7. Let $\mathfrak{H}$ be a subspace in a Hilbert space $\widetilde{\mathfrak{H}}$. The relation $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ is called $\mathfrak{H}$-minimal if

$$\operatorname{span}\{\mathfrak{H}, (\widetilde{A} - \lambda)^{-1}\mathfrak{H} : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \widetilde{\mathfrak{H}}.$$

Definition 3.8. The relations $T_j \in \widetilde{\mathcal{C}}(\mathfrak{H}_j)$, $j \in \{1, 2\}$, are said to be unitary equivalent (by means of a unitary operator $U \in [\mathfrak{H}_1, \mathfrak{H}_2]$) if $T_2 = \widetilde{U}T_1$ with $\widetilde{U} = U \oplus U \in [\mathfrak{H}_1^2, \mathfrak{H}_2^2]$.

Recall further the following definition.

Definition 3.9. Let A be a symmetric relation in a Hilbert space $\mathfrak{H}$. The operator functions $R(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathfrak{H}]$ and $F(\cdot) : \mathbb{R} \to [\mathfrak{H}]$ are called the generalized resolvent and the spectral function of A respectively if there exist a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ and a self-adjoint relation $\widetilde{A} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ such that $A \subset \widetilde{A}$ and the following equalities hold:

$$R(\lambda) = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{3.17}$$

$$F(t) = P_{\mathfrak{H}} E((-\infty, t)) \upharpoonright \mathfrak{H}, \quad t \in \mathbb{R} \tag{3.18}$$

(in formula (3.18) $E(\cdot)$ is the spectral measure of $\widetilde{A}$).

The relation $\widetilde{A} \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ in (3.17) is called an exit space self-adjoint extension of A.

According to [20] each generalized resolvent of A is generated by some $\mathfrak{H}$-minimal exit space extension $\widetilde{A}$ of A. Moreover, if the $\mathfrak{H}$-minimal exit space extensions $\widetilde{A}_1 \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}}_1)$ and $\widetilde{A}_2 \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}}_2)$ of A induce the same generalized resolvent $R(\lambda)$, then there exists a unitary operator $V \in [\widetilde{\mathfrak{H}}_1 \ominus \mathfrak{H}, \widetilde{\mathfrak{H}}_2 \ominus \mathfrak{H}]$ such that $\widetilde{A}_1$ and $\widetilde{A}_2$ are unitarily equivalent by means of $U = I_{\mathfrak{H}} \oplus V$. By using this fact we suppose in the following that the exit space extension $\widetilde{A}$ in (3.17) is $\mathfrak{H}$-minimal, so that $\widetilde{A}$ is defined by (3.17) uniquely up to the unitary equivalence.

Definition 3.10. The generalized resolvent (3.17) is called canonical if $\widetilde{\mathfrak{H}} = \mathfrak{H}$, i.e., if $R(\lambda) = (\widetilde{A} - \lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is the resolvent of the extension $\widetilde{A} = \widetilde{A}^* \in \widetilde{\mathcal{C}}(\mathfrak{H})$ of A.

As is known, canonical resolvents exist if and only if $n_+(A) = n_-(A)$, while generalized resolvents exist for any symmetric relation A.

Theorem 3.11. *Let A be a closed symmetric linear relation in $\mathfrak{H}$ and let $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. If $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$ is a collection of holomorphic pairs* (2.9)*, then for every $g \in \mathfrak{H}$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the abstract boundary value problem*

$$\{f, \lambda f + g\} \in A^*, \tag{3.19}$$

$$C_0(\lambda)\Gamma_0\{f, \lambda f + g\} - C_1(\lambda)\Gamma_1\{f, \lambda f + g\} = 0, \quad \lambda \in \mathbb{C}_+, \tag{3.20}$$

$$D_0(\lambda)\Gamma_0\{f, \lambda f + g\} - D_1(\lambda)\Gamma_1\{f, \lambda f + g\} = 0, \quad \lambda \in \mathbb{C}_- \tag{3.21}$$

has a unique solution $f = f(g, \lambda)$ and the equality $R(\lambda)g := f(g, \lambda)$ defines a generalized resolvent $R(\lambda) = R_\tau(\lambda)$ of A. Moreover, $0 \in \rho(\tau_+(\lambda) + M_+(\lambda))$ if $\alpha = +1$, $0 \in \rho(\tau_-(\lambda) + M_-(\lambda))$ if $\alpha = -1$ and the following Krein-Naimark formulas for resolvents are valid:

(i) *in the case $\alpha = +1$*

$$R_\tau(\lambda) = (A_0 - \lambda)^{-1} - \gamma_+(\lambda)(\tau_+(\lambda) + M_+(\lambda))^{-1}\gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+; \tag{3.22}$$

(ii) *in the case $\alpha = -1$*

$$R_\tau(\lambda) = (A_0 - \lambda)^{-1} - \gamma_-(\lambda)(\tau_-(\lambda) + M_-(\lambda))^{-1}\gamma_+^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_-. \tag{3.23}$$

Conversely, for each generalized resolvent $R(\lambda)$ of A there exists a unique $\tau \in \widetilde{R}_\alpha(\mathcal{H}_0, \mathcal{H}_1)$ such that $R(\lambda) = R_\tau(\lambda)$ and, consequently, the equalities (3.22) *and* (3.23) *are valid.*

Moreover, $R_\tau(\lambda)$ is a canonical resolvent of A if and only if $\tau \in \widetilde{R}^0_\alpha(\mathcal{H}_0,\mathcal{H}_1)$. In this case formula (3.22) *takes the form*

$$(A_\theta-\lambda)^{-1}=(A_0-\lambda)^{-1}+\gamma_+(\lambda)(\theta-M_+(\lambda))^{-1}\gamma_-^*(\overline{\lambda}),\quad \lambda\in\mathbb{C}_+, \tag{3.24}$$

where $\theta\in \mathrm{Self}_{+1}(\mathcal{H}_0,\mathcal{H}_1)$, $\theta\equiv-\tau_\pm(\lambda)$, $\lambda\in\mathbb{C}_\pm$.

Remark 3.12. It follows from Theorem 3.11 that the boundary value problem (3.19)–(3.21) as well as formulas for resolvents (3.22) and (3.23) give a parametrization of all generalized resolvents

$$R(\lambda)=R_\tau(\lambda)=P_{\mathfrak{H}}(\widetilde{A}^\tau-\lambda)^{-1}\upharpoonright\mathfrak{H},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{3.25}$$

and, consequently, all ($\mathfrak{H}$-minimal) exit space self-adjoint extensions $\widetilde{A}=\widetilde{A}^\tau$ of A by means of an abstract boundary parameter $\tau\in\widetilde{R}_\alpha(\mathcal{H}_0,\mathcal{H}_1)$.

Remark 3.13. (1) For the case $\alpha=+1$ definition of the boundary triplet $\Pi_\alpha=\Pi_+$ and the results of Subsections 3.1 and 3.2 are contained in [25]. The same results for the case $\alpha=-1$ can be derived from the obvious equivalence

$$\begin{aligned}\Pi_\alpha=&\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}\ \text{ is a boundary triplet for }\ A^*\\ &\iff \Pi_{-\alpha}=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0W,-\Gamma_1W\}\ \text{ is a boundary triplet for }\ (-A)^*,\end{aligned} \tag{3.26}$$

where $W\{f,f'\}=\{f,-f'\}$, $\{f,f'\}\in(-A)^*$.

(2) If $\mathcal{H}_0=\mathcal{H}_1:=\mathcal{H}$, then the triplet Π_α turns into the boundary triplet (boundary value space) $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* in the sense of [12, 22]. In this case $n_+(A)=n_-(A)=\dim\mathcal{H}$, $A_0(=\ker\Gamma_0)$ is a self-adjoint extension of A and according to [8, 22, 9] the equality

$$\Gamma_1\upharpoonright\widehat{\mathfrak{N}}_\lambda(A)=M(\lambda)\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_\lambda(A),\quad \lambda\in\rho(A_0),$$

defines the function $M(\cdot)\in R[\mathcal{H}]$, which is called the Weyl function of the triplet Π. Moreover, in this case the boundary parameter τ in Theorem 3.11 is a Nevanlinna operator pair $\tau\in\widetilde{R}(\mathcal{H})$ of the form (2.19). Observe also that for the triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ all the results in Subsections 3.1 and 3.2 were obtained in [8, 22, 9, 6]. In the following a boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ in the sense of [12, 22] will be sometimes called an ordinary boundary triplet for A^*.

4. Characterization of exit space extensions

4.1. **Auxiliary results.** Let A be a closed symmetric linear relation in $\mathfrak{H}$ and let $\mathfrak{H}$ be decomposed as

$$\mathfrak{H}=\mathfrak{H}_s\oplus \mathrm{mul}\,A, \tag{4.1}$$

where $\mathfrak{H}_s=\mathfrak{H}\ominus\mathrm{mul}\,A$. The decomposition (4.1) induces the orthogonal decomposition

$$A=\mathrm{gr}A_s\oplus\widehat{\mathrm{mul}}\,A,\quad \widehat{\mathrm{mul}}\,A=\{0\}\oplus\mathrm{mul}\,A, \tag{4.2}$$

where A_s is a closed symmetric operator in $\mathfrak{H}_s$ (the operator part of A). It follows from (4.2) that $\mathrm{dom}\,A_s=\mathrm{dom}\,A$.

Next assume that $\widetilde{A}$ is a maximal symmetric (in particular, self-adjoint) extension of A. Then $\mathrm{mul}\,A\subset\mathrm{mul}\,\widetilde{A}$ and, therefore,

$$\widetilde{A}=\widetilde{A}'\oplus\widehat{\mathrm{mul}}\,A,$$

where $\widetilde{A}'\in\widetilde{\mathcal{C}}(\mathfrak{H}_s)$ is a maximal symmetric extension of A_s.

The following proposition is obvious.

Proposition 4.1. *Let $A\in\widetilde{\mathcal{C}}(\mathfrak{H})$ be a symmetric relation and let A_s be the operator part of A. Then*

(1) *If $\widetilde{A}\in\widetilde{\mathcal{C}}(\mathfrak{H})$ is a maximal symmetric extension of A, then $\widetilde{A}'$ is an operator if and only if* $\mathrm{mul}\,\widetilde{A}=\mathrm{mul}\,A$.

(2) *The following statements are equivalent:* (i) A_s *is densely defined (that is,* $\overline{\operatorname{dom} A_s} = \overline{\operatorname{dom} A} = \mathfrak{H}_s$*);* (ii) $\operatorname{mul} A = \operatorname{mul} A^*$*;* (iii) $\operatorname{mul} \widetilde{A} = \operatorname{mul} A$ *for any exit space extension* $\widetilde{A} = \widetilde{A}^*$ *of* A.

(3) *If* A *is maximal symmetric (self-adjoint), then* $\operatorname{mul} A = \operatorname{mul} A^*$ *and* A_s *is a maximal symmetric (resp. self-adjoint) operator in* $\mathfrak{H}_s$.

Assume that A is a symmetric relation in $\mathfrak{H}$ and $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^* with $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$. Moreover, let A_r be a (densely defined) maximal symmetric operator in a Hilbert space $\mathfrak{H}_r$ and let $n_+(A_r) = 0$, $n_-(A_r) = \dim \mathcal{H}_2$. Then by Proposition 3.2 there exists a surjective linear mapping $\Gamma_r : A_r^* \to \mathcal{H}_2$ such that

$$(f_r', g_r) - (f_r, g_r') = -i(\Gamma_r \widehat{f}_r, \Gamma_r \widehat{g}_r), \quad \widehat{f}_r = \{f_r, f_r'\}, \quad \widehat{g}_r = \{g_r, g_r'\} \in A_r^*, \tag{4.3}$$

and Proposition 3.3, (1) yields $\ker \Gamma_r = A_r$. Let $\mathfrak{H}_e := \mathfrak{H} \oplus \mathfrak{H}_r$ and let $A_e := A \oplus A_r$. Clearly, A_e is a symmetric relation in $\mathfrak{H}_e$ and $A_e^* := A^* \oplus A_r^*$. Introduce also the operators $\Gamma_j^e : A_e^* \to \mathcal{H}_0$, $j \in \{0, 1\}$ by setting

$$\Gamma_0^e \widehat{f}_e = \{P_1 \Gamma_0 \widehat{f},\ P_2 \Gamma_0 \widehat{f} + \Gamma_r \widehat{f}_r\} \quad (\in \mathcal{H}_1 \oplus \mathcal{H}_2), \tag{4.4}$$

$$\Gamma_1^e \widehat{f}_e = \{\Gamma_1 \widehat{f},\ \tfrac{i}{2}(P_2 \Gamma_0 \widehat{f} - \Gamma_r \widehat{f}_r)\} \quad (\in \mathcal{H}_1 \oplus \mathcal{H}_2), \quad \widehat{f}_e = \{\widehat{f}, \widehat{f}_r\} \in A^* \oplus A_r^*. \tag{4.5}$$

Proposition 4.2. *Let the above assumptions be satisfied and let* $M_\pm(\cdot)$ *be the Weyl functions of the triplet* Π_+ *represented as in* (3.9) *and* (3.10). *Then the triplet* $\Pi^e = \{\mathcal{H}_0, \Gamma_0^e, \Gamma_1^e\}$ *is an ordinary boundary triplet for* A_e^* *and the Weyl function* $\mathcal{M}(\cdot)$ *of* Π^e *is*

$$\mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & N_+(\lambda) \\ 0 & \frac{i}{2} I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_+, \tag{4.6}$$

$$\mathcal{M}(\lambda) = \begin{pmatrix} M(\lambda) & 0 \\ N_-(\lambda) & -\frac{i}{2} I_{\mathcal{H}_2} \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2, \quad \lambda \in \mathbb{C}_-. \tag{4.7}$$

Proof. The immediate checking with taking (3.1) and (4.3) into account gives the identity (3.1) for the operators Γ_0^e and Γ_1^e. Moreover, the mapping $\Gamma^e = (\Gamma_0^e, \Gamma_1^e)^\top$ is surjective, because so are $\Gamma = (\Gamma_0, \Gamma_1)^\top$ and Γ_r. Hence $\Pi^e = \{\mathcal{H}_0, \Gamma_0^e, \Gamma_1^e\}$ is an ordinary boundary triplet for A_e^*. Next, for each $\lambda \in \mathbb{C}_+$ one has $\widehat{\mathfrak{N}}_\lambda(A_r) = \{0\}$. Therefore a vector $\widehat{f}_{\lambda,e} \in \widehat{\mathfrak{N}}_\lambda(A_e)$ admits the representation $\widehat{f}_{\lambda,e} = \{\widehat{f}_\lambda, 0\}$ with $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(A)$ and the equalities (4.4) and (4.5) yield

$$\Gamma_0^e \widehat{f}_{\lambda,e} = \{P_1 \Gamma_0 \widehat{f}_\lambda, P_2 \Gamma_0 \widehat{f}_\lambda\} (\in \mathcal{H}_1 \oplus \mathcal{H}_2), \quad \Gamma_1^e \widehat{f}_{\lambda,e} = \{\Gamma_1 \widehat{f}_\lambda, \tfrac{i}{2} P_2 \Gamma_0 \widehat{f}_\lambda\} (\in \mathcal{H}_1 \oplus \mathcal{H}_2). \tag{4.8}$$

Let $\mathcal{M}(\lambda)$ be defined by (4.6) and (4.7). Then by (4.8) and (3.7) for each $\lambda \in \mathbb{C}_+$ one has

$$\mathcal{M}(\lambda) \Gamma_0^e \widehat{f}_{\lambda,e} = \{M_+(\lambda) \Gamma_0 \widehat{f}_\lambda, \tfrac{i}{2} P_2 \Gamma_0 \widehat{f}_\lambda\} = \{\Gamma_1 \widehat{f}_\lambda, \tfrac{i}{2} P_2 \Gamma_0 \widehat{f}_\lambda\} = \Gamma_1^e \widehat{f}_{\lambda,e}, \quad \widehat{f}_{\lambda,e} \in \widehat{\mathfrak{N}}_\lambda(A_e),$$

and (3.11) yields the equality $\mathcal{M}(\lambda) = \mathcal{M}^*(\overline{\lambda})$, $\lambda \in \mathbb{C}_-$. Therefore $\mathcal{M}(\cdot)$ is the Weyl function of the triplet Π^e. $\square$

In the following proposition we provide a connection between different boundary triplets and the corresponding Weyl functions.

Proposition 4.3. *Assume that* $\Pi_\alpha = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^*, $\widetilde{\mathcal{H}}_0$ *is a Hilbert space,* $\widetilde{\mathcal{H}}_1$ *is a subspace in* $\widetilde{\mathcal{H}}_0$ *and let*

$$\begin{gathered} J_\alpha = \begin{pmatrix} -\alpha i P_2 & -I_{\mathcal{H}_1} \\ P_1 & 0 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1, \\ J_\alpha = \begin{pmatrix} -\alpha i \widetilde{P}_2 & -I_{\widetilde{\mathcal{H}}_1} \\ \widetilde{P}_1 & 0 \end{pmatrix} : \widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1. \end{gathered} \tag{4.9}$$

Then

(1) *The the equality*

$$\begin{pmatrix}\widetilde{\Gamma}_0\\ \widetilde{\Gamma}_1\end{pmatrix}=\begin{pmatrix}X_{00} & X_{01}\\ X_{10} & X_{11}\end{pmatrix}\begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix} \tag{4.10}$$

establishes a bijective correspondence between all boundary triplets $\widetilde{\Pi}_\alpha=\{\widetilde{\mathcal{H}}_0\oplus\widetilde{\mathcal{H}}_1,\widetilde{\Gamma}_0,\widetilde{\Gamma}_1\}$ *for* A^* *and all operators* $X=(X_{ij})_{i,j=0}^1\in[\mathcal{H}_0\oplus\mathcal{H}_1,\widetilde{\mathcal{H}}_0\oplus\widetilde{\mathcal{H}}_1]$ *such that* $X^*\widetilde{J}_\alpha X=J_\alpha$ *and* $XJ_\alpha X^*=\widetilde{J}_\alpha$.

(2) *If* $M_\pm(\cdot)$ *are the Weyl functions of the triplet* Π_α, *then the Weyl functions* $\widetilde{M_\pm}(\cdot)$ *corresponding to the triplet* $\widetilde{\Pi}_\alpha$ *are of the form:*

(i) *in the case* $\alpha=+1$

$$\widetilde{M_+}(\lambda)=(X_{10}+X_{11}M_+(\lambda))(X_{00}+X_{01}M_+(\lambda))^{-1},\quad \lambda\in\mathbb{C}_+, \tag{4.11}$$

(ii) *in the case* $\alpha=-1$

$$\widetilde{M_-}(z)=(X_{10}+X_{11}M_-(z))(X_{00}+X_{01}M_-(z))^{-1},\quad z\in\mathbb{C}_-. \tag{4.12}$$

In the case $\alpha=+1$ and $\widetilde{\mathcal{H}}_j=\mathcal{H}_j,\ j\in\{0,1\}$, the proof of Proposition 4.3 can be found in [25]. In general case the proof is similar.

Corollary 4.4. *Let* $\Pi_+=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ *be a boundary triplet for* A^* *and let* $M_+(\lambda)=(M(\lambda),N_+(\lambda))$ *and* $M_-(z)=(M(z),N_-(z))^\top$ *be the corresponding Weyl functions* (3.9) *and* (3.10). *Then*

(1) *The triplet* $\widehat{\Pi}_+=\{\mathcal{H}_0\oplus\mathcal{H}_1,\widehat{\Gamma}_0,\widehat{\Gamma}_1\}$, *where*

$$\widehat{\Gamma}_0\widehat{f}=-\Gamma_1\widehat{f}+P_2\Gamma_0\widehat{f},\quad \widehat{\Gamma}_1\widehat{f}=P_1\Gamma_0\widehat{f},\quad \widehat{f}\in A^*, \tag{4.13}$$

is a boundary triplet for A^* *with* $\widehat{A}_0(=\ker\widehat{\Gamma}_0)=A_1$.

(2) *If* $\tau=\{\tau_+,\tau_-\}\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ *is a collection* (2.9)–(2.11) *and* $\widetilde{A}=\widetilde{A}^\tau$, *then for the triplet* $\widehat{\Pi}_+$ *one has* $\widetilde{A}=\widetilde{A}^{\widehat{\tau}}$, *where* $\widehat{\tau}=\{\widehat{\tau}_+,\widehat{\tau}_-\}\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ *is given by*

$$\widehat{\tau}_+(\lambda)=\{(\widehat{C}_0(\lambda),\widehat{C}_1(\lambda));\mathcal{K}_+\},\ \lambda\in\mathbb{C}_+;\quad \widehat{\tau}_-(\lambda)=\{(\widehat{D}_0(\lambda),\widehat{D}_1(\lambda));\mathcal{K}_-\},\ \lambda\in\mathbb{C}_-, \tag{4.14}$$

$$\widehat{C}_0(\lambda)=(C_1(\lambda),C_{02}(\lambda)):\mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{K}_+,\qquad \widehat{C}_1(\lambda)=-C_{01}(\lambda),\quad \lambda\in\mathbb{C}_+, \tag{4.15}$$

$$\widehat{D}_0(\lambda)=(D_1(\lambda),D_{02}(\lambda)):\mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{K}_-,\qquad \widehat{D}_1(\lambda)=-D_{01}(\lambda),\quad \lambda\in\mathbb{C}_-. \tag{4.16}$$

(3) *The Weyl functions of the triplet* $\widehat{\Pi}_+$ *are*

$$\widehat{M_+}(\lambda)=(-M^{-1}(\lambda),\ -M^{-1}(\lambda)N_+(\lambda)):\mathcal{H}_1\oplus\mathcal{H}_2\to\mathcal{H}_1,\quad \lambda\in\mathbb{C}_+, \tag{4.17}$$

$$\widehat{M_-}(z)=(-M^{-1}(z),\ -N_-(z)M^{-1}(z))^\top:\mathcal{H}_1\to\mathcal{H}_1\oplus\mathcal{H}_2,\quad z\in\mathbb{C}_-. \tag{4.18}$$

Proof. (1) Assume that $I_{\mathcal{H}_1,\mathcal{H}_0}\in[\mathcal{H}_1,\mathcal{H}_0]$ is the operator given for each $h_1\in\mathcal{H}_1$ by $I_{\mathcal{H}_1,\mathcal{H}_0}h_1=h_1$ (i.e., $I_{\mathcal{H}_1,\mathcal{H}_0}$ is the "embedding operator" from $\mathcal{H}_1(\subset\mathcal{H}_0)$ to $\mathcal{H}_0$) and let

$$X=\begin{pmatrix}X_{00} & X_{01}\\ X_{10} & X_{11}\end{pmatrix}=\begin{pmatrix}P_2 & -I_{\mathcal{H}_1,\mathcal{H}_0}\\ P_1 & 0\end{pmatrix}:\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{H}_0\oplus\mathcal{H}_1. \tag{4.19}$$

Then $X^*J_{+1}X=J_{+1},\ XJ_{+1}X^*=J_{+1}$ and the equality (4.10) holds with $\widetilde{\Gamma}_j=\widehat{\Gamma}_j$ and X_{ij} taken from (4.19). Therefore by Proposition 4.3, (1) $\widehat{\Pi}_+$ is a boundary triplet for A^*. Moreover, the equality $\widehat{A}_0=A_1$ is implied by (4.13) and the second equality in (3.5).

(2) The immediate checking shows that

$$\widehat{C}_0(\lambda)\widehat{\Gamma}_0-\widehat{C}_1(\lambda)\widehat{\Gamma}_1=C_0(\lambda)\Gamma_0-C_1(\lambda)\Gamma_1,\quad \widehat{D}_0(\lambda)\widehat{\Gamma}_0-\widehat{D}_1(\lambda)\widehat{\Gamma}_1=D_0(\lambda)\Gamma_0-D_1(\lambda)\Gamma_1.$$

Hence the boundary value problem (3.19)–(3.21) defines the same generalized resolvent $R(\lambda)$ as the problem (3.19)–(3.21) with $\widehat{C}_j(\cdot),\widehat{D}_j(\cdot)$ and $\widehat{\Gamma}_j$ instead of $C_j(\cdot),\ D_j(\cdot)$ and $\Gamma_j,\ j\in\{0,1\}$.

(3) It follows from (4.11) and (4.19) that the Weyl function of the triplet $\widehat{\Pi}$ is

$$\widehat{M_+}(\lambda) = P_1(P_2 - M_+(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+,$$

where $M_+(\lambda)$ is considered as the operator in $\mathcal{H}_0$. Moreover, the immediate checking shows that

$$(P_2 - M_+(\lambda))^{-1} = -M^{-1}(\lambda)P_1 - M^{-1}(\lambda)N_+(\lambda)P_2 + P_2.$$

This yields (4.17) and (4.18). □

4.2. **The case $n_-(A) \le n_+(A)$.** We start with the following basic theorem implied by the results of [18, 20, 22].

Theorem 4.5. *Let $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be a symmetric relation with equal deficiency indices $n_+(A) = n_-(A)$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* and let $M(\cdot)$ be the corresponding Weyl function. Then*

(1) *The extension $A_0 = A_0^*(= \ker \Gamma_0)$ of A satisfies* $\operatorname{mul} A_0 = \operatorname{mul} A$ *if and only if*

$$\mathcal{B}_M(= s - \lim_{y\to\infty} \tfrac{1}{iy} M(iy)) = 0. \tag{4.20}$$

(2) *The equality* $\operatorname{mul} A = \operatorname{mul} A^*$ *holds if and only if* (4.20) *is satisfied and*

$$\lim_{y\to\infty} y \operatorname{Im}(M(iy)h, h) = +\infty, \quad h \in \mathcal{H}, \quad h \ne 0. \tag{4.21}$$

Generalization of Theorem 4.5 to the case of possibly unequal deficiency indices $n_-(A) \le n_+(A)$ is given in the following theorem.

Theorem 4.6. *Let $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be a closed symmetric linear relation in $\mathfrak{H}$ with $n_-(A) \le n_+(A)$, let $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, let $M_+(\cdot)$ be the Weyl function of Π_+ and let $M(\cdot)(\in R[\mathcal{H}_1])$ be the operator function defined by* (3.9) *and* (3.10)*. Then*

(1) *The maximal symmetric extension $A_0(= \ker \Gamma_0)$ of A satisfies* $\operatorname{mul} A_0 = \operatorname{mul} A$ *if and only if* (4.20) *holds.*

(2) *The equality* $\operatorname{mul} A = \operatorname{mul} A^*$ *holds if and only if* (4.20) *is satisfied and*

$$\lim_{y\to+\infty} y \left(\operatorname{Im}(M_+(iy)h_0, h_0)_{\mathcal{H}_0} + \tfrac{1}{2}||P_2 h_0||^2\right) = +\infty, \quad h_0 \in \mathcal{H}_0, \quad h_0 \ne 0. \tag{4.22}$$

If, in addition, $\operatorname{mul} A = \{0\}$ *(i.e., A is the operator), then:* (i) *A_0 is the operator if and only if* (4.20) *holds;* (ii) *A is densely defined if and only if* (4.20) *and* (4.22) *hold.*

Proof. Let A_r be a maximal symmetric operator in $\mathfrak{H}_r$ with $n_+(A_r) = 0$ and $n_-(A_r) = \dim \mathcal{H}_2$ and let $\Gamma_r : A^* \to \mathcal{H}_r$ be a surjective linear mapping satisfying (4.3). Moreover, let $\mathfrak{H}_e := \mathfrak{H} \oplus \mathfrak{H}_r$ and let $A_e := A \oplus A_r$ (see the reasonings before Proposition 4.2). Then according to Proposition 4.2 the operators (4.4) and (4.5) form a boundary triplet $\Pi^e = \{\mathcal{H}_0, \Gamma_0^e, \Gamma_1^e\}$ for $A_e^*(= A^* \oplus A_r^*)$ and the corresponding Weyl function $\mathcal{M}(\cdot)$ is given by (4.6) and (4.7).

Since A_0 is a maximal symmetric relation in $\mathfrak{H}$ and A_r is a maximal symmetric operator in $\mathfrak{H}_r$, it follows from Proposition 4.1, (3) that $\operatorname{mul} A_0 = \operatorname{mul} A_0^*$, $\operatorname{mul} A_r = \operatorname{mul} A_r^* = \{0\}$ and, consequently,

$$\operatorname{mul} A_e = \operatorname{mul} A, \quad \operatorname{mul} A_e^* = \operatorname{mul} A^*, \tag{4.23}$$

$$\operatorname{mul}(A_0 \oplus A_r) = \operatorname{mul}(A_0 \oplus A_r)^*(= \operatorname{mul} A_0). \tag{4.24}$$

Let $A_{0,e} = A_{0,e}^* \in Ext_{A_e}$ be given by $A_{0,e} = \ker \Gamma_0^e$. Since $\ker \Gamma_0 = A_0$ and $\ker \Gamma_r = A_r$, it follows from (4.4) that $A_0 \oplus A_r \subset A_{0,e}$. Therefore by (4.24) and Proposition 4.1, (2) $\operatorname{mul} A_{0,e} = \operatorname{mul}(A_0 \oplus A_r) = \operatorname{mul} A_0$, which together with the first equality in (4.23) yields

$$\operatorname{mul} A = \operatorname{mul} A_0 \iff \operatorname{mul} A_e = \operatorname{mul} A_{0,e}. \tag{4.25}$$

Moreover, applying Theorem 4.5, (1) to the boundary triplet Π^e for A_e^* one obtains

$$\text{mul}\, A_e = \text{mul}\, A_{0,e} \iff \mathcal{B}_{\mathcal{M}} = 0 \tag{4.26}$$

and by Proposition 2.2 one has $\mathcal{B}_{\mathcal{M}} = 0 \iff \mathcal{B}_M = 0$. Combining this equivalence with (4.25) and (4.26) we arrive at the statement (1) of the theorem.

To prove statement (2) note that in view of (4.23) $\text{mul}\, A = \text{mul}\, A^*$ if and only if $\text{mul}\, A_e = \text{mul}\, A_e^*$. Therefore by Theorem 4.5, (2) applied to the triplet Π^e the equality $\text{mul}\, A = \text{mul}\, A^*$ holds if and only if (4.20) is satisfied and

$$\lim_{y\to+\infty} y\, \text{Im}(\mathcal{M}(iy)h_0, h_0) = +\infty, \quad h_0 \in \mathcal{H}_0, \quad h_0 \neq 0. \tag{4.27}$$

Moreover, in view of (4.6) one has

$$(\mathcal{M}(\lambda)h_0, h_0) = (M_+(\lambda)h_0, h_0)_{\mathcal{H}_0} + \tfrac{i}{2}||P_2 h_0||^2, \quad h_0 \in \mathcal{H}_0, \quad \lambda \in \mathbb{C}_+,$$

so that the condition (4.27) can be represented in the form (4.22). This yields statement (2). Finally, the last statement of the theorem is obvious. □

Our next goal is to characterize exit space self-adjoint extensions in terms of a boundary parameter τ and the Weyl function. To this end we first prove the following theorem.

Theorem 4.7. *Assume that* $n_+(A) = n_-(A)$, $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^*, $M_\pm(\cdot)$ *are the Weyl functions of* Π_+ *and* A_0 *is the extension* (3.5) *of* A. *Moreover, let* $\theta \in \text{Self}_{+1}(\mathcal{H}_0, \mathcal{H}_1)$, *let* $A_\theta = A_\theta^* \in \widetilde{\mathcal{C}}(\mathfrak{H})$ *be an extension of* A *and let*

$$\Phi_\theta(\lambda) := P_1(\theta - M_+(\lambda))^{-1}, \;\; \lambda \in \mathbb{C}_+; \quad \Phi_\theta(\lambda) := \Phi_\theta^*(\overline{\lambda}) = (\theta^* - M_-(\lambda))^{-1} \upharpoonright \mathcal{H}_1, \;\; \lambda \in \mathbb{C}_-.$$

Then $\Phi_\theta(\cdot) \in R[\mathcal{H}_1]$ *and the following equivalence holds:*

$$\text{mul}\, A_\theta \subset \text{mul}\, A_0 \iff \mathcal{B}_{\Phi_\theta} (= s - \lim_{y\to+\infty} \tfrac{1}{iy} P_1(\theta - M_+(iy))^{-1}) = 0. \tag{4.28}$$

Proof. It follows from [24, Proposition 3.6] that θ admits the representation $\theta = \{(C_0, C_1); \mathcal{H}_1\}$ with operators $C_0 = (C_{01}, C_{02}) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$ and $C_1 \in [\mathcal{H}_1]$ satisfying

$$-C_1 C_{01}^* + C_{01} C_1^* + iC_{02} C_{02}^* = 0, \qquad C_1 C_1^* + C_{01} C_{01}^* + C_{02} C_{02}^* = I_{\mathcal{H}_1}, \tag{4.29}$$

$$C_1^* C_{01} - C_{01}^* C_1 = 0, \qquad C_1^* C_1 + C_{01}^* C_{01} = I_{\mathcal{H}_1}, \tag{4.30}$$

$$2C_{02}^* C_{02} = I_{\mathcal{H}_2}, \qquad (C_{01}^* - iC_1^*)C_{02} = 0. \tag{4.31}$$

Using such a representation of θ introduce the operator

$$X = \begin{pmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{pmatrix} := \left(\begin{array}{cc|c} C_{01} & C_{02} & C_1 \\ \hline -C_1 & iC_{02} & C_{01} \end{array}\right) : (\overbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}^{\mathcal{H}_0}) \oplus \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_1. \tag{4.32}$$

Let $\widetilde{\Gamma}_j : A^* \to \mathcal{H}_1$, $j \in \{0, 1\}$, be the mappings defined by (4.10) or, equivalently, by

$$\widetilde{\Gamma}_0 = C_0 \Gamma_0 + C_1 \Gamma_1, \quad \widetilde{\Gamma}_1 = (-C_1 P_1 + iC_{02} P_2)\Gamma_0 + C_{01}\Gamma_1. \tag{4.33}$$

Then a collection $\widetilde{\Pi} = \{\mathcal{H}_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ forms an ordinary boundary triplet for A^*. Indeed, in this case the operators (4.9) take the form

$$J_{+1} = \left(\begin{array}{cc|c} 0 & 0 & -I_{\mathcal{H}_1} \\ 0 & -iI_{\mathcal{H}_2} & 0 \\ \hline I_{\mathcal{H}_1} & 0 & 0 \end{array}\right) : (\overbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}^{\mathcal{H}_0}) \oplus \mathcal{H}_1 \to (\overbrace{\mathcal{H}_1 \oplus \mathcal{H}_2}^{\mathcal{H}_0}) \oplus \mathcal{H}_1,$$

$$\widetilde{J} = \widetilde{J}_\alpha = \begin{pmatrix} 0 & -I_{\mathcal{H}_1} \\ I_{\mathcal{H}_1} & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_1$$

and the immediate checking with taking (4.29)-(4.31) into account shows that $X^* \widetilde{J} X = J_{+1}$ and $X J_{+1} X^* = \widetilde{J}$. This and Proposition 4.3, (1) give the required statement concerning $\widetilde{\Pi}$.

It follows from (4.33) and definition (3.4) of A_θ that $\widetilde{A}_0 (= \ker \widetilde{\Gamma}_0) = A_\theta$. Moreover, by (4.32) and Proposition 4.3, (2) the Weyl function of the triplet $\widetilde{\Pi}$ is

$$\widetilde{M}(\lambda) = (-C_1 P_1 + iC_{02}P_2 + C_{01}M_+(\lambda))(C_0 + C_1 M_+(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{4.34}$$

Let us show that $\widetilde{M}(\lambda)$ satisfies

$$B + C_1^* \widetilde{M}(\lambda) C_1 = \Phi_\theta(\lambda) (= P_1(\theta - M_+(\lambda))^{-1}), \quad \lambda \in \mathbb{C}_+, \tag{4.35}$$

with some $B = B^* \in [\mathcal{H}_1]$. Since $\theta = \theta^\times$, it follows from [24, Proposition 3.1] that

$$\theta = \{\{-(C_1^* + iC_{02}^*)h_1, C_{01}^* h_1\} : h_1 \in \mathcal{H}_1\}.$$

Moreover, by [25, Proposition 4.1] one has

$$0 \in \rho(C_0 + C_1 M_+(\lambda)) \cap \rho(C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*))$$

and Lemma 2.1 in [23] yields

$$\begin{aligned} (\theta - M_+(\lambda))^{-1} &= -(C_0 + C_1 M_+(\lambda))^{-1} C_1 \\ &= -(C_1^* + iC_{02}^*)(C_{01}^* + M_+(\lambda) = (C_1^* + iC_{02}^*))^{-1}, \quad \lambda \in \mathbb{C}_+. \end{aligned} \tag{4.36}$$

Combining of (4.34) and (4.36) gives

$$\begin{aligned} &C_1^* \widetilde{M}(\lambda) C_1 \\ &= C_1^*(-C_1 P_1 + iC_{02}P_2 + C_{01}M_+(\lambda))(C_1^*! + iC_{02}^*)(C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*))^{-1} \\ &= C_1^*[-C_1 C_1^* - C_{02}C_{02}^* + C_{01}M_+(\lambda)(C_1^* + iC_{02}^*)](C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*))^{-1}. \end{aligned} \tag{4.37}$$

It follows from the first equality in (4.30) that the operator $B := -C_1^* C_{01}$ satisfies $B = B^*$. Now by using first (4.37) and then the second equality in (4.29) one obtains

$$\begin{aligned} B + C_1^* \widetilde{M}(\lambda) C_1 &= -C_1^* C_{01} + C_1^*[-C_1 C_1^* - C_{02}C_{02}^* + C_{01}M_+(\lambda)(C_1^* + iC_{02}^*)] \\ &\times (C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*))^{-1} = -C_1^*[C_{01}(C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*)) \\ &+ (C_1 C_1^* + C_{02}C_{02}^* - C_{01}M_+(\lambda))(C_1^* + iC_{02}^*)](C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*))^{-1} \\ &= -C_1^*(C_{01}C_{01}^* + C_1 C_1^* + C_{02}C_{02}^*)(C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*))^{-1} \\ &= -C_1^*(C_{01}^* + M_+(\lambda)(C_1^* + iC_{02}^*))^{-1}. \end{aligned}$$

This and (4.36) yield the equality (4.35).

Next assume that A' is a symmetric extension of A given by

$$A' = A_\theta \cap A_0 = \{\widehat{f} \in A^* : \Gamma_0 \widehat{f} = 0 \text{ and } C_1 \Gamma_1 \widehat{f} = 0\}. \tag{4.38}$$

Moreover, let $\mathcal{H}_1'$ be a closure of $\operatorname{ran} C_1$ and let $\mathcal{H}_1'' = \ker C_1^*$, so that

$$\mathcal{H}_1 = \mathcal{H}_1' \oplus \mathcal{H}_1''. \tag{4.39}$$

Let us prove the equality

$$A' = \{\widehat{f} \in A^* : \widetilde{\Gamma}_0 \widehat{f} = 0 \text{ and } \widetilde{\Gamma}_1 \widehat{f} \in \mathcal{H}_1''\}. \tag{4.40}$$

Let $\widehat{f} \in A'$, so that $\Gamma_0 \widehat{f} = 0$ and $C_1 \Gamma_1 \widehat{f} = 0$. Then by (4.33) $\widetilde{\Gamma}_0 \widehat{f} = 0$, $\widetilde{\Gamma}_1 \widehat{f} = C_{01} \Gamma_1 \widehat{f}$ and in view of the first equality in (4.30) one has

$$C_1^* \widetilde{\Gamma}_1 \widehat{f} = C_1^* C_{01} \Gamma_1 \widehat{f} = C_{01}^* C_1 \Gamma_1 \widehat{f} = 0.$$

Hence $\widetilde{\Gamma}_1 \widehat{f} \in \mathcal{H}_1''$. Conversely, let $\widehat{f} \in A^*$ satisfies $\widetilde{\Gamma}_0 \widehat{f} = 0$ and $\widetilde{\Gamma}_1 \widehat{f} \in \mathcal{H}_1''$. Then $C_1^* \widetilde{\Gamma}_1 \widehat{f} = 0$ and the first equality in (4.29) yields $C_{02}^* \widetilde{\Gamma}_1 \widehat{f} = 0$. Therefore by (4.29)

$$C_1 C_{01}^* \widetilde{\Gamma}_1 \widehat{f} = 0 \quad \text{and} \quad C_{01} C_{01}^* \widetilde{\Gamma}_1 \widehat{f} = \widetilde{\Gamma}_1 \widehat{f}. \tag{4.41}$$

Since the mapping $\Gamma = (\Gamma_0, \Gamma_1)^\top$ is surjective, there exists $\widehat{g} \in A^*$ such that $\Gamma_0 \widehat{g} = 0$ and $\Gamma_1 \widehat{g} = C_{01}^* \widetilde{\Gamma}_1 \widehat{f}$. It follows from the first equality in (4.41) that $C_1 \Gamma_1 \widehat{g} = 0$ and , therefore, $\widehat{g} \in A'$. Moreover, combining of (4.33) with the second equality in (4.41) yields

$$\widetilde{\Gamma}_0 \widehat{g} = 0 = \widetilde{\Gamma}_0 \widehat{f}, \quad \widetilde{\Gamma}_1 \widehat{g} = C_{01} \Gamma_1 \widehat{g} = C_{01} C_{01}^* \widetilde{\Gamma}_1 \widehat{f} = \widetilde{\Gamma}_1 \widehat{f}.$$

Hence $\widehat{f} = \widehat{g} + \widehat{\varphi}$ with some $\widehat{\varphi} \in A \subset A'$ and, consequently, $\widehat{f} \in A'$. This completes the proof of (4.40).

Now applying [6, Proposition 4.1] to the boundary triplet $\widetilde{\Pi} = \{\mathcal{H}_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for A^* and decomposition (4.39) of $\mathcal{H}_1$ one obtains the following assertion: there exists an ordinary boundary triplet $\Pi' = \{\mathcal{H}_1', \Gamma_0', \Gamma_1'\}$ for $(A')^*$ such that $\ker \Gamma_0' = \ker \widetilde{\Gamma}_0 = A_\theta$ and the corresponding Weyl function $M'(\cdot)$ is

$$M'(\lambda) = P_{\mathcal{H}_1'} \widetilde{M}(\lambda) \upharpoonright \mathcal{H}_1', \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.42}$$

Moreover, application of Theorem 4.5 to a symmetric relation A' and the boundary triplet Π' for $(A')^*$ yields the equivalence

$$\operatorname{mul} A_\theta = \operatorname{mul} A' \iff \mathcal{B}_{M'} (= s - \lim_{y\to\infty} \tfrac{1}{iy} M'(iy)) = 0. \tag{4.43}$$

In view of (4.42) and the equality $\overline{\operatorname{ran} C_1} = \mathcal{H}_1'$ one may rewrite (4.35) as

$$B + C_1^* M'(\lambda) C_1 = \Phi_\theta(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.44}$$

It follows from (4.44) that $\Phi_\theta(\cdot) \in R[\mathcal{H}_1]$ and the equivalence

$$\mathcal{B}_{M'} = 0 \iff \mathcal{B}_{\Phi_\theta} = 0 \tag{4.45}$$

is valid. Observe also that in view of (4.38) $\operatorname{mul} A' = \operatorname{mul} A_\theta \cap \operatorname{mul} A_0$, so that the equality $\operatorname{mul} A_\theta = \operatorname{mul} A'$ is equivalent to $\operatorname{mul} A_\theta \subset \operatorname{mul} A_0$. Combining this fact with (4.43) and (4.45) we arrive at the required equivalence (4.28) □

In the following theorem we extend statement of Theorem 4.7 to exit space extensions.

Theorem 4.8. *Assume that A is a closed symmetric linear relation in $\mathfrak{H}$ with $n_-(A) \leq n_+(A)$, $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for A^*, $M_\pm(\cdot)$ are the Weyl functions* (3.9) *and* (3.10) *and A_0 is the maximal symmetric extension* (3.5) *of A. Moreover, let $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ be a collection of holomorphic pairs* (2.9) *and let $\widetilde{A}^\tau \in \widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ be the corresponding exit space self-adjoint extension of A (see remark 3.12). Then*

(1) *The equalities*

$$\Phi_\tau(\lambda) := -P_1(\tau_+(\lambda) + M_+(\lambda))^{-1} = P_1(C_0(\lambda) - C_1(\lambda) M_+(\lambda))^{-1} C_1(\lambda), \quad \lambda \in \mathbb{C}_+, \tag{4.46}$$

$$\begin{aligned}\Phi_\tau(\lambda) :&= -(\tau_+^*(\overline{\lambda}) + M_-(\lambda))^{-1} \upharpoonright \mathcal{H}_1 \\ &= (D_{01}(\lambda) - D_1(\lambda) M(\lambda) - i D_{02}(\lambda) N_-(\lambda))^{-1} D_1(\lambda), \quad \lambda \in \mathbb{C}_-,\end{aligned} \tag{4.47}$$

where $D_{0j}(\lambda)$, $j \in \{1, 2\}$, are taken from (2.11) *define the operator function $\Phi_\tau(\cdot) \in R[\mathcal{H}_1]$. Hence there exists the strong limit*

$$\begin{aligned}\mathcal{B}_\tau := \mathcal{B}_{\Phi_\tau} &= s - \lim_{y\to+\infty} \tfrac{1}{iy} P_1(C_0(iy) - C_1(iy) M_+(iy))^{-1} C_1(iy) \\ &= s - \lim_{y\to-\infty} \tfrac{1}{iy} (D_{01}(iy) - D_1(iy) M(iy) - i D_{02}(iy) N_-(iy))^{-1} D_1(iy).\end{aligned} \tag{4.48}$$

(2) *If $\widetilde{\mathfrak{H}}$ is decomposed as $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{H}_1$ (with $\mathfrak{H}_1 := \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$), then the following equivalence holds:*

$$\operatorname{mul} \widetilde{A}^\tau \subset \operatorname{mul} A_0 \oplus \mathfrak{H}_1 \iff \mathcal{B}_\tau = 0. \tag{4.49}$$

Proof. Put $\widetilde{\mathcal{H}}_0 = \mathcal{H}_0 \oplus \mathfrak{H}_1$ and $\widetilde{\mathcal{H}}_1 = \mathcal{H}_1 \oplus \mathfrak{H}_1$. According to [25, Theorem 4.4] the adjoint linear relation of A in the space $\widetilde{\mathfrak{H}}$ is

$$A^*_{\widetilde{\mathfrak{H}}} = A^* \oplus \mathfrak{H}_1^2;$$

and the operators

$$\widetilde{\Gamma}_0 = \begin{pmatrix} \Gamma_0 & 0 \\ 0 & G_0 \end{pmatrix} \in [A^* \oplus \mathfrak{H}_1^2, \mathcal{H}_0 \oplus \mathfrak{H}_1], \quad \widetilde{\Gamma}_1 = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & G_1 \end{pmatrix} \in [A^* \oplus \mathfrak{H}_1^2, \mathcal{H}_1 \oplus \mathfrak{H}_1]$$

with $G_0\{h_1,h_1'\} = h_1$ and $G_1\{h_1,h_1'\} = h_1'$, $\{h_1,h_1'\} \in \mathfrak{H}_1^2$, form a boundary triplet $\widetilde{\Pi}_+ = \{\widetilde{\mathcal{H}}_0 \oplus \widetilde{\mathcal{H}}_1, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ for $A^*_{\widetilde{\mathfrak{H}}}$. Moreover, for this triplet

$$\widetilde{A}_0 (= \ker \widetilde{\Gamma}_0) = A_0 \oplus (\{0\} \oplus \mathfrak{H}_1) \tag{4.50}$$

and the corresponding γ-fields are

$$\widetilde{\gamma}_+(\lambda) = \begin{pmatrix} \gamma_+(\lambda) & 0 \\ 0 & I_{\mathfrak{H}_1} \end{pmatrix}, \quad \lambda \in \mathbb{C}_+; \quad \widetilde{\gamma}_-(z) = \begin{pmatrix} \gamma_-(z) & 0 \\ 0 & I_{\mathfrak{H}_1} \end{pmatrix}, \quad z \in \mathbb{C}_-. \tag{4.51}$$

Next, according to Proposition 3.3 the extension $\widetilde{A}^\tau$ is parametrized in the triplet $\widetilde{\Pi}$ as $\widetilde{A}^\tau = A_{\widetilde{\theta}}$ with some $\widetilde{\theta} \in \mathrm{Self}_{+1}(\widetilde{\mathcal{H}}_0, \widetilde{\mathcal{H}}_1)$. It follows from the formula (3.24) for the triplet $\widetilde{\Pi}$ that the canonical resolvent $(\widetilde{A}^\tau - \lambda)^{-1} (\in [\widetilde{\mathfrak{H}}])$ admits the representation

$$(\widetilde{A}^\tau - \lambda)^{-1} = (\widetilde{A}_0 - \lambda)^{-1} + \widetilde{\gamma}_+(\lambda)(\widetilde{\theta} - \widetilde{M}_+(\lambda))^{-1} \widetilde{\gamma}_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+, \tag{4.52}$$

where $\widetilde{M}_+(\cdot)$ is the Weyl function of the triplet $\widetilde{\Pi}$. Moreover, (4.50) yields

$$(\widetilde{A}_0 - \lambda)^{-1} = \begin{pmatrix} (A_0 - \lambda)^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \mathfrak{H} \oplus \mathfrak{H}_1 \to \mathfrak{H} \oplus \mathfrak{H}_1. \tag{4.53}$$

Now combining (4.52) with (4.51) and (4.53) one gets

$$P_{\mathfrak{H}_1} (\widetilde{A}^\tau - \lambda)^{-1} \upharpoonright \mathfrak{H}_1 = P_{\mathfrak{H}_1} (\widetilde{\theta} - \widetilde{M}_+(\lambda))^{-1} \upharpoonright \mathfrak{H}_1, \quad \lambda \in \mathbb{C}_+. \tag{4.54}$$

It was shown in the proof of Theorem 4.4 in [25] that

$$P_{\mathcal{H}_0} (\widetilde{\theta} - \widetilde{M}_+(\lambda))^{-1} \upharpoonright \mathcal{H}_1 = -(\tau_+(\lambda) + M_+(\lambda))^{-1}, \quad \lambda \in \mathbb{C}_+.$$

Therefore by (4.46) one has

$$P_{\mathcal{H}_1} (\widetilde{\theta} - \widetilde{M}_+(\lambda))^{-1} \upharpoonright \mathcal{H}_1 = \Phi_\tau(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.55}$$

By using (4.54) and (4.55) we obtain

$$\Phi_{\widetilde{\theta}}(\lambda) := P_{\widetilde{\mathcal{H}}_1} (\widetilde{\theta} - \widetilde{M}_+(\lambda))^{-1} = \begin{pmatrix} \Phi_\tau(\lambda) & * \\ * & P_{\mathfrak{H}_1} (\widetilde{A}^\tau - \lambda)^{-1} \upharpoonright \mathfrak{H}_1 \end{pmatrix} : \underbrace{\mathcal{H}_1 \oplus \mathfrak{H}_1}_{\widetilde{\mathcal{H}}_1} \to \underbrace{\mathcal{H}_1 \oplus \mathfrak{H}_1}_{\widetilde{\mathcal{H}}_1} \tag{4.56}$$

for all $\lambda \in \mathbb{C}_+$(the entries $*$ do not matter). Since $\Phi_{\widetilde{\theta}}(\cdot) \in R[\widetilde{\mathcal{H}}_1]$, it follows from (4.56) that $\mathrm{Im}\Phi_\tau(\lambda) \geq 0$, $\lambda \in \mathbb{C}_+$. Moreover, by the first equality in (4.47) one has $\Phi_\tau(\lambda) = \Phi_\tau^*(\overline{\lambda})$, $\lambda \in \mathbb{C}_-$. Therefore $\Phi_\tau(\cdot) \in R[\mathcal{H}_1]$.

Using formula (2.3) from [26] one can easily prove that

$$\tau_+^*(\overline{\lambda}) = \{(D_{01}(\lambda), D_1(\lambda)P_1 + iD_{02}(\lambda)P_2); \mathcal{K}_-\}, \quad \lambda \in \mathbb{C}_-. \tag{4.57}$$

Applying Lemma 2.1, (2) from [23] to the first equality in (2.9) and (4.57) one obtains the second equalities in (4.46) and (4.47).

Next, combining (4.56) with the known equality

$$s - \lim_{y \to +\infty} \tfrac{1}{iy} P_{\mathfrak{H}_1} (\widetilde{A}^\tau - iy)^{-1} \upharpoonright \mathfrak{H}_1 = 0$$

and taking Proposition 2.2 into account one gets the equivalence

$$\mathcal{B}_{\Phi_{\widetilde{\theta}}} = 0 \iff \mathcal{B}_\tau (= \mathcal{B}_{\Phi_\tau}) = 0. \tag{4.58}$$

Moreover, by (4.50) $\mathrm{mul}\, \widetilde{A}_0 = \mathrm{mul}\, A_0 \oplus \mathfrak{H}_1$ and application of Theorem 4.7 to the triplet $\widetilde{\Pi}$ and the extension $\widetilde{A}^\tau = A_{\widetilde{\theta}}$ yields

$$\mathrm{mul}\, \widetilde{A}^\tau \subset \mathrm{mul}\, A_0 \oplus \mathfrak{H}_1 \iff \mathcal{B}_{\Phi_{\widetilde{\theta}}} = 0. \tag{4.59}$$

Now combining (4.58) and (4.59) we arrive at the equivalence (4.49). □

In the following theorem we describe in terms of the boundary parameter τ exit space self-adjoint extensions $\widetilde{A}^\tau$ of A satisfying $\mathrm{mul}\, \widetilde{A}^\tau = \mathrm{mul}\, A$.

Theorem 4.9. *Let the assumptions of Theorem 4.8 be satisfied. Then*

(1) *The equality*

$$\widehat{\Phi}_\tau(\lambda) := M(\lambda)(D_{01}(\lambda) - D_1(\lambda)M(\lambda) - iD_{02}(\lambda)N_-(\lambda))^{-1}D_{01}(\lambda), \quad \lambda \in \mathbb{C}_-, \tag{4.60}$$

where $D_{0j}(\lambda),\ j \in \{1,2\}$, *are taken from* (2.11) *defines the holomorphic operator function* $\widehat{\Phi}_\tau(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1]$ *such that* $\mathrm{Im}\widehat{\Phi}_\tau(\lambda) \le 0,\ \lambda \in \mathbb{C}_-$. *Hence there exists the strong limit*

$$\widehat{\mathcal{B}}_\tau := s - \lim_{y\to-\infty} \tfrac{1}{iy} M(iy)(D_{01}(iy) - D_1(iy)M(iy) - iD_{02}(iy)N_-(iy))^{-1}D_{01}(iy). \tag{4.61}$$

(2) *The exit space extension* $\widetilde{A}^\tau$ *satisfies* $\mathrm{mul}\,\widetilde{A}^\tau = \mathrm{mul}\,A$ *if and only if* $\mathcal{B}_\tau = \widehat{\mathcal{B}}_\tau = 0$ *(here* $\mathcal{B}_\tau$ *is defined by* (4.48)*).*

(3) *If , in addition,* $\mathrm{mul}\,A_0 = \mathrm{mul}\,A$, *then*

$$\mathrm{mul}\,\widetilde{A}^\tau = \mathrm{mul}\,A \iff \mathcal{B}_\tau = 0. \tag{4.62}$$

(4) *If* $\mathrm{mul}\,A_1 = \mathrm{mul}\,A$, *then*

$$\mathrm{mul}\,\widetilde{A}^\tau = \mathrm{mul}\,A \iff \widehat{\mathcal{B}}_\tau = 0. \tag{4.63}$$

Proof. Let $\widehat{\Pi}_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ be the boundary triplet (4.13) for A^*. Applying to this triplet Theorem 4.8 and taking Corollary 4.4 into account one gets the following assertions:

(i) The equality

$$\widehat{\Phi}_\tau(\lambda) = -(D_1(\lambda) - D_{01}(\lambda)M^{-1}(\lambda) + iD_{02}(\lambda)N_-(\lambda)M^{-1}(\lambda))^{-1}D_{01}(\lambda), \quad \lambda \in \mathbb{C}_-,$$

defines the holomorphic operator function $\widehat{\Phi}_\tau(\cdot) : \mathbb{C}_- \to [\mathcal{H}_1]$ such that $\mathrm{Im}\widehat{\Phi}_\tau(\lambda) \le 0$, $\lambda \in \mathbb{C}_-$. Therefore there exists the limit $\widehat{\mathcal{B}}_\tau := \lim_{y\to-\infty} \frac{1}{iy}\widehat{\Phi}_\tau(iy)$;

(ii) The following equivalence holds:

$$\mathrm{mul}\,\widetilde{A}^\tau \subset \mathrm{mul}\,A_1 \oplus \mathfrak{H}_1 \iff \widehat{\mathcal{B}}_\tau = 0. \tag{4.64}$$

The assertion (i) gives statement (1) of the theorem.

Next, combining of (4.49) and (4.64) yields

$$\mathrm{mul}\,\widetilde{A}^\tau \subset (\mathrm{mul}\,A_0 \oplus \mathfrak{H}_1) \cap (\mathrm{mul}\,A_1 \oplus \mathfrak{H}_1) \iff \mathcal{B}_\tau = \widehat{\mathcal{B}}_\tau = 0. \tag{4.65}$$

Since $\mathrm{mul}\,A_0 \subset \mathfrak{H}$ and $\mathrm{mul}\,A_1 \subset \mathfrak{H}$, it follows that

$$(\mathrm{mul}\,A_0 \oplus \mathfrak{H}_1) \cap (\mathrm{mul}\,A_1 \oplus \mathfrak{H}_1) = (\mathrm{mul}\,A_0 \cap \mathrm{mul}\,A_1) \oplus \mathfrak{H}_1 = \mathrm{mul}\,(A_0 \cap A_1) \oplus \mathfrak{H}_1.$$

Moreover, by (3.5) and Proposition 3.3, (1) one has $A_0 \cap A_1 \subset A$ and hence $A_0 \cap A_1 = A$. Therefore the equivalence (4.65) can be written as

$$\mathrm{mul}\,\widetilde{A}^\tau \subset \mathrm{mul}\,A \oplus \mathfrak{H}_1 \iff \mathcal{B}\tau = \widehat{\mathcal{B}}_\tau = 0. \tag{4.66}$$

Since the extension $\widetilde{A}^\tau$ is $\mathfrak{H}$-minimal, the equality $\mathrm{mul}\,\widetilde{A}^\tau \cap \mathfrak{H}_1 = \{0\}$ is valid. This and the inclusion $\mathrm{mul}\,A \subset \mathrm{mul}\,\widetilde{A}^\tau$ yield the equivalence

$$\mathrm{mul}\,\widetilde{A}^\tau \subset \mathrm{mul}\,A \oplus \mathrm{mul}\,\mathfrak{H}_1 \iff \mathrm{mul}\,\widetilde{A}^\tau = \mathrm{mul}\,A. \tag{4.67}$$

Now combining (4.66) and (4.67) we arrive at statement (2) of the theorem.

Statement (3) follows from (4.49) and (4.67). Finally, statement (4) is statement (3) for the triplet $\widehat{\Pi}$. □

The following corollary is immediate from Theorem 4.9.

Corollary 4.10. *Let the assumptions of Theorem 4.9 be satisfied and let* $\mathcal{B}_\tau$ *and* $\widehat{\mathcal{B}}_\tau$ *be given by* (4.48) *and* (4.61) *respectively. Assume also that* A *is the operator. Then*

(1) $\widetilde{A}^\tau$ *is the operator if and only if* $\mathcal{B}_\tau = \widehat{\mathcal{B}}_\tau = 0$.

(2) *If , in addition,* A_0 *is the operator, then* $\widetilde{A}^\tau$ *is the operator if and only if* $\mathcal{B}_\tau = 0$.

(3) *If* A_1 *is the operator, then* $\widetilde{A}^\tau$ *is the operator if and only if* $\widehat{\mathcal{B}}_\tau = 0$.

Remark 4.11. If $n_+(A) = n_-(A)$ and $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for A^*, then the boundary parameter $\tau(\cdot)$ is a Nevanlinna operator pair (2.19), $\widehat{\tau}(\lambda) = -\tau^{-1}(\lambda)$, $\widehat{M}(\lambda) = -M^{-1}(\lambda)$ and the equalities (4.48) and (4.61) take the form

$$\mathcal{B}_\tau = \lim_{y\to\infty} (-\tfrac{1}{iy}(\tau(iy) + M(iy))^{-1}) = \lim_{y\to\infty} \tfrac{1}{iy}(C_0(iy) - C_1(iy)M(iy))^{-1}C_1(iy),$$

$$\widehat{\mathcal{B}}_\tau = \lim_{y\to\infty} \tfrac{1}{iy}(\tau^{-1}(iy) + M^{-1}(iy))^{-1} = \lim_{y\to\infty} \tfrac{1}{iy}M(iy)(C_0(iy) - C_1(iy)M(iy))^{-1}C_0(iy),$$

where $M(\cdot)$ is the Weyl function of the triplet Π and all the limits are understood in the sense of the strong operator convergence. Note that for this case Theorem 4.9 was proved in [6, 7].

4.3. **The case $n_+(A) \le n_-(A)$.** By using (3.26) and the results of the previous subsection one can easily prove the following two theorems for the case $n_+(A) \le n_-(A)$.

Theorem 4.12. *Let $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ be a symmetric relation with $n_+(A) \le n_-(A)$, let $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, let $M_-(\cdot)$ be the Weyl functions of Π_- and let $M(\cdot)(\in R[\mathcal{H}_1])$ be the operator function defined by* (3.15) *and* (3.16). *Then*

(1) *The extension $A_0(= \ker\Gamma_0)$ satisfies* $\operatorname{mul} A_0 = \operatorname{mul} A$ *if and only if* (4.20) *holds.*

(2) $\operatorname{mul} A = \operatorname{mul} A^*$ if and only if (4.20) is satisfied and

$$\lim_{y\to-\infty} y\left(\operatorname{Im}(M_-(iy)h_0, h_0)_{\mathcal{H}_0} - \tfrac{1}{2}||P_2 h_0||^2\right) = +\infty, \quad h_0 \in \mathcal{H}_0, \quad h_0 \ne 0.$$

Theorem 4.13. *Let A and Π_- be the same as in Theorem 4.12, let $M_\pm(\cdot)$ be the Weyl functions* (3.15) *and* (3.16), *let $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}_-(\mathcal{H}_0, \mathcal{H}_1)$ be a collection of holomorphic pairs* (2.9)-(2.11) *and let $\widetilde{A}^\tau$ be the corresponding exit space self-adjoint extension of A. Then*

(1) *There exist the strong limits*

$$\begin{aligned}\mathcal{B}_\tau :&= s-\lim_{y\to+\infty} \tfrac{1}{iy}(C_{01}(iy) - C_1(iy)M(iy) + iC_{02}(iy)N_+(iy))^{-1}C_1(iy)\\ &= s-\lim_{y\to-\infty} \tfrac{1}{iy}P_1(D_0(iy) - D_1(iy)M_-(iy))^{-1}D_1(iy),\\ \widehat{\mathcal{B}}_\tau :&= s-\lim_{y\to+\infty} \tfrac{1}{iy}M(iy)(C_{01}(iy) - C_1(iy)M(iy) + iC_{02}(iy)N_+(iy))^{-1}C_{01}(iy).\end{aligned}$$

(2) *The equality* $\operatorname{mul}\widetilde{A}^\tau = \operatorname{mul} A$ *holds if and only if* $\mathcal{B}_\tau = \widehat{\mathcal{B}}_\tau = 0$.

(3) *If in addition* $\operatorname{mul} A_0 = \operatorname{mul} A$ *(resp.* $\operatorname{mul} A_1 = \operatorname{mul} A$*), then equivalence* (4.62) *(resp.* (4.63)*) is valid.*

5. Applications to symmetric systems

5.1. **Preliminary facts about symmetric systems.** In this subsection we recall briefly some results on symmetric systems from [1].

Assume that H and $\widehat{H}$ are finite dimensional Hilbert spaces, let

$$H_0 := H \oplus \widehat{H}, \quad \mathbb{H} := H_0 \oplus H = H \oplus \widehat{H} \oplus H, \tag{5.1}$$

and let $J \in [\mathbb{H}]$ be operator (1.15). A first order symmetric system of differential equations on an interval $\mathcal{I} = [a, b\rangle, -\infty < a < b \le \infty$, (with the regular endpoint a) is of the form

$$Jy'(t) - B(t)y(t) = \Delta(t)f(t), \quad t \in \mathcal{I}, \tag{5.2}$$

where $B(t) = B^*(t)$ and $\Delta(t) \ge 0$ are the $[\mathbb{H}]$-valued functions on $\mathcal{I}$ integrable on each compact interval $[a, \beta] \subset \mathcal{I}$. Below we assume that the system (5.2) is definite. The latter means that for any $\lambda \in \mathbb{C}$ each common solution of the equations

$$Jy'(t) - B(t)y(t) = \lambda\Delta(t)y(t) \tag{5.3}$$

and $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) is trivial, i.e., $y(t) = 0,\ t \in \mathcal{I}$.

Denote by $\mathcal{L}^2_\Delta(\mathcal{I})$ the semi-Hilbert space of Borel measurable functions $f(\cdot):\mathcal{I}\to\mathbb{H}$ such that $\int_{\mathcal{I}}(\Delta(t)f(t),f(t))_{\mathbb{H}}\,dt<\infty$ and let $\mathfrak{H}$ be the Hilbert space of all equivalence classes in $\mathcal{L}^2_\Delta(\mathcal{I})$. Denote also by π the quotient map from $\mathcal{L}^2_\Delta(\mathcal{I})$ onto $\mathfrak{H}$.

With each system (5.2) one associates the minimal and maximal linear relations $\mathcal{T}_{\min}$ and $\mathcal{T}_{\max}$ in $\mathcal{L}^2_\Delta(\mathcal{I})$, which generate in turn the relations $T_{\min}=(\pi\oplus\pi)\mathcal{T}_{\min}$ and $T_{\max}=(\pi\oplus\pi)\mathcal{T}_{\max}$ in $\mathfrak{H}$ [13, 21, 28]. It turns out that $T_{\min}$ is a closed symmetric relation with finite not necessarily equal deficiency indices $n_\pm(T_{\min})$ and $T_{\max}=T^*_{\min}$.

Next assume that

$$U=\begin{pmatrix}u_1 & u_2 & u_3\\ u_4 & u_5 & u_6\end{pmatrix}:H\oplus\widehat{H}\oplus H\to\widehat{H}\oplus H \tag{5.4}$$

is the operator such that $\operatorname{ran}U=\widehat{H}\oplus H$ and

$$iu_2u_2^*-u_1u_3^*+u_3u_1^*=iI_{\widehat{H}},\quad iu_5u_2^*-u_4u_3^*+u_6u_1^*=0,\quad iu_5u_5^*+u_6u_4^*-u_4u_6^*=0.$$

One can prove that the operator (5.4) admits an extension to the J-unitary operator

$$\widetilde{U}=\begin{pmatrix}u_7 & u_8 & u_9\\ \hline u_1 & u_2 & u_3\\ u_4 & u_5 & u_6\end{pmatrix}:H\oplus\widehat{H}\oplus H\to H\oplus\widehat{H}\oplus H. \tag{5.5}$$

For each function $y\in\operatorname{dom}\mathcal{T}_{\max}$ decomposed as $y(t)=\{y_0(t),\,\widehat{y}(t),\,y_1(t)\}(\in H\oplus\widehat{H}\oplus H)$, $\quad t\in\mathcal{I}$, we let

$$\Gamma_{0a}y=u_7y_0(a)+u_8\widehat{y}(a)+u_9y_1(a), \tag{5.6}$$

$$\widehat{\Gamma}_a y=u_1y_0(a)+u_2\widehat{y}(a)+u_3y_1(a),\quad \Gamma_{1a}y=u_4y_0(a)+u_5\widehat{y}(a)+u_6y_1(a). \tag{5.7}$$

Clearly, $\widehat{\Gamma}_a y(\in\widehat{H})$ and $\Gamma_{1a}y(\in H)$ are determined by the operator U, while $\Gamma_{0a}y(\in H)$ is determined by the extension $\widetilde{U}$. Moreover,with the operator U we associate the operator solution $\varphi(\cdot,\lambda)=\varphi_U(\cdot,\lambda)(\in[H_0,\mathbb{H}])$, $\lambda\in\mathbb{C}$, of Eq. (5.3) with the initial data

$$\varphi_U(a,\lambda)=\begin{pmatrix}u_6^* & iu_3^*\\ -iu_5^* & u_2^*\\ -u_4^* & -iu_1^*\end{pmatrix}:\underbrace{H\oplus\widehat{H}}_{H_0}\to\underbrace{H\oplus\widehat{H}\oplus H}_{\mathbb{H}}.$$

One can easily verify that $\widetilde{U}\varphi_U(a,\lambda)=(I_{H_0},0)^\top:H_0\to H_0\oplus H$ for each J-unitary extension $\widetilde{U}$ of U.

In the following we suppose that $n_-(T_{\min})\le n_+(T_{\min})$. In this case there exist a finite dimensional Hilbert space $\mathcal{H}_{0b}$, a subspace $\mathcal{H}_{1b}\subset\mathcal{H}_{0b}$ and a surjective linear mapping

$$\Gamma_b=(\Gamma_{0b},\,\widehat{\Gamma}_b,\,\Gamma_{1b})^\top:\operatorname{dom}\mathcal{T}_{\max}\to\mathcal{H}_{0b}\oplus\widehat{H}\oplus\mathcal{H}_{1b} \tag{5.8}$$

such that for all $y,z\in\operatorname{dom}\mathcal{T}_{\max}$ the following identity is valid:

$$\lim_{t\uparrow b}(Jy(t),z(t)))=(\Gamma_{0b}y,\Gamma_{1b}z)-(\Gamma_{1b}y,\Gamma_{0b}z)+i(P_{2b}\Gamma_{0b}y,P_{2b}\Gamma_{0b}z)+i(\widehat{\Gamma}_b y,\widehat{\Gamma}_b z)$$

(here P_{2b} is the orthoprojector on $\mathcal{H}_{0b}$ onto $\mathcal{H}_{2b}:=\mathcal{H}_{0b}\ominus\mathcal{H}_{1b}$).

By using (5.6), (5.7) and (5.8) one constructs a boundary triplet $\Pi_+=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ for $T_{\max}$, where $\mathcal{H}_j=H_0\oplus\mathcal{H}_{jb},\ j\in\{0,1\}$, and

$$\Gamma_0\{\widetilde{y},\widetilde{f}\}=\{-\Gamma_{1a}y+i(\widehat{\Gamma}_a-\widehat{\Gamma}_b)y,\ \Gamma_{0b}y\}(\in H_0\oplus\mathcal{H}_{0b}), \tag{5.9}$$

$$\Gamma_1\{\widetilde{y},\widetilde{f}\}=\{\Gamma_{0a}y+\tfrac{1}{2}(\widehat{\Gamma}_a+\widehat{\Gamma}_b)y,\ -\Gamma_{1b}y\}(\in H_0\oplus\mathcal{H}_{1b}),\quad \{\widetilde{y},\widetilde{f}\}\in T_{\max} \tag{5.10}$$

(in [1] such a triplet is called decomposing). Moreover, the equalities

$$\begin{gathered}T=\{\{\widetilde{y},\widetilde{f}\}\in T_{\max}:\Gamma_{1a}y=0,\ \widehat{\Gamma}_a y=\widehat{\Gamma}_b y,\ \Gamma_{0b}y=\Gamma_{1b}y=0\},\\ T^*=\{\{\widetilde{y},\widetilde{f}\}\in T_{\max}:\Gamma_{1a}y=0,\ \widehat{\Gamma}_a y=\widehat{\Gamma}_b y\}\end{gathered} \tag{5.11}$$

define a symmetric extension T of $T_{\min}$ and its adjoint T^* and the collection $\dot{\Pi}_+ = \{\mathcal{H}_{0b} \oplus \mathcal{H}_{1b}, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ with

$$\dot{\Gamma}_0\{\widetilde{y}, \widetilde{f}\} = \Gamma_{0b} y, \quad \dot{\Gamma}_1\{\widetilde{y}, \widetilde{f}\} = -\Gamma_{1b} y, \quad \{\widetilde{y}, \widetilde{f}\} \in T^* \tag{5.12}$$

forms a boundary triplet for T^*.

Definition 5.1. A boundary parameter τ (at the endpoint b) is a collection $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}_+(\mathcal{H}_{0b}, \mathcal{H}_{1b})$ of holomorphic operator pairs

$$\tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}_{0b}\}, \ \ \lambda \in \mathbb{C}_+; \quad \tau_-(\lambda) = \{(D_0(\lambda), D_1(\lambda)); \mathcal{H}_{1b}\}, \ \ \lambda \in \mathbb{C}_- \tag{5.13}$$

with $C_0(\lambda) \in [\mathcal{H}_{0b}]$, $C_1(\lambda) \in [\mathcal{H}_{1b}, \mathcal{H}_{0b}]$, $D_0(\lambda) \in [\mathcal{H}_{0b}, \mathcal{H}_{1b}]$ and $D_1(\lambda) \in [\mathcal{H}_{1b}]$.

Application of Theorem 3.11 to the boundary triplet $\dot{\Pi}_+$ gives the following theorem.

Theorem 5.2. ([1]). *Assume that U is the operator* (5.4)*, $\widehat{\Gamma}_a y$ and $\Gamma_{1a} y$ are defined by* (5.7)*, Γ_b is the mapping* (5.8) *and T is the symmetric relation* (5.11)*. If $\tau = \{\tau_+, \tau_-\}$ is a boundary parameter* (5.13)*, then for every $f \in \mathcal{L}^2_\Delta(\mathcal{I})$ the boundary value problem*

$$Jy' - B(t)y = \lambda \Delta(t) y + \Delta(t) f(t), \quad t \in \mathcal{I}, \tag{5.14}$$

$$\Gamma_{1a} y = 0, \quad \widehat{\Gamma}_a y = \widehat{\Gamma}_b y, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{5.15}$$

$$C_0(\lambda)\Gamma_{0b} y + C_1(\lambda)\Gamma_{1b} y = 0, \quad \lambda \in \mathbb{C}_+, \tag{5.16}$$

$$D_0(\lambda)\Gamma_{0b} y + D_1(\lambda)\Gamma_{1b} y = 0, \quad \lambda \in \mathbb{C}_- \tag{5.17}$$

has a unique solution $y(t, \lambda) = y_f(t, \lambda)$ and the equality

$$R(\lambda)\widetilde{f} = \pi(y_f(\cdot, \lambda)), \quad \widetilde{f} \in L^2_\Delta(\mathcal{I}), \quad f \in \widetilde{f}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

defines a generalized resolvent $R(\lambda) =: R_\tau(\lambda)$ of T. Conversely, for each generalized resolvent $R(\lambda)$ of T there exists a unique boundary parameter τ such that $R(\lambda) = R_\tau(\lambda)$.

According to Remark 3.12 boundary value problem (5.14)–(5.17) gives a parametrization of all exit space self-adjoint extensions $\widetilde{T} = \widetilde{T}^\tau$ of T by means of a boundary parameter $\tau \in \widetilde{R}_+(\mathcal{H}_{0b}, \mathcal{H}_{1b})$. Denote also by $F_\tau(\cdot)$ the spectral function of T generated by $\widetilde{T}^\tau$.

Definition 5.3. Let $\mathfrak{H}_b$ be the set of all $\widetilde{f} \in \mathfrak{H}$ such that $\Delta(t) f(t) \equiv 0$ on some interval $[\beta, b) \subset \mathcal{I}$ (depending on $\widetilde{f}$) and let τ be a boundary parameter. A nondecreasing left-continuous operator function $\Sigma_\tau(\cdot) : \mathbb{R} \to [H_0]$ is called a spectral function of the boundary value problem (5.14)–(5.17) if, for each $\widetilde{f} \in \mathfrak{H}_b$, the Fourier transform

$$\widehat{f}(s) = \int_{\mathcal{I}} \varphi_U^*(t, s)\Delta(t) f(t)\, dt, \quad f \in \widetilde{f}, \tag{5.18}$$

satisfies

$$((F_\tau(\beta) - F_\tau(\alpha))\widetilde{f}, \widetilde{f})_{\mathfrak{H}} = \int_{[\alpha, \beta)} (d\Sigma_\tau(s)\widehat{f}(s), \widehat{f}(s)), \quad [\alpha, \beta) \subset \mathbb{R}. \tag{5.19}$$

It follows from (5.19) that the mapping $V f = \widehat{f}$, originally defined by (5.18) for $\widetilde{f} \in \mathfrak{H}_b$, admits a continuous extension to a contractive map $V : \mathfrak{H} \to L^2(\Sigma_\tau; H_0)$ (for definition of the Hilbert space $L^2(\Sigma_\tau; H_0)$ see [10, Chapter 13.5]). Moreover, $V \upharpoonright \operatorname{mul} T = \{0\}$, so that $\mathfrak{H}_0 := \mathfrak{H} \ominus \operatorname{mul} T$ is the maximally possible subspace of $\mathfrak{H}$ on which the Fourier transform V may be isometric.

Definition 5.4. [1] A spectral function $\Sigma_\tau(\cdot)$ of the boundary value problem (5.14)–(5.17) is referred to the class SF_0 if the operator $V \upharpoonright \mathfrak{H}_0$ is an isometry from $\mathfrak{H}_0$ to $L^2(\Sigma_\tau; H_0)$.

The class SF_0 seems to be especially interesting, because in the case $\Sigma_\tau(\cdot) \in SF_0$ for each $\widetilde{f} \in \mathfrak{H}_0$ the inverse Fourier transform can be calculated by

$$\widetilde{f} = \pi\left(\int_{\mathbb{R}} \varphi_U(\cdot, s)\, d\Sigma_\tau(s)\widehat{f}(s)\right).$$

5.2. **Description of spectral functions.** According to [1] for each boundary parameter τ there exists a unique spectral function $\Sigma_\tau(\cdot)$ of the boundary value problem (5.14)–(5.17).

In the following theorem we give a parametrization of all spectral functions $\Sigma_\tau(\cdot)$ (in particular, of the class SF_0) immediately in terms of the boundary parameter τ.

Theorem 5.5. *Let the assumptions of Theorem 5.2 be satisfied. Assume also that $\widetilde{U}$ is a J-unitary extension* (5.5) *of U, that $\Gamma_{0a}y$ is defined by* (5.6)*, that $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is the decomposing boundary triplet* (5.9), (5.10) *for $T_{\max}$ and that*

$$M_+(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2+}(\lambda) \\ M_{3+}(\lambda) & M_{4+}(\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_{0b} \to H_0 \oplus \mathcal{H}_{1b}, \quad \lambda \in \mathbb{C}_+, \tag{5.20}$$

$$M_-(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2-}(\lambda) \\ M_{3-}(\lambda) & M_{4-}(\lambda) \end{pmatrix} : H_0 \oplus \mathcal{H}_{1b} \to H_0 \oplus \mathcal{H}_{0b}, \quad \lambda \in \mathbb{C}_- \tag{5.21}$$

are the block matrix representations of the corresponding Weyl functions. Then, for each boundary parameter $\tau = \{\tau_+, \tau_-\}$ of the form (5.13)*, the equalities*

$$m_\tau(\lambda) = m_0(\lambda) + M_{2+}(\lambda)(C_0(\lambda) - C_1(\lambda)M_{4+}(\lambda))^{-1}C_1(\lambda)M_{3+}(\lambda), \quad \lambda \in \mathbb{C}_+, \tag{5.22}$$

$$\Sigma_\tau(s) = \lim_{\delta\to+0}\lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{-\delta}^{s-\delta} \operatorname{Im} m_\tau(\sigma + i\varepsilon)\, d\sigma \tag{5.23}$$

define a spectral function $\Sigma_\tau(\cdot)$ of the boundary value problem (5.14)–(5.17) *and the following statements are valid:*

(1) *Let $\mathcal{H}_{0b}$ be decomposed as $\mathcal{H}_{0b} = \mathcal{H}_{1b} \oplus \mathcal{H}_{2b}$ with $\mathcal{H}_{2b} := \mathcal{H}_{0b} \ominus \mathcal{H}_{1b}$, let P_{jb} be the orthoprojector in $\mathcal{H}_{0b}$ onto $\mathcal{H}_{jb}$, $j \in \{1, 2\}$, and let*

$$D_0(\lambda) = (D_{01}(\lambda), D_{02}(\lambda)) : \mathcal{H}_{1b} \oplus \mathcal{H}_{2b} \to \mathcal{H}_{1b}, \quad \lambda \in \mathbb{C}_-,$$
$$M_{4-}(\lambda) = (M_4(\lambda), N_{4-}(\lambda))^\top : \mathcal{H}_{1b} \to \mathcal{H}_{1b} \oplus \mathcal{H}_{2b}, \quad \lambda \in \mathbb{C}_-$$

be the block representations of $D_0(\cdot)$ (see (5.13)*) and $M_{4-}(\cdot)$. Then there exist limits*

$$\begin{aligned} \mathcal{B}_\tau &= \lim_{y\to+\infty} \tfrac{1}{iy} P_{1b}(C_0(iy) - C_1(iy)M_{4+}(iy))^{-1}C_1(iy) \\ &= \lim_{y\to-\infty} \tfrac{1}{iy}(D_{01}(iy) - D_1(iy)M_4(iy) - iD_{02}(iy)N_{4-}(iy))^{-1}D_1(iy), \\ \widehat{\mathcal{B}}_\tau &= \lim_{y\to-\infty} \tfrac{1}{iy} M_4(iy)(D_{01}(iy) - D_1(iy)M_4(iy) - iD_{02}(iy)N_{4-}(iy))^{-1}D_{01}(iy) \end{aligned}$$

and the following equivalence holds:

$$\Sigma_\tau(\cdot) \in SF_0 \iff \mathcal{B}_\tau = \widehat{\mathcal{B}}_\tau = 0. \tag{5.24}$$

(2) *If in addition*

$$\lim_{y\to\infty} \tfrac{1}{iy} M_4(iy) = 0, \tag{5.25}$$

then the equivalence $\Sigma_\tau(\cdot) \in SF_0 \iff \mathcal{B}_\tau = 0$ is valid.

(3) *Each spectral function $\Sigma_\tau(\cdot)$ belongs to SF_0 if and only if* (5.25) *is satisfied and*

$$\lim_{y\to+\infty} y\left(\operatorname{Im}(M_{4+}(iy)h, h)_{\mathcal{H}_{0b}} + \tfrac{1}{2}||P_{2b}h||^2\right) = +\infty, \quad h \in \mathcal{H}_{0b}, \quad h \neq 0.$$

Proof. Formulas (5.22) and (5.23) are implied by Theorems 5.5 and 6.5 in [1].

Next assume that $\widetilde{T}^{\tau}$ is the exit space self-adjoint extension of T corresponding to the boundary parameter τ. Then according to [1] one has

$$\Sigma_{\tau}(\cdot) \in SF_0 \iff \operatorname{mul}\widetilde{T}^{\tau} = \operatorname{mul} T. \tag{5.26}$$

Consider the boundary triplet $\dot{\Pi}_+ = \{\mathcal{H}_{0b} \oplus \mathcal{H}_{1b}, \dot{\Gamma}_0, \dot{\Gamma}_1\}$ for T^* given by (5.12). Since the Weyl functions $M_{\pm}(\cdot)$ of the decomposing boundary triplet Π_+ have the block representations (5.20) and (5.21), it follows from [1, Proposition 2.10] that the Weyl functions of the triplet $\dot{\Pi}_+$ are $\dot{M}_{\pm}(\lambda) = M_{4\pm}(\lambda)$, $\lambda \in \mathbb{C}_{\pm}$. Now applying Theorems 4.9 and 4.6 (1) to the boundary triplet $\dot{\Pi}_+$ and taking (5.26) into account one obtains statements (1) and (2) of the theorem. Finally, statement (3) follows from Theorem 4.6 (2), Proposition 4.1 (2) and equivalence (5.26). □

Remark 5.6. (1) According to [1, Proposition 4.4] the operator functions $m_0(\cdot)$ and $M_{j\pm}(\cdot)$, $j \in \{2, 3, 4\}$, in (5.20)–(5.22) are expressed in terms of the boundary values of respective operator solutions of (5.3).

(2) The operator function $m_{\tau}(\cdot)$ in (5.22) is the m-function of the boundary value problem (5.14)– (5.17) and hence $m_{\tau}(\cdot) \in R[H_0]$ (for definition of the m-function for the system (5.2) and its properties see [1]). Moreover, (5.23) is the Stieltjes inversion formula for the function $m_{\tau}(\cdot)$. Observe also that $m_0(\cdot)$ is the m-function of the boundary value problem (5.14)– (5.17) with $C_0(\lambda) \equiv I_{\mathcal{H}_{0b}}$, $C_1(\lambda) \equiv 0$ and $D_0(\lambda) \equiv P_{1b}$, $D_1(\lambda) \equiv 0$.

(3) In the case of equal deficiency indices $n_+(T_{\min}) = n_-(T_{\min})$ Theorem 5.5 was proved in [1].

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