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LOGARITHMIC SOBOLEV INEQUALITY FOR A CLASS OF MEASURES ON CONFIGURATION SPACES

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Dedicated to Yuri Kondratiev on occasion of his 60th Birthday.

Abstract. We study a class of measures on the space Γ_X of locally finite configurations in $X = \mathbb{R}^d$, obtained as images of "lattice" Gibbs measures on $X^{\mathbb{Z}^d}$ with respect to an embedding $\mathbb{Z}^d \subset \mathbb{R}^d$. For these measures, we prove the integration by parts formula and log-Sobolev inequality.

1. Introduction

Let $X = \mathbb{R}^d$ be a d-dimensional Euclidean space, and consider the space

$$\Gamma_X := \{\gamma \subset X : |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}$$

of all locally finite subsets (configurations) in X. Here $|A|$ denotes the cardinality of the set A. Observe that Γ_X can be embedded into the space of all Radon measures $\mathcal{M}(X)$ on X via the map $\gamma \mapsto \sum_{x\in\gamma} \delta(x)$, where $\delta(x)$ is the Dirac measure at $x \in X$. We will denote by $\ddot{\Gamma}_X$ the space of all integer-valued Radon measures on X, which can be interpreted as the space of locally finite configurations with finite multiplicities, so that we have the inclusions

$$\Gamma_X \subset \ddot{\Gamma}_X \subset \mathcal{M}(X).$$

Configuration spaces Γ_X and $\ddot{\Gamma}_X$ are endowed with the topology induced by the weak topology on $\mathcal{M}(X)$ and is called the vague topology, which makes them Polish spaces, see e.g. [12]. We denote by $\mathcal{B}(\Gamma_X)$ and $\mathcal{B}(\ddot{\Gamma}_X)$ the corresponding Borel σ-algebras.

Interest to stochastic analysis on Γ_X has been growing in recent times due to rich applications in the study of multi-component stochastic systems, which arise in mathematical physics, mathematical biology and other sciences, see e.g. [7], [8], [9] and references therein. An important task in the development of such analysis is construction and study of probability measures on Γ_X (also called point processes in X) that satisfy certain analytic properties, like finiteness of moments and integration by parts formulae. These measures can in turn be used in various constructions on Γ_X, including Laplace-type operators and stochastic dynamics. Most studies in this respect are concerned with Poisson and Gibbs measures on Γ_X, see e.g. [2], [3] and references given there. Cluster point processes in X have been considered from this point of view in [5].

In the present work we explore another class of measures on Γ_X, obtained as push-forwards of "lattice" Gibbs measures in $X^{\mathbb{Z}^d}$ with respect to a special embedding $\mathbb{Z}^d \subset \mathbb{R}^d$, where $\mathbb{Z}^d$ is the d-dimensional integer lattice. These measures present interesting properties, including the log-Sobolev inequality, which is not typical for measures on Γ_X (note that neither Poisson nor Gibbs measures on Γ_X satisfy this inequality).

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The construction is as follows. Consider the infinite Cartesian product

$$X^{\mathbb{Z}^d} = \underset{k\,\in\,\mathbb{Z}^d}{\times} X_k, \quad X_k \,=\, X,$$

of identical copies of X, endowed with the product topology and the corresponding Borel structure. For any $\mathbf{x} = (x_k)_{k\,\in\,\mathbb{Z}^d} \in X^{\mathbb{Z}^d}$ define a map $\mathfrak{p} : X^{\mathbb{Z}^d} \to \mathcal{M}(X)$ by setting

$$\mathfrak{p}(\mathbf{x}) = \sum_{k\in\mathbb{Z}^d} \delta(x_k + \alpha(k)),$$

where α is a "correctly rescaled" embedding $\mathbb{Z}^d \subset \mathbb{R}^d$ (which should satisfy the condition $\sum_{k\in\mathbb{Z}^d} |\alpha(k)|^{-2} < \infty$). To be more concrete, we set

$$\alpha(k) := |k|^{d-1}k, \quad |k| := \sum_{m=1}^{d} |k_m|, \tag{1}$$

$k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$.

In general, $\mathfrak{p}(\mathbf{x})$ is not locally finite and therefore belongs neither to Γ_X nor to $\ddot{\Gamma}_X$. However, it is possible to construct a dense Borel subset $\mathcal{H}_- \subset X^{\mathbb{Z}^d}$, which consists of "tempered" sequences, and such that (i) $\mathfrak{p}(\mathcal{H}_-) \subset \ddot{\Gamma}_X$, and (ii) $\mathcal{H}_-$ supports a wide class of probability measures θ on $X^{\mathbb{Z}^d}$.

Next, given such measure θ, we can define the push-forward measure $\nu_\theta = \mathfrak{p}^*\,\theta$ on $\ddot{\Gamma}_X$ by the formula

$$\nu_\theta(A) = \theta\left(\mathfrak{p}^{-1}(A)\right), \quad A \in \mathcal{B}(\ddot{\Gamma}_X).$$

If we assume in addition that θ has "off-diagonal" support, the measure ν_θ will live on the space of proper configurations Γ_X. The main example of θ (and thus ν_θ) is given by a Gibbs measure on $X^{\mathbb{Z}^d}$. It turns out that ν_θ inherits many important properties of the underlaying measure θ, including finiteness of moments, the integration by parts formula and log-Sobolev inequality.

2. Gelfand triple associated with $X^{\mathbb{Z}^d}$

Let $X_0^{\mathbb{Z}^d} \subset X^{\mathbb{Z}^d}$ be the set of all finite sequences of elements of X. Define inner products $(\cdot,\cdot)_+$, $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_-$ on $X_0^{\mathbb{Z}^d}$ by the formulae

$$(\mathbf{u},\mathbf{v})_+ := \sum_{k\in\mathbb{Z}^d} u_k\, v_k (1+|k|^d)^2,$$
$$(\mathbf{u}\,,\,\mathbf{v})_0 := \sum_{k\in\mathbb{Z}^d} u_k\, v_k,$$
$$(\mathbf{u},\mathbf{v})_- := \sum_{k\in\mathbb{Z}^d} u_k\, v_k (1+|k|^d)^{-2},$$

$\mathbf{u} = (u_k)_{k\,\in\,\mathbb{Z}^d}$, $\mathbf{v} = (v_k)_{k\,\in\,\mathbb{Z}^d} \in X_0^{\mathbb{Z}^d}$, and introduce Hilbert spaces $\mathcal{H}_+$, $\mathcal{H}_0$ and $\mathcal{H}_-$ as the completions of $X_0^{\mathbb{Z}^d}$ in the corresponding norms $\|\cdot\|_+$, $\|\cdot\|_0$ and $\|\cdot\|_-$, respectively. Thus we have the following chain of spaces:

$$X_0^{\mathbb{Z}^d} \subset \mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_- \subset X^{\mathbb{Z}^d}.$$

Observe that the inner product $(\cdot,\cdot)_0$ establishes the duality within the pairs of spaces $\left(X_0^{\mathbb{Z}^d}, X^{\mathbb{Z}^d}\right)$ and $(\mathcal{H}_+,\mathcal{H}_-)$. Also, the embeddings $\mathcal{H}_+ \subset \mathcal{H}_0$ and $\mathcal{H}_0 \subset \mathcal{H}_-$ are Hilbert-Schmidt, so that $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$ is a Gelfand triple.

The following technical results can be proved by direct (although quite lengthy) calculations.

Lemma 1. *For any* $\mathbf{y} \in \mathcal{H}_-$ *and* $R \in \mathbb{R}_+$, *there exists* $N \in \mathbb{N}$ *such that for any* $\mathbf{x} \in B_{\mathcal{H}_-}(\mathbf{y}, \frac{1}{4})$ *and* $k \in \mathbb{Z}^d$ *with* $|k| > N$ *we have*

$$|x_k + \alpha(k)| > R,$$

where $\alpha(k)$ *is defined by formula* (1) *and* $B_{\mathcal{H}_-}(\mathbf{y}, r)$ *is the open ball in* $\mathcal{H}_-$ *centered at* $\mathbf{y}$ *and of radius* r.

Lemma 2. *Let* μ *be a probability measure on* X *such that* $\int_X |y|^s \mu(dy) < \infty$ *for all* $s \in \mathbb{N}$. *Then*

$$\sum_{k \in \mathbb{Z}^d} \mu(A - \alpha(k))^{1/p}(1 + |k|^m)^n < \infty$$

for any bounded Borel set $A \subset X$ *and all* $p, m, n \in \mathbb{N}$.

Recall that the vague topology on Γ_X (resp. $\ddot{\Gamma}_X$) is the weakest topology that makes the mappings

$$\gamma \mapsto \langle f, \gamma \rangle := \sum_{x \in \gamma} f(x), \quad f \in C_0(X),$$

continuous.

Theorem 1. *We have the inclusion*

$$\mathfrak{p}(\mathcal{H}_-) \subset \ddot{\Gamma}_X, \tag{2}$$

and the restriction of the map $\mathfrak{p}$ *to* $\mathcal{H}_-$ *is continuous.*

Proof. Formula (2) follows from Lemma 1 applied to arbitrary $\mathbf{x} = \mathbf{y} \in \mathcal{H}_-$. To prove the continuity of the map $\mathfrak{p}$ we fix a function $f \in C_0(X)$ and a sequence $\mathbf{x}^{(n)} \to \mathbf{x}$, $n \to \infty$ in $\mathcal{H}_-$. Since all $\mathbf{x}^{(n)} \in B_{\mathcal{H}_-}(\mathbf{x}, \frac{1}{4})$ for n big enough, Lemma 1 implies that there exists $N \in \mathbb{N}$ such that $f(\mathfrak{p}(\mathbf{x}^{(k)})) = 0$ if $|k| > N$, which in turn implies that the map $\mathcal{H}_- \ni \mathbf{x} \mapsto \langle \mathfrak{p}(\mathbf{x}), f \rangle$ is continuous, and the statement follows from the definition of the topology of $\ddot{\Gamma}_X$. □

We preserve the notation $\mathfrak{p}$ for the corresponding (restricted) map

$$\mathfrak{p} : \mathcal{H}_- \to \ddot{\Gamma}_X. \tag{3}$$

3. Push-forward measures on Γ_X: definition, support and finiteness of moments

Let θ be a Borel probability measure on $\mathcal{H}_-$ satisfying the following conditions:

(1) the measure θ has off-diagonal support, that is, $\theta\,(\mathrm{Diag}(\mathcal{H}_-)) = 0$, where

$$\mathrm{Diag}(\mathcal{H}_-) := \{\mathbf{x} \in \mathcal{H}_- : \ \exists\, k, j \in \mathbb{Z}^d \text{ s.t. } x_k - x_j \in \mathbb{Z}^d\}; \tag{4}$$

(2) the measure θ is invariant with respect to the lattice shifts

$$S_j : (x_k)_{k \in \mathbb{Z}^d} \longmapsto (x_{k+j})_{k \in \mathbb{Z}^d}\,, \quad j \in \mathbb{Z}^d; \tag{5}$$

(3) all moments of the measure θ are finite, that is,

$$\int_{\mathcal{H}_-} |x_k|^s \theta(d\mathbf{x}) < \infty \quad \text{for all} \quad s \in \mathbb{N} \quad \text{and} \quad k \in \mathbb{Z}^d. \tag{6}$$

Let us define the push-forward measure $\nu_\theta := \mathfrak{p}^*\theta$ on $\ddot{\Gamma}_X$. That is, for any $A \in \mathcal{B}(\ddot{\Gamma}_X)$ we have

$$\nu_\theta(A) = \theta\left(\mathfrak{p}^{-1}(A)\right). \tag{7}$$

Theorem 1 implies that $\mathfrak{p} : \mathcal{H}_- \to \ddot{\Gamma}_X$ is measurable, so that ν_θ is a Borel probability measure on $\ddot{\Gamma}_X$. Moreover, it follows from condition (4) that ν_θ is supported on the space of configurations without multiple points, that is, $\nu_\theta(\Gamma_X) = 1$.

We say that a measure ν on Γ_X has finite n-th moments if

$$\int_{\Gamma_X} |\langle f,\gamma\rangle|^n \, \nu(d\gamma) < \infty \quad \text{for any} \quad f \in C_0(X).$$

The class of all such measures will be denoted by $\mathfrak{M}^n(\Gamma_X)$.

Theorem 2. *We have the inclusion $\nu_\theta \in \mathfrak{M}^n(\Gamma_X)$ for any $n \in \mathbb{N}$.*

Proof. The result follows from Lemma 2 applied to the projection θ_k of the measure θ onto $X_k = X$ (which is independent of $k \in \mathbb{Z}^d$) and multiple Cauchy-Schwarz inequality. □

4. Integration by parts formula on $X^{\mathbb{Z}^d}$

In this section, we first recall main definitions related to the integration by parts (IBP) formula on the space $X^{\mathbb{Z}^d}$ (following [1]). Next, we prove the IBP formula for a special class of vector fields (needed in the next section).

Let us denote by $\mathcal{F}C(X^{\mathbb{Z}^d})$ the set of functions $f : X^{\mathbb{Z}^d} \to \mathbb{R}$ of the form

$$f(\mathbf{x}) = f_N(x_{m_1}, \ldots, x_{m_N}), \quad \mathbf{x} = (x_k)_{k\in\mathbb{Z}^d} \in X^{\mathbb{Z}^d},$$

for some $N \in \mathbb{N}$, $m_1, \ldots, m_N \in \mathbb{Z}^d$, and f_N is a bounded smooth function on $\mathbb{R}^N$ (which may depend on f). For $f \in \mathcal{F}C(X^{\mathbb{Z}^d})$ define the gradient $\nabla f(\mathbf{x})$ by the formula

$$X^{\mathbb{Z}^d} \ni \mathbf{x} \longmapsto \nabla f(\mathbf{x}) = (\nabla_k f(\mathbf{x}))_{k\,\in\,\mathbb{Z}^d} \in X_0^{\mathbb{Z}^d},$$

where

$$\nabla_k f(\mathbf{x}) = \frac{\partial}{\partial x_k} f_N(x_{m_1}, \ldots, x_{m_N}).$$

We assume that the measure θ satisfies the IBP formula

$$\int_{\mathcal{H}_-} (\nabla f(\mathbf{x}), \phi)_0 \, \theta(d\mathbf{x}) = -\int_{\mathcal{H}_-} f(\mathbf{x}) \beta_\theta^\phi(\mathbf{x}) \theta(d\mathbf{x}) \tag{8}$$

for any $\phi \in X_0^{\mathbb{Z}^d}$ and $f \in \mathcal{F}C(X^{\mathbb{Z}^d})$, where $\beta_\theta^\phi \in L^1(\mathcal{H}_-, \theta)$ is the logarithmic derivative of the measure θ in the direction of ϕ. It has the form $\beta_\theta^\phi(\mathbf{x}) = (\beta_\theta(\mathbf{x}), \phi)_0$, where $\beta_\theta : \mathcal{H}_- \to \mathcal{H}_-$ is the vector logarithmic derivative of θ. We assume that it satisfies the condition

$$\int_{\mathcal{H}_-} \|\beta_\theta(\mathbf{x})\|_-^4 \, \theta(d\mathbf{x}) < \infty. \tag{9}$$

It is known that IBP formula (8) can be extended to non-constant vector fields $V \in C_b^1(\mathcal{H}_-, \mathcal{H}_+)$ (see e.g. [6]; here C_b^1 stands as usual for "bounded continuously differentiable"). The logarithmic derivative takes the form

$$\beta_\theta^V(\mathbf{x}) = (\beta_\theta(\mathbf{x}), V(\mathbf{x}))_0 + \mathbf{div} V(\mathbf{x}),$$

where

$$\mathbf{div} V(\mathbf{x}) := \mathrm{Tr} V'(\mathbf{x}) = \sum_{k\in\mathbb{Z}^d} \mathrm{div}_k V_k(\mathbf{x}),$$

and $\mathrm{div}_k V_k$ is the divergence of the k-th component V_k of V with respect to x_k.

In what follows we would like to establish the IBP formula for a special class of vector fields on $\mathcal{H}_-$. Let $v \in C_0^\infty(X)$ and define a map $\widehat{v} : X^{\mathbb{Z}^d} \to X^{\mathbb{Z}^d}$ by setting

$$\widehat{v}_k(\mathbf{x}) = v(x_k + \alpha(k))_{k\in\mathbb{Z}^d},$$

where $\alpha(k) = |k|^{d-1}k$. It is clear that $\widehat{v} \notin C_b^1(\mathcal{H}_-, \mathcal{H}_+)$. However, it possesses the following regularity properties.

Proposition 1. *We have the following:*
(i) $\widehat{v}:\mathcal{H}_-\to X_0^{\mathbb{Z}^d}$ *and* $\int_{\mathcal{H}_-}\|\widehat{v}(\mathbf{x})\|_+^2\,\theta(d\mathbf{x})<\infty$,
(ii) $\mathbf{div}\widehat{v}\;(\mathbf{x})<\infty$, $\mathbf{x}\in\mathcal{H}_-$, *and* $\int_{\mathcal{H}_-}|\mathbf{div}\widehat{v}(\mathbf{x})|\;\theta(d\mathbf{x})<\infty$.

Let us introduce the Sobolev space $H^{1,2}(\mathcal{H}_-,\theta)$ as a completion of the space $\mathcal{F}C(X^{\mathbb{Z}^d})$ in the norm $\|\cdot\|_{1,2}$ given by the formula

$$\|\mathbf{h}\|_{1,2}^2=\int_{\mathcal{H}_-}|\mathbf{h}(\mathbf{x})|^2\;\theta(d\mathbf{x})+\int_{\mathcal{H}_-}\|\nabla\mathbf{h}(\mathbf{x})\|_0^2\;\theta(d\mathbf{x}).$$

Theorem 3. *(i) For the vector field* $\widehat{v}$*, the following IBP formula holds:*

$$\int_{\mathcal{H}_-}(\nabla f(\mathbf{x}),\widehat{v}(\mathbf{x}))_0\,\theta(d\mathbf{x})=-\int_{\mathcal{H}_-}f(\mathbf{x})\,\beta_\theta^{\widehat{v}}(\mathbf{x})\theta(d\mathbf{x}),\quad f\in\mathcal{F}C_b^\infty(X^{\mathbb{Z}^d}),\tag{10}$$

where the logarithmic derivative $\beta_\theta^{\widehat{v}}(\mathbf{x})$ *of the measure* θ *in the direction of* $\widehat{v}$ *has the form*

$$\beta_\theta^{\widehat{v}}(\mathbf{x})=(\beta_\theta(\mathbf{x}),\widehat{v}(\mathbf{x}))_0+\mathbf{div}\widehat{v}(\mathbf{x}).$$

Moreover, $\beta_\theta^{\widehat{v}}\in L^2(\mathcal{H}_-,\theta)$.
(ii) IBP formula (10) can be extended to any $f\in H^{1,2}(\mathcal{H}_-,\theta)$.

Proof. The first statement can be verified using the approximation of $\widehat{v}$ by cut-off vector fields $\widehat{v}^{(N)}\in C_b^1(\mathcal{H}_-,\mathcal{H}_+)$ given by $\widehat{v}_k^{(N)}=\widehat{v}_k$, $|k|\le N$, and $\widehat{v}_k^{(N)}=0$ otherwise. The second statement is standard. □

5. Integration by parts formula on Γ_X

The aim of this section is to prove an IBP formula for the measure ν_θ on Γ_X. First we need to introduce certain classes of functions on Γ_X. For $\gamma\in\Gamma_X$ and $x\in\gamma$, denote by $\mathcal{O}_{\gamma,x}$ an open neighborhood of x in X such that $\mathcal{O}_{\gamma,x}\cap\gamma=x$. For any measurable function $F:\Gamma_X\to\mathbb{R}$, define the function $F_x(\gamma,\boldsymbol{.}):\mathcal{O}_{\gamma,x}\to\mathbb{R}$ by

$$F_x(\gamma,y):=F((\gamma\diagdown x)\cup y)$$

and set

$$\nabla_xF(\gamma):=\nabla F_x(\gamma,y)|_{y=x}\,,\quad x\in X,$$

provided $F_x(\gamma,\boldsymbol{.})$ is differentiable at x. Here ∇ means usual gradient on $X=\mathbb{R}^d$.

Denote by $\mathcal{F}C(\Gamma_X)$ the class of functions on Γ_X of the form

$$F(\gamma)=f\big(\langle\phi_1,\gamma\rangle,\dots,\langle\phi_k,\gamma\rangle\big)\,,\quad\gamma\in\Gamma_X,$$

where $k\in\mathbb{N}$, $f\in C_b^\infty(\mathbb{R}^k)$, and $\phi_1,\dots,\phi_k\in C_0^\infty(X)$.

For $F:\Gamma_X\to\mathbb{R}$, define the function $\mathcal{I}F:=F\circ\mathfrak{p}$, that is

$$\mathcal{I}F(\mathbf{x})\;=\;F(\mathfrak{p}(\mathbf{x})),\quad\mathbf{x}\in\mathcal{H}_-,\tag{11}$$

where $\mathfrak{p}:\mathcal{H}_-\to\Gamma_X$ is the map defined by the formula (3). Clearly, $\mathcal{I}F$ is a function on $\mathcal{H}_-$. It is immediate that the operator $\mathcal{I}$ is an isometry from $L^2(\Gamma_X,\nu_\theta)$ to $L^2(\mathcal{H}_-,\theta)$.

Remark 1. *The operator* $\mathcal{I}$ *is not an isomorphism. Indeed, the function* $\mathcal{I}F(\mathbf{x})$ *is symmetric with respect to permutations of the components of* $\mathbf{x}=(x_k)_{k\in\mathbb{Z}^d}$*, which implies that* $\mathcal{I}:L^2(\Gamma_X,\nu_\theta)\to L^2(X^{\mathbb{Z}^d},\theta)$ *is not surjective.*

A direct check shows that the following result holds.

Lemma 3. *Let* $F\in\mathcal{F}C(\Gamma_X)$*. Then we have* $\mathcal{I}F\in H^{1,2}(\mathcal{H}_-,\theta)$.

Let $\mathcal{I}^*:L^2(X^{\mathbb{Z}^d},\theta)\to L^2(\Gamma_X,\nu_\theta)$ be the adjoint operator of the isometry $\mathcal{I}$. We are now in a position to state the main result of this section.

Theorem 4. *Let $v \in \mathrm{Vect}_0(X)$ and $F \in \mathcal{F}C(\Gamma_X)$. Then the measure ν_θ on Γ_X given by* (7) *satisfies the IBP formula*

$$\int_{\Gamma_X} \sum_{x\in\gamma} \nabla_x F(\gamma) \cdot v(x)\nu_\theta(d\gamma) = \int_{\Gamma_X} F(\gamma)\beta^v_{\nu_\theta}(\gamma)\nu_\theta(d\gamma), \tag{12}$$

where

$$\beta^v_{\nu_\theta} := \mathcal{I}^* \beta^{\widehat{v}}_\theta \in L^2(\Gamma_X, \nu_\theta).$$

Proof. The result follows from Theorem 3 applied to the function $\mathcal{I}F$. □

6. Logarithmic Sobolev inequality

Let us introduce a pre-Dirichlet form $\mathcal{E}_{\nu_\theta}$ associated with the measure ν_θ, defined on functions $F \in \mathcal{F}C(\Gamma_X) \subset L^2(\Gamma_X, \nu_\theta)$ by the expression

$$\mathcal{E}_{\nu_\theta}(F, F) = \int_{\Gamma_X} \sum_{x\in\gamma} |\nabla_x F(\gamma)|^2 \,\nu_\theta(d\gamma).$$

It follows from the general theory of (pre-)Dirichlet forms associated with measures from the class $\mathfrak{M}^2(\Gamma_X)$ (see [3], [13]) that satisfy IBP formula (12) that:

- the pre-Dirichlet form $\mathcal{E}_{\nu_\theta}$ is well-defined, i.e. $\mathcal{E}_{\nu_\theta}(F, F) < \infty$ for all $F \in \mathcal{F}C(\Gamma_X)$;
- $\mathcal{E}_{\nu_\theta}$ is closable and its closure is a quasi-regular local Dirichlet form on $\ddot{\Gamma}_X$;
- there exists a conservative diffusion process $\mathbf{X} = (\mathbf{X}_t, t \geqslant 0)$ on $\ddot{\Gamma}_X$, properly associated with the Dirichlet form $\mathcal{E}_{\nu_\theta}$.

Let θ be a probability measure on $\mathcal{H}_-$ satisfying conditions (1)–(3) of Section 3, IBP formula (8) and condition (9). Consider its pre-Dirichlet form $\mathcal{E}_\theta$ defined by the formula

$$\mathcal{E}_\theta(f, f) = \int_{X^{\mathbb{Z}^d}} \|\nabla f(\mathbf{x})\|_0^2 \,\theta(d\mathbf{x}),$$

where $f \in \mathcal{F}C(X^{\mathbb{Z}^d})$. The form $(\mathcal{E}_\theta, \mathcal{F}C(X^{\mathbb{Z}^d}))$ is closable (see [1]). We denote its closure by $(\mathcal{E}_\theta, D(\mathcal{E}_\theta))$. By the definition, $D(\mathcal{E}_\theta)$ is the completion of $\mathcal{F}C(X^{\mathbb{Z}^d})$ in the norm $\|\cdot\|_{\mathcal{E}_\theta}$ given by the formula

$$\begin{aligned}\|f\|^2_{\mathcal{E}_\theta} &:= \int_{\mathcal{H}_-} f^2(\mathbf{x})\theta(d\mathbf{x}) + \int_{\mathcal{H}_-} \|\nabla f(\mathbf{x})\|_0^2\theta(d\mathbf{x}) \\ &= \|f\|^2_{H^{1,2}(\mathcal{H}_-,\theta)},\end{aligned}$$

and therefore

$$D(\mathcal{E}_\theta) = H^{1,2}(\mathcal{H}_-, \theta).$$

Theorem 5. *For any $F \in D(\mathcal{E}_{\nu_\theta})$ we have $\mathcal{I}F \in D(\mathcal{E}_\theta)$ and*

$$\mathcal{E}_{\nu_\theta}(F, F) = \mathcal{E}_\theta(\mathcal{I}F, \mathcal{I}F),$$

where $\mathcal{I}F$ is defined in (11).

Proof. The statement immediately follows from Lemma 3 and the definition of the forms $\mathcal{E}_{\nu_\theta}$ and $\mathcal{E}_\theta$. □

It is said the measure θ satisfies the log-Sobolev inequality if

$$C_{LS}\, \mathcal{E}_\theta(f, f) \geqslant \int_{\mathcal{H}_-} |f(\mathbf{x})|^2 \,\log |f(\mathbf{x})|\theta(d\mathbf{x}) - \|f\|^2_{L^2(\mathcal{H}_-,\theta)} \,\log\|f\|_{L^2(\mathcal{H}_-,\theta)} \tag{13}$$

for some constant $C_{LS} > 0$ and any $f \in D(\mathcal{E}_\theta)$. This is an important inequality which has been extensively studied (see e.g. [10], [11] and references given there).

Theorem 6. *Let us assume that θ satisfies log-Sobolev inequality (13). Then the measure ν_θ satisfies the log-Sobolev inequality with the same constant C_{LS}, i.e.*

$$C_{LS}\,\mathcal{E}_{\nu_\theta}(F,F) \geqslant \int_{\Gamma_X} |F(\gamma)|^2 \,\log|F(\gamma)|\nu_\theta(d\gamma) - \|F\|^2_{L^2(\Gamma_X,\nu_\theta)} \log\|F\|_{L^2(\Gamma_X,\nu\theta)},$$

for any $F \in D(\mathcal{E}_{\nu_\theta})$.

Proof. The result follows from Theorem 5 and LSI (13) applied to the function $\mathcal{I}F$. □

7. Examples

Here we give some examples of measures θ on $X^{\mathbb{Z}^d}$ that satisfy conditions (4)–(6), (9) and (13) and therefore give rise to push-forward measures ν_θ satisfying IBP formula (12) and log-Sobolev inequality (6).

Example 1. ***Product Measures***. *Let μ be a probability measure on X that is absolutely continuous with respect to the Lebesgue measure with density e^{-V} and assume that the function $V : X \to \mathbb{R}$ has polynomial growth and its second derivative satisfies the bound $V''(x) \geq C\,\mathrm{Id}$ for some constant $C > 0$. Set $\theta := \bigotimes_{k\in\mathbb{Z}^d} \mu_k,\ \mu_k = \mu$. Conditions (4)–(6) and (9) can be verified by a direct calculation. The Bakry-Emery criterion implies that μ satisfies the log-Sobolev inequality with $C_{LS} = C$, which implies that θ satisfies log-Sobolev inequality (13) with the same constant (see* [10]*).*

Example 2. ***Gaussian Measures.*** *Consider a bounded positive linear operator A in $\mathcal{H}_0$ such that $AS_k = S_kA$ for all $k \in \mathbb{Z}^d$. Let θ be the Gaussian measure with correlation operator A^{-1} and zero mean. Conditions (4)–(6) can be checked by a direct calculation. For the proof of the integration by parts formula (8) and condition (9) see* [4] *and* [6]*. Log-Sobolev inequality (13) holds due to* [1] *and* [10]*.*

Example 3. ***Gibbs Measures.*** *Let $V : X \to \mathbb{R}^1$ be a continuous function satisfying the quadratic growth estimate $V(x) \geq a\,|x|^2 - b$, $a, b > 0$, and define θ to be a Gibbs measure with energy function $E(\mathbf{x}) = (A\mathbf{x},\mathbf{x})_0 + \varepsilon V(\mathbf{x})$, $\mathbf{x} \in X_0^{\mathbb{Z}^d}$. Conditions of the existence of θ are well-known (see* [1] *and references therein); in particular, they are fulfilled for ε small enough. Once again, conditions (4)–(6) can be checked directly. The integration by parts formula (8), condition (9) and log-Sobolev inequality (13) (for small ε) have been proved in* [1]*.*

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