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ON FINITE DIMENSIONAL LIE ALGEBRAS OF PLANAR VECTOR FIELDS WITH RATIONAL COEFFICIENTS

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Abstract. The Lie algebra of planar vector fields with coefficients from the field of rational functions over an algebraically closed field of characteristic zero is considered. We find all finite-dimensional Lie algebras that can be realized as subalgebras of this algebra.

Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $R = \mathbb{K}(x, y)$ be the field of rational functions. Recall that a $\mathbb{K}$-linear mapping $D : R \to R$ is called a $\mathbb{K}$-derivation if $D(fg) = D(f)g + fD(g)$ for all $f, g \in R$. We denote by $\widetilde{W_2}(\mathbb{K})$ the Lie algebra of all $\mathbb{K}$-derivations of R, this algebra is a two-dimensional vector space over R, its basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ will be called standard. In geometric terms, a derivation D is a vector field with rational coefficients and $\widetilde{W_2}(\mathbb{K})$ is the Lie algebra of all vector fields on $\mathbb{K}^2$ with rational coefficients. The Lie algebra $\widetilde{W_2}(\mathbb{K})$ is closely connected with the automorphism group $\mathrm{Aut}(R)$ of the field R (for example if D is a locally nilpotent derivation of R, then $\exp D$ is an automorphism of R). The group $\mathrm{Aut}(R)$ was intensively studied by many authors (see, for example [3]). A question about finite subgroups of $\mathrm{Aut}(R)$ is of special interest, the description of such subgroups was recently completed by I. Dolgachev and V. Iskovskikh [3]. So, it is of interest to study finite dimensional subalgebras of the Lie algebra $\mathrm{Der}(R) = \widetilde{W_2}(\mathbb{K})$ which corresponds in some sense to $\mathrm{Aut}(R)$.

In this paper, we give a description of finite dimensional subalgebras of $\widetilde{W_2}(\mathbb{K})$ up to isomorphism as Lie algebras using only algebraic tools. The advantage of this approach is that many results of the paper can be transferred on Lie algebras of derivations of commutative and associative algebras over fields (in [8] we have obtained estimations for derived length of solvable Lie algebras of derivations in a similar way). Such a description over the field of complex numbers can also be obtained using analytical and geometric methods; it can be deduced from results of S. Lie (see [7], 71–73). There are many papers devoted to such subalgebras, see for example [2], [4], [10], [9], [6], [11]. The main result of the paper is Theorem 1 where all types of finite dimensional subalgebras of $\widetilde{W_2}(\mathbb{K})$ are listed. From this description one can easily obtain all possible types of finite dimensional subalgebras of the Lie algebra $W_2(\mathbb{K}) = \mathrm{Der}\mathbb{K}[x, y]$ (up to isomorphism as Lie algebras).

We use standard notations, the ground field $\mathbb{K}$ is algebraically closed of characteristic zero (some results are valid for any field of characteristic 0). If $D_1, \ldots, D_n$ are elements of $\widetilde{W_2}(\mathbb{K})$, then we denote by $\mathbb{K}\langle D_1, \ldots, D_n\rangle$ or simply $\langle D_1, \ldots, D_n\rangle$ the linear span of elements $D_1, \ldots, D_n$ over the field $\mathbb{K}$. The field $\mathbb{K}(x, y)$ of rational functions will be denoted by R, every nonzero $\mathbb{K}$-subspace of $\widetilde{W_2}(\mathbb{K})$ has rank 1 or 2 over R as a system of elements of the two-dimensional vector space $\widetilde{W_2}(\mathbb{K})$ over R.

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1. Preliminaries

Lemma 1. *Suppose that* $D_1, D_2 \in \widetilde{W}_2(\mathbb{K})$. *Then*

(1) *For any* $a, b \in R$ *it holds* $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.

(2) *If* D_1, D_2 *are linearly independent over* R *and* $D_1(c) = D_2(c) = 0$ *for some* $c \in R$, *then* $c \in \mathbb{K}$.

Proof. 1. Straightforward calculation.

2. Note that $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \in \widetilde{W}_2(\mathbb{K})$ are linear combinations of D_1 and D_2 with coefficients in R. Then $\frac{\partial}{\partial x}(c) = \frac{\partial}{\partial x}(c) = 0$ which implies $c \in \mathbb{K}$. □

Lemma 2. *Let* L *be a finite dimensional subalgebra of the Lie algebra* $\widetilde{W}_2(\mathbb{K})$. *If* L *is of rank* 1 *over* R, *then there exists an element* $D_1 \in \widetilde{W}_2(\mathbb{K})$ *such that* L *is one of the following algebras:*

(1) $L = \langle D_1, a_1D_1, \dots, a_nD_1\rangle$ *for some* $a_i \in R$ *such that* $D_1(a_i) = 0$ *for all* i. *The algebra* L *is abelian.*

(2) $L = \langle D_1, a_1D_1, \dots, a_{n-1}D_1, bD_1\rangle$ *for some* $a_i, b \in R$ *such that* $D_1(a_i) = 0$ *for all* i, $D_1(b) = -1$. L *is metabelian.*

(3) $L = \langle D_1, -a^2D_1, -2aD_1\rangle$ *for some* $a \in R$ *with* $D_1(a) = 1$. *The algebra* L *is isomorphic to* $sl_2(\mathbb{K})$.

Proof. Replacing the polynomial ring $\mathbb{K}[x, y]$ by the field $R = \mathbb{K}(x, y)$ in the proof of Theorem 1 in [1] one can show that a finite dimensional subalgebra of rank 1 over R from $\widetilde{W}_2(\mathbb{K})$ is either abelian, or metabelian of the form $L = \langle b\rangle \ltimes A, [b, a] = a$ for all $a \in A$ with abelian A, or $L \simeq sl_2(\mathbb{K})$. Consider all these cases. If L is abelian with $\mathbb{K}$-basis $\{D_1, a_1D_1, \dots, a_nD_1\}$ then $[D_1, a_iD_1] = 0 = D_1(a_i)D_1$ for all i. Hence $D_1(a_i) = 0$ for all i and L is of type 1. Let $L = \langle b\rangle \ltimes A$ with abelian subalgebra $A = \{D_1, a_1D_1, \dots, a_{n-1}D_1\}$. Then as above $D_1(a_i) = 0$ for all i and since $[bD_1, D_1] = D_1$ we get $D_1(b) = -1$. Thus L is of type 2. Finally, let $L \simeq sl_2(\mathbb{K})$. Choose the standard basis $\{e, f, h\}$ of L over $\mathbb{K}$. Without loss of generality we may put $e = D_1, f = bD_1, h = aD_1$ for some $a, b \in R$. Then

$$[aD_1, D_1] = 2D_1, \quad [aD_1, bD_1] = -2bD_1, \quad [D_1, bD_1] = aD_1,$$

so using Lemma 1 we get from the first equality that $D_1(a) = -2$. The second equality implies $aD_1(b) + 2b = -2b$ and therefore $D_1(b) = -4b/a$. The third equality yields $D_1(b) = a$. So, $a = -4b/a$ and $a^2 = -4b$, i.e. $b = -a^2/4$. We get the basis $\{D_1, -a^2/4D_1, aD_1\}$ of the algebra L. Replacing here a by $-a/2$ we obtain a basis $\{D_1, -a^2D_1, -2aD_1\}$ where $D_1(a) = 1$. □

Remark 1. One can easily point out realizations for Lie algebras from the previous Lemma: 1. $D_1 = \frac{\partial}{\partial x}, a_i = y^i, i = 1, \dots, n$; 2. $D_1 = \frac{\partial}{\partial x}, a_i = y^i, i = 1, \dots, n-1, b = -x$; 3. $D_1 = \frac{\partial}{\partial x}, a = x$.

Lemma 3. *Let* $L \neq 0$ *be a finite dimensional solvable subalgebra of the Lie algebra* $\widetilde{W}_2(\mathbb{K})$ *and let* $\langle D_1\rangle$ *be its arbitrary one-dimensional ideal. Then*

(1) *The set* $I = RD_1 \cap L$ *is an ideal of* L.

(2) $\dim L/I \le 2$ *and if* $\dim L/I = 2$, *then the quotient algebra* L/I *is nonabelian.*

(3) *If* $\dim L \ge 5$, *then the ideal* I *contains all ideals of rank 1 over* R *from* L.

Proof. 1. Take any element $D \in L$. Since $\langle D_1\rangle$ is an ideal of L we have $[D, D_1] = \lambda D_1$ for some $\lambda \in \mathbb{K}$ depending on D. Then for any element $aD_1 \in I$ it holds

$$[D, aD_1] = D(a)D_1 + a[D, D_1] = (D(a) + \lambda a)D_1 \in I.$$

Therefore I is an ideal of L.

2. We can obviously assume that $I \neq L$. Choose a one-dimensional ideal $\langle D_2 + I\rangle$ of the quotient algebra L/I. As $D_2 \notin I$ the elements D_1, D_2 are linearly independent over R. It suffices to show that the ideal $J = I + \langle D_2\rangle$ of the algebra L is of codimension ≤ 1 in L. Take arbitrary elements $D_3 = a_3D_1 + b_3D_2$, $D_4 = a_4D_1 + b_4D_2$ with $a_3, a_4, b_3, b_4 \in R$ from the set $L \setminus J$. Since

$$[D_1, D_3] = D_1(a_3)D_1 + D_1(b_3)D_2 + b_3\lambda D_1 \in \langle D_1\rangle$$

(here $[D_2, D_1] = \lambda D_1$) we get $D_1(b_3) = 0$. Analogously from the relation $[D_2, D_3] \in J$ we have $D_2(b_3) = c_3 \in \mathbb{K}$. Similar calculations yield $D_1(b_4) = 0, D_2(b_4) = c_4 \in \mathbb{K}$. It can be easily shown that $c_3 \neq 0, c_4 \neq 0$. Indeed, let to the contrary $c_3 = 0$. Then the equalities $D_1(b_3) = 0, D_2(b_3) = c_3 = 0$ imply by Lemma 1 that $b_3 \in \mathbb{K}$. This means that $a_3D_1 \in L$ and as $a_3D_1 \in I$ we get $D_3 \in J$. The latter contradicts to the choice of D_3. Analogously one can show that $c_4 \neq 0$. Consider the element $c_4D_3 - c_3D_4$ of L and write it in the form

$$(c_4a_3 - c_3a_4)D_1 + (c_4b_3 - c_3b_4)D_2 = r_1D_1 + r_2D_2.$$

Straightforward calculation shows that $D_2(r_2) = 0$. As also $D_1(r_2) = 0$, the element r_2 belongs to $\mathbb{K}$ by Lemma 1. Therefore $c_4D_3 - c_3D_4 \in J$ and D_3, D_4 are linearly dependent over J, i.e. $\dim L/J \leq 1$.

Now let $\dim L/I = 2$ and $\{D_2 + I, aD_1 + bD_2 + I\}$ be a basis of L/I. Suppose that L/I is abelian. Then $[D_2, aD_1 + bD_2] \in I$ and therefore $D_2(b) = 0$. From the relation $[aD_1 + bD_2, D_1] \in \langle D_1\rangle$ it follows that $D_1(b) = 0$. But then Lemma 1 yields $b \in \mathbb{K}$ which implies $aD_1 \in L$. This means that $aD_1 \in I$ and $aD_1 + bD_2 \in I + \langle D_2\rangle$. The latter is impossible because the elements D_2 and $aD_1 + bD_2$ are linearly independent over I. This contradiction shows that L/I is nonabelian.

3. Finally, let $\dim L \geq 5$, $I = RD_1 \cap L$ and $T = RD_2 \cap L$ for some ideals $\langle D_1\rangle$ and $\langle D_2\rangle$. Suppose that elements D_1 and D_2 linearly independent over R. Since $\dim L/I \leq 2$ and $dim L/T \leq 2$ (by the proved above) and $I \cap T = 0$ we get $\dim L \leq 4$ which contradicts to our assumption. Thus, I contains all ideals of rank 1 over R. □

We need also some elementary properties of rational functions in a single variable. These properties seem to be known but having no reference we supply them with complete proofs. For a rational function $\varphi \in \mathbb{K}(t)$ we will denote $\varphi' = \frac{d\varphi}{dt}$. If $p(t) \in \mathbb{K}[t]$ is an irreducible polynomial, then $\mathrm{ord}_p\varphi$ denotes as usually the integer α from the decomposition of φ into the product of the form $\varphi = p^\alpha\psi$, where neither numerator nor the denominator of ψ is divisible by p.

Lemma 4. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Then*

(1) *If $\varphi(t) \in \mathbb{K}(t)\backslash\mathbb{K}$, then there does not exist any function $\psi \in \mathbb{K}(t)$ such that $\psi' = \frac{\varphi'}{\varphi}$.*

(2) *Let $\varphi, \psi \in \mathbb{K}(t)\backslash\mathbb{K}$ be such functions that $\mu\varphi'\psi - \varphi\psi' = 0$ for some $\mu \in \mathbb{K}$. Then $\mu \in \mathbb{Q}$, $\mu = \frac{m}{n}$, and $\varphi^m = c\psi^n$ for some $c \in \mathbb{K}$. Moreover, there exists $\theta \in \mathbb{K}(t)$ such that $\varphi = c_1\theta^s$, $\psi = c_2\theta^t$ for some $c_1, c_2 \in \mathbb{K}$, $s, t \in \mathbb{Z}$.*

Proof. 1. Suppose on the contrary that there exists $\psi \in \mathbb{K}(t)$ such that $\psi' = \frac{\varphi'}{\varphi}$. Let $p \in \mathbb{K}[t]$ be an irreducible polynomial such that $\mathrm{ord}_p(\varphi) \neq 0$. Put $\alpha = \mathrm{ord}_p(\varphi)$. Then $\varphi = p^\alpha q$ and $\varphi' = \alpha p^{\alpha-1}p'q + p^\alpha q'$. Therefore

$$\frac{\varphi'}{\varphi} = \frac{\alpha p'p^{\alpha-1}q + p^\alpha q'}{p^\alpha q} = \frac{\alpha qp' + pq'}{pq}.$$

Since $\mathrm{ord}_p(\alpha qp' + pq') = 0$ it holds $\mathrm{ord}_p\left(\frac{\varphi'}{\varphi}\right) = -1$ (note that $\mathrm{ord}_p(q) = 0$). Now put $\beta = \mathrm{ord}_p(\psi)$, $\psi = p^\beta r$. Then $\psi' = \beta p'p^{\beta-1}r + p^\beta r'$. If $\beta = 0$, then $\psi' = r'$ and

$\mathrm{ord}_p(\psi') = \mathrm{ord}_p(r') \geq 0$. Suppose that $\beta \neq 0$. Then

$$\mathrm{ord}_p(\psi') = \mathrm{ord}_p(\beta p' p^{\beta-1} r + p^{\beta} r') = \mathrm{ord}_p(\beta p' p^{\beta-1} r) = \beta - 1.$$

Therefore in any case $\mathrm{ord}_p \psi' \neq -1$, which contradicts to the equality $\mathrm{ord}_p \left(\frac{\varphi'}{\varphi}\right) = -1$. Hence there does not exist such a polynomial ψ that $\psi' = \frac{\varphi'}{\varphi}$.

2. Take any functions φ, ψ from $\mathbb{K}(t) \backslash \mathbb{K}$ satisfying the condition

$$\mu \varphi' \psi - \varphi \psi' = 0. \tag{1}$$

It can be easily shown that there exists a point $t_0 \in \mathbb{K}$ such that $\mathrm{ord}_{t-t_0} \varphi \neq 0$ (because the field $\mathbb{K}$ is algebraically closed). Without loss of generality we can assume that the field $\mathbb{K}(t)$ is embedded to the field $\mathbb{K}((t))$ of Laurent series at the point t_0. Put

$$\varphi = \sum_{i=m}^{\infty} \alpha_i (t - t_0)^i, \quad \psi = \sum_{i=n}^{\infty} \beta_i (t - t_0)^i, \quad \text{where} \quad m, n \in \mathbb{Z}, \quad \alpha_m \beta_n \neq 0.$$

Since $\mathrm{ord}_{t-t_0} \varphi \neq 0$, it holds $m \neq 0$, We can assume that $\alpha_m = \beta_n = 1$, because the equation (1) is homogeneous. Computing coefficients at t^{m+n-1} in both sides of the equation (1) we obtain $\mu m = n$. Therefore $\mu = n/m \in \mathbb{Q}$. Further,

$$\left(\frac{\varphi^n}{\psi^m}\right)' = \frac{n\varphi^{n-1}\varphi'\psi^m - m\varphi^n\psi^{m-1}\psi'}{\psi^{2m}} = \frac{\varphi^{n-1}\psi^{m-1}(n\varphi'\psi - m\varphi\psi')}{\psi^{2m}} = 0,$$

because $n\varphi'\psi - m\varphi\psi' = m(\mu\varphi'\psi - \varphi\psi') = 0$. Hence, $\frac{\varphi^n}{\psi^m} \in \mathbb{K}$ i.e. $\varphi^n = c\psi^m$ for some $c \in \mathbb{K}$.

The functions φ and ψ can be written as products of irreducible factors with (nonzero) integer powers

$$\varphi = \prod_{i=1}^{s} u_i^{k_i}, \quad \psi = \prod_{j=1}^{k} v_i^{l_i}.$$

Using the equality $\varphi^n = c\psi^m$ we get $k = s$ and after renumbering the factors we can write down $u_i = \gamma_i v_i$ for some $\gamma_i \in \mathbb{K}$. Hence we have

$$\left(\prod_{i=1}^{k} u_i^{k_i}\right)^n = c \left(\prod_{i=1}^{k} (\gamma_i u_i)^{l_i}\right)^m.$$

This equality implies that $nk_i = ml_i$ for all $i = 1, \ldots, k$. Denote $d = \gcd(m, n)$ and $m = m_1 d$, $n = n_1 d$. We obtain equalities $n_1 d k_i = m_1 d l_i$, $i = 1, \ldots, k$, and therefore $n_1 k_i = m_1 l_i$. Since $\gcd(m_1, n_1) = 1$ we obtain that l_i is divisible by n_1, k_i is divisible by m_1, $i = 1, \ldots, k$. Denote $\frac{l_i}{n_1} = \frac{k_i}{m_1} = r_i$ and $\theta = \prod_{i=1}^{s} u_i^{r_i}$. Then $\varphi = \theta^{m_1}$, $c_1 \psi = \theta^{n_1}$ for some $c_1 \in \mathbb{K}^*$. This completes the proof of Lemma. □

Lemma 5. *Let D_1 and D_2 be elements of $\widetilde{W_2}(\mathbb{K})$ linearly independent over R such that $[D_2, D_1] = \nu D_1$ for some $\nu \in \mathbb{K}$. Let b_1, b_2 be linearly independent over $\mathbb{K}$ elements of $R \backslash \mathbb{K}$ such that $D_1(b_i) = 0, i = 1, 2$. Then*

(1) If $[D_2, b_i D_1] = \lambda_i b_i D_1$ for some $\lambda_i \in \mathbb{K}, i = 1, 2$, then $\lambda_1 \neq \lambda_2$. If $\lambda_1 \neq \nu$, then $\frac{\lambda_2 - \nu}{\lambda_1 - \nu} \in \mathbb{Q}$.

(2) If $[D_2, b_1 D_1] = \lambda b_1 D_1, [D_2, b_2 D_1] = \lambda b_2 D_1 + b_1 D_1$ for some $\lambda \in \mathbb{K}$, then $\lambda = \nu$.

Proof. 1. Using the condition $[D_2, b_i D_1] = \lambda_i b_i D_1$ we get

$$D_2(b_i) = (\lambda_i - \nu) b_i, \quad i = 1, 2. \tag{2}$$

Suppose that $\lambda_1 = \lambda_2$. Then $D_2\left(\frac{b_1}{b_2}\right) = \frac{D_2(b_1) b_2 - b_1 D_2(b_2)}{b_2^2} = 0$. Besides, $D_1(\frac{b_1}{b_2}) = 0$ by conditions of Lemma. Then using linear independence of elements D_1, D_2 we obtain by Lemma 1 the inclusion $\frac{b_1}{b_2} \in \mathbb{K}$. The latter is impossible because of linear independence of elements b_1, b_2 over $\mathbb{K}$. Hence $\lambda_1 \neq \lambda_2$.

Let now $\lambda_1 \neq \nu$. Since $b_1, b_2 \in R\backslash\mathbb{K}$, the subfield $\ker(D_1)$ of R is of transcendence degree 1 over $\mathbb{K}$ (it is obvious that this degree cannot be equal to 2). Hence $\ker D_1$ is generated by a single element (see, for example, [12], Th. 3). Denote this element by θ. Then $b_1 = \varphi_1(\theta)$, $b_2 = \varphi_2(\theta)$ for some rational functions $\varphi_1(t)$, $\varphi_2(t) \in \mathbb{K}(t)$. Using the relation $[D_2, D_1] = \nu D_1$ we see that $D_2(\theta) \in \ker(D_1)$. Denote also $D_2(\theta) = f(\theta)$, $f \in \mathbb{K}(t)$. The conditions (2) imply

$$\varphi_1'(\theta)f(\theta) = (\lambda_1 - \nu)\varphi_1(t), \quad \varphi_2'(\theta)f(\theta) = (\lambda_2 - \nu)\varphi_2(\theta).$$

Since φ_i are not constants and $\lambda_1 - \nu \neq 0$ we have

$$\varphi_1\varphi_2' - \mu\varphi_1'\varphi_2 = 0, \quad \text{where} \quad \mu = \frac{\lambda_2 - \nu}{\lambda_1 - \nu}.$$

Now Lemma 4 yields the inclusion $\mu \in \mathbb{Q}$.

2. By the condition (2) of Lemma we have

$$D_2(b_1) = (\lambda - \nu)b_1, \quad D_2(b_2) = (\lambda - \nu)b_2 + b_1. \tag{3}$$

As above we can show that $b_1 = \psi_1(\theta)$, $b_2 = \psi_2(\theta)$, where θ is a generator of the subfield $\ker D_1$ and $D_2(\theta) = g(\theta)$ for some rational functions $\psi_1, \psi_2, g \in \mathbb{K}(t)$. Using (3) one can easily show that

$$\psi_1' g = (\lambda - \nu)\psi_1, \quad \psi_2' g = (\lambda - \nu)\psi_2 + \psi_1. \tag{4}$$

Since $b_1 \in R\backslash\mathbb{K}$ it holds $\psi_1' \neq 0$. The equality (4) implies the next relations

$$\frac{\psi_1'}{\psi_1} = \frac{(\lambda - \nu)\psi_2'}{(\lambda - \nu)\psi_2 + \psi_1} = \left(\frac{(\lambda - \nu)\psi_2}{\psi_1}\right)' \tag{5}$$

(note that $(\lambda - \nu)\psi_2 + \psi_1 \neq 0$ because ψ_1 and ψ_2 are linearly independent over $\mathbb{K}$). But the relation (5) is impossible if $\lambda \neq \nu$ by Lemma 4. This contradiction shows that $\lambda = \nu$. □

The next statement can be easily deduced from the theorem of S. Lie about solvable Lie algebras.

Lemma 6. *Let V be a finite dimensional vector space over the field $\mathbb{K}$ and T, S be linear operators on V. If $[T, S] = S$, then the operator S is nilpotent.*

2. Finite dimensional solvable subalgebras of $\widetilde{W}_2(\mathbb{K})$

Lemma 7. *Let L be a finite dimensional solvable subalgebra of rank 2 over R of $\widetilde{W}_2(\mathbb{K})$ and let $\langle D_1\rangle$ be its arbitrary one dimensional ideal. Denote $I = RD_1 \cap L$. If the ideal I is abelian, then there exists an element $D_2 \in L\backslash I$ such that L is one of the following algebras:*

(1) $L = \langle D_1, aD_1, \ldots, (a^n/n!)D_1, D_2\rangle$, where $a \in R$ such that $D_1(a) = 0, D_2(a) = 1, [D_2, D_1] = \lambda D_1$ and $\lambda = 0$ or $\lambda = 1$, $n \geq 1$. If $n = 0$, we put $L = \langle D_1, D_2\rangle$.

(2) $L = \langle D_1, a_1D_1, \ldots, a_nD_1, D_2\rangle$, where $a_i \in R, [D_2, D_1] = D_1, D_1(a_i) = 0, D_2(a_i) = \beta m_i a_i, m_i \in \mathbb{Z}$ for all $i, \beta \in \mathbb{K}^\star, m_i \neq m_j$ for $i \neq j$, $n \geq 1$.

(3) $L = \langle D_1, aD_1, \ldots, (a^n/n!)D_1, D_2, bD_1 + aD_2\rangle$, where $a, b \in R$ such that $D_1(a) = 0, D_1(b) = \beta, \beta \in \mathbb{K}$, $[D_2, D_1] = 0, D_2(a) = 1, D_2(b) = (n+1)\gamma a^n, \gamma \in \mathbb{K}$, $n \geq 1$ (if $n = 0$ we put $L = \langle D_1, D_2, bD_1 + aD_2\rangle$).

Proof. The set $I = RD_1 \cap L$ is an ideal of L by Lemma 3. We can write $I = \langle D_1, a_1D_1, \ldots, a_nD_1\rangle$ for some elements $a_i \in R$ and $n \geq 1$ (if $n = 0$ we put $I = \langle D_1\rangle$). Since the ideal I is abelian we have $D_1(a_i) = 0$, $i = 1, \ldots, n$. We consider two cases depending on $\dim L/I$ (recall that $\dim L/I \leq 2$ by Lemma 3).

Case 1. $\dim L/I = 1$. Take any element $D_2 \in L\backslash I$. As $\langle D_1\rangle$ is an ideal of L we have $[D_2, D_1] = \nu D_1$ for some $\nu \in \mathbb{K}$. The elements D_1 and D_2 are linearly independent over R

by the choice of the ideal I. First, let the linear operator $\operatorname{ad} D_2$ have the only eigenvalue ν on the vector space I (recall that $[D_2, D_1] = \nu D_1$). If $aD_1, bD_1 \in I$ are eigenvectors of $\operatorname{ad} D_2$, i.e. $[D_2, aD_1] = \nu a D_1$, $[D_2, bD_1] = \nu b D_1$, then the elements aD_1, bD_1 are linearly dependent over $\mathbb{K}$ by Lemma 5. Hence D_1 is the unique eigenvector of $\operatorname{ad} D_2$ on I (up to multiplication by a nonzero scalar). But then the linear operator $\operatorname{ad} D_2$ has a Jordan basis in I of the form $\{D_1, a_1 D_1, \ldots, a_n D_1\}$, $a_i \in R$ such that

$$[D_2, a_i D_1] = \nu a_i D_1 + a_{i-1} D_1, \quad i = 1, \ldots, n, \quad [D_2, D_1] = \nu D_1$$

(in case $n = 0$ we have $I = \langle D_1 \rangle$). The last relations imply the equalities $D_2(a_i) = a_{i-1}, i = 2, \ldots, n$, $D_2(a_1) = 1$. Denoting $a = a_1$ we have $D_2(a_2 - a^2/2!) = 0$ and taking into account the relation $D_1(a_2 - a^2/2!) = 0$ we see by Lemma 1 that $a_2 - a^2/2! \in \mathbb{K}$. But then without loss of generality we can take $a_2 = a^2/2!$. Analogously $D_2(a_3 - a^3/3!) = a_2 - a_2 = 0$ and $D_1(a_3 - a^3/3!) = 0$, so we can put $a_3 = a^3/3!$. Repeating these considerations we get a $\mathbb{K}$-basis $\{D_1, aD_1, \ldots, (a^n/n!)D_1\}$ of the ideal I (recall that $I = \langle D_1 \rangle$ in case $n = 0$). The algebra Lie L is of type 1 because we always can assume that $\nu = 0$ or $\nu = 1$ choosing a convenient multiple of the element D_2.

Now let $\operatorname{ad} D_2$ have on I at least two different eigenvalues. Our aim is to show that $\operatorname{ad} D_2$ is a diagonalizable operator on I. Denote by $I(\lambda)$ the root space of $\operatorname{ad} D_2$ corresponding to the eigenvalue $\lambda, \lambda \neq \nu$. Since $\operatorname{ad} D_2$ has on $I(\lambda)$ the only eigenvalue λ it follows from the previous considerations that $\operatorname{ad} D_2$ has on $I(\lambda)$ a Jordan basis such that the matrix of $\operatorname{ad} D_2$ in this basis is a single Jordan block. Therefore if $\dim I(\lambda) > 1$ then there exist elements $aD_1, bD_1 \in I$ such that

$$[D_2, aD_1] = \lambda a D_1, \quad [D_2, bD_1] = \lambda b D_1 + a D_1.$$

The latter is impossible by Lemma 5 and therefore $\dim I(\lambda) = 1$. Choosing any element $D_1' \in I$ with property $[D_2, D_1'] = \lambda D_1'$ instead of the element D_1 and using Lemma 5 we can analogously show that $\dim I(\nu) = 1$, where $I(\nu)$ is the root space corresponding to the eigenvalue ν of $\operatorname{ad} D_2$ on I. Therefore all the root spaces are one-dimensional and $\operatorname{ad} D_2$ is diagonalizable on I.

Since at least one of the eigenvalues of $\operatorname{ad} D_2$ on I is nonzero we can choose elements D_1 and D_2 in such a way that

$$[D_2, D_1] = D_1, I = \langle D_1, a_1 D_1, \ldots, a_n D_1 \rangle,$$

where $[D_2, a_i D_1] = \lambda_i a_i D_1, \lambda_i \neq \lambda_j$ if $i \neq j$ and $\lambda_i \neq 1, i = 1, \ldots, n$.

Applying Lemma 5 (with $\nu = 1$) we can easily show that $\frac{\lambda_i - 1}{\lambda_1 - 1} = \tau_i \in \mathbb{Q}$, $i = 2, \ldots, n$. Denote $\tau_i = \frac{k_i}{l_i}, k_i, l_i \in \mathbb{Z}, i = 2, \ldots, n$. If l is the least common multiple of $l_2, \ldots, l_n$, then one can write $\tau_i = \frac{m_i}{l}$ and $\lambda_i = m_i \beta + 1$, where $\beta = \frac{\lambda_1 - 1}{l}$ (note that $\lambda_i - 1 = \tau_i(\lambda_1 - 1)$). Thus, L is an algebra of type 2 of Lemma.

Case 2. $\dim L/I = 2$. The quotient algebra L/I is nonabelian by Lemma 3, so it contains a noncentral one-dimensional ideal $\langle D_2 + I \rangle$. Then there exists an element $bD_1 + cD_2 \in L$ such that

$$[bD_1 + cD_2 + I, D_2 + I] = D_2 + I.$$

This means that $[bD_1 + cD_2, D_2] = D_2 + gD_1$ for some element $gD_1 \in I$. Since the ideal I is abelian it is obvious that $\operatorname{ad} D_2 = \operatorname{ad}(D_2 + gD_1)$ on the vector space I over $\mathbb{K}$. We obtain the following relation for linear operators on I :

$$[\operatorname{ad}(bD_1 + cD_2), \operatorname{ad} D_2] = \operatorname{ad}(D_2 + gD_1) = \operatorname{ad} D_2.$$

But then $\operatorname{ad} D_2$ acts nilpotently on I by Lemma 6. In case $\dim I = 1$ we get (after direct calculations) the Lie algebra of type 3 with $n = 0$. Let $\dim I \geq 2$. Since $[D_2, D_1] = 0$ one can easily show (using Lemma 3) that the ideal I can be written in the form $I = \langle D_1, aD_1, \ldots, (a^n/n!)D_1 \rangle$ for some $a \in R, D_2(a) = 1, n \geq 1$.

Returning now to the above mentioned element $bD_1 + cD_2 \in L$ we see that

$$[D_1, bD_1 + cD_2] = D_1(b)D_1 + D_1(c)D_2 \in \langle D_1\rangle$$

and therefore $D_1(c) = 0, D_1(b) \in \mathbb{K}$. Further, from the equality

$$[D_2, bD_1 + cD_2] = D_2(b)D_1 + D_2(c)D_2 \in I + \langle D_2\rangle$$

we obtain $D_2(c) = \gamma \in \mathbb{K}$, $D_2(b) \in \langle 1, a, a^2/2!, \dots, a^n/n!\rangle$. From the relations $D_2(c) = \gamma \in \mathbb{K}$ and $D_2(a) = 1$ it follows that $D_2(\gamma a - c) = 0$. Then Lemma 1 yields $\gamma a - c \in \mathbb{K}$, i.e. $c = \gamma a + b$ for some $\gamma, \beta \in \mathbb{K}$.

The element $D_3 = \gamma^{-1}(bD_1 + cD_2 - \beta D_2)$ of the algebra L can be written in the form $D_3 = b_1D_1 + aD_2$ for some $b_1 \in R$. As $D_2(b_1) \in \langle 1, a, a^2/2!, \dots, a^n/n!\rangle$ we can subtract from $b_1D_1 + aD_2$ a suitable linear combination of the elements $D_1, aD_1, a^2/2!D_1, \dots, a^n/n!D_1$ and assume without loss of generality that $D_2(b_1) = (n+1)\gamma a^n$ for some $\gamma \in \mathbb{K}$. Denoting $b = b_1$, $\beta = D_1(b) \in \mathbb{K}$ we see that L is of type 3 of this Lemma. □

Remark 2. For each type of Lie algebras from Lemma 7 one can easily point out a realization

1. $\lambda = 0, D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}, a = y$. $\lambda = 1, D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial x}, a = y$.
2. $D_1 = \frac{\partial}{\partial x}, a_i = y^{m_i}, D_2 = \beta y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}, \beta \in \mathbb{K}$.
3. $D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}$, $a = y, b = \beta x + \gamma y^{n+1}, \beta, \gamma \in \mathbb{K}$.

Lemma 8. *Let L be a subalgebra of $\widetilde{W_2}(\mathbb{K})$ satisfying all the conditions of the previous Lemma with the exception of that the ideal I is abelian. If I is nonabelian, then there exist elements $D_1 \in I, D_2 \in L\backslash I$ such that L is one of the following algebras:*

(1) *$L = \langle D_1, aD_1, \dots, (a^{n-1}/(n-1)!)D_1, bD_1, D_2\rangle$, where $a, b \in R$ such that $D_1(a) = 0, D_2(a) = 1, D_1(b) = -1, D_2(b) = 0, [D_2, D_1] = 0$.*

(2) *$L = \langle D_1, a_1D_1, \dots, a_{n-1}D_1, bD_1, D_2\rangle$, where $a_i, b \in R$ such that $[D_2, D_1] = D_1, D_1(a_i) = 0, D_1(b) = -1, D_2(b) = -b, D_2(a_i) = \beta m_i a_i$ for some $m_i \in \mathbb{Z}, \beta \in \mathbb{K}^\star$ and $m_i \neq m_j$ if $i \neq j$.*

(3) *$L = \langle D_1, aD_1, \dots, (a^{n-1}/(n-1)!)D_1, (v - \alpha a^n)D_1, D_2, (-\beta v + \gamma(a^n/n!))D_1 - aD_2\rangle$, where $a, v \in R$ such that $[D_1, D_2] = 0, D_1(a) = 0, D_2(a) = 1, D_1(v) = -1, D_2(v) = 0, \alpha, \beta \in \mathbb{K}$, and $\gamma = \alpha(\beta - n)$.*

Proof. Let $\langle D_1\rangle$ be the one-dimensional ideal of L lying in I. The ideal I has by Lemma 2 a basis over $\mathbb{K}$ of the form $\{D_1, a_1D_1, \dots, a_{n-1}D_1, bD_1\}$, where $D_1(a_i) = 0, D_1(b) = -1, i = 1, \dots, n-1$ (for $n = 0$ we put $I = \langle D_1, bD_1\rangle$ with $D_1(b) = -1$). Suppose that $n = 0$, i.e. $\dim I = 2$. If $\dim L/I = 1$, then $L = \langle D_1, bD_1, D_2\rangle$ is of type 2 or 3. If $\dim L/I = 2$, then L/I is nonabelian by Lemma 3 and taking into account that L/I is nonabelian we have $L = I \oplus J$ for nonabelian ideal J of dimension 2. Then L is of type 3. So we may assume that $\dim I \geq 3$. As in the previous Lemma we divide the proof into following cases:

<u>Case 1.</u> $\dim L/I = 1$. Take any element $D_2 \in L\backslash I$. Then $[D_2, bD_1] = \lambda bD_1 + cD_1$, where $cD_1 \in I' = [I, I]$ because $\dim L/I' = 2$ and $\langle bD_1 + I'\rangle$ is a one-dimensional ideal of L/I'. If $\lambda \neq 0$, then we may assume without loss of generality that $\lambda = 1$, and then

$$[\operatorname{ad} D_2, \operatorname{ad}(bD_1)] = \operatorname{ad}(bD_1 + cD_1) = \operatorname{ad}(bD_1)$$

on I' because I' is an abelian ideal of L. But then the linear operator $\operatorname{ad}(bD_1)$ acts nilpotently on I' by Lemma 6. The latter is impossible and therefore $\lambda = 0$. This means that L/I' is an abelian Lie algebra of dimension 2. As $[D_2, bD_1] = cD_1$ for some element $cD_1 \in I'$ we get $[D_2 + cD_1, bD_1] = 0$ (recall that $[bD_1, cD_1] = cD_1$ for all $cD_1 \in I'$). So, we can choose the element D_2 in such a way that $[D_2, bD_1] = 0$. If the linear operator $\operatorname{ad} D_2$ has on $I' = \langle D_1, \dots, a_{n-1}D_1\rangle$ at least two different eigenvalues, then there exists by Lemma 5 a basis $\{D_1, \dots, a_{n-1}D_1\}$ of I' such that $D_2(a_i) = m_i\beta a_i$, for some $m_i \in \mathbb{Z}, \beta \in \mathbb{K}^\star$, $m_i \neq m_j$ if $i \neq j, [D_2, D_1] = D_1$. Then from the relation $[D_2, bD_1] = 0$ it follows $D_2(b) = -b$. The algebra L is of type 2 of Lemma.

Now let $\operatorname{ad} D_2$ have the only eigenvalue μ on I'. If $\mu = 0$, then L is obviously the Lie algebra of type 1 of Lemma. Let $\mu \neq 0$. Taking a suitable multiple of D_2 we may assume that $\mu = 1$. Then replacing the element D_2 by the element $D_2 - bD_1$ we get the case $\mu = 0$ and L is again of type 1 of Lemma.

Case 2. $\dim L/I = 2$. As in the case 1 take a one-dimensional ideal $\langle D_1 \rangle$ of L lying in I' and a basis of I of the form $\{D_1, a_1D_1, \dots, a_{n-1}D_1, bD_1\}$ such that $D_1(a_i) = 0$, $D_1(b) = -1$, $i = 0, \dots, n-1$. Let $\langle D_2 + I \rangle$ be the one-dimensional ideal of the nonabelian quotient algebra L/I. Accordingly to Case 1 the algebra $\langle D_2 \rangle + I$ is of type 1 or type 2 of this Lemma. Let us show that $\langle D_2 \rangle + I$ is of type 1 of this Lemma, i. e. the linear operator $\operatorname{ad} D_2$ acts nilpotently on I'. Really since $\langle bD_1 + I' \rangle$ is an ideal of the algebra L/I' and $\operatorname{ad}(bD_1)$ acts on I' as the identity operator the ideal $\langle bD_1 + I' \rangle$ lies in the center of L/I' (because of Lemma 6), i. e. $[D, bD_1] \in I'$ for any element $D \in L$. Take any element $cD_1 + dD_2 \in L \backslash I$ such that $[cD_1 + dD_2, D_2] = D_2 + rD_1$ for some element $rD_1 \in I$. The element rD_1 can be written in the form $rD_1 = \mu bD_1 + r_1D_1$, where $\mu \in \mathbb{K}$, $r_1D_1 \in I'$. But then we obtain

$$[cD_1 + bD_2, D_2 + \mu bD_1] = (D_2 + \mu bD_1) + r_2D_1$$

for some element $r_2D_1 \in I'$ The latter means that $\operatorname{ad}(D_2 + \mu bD_1)$ acts nilpotently on I' (by Lemma 6). Replacing the element D_2 by the element $D_2 + \mu bD_1$ we can assume without loss of generality that $\operatorname{ad} D_2$ is a nilpotent linear operator on I'. So, the subalgebra $\langle D_2 \rangle + I$ is of type 1 of this Lemma and hence $I' + \langle D_2 \rangle$ can be written in the form

$$I' + \langle D_2 \rangle = \langle D_1, aD_1, \dots, \frac{a^{n-1}}{(n-1)!} D_1, D_2, \rangle$$

where $[D_2, D_1] = 0$, $D_1(a) = 0$, $D_2(a) = 1$.

Further, it follows from the above mentioned equality

$$[cD_1 + dD_2, D_2] = D_2 + r_2D_1 \tag{6}$$

that $D_2(d) = -1$. Analogously we obtain $D_1(d) = 0$, $D_1(c) = \beta_1 \in \mathbb{K}$ from the relation $[cD_1 + dD_2, D_1] \in \langle D_1 \rangle$. Since $D_2(a) = 1$ and $D_2(d) = -1$ we have $D_2(a+d) = 0$. Taking into account the equality $D_1(a + d) = 0$ we obtain by Lemma 1 that $a + d = \alpha_1 \in \mathbb{K}$. But then $d = -a + \alpha_1$ and without loss of generality we can choose $cD_1 - aD_2$ instead of the element $cD_1 + dD_2$.

Since $[D_2, bD_1] \in I'$ (as we have proved before) we see that

$$D_2(b) = \alpha_0 + \alpha_1 a + \dots + \alpha_{n-1} \frac{a^{n-1}}{(n-1)!}$$

for some $\alpha_i \in \mathbb{K}$. Put $v = b - \alpha_0 a - \alpha_1 \frac{a^2}{2!} - \dots - \alpha_{n-1} \frac{a^n}{n!}$. Then $D_1(v) = D_1(b) = -1, D_2(v) = 0$. Subtracting the element $(\alpha_0 a + \alpha_1 \frac{a^2}{2!} + \dots + \alpha_{n-2} \frac{a^{n-1}}{(n-1)!})D_1 \in I'$ from the element bD_1 we can assume without loss of generality that $b = v - \alpha_{n-1} \frac{a^n}{n!}$ for some $\alpha_{n-1} \in \mathbb{K}$. Then $D_1(b) = -1$, $D_2(b) = \alpha_{n-1} \frac{a^{n-1}}{(n-1)!}$. Further, recall that for the basic element $cD_1 - aD_2$ we have $D_1(c) = \beta_1 \in \mathbb{K}$.

Rewriting the relation 6 in the form $[cD_1 - aD_2, D_2] = D_2 + r_2D_1$ we obtain that

$$D_2(c) = \gamma_0 + \gamma_1 a + \dots + \gamma_{n-1} \frac{a^{n-1}}{(n-1)!} \quad \text{for some} \quad \gamma_i \in \mathbb{K}, \quad i = 1, \dots, n-1.$$

Subtracting the element $(\gamma_0 a + \gamma_1 \frac{a^2}{2!} + \dots + \gamma_{n-2} \frac{a^{n-1}}{(n-1)!})D_1 \in I'$ from the element $cD_1 - aD_2$ we may assume without loss of generality that $D_2(c) = \gamma_{n-1} \frac{a^{n-1}}{(n-1)!}$. Suppose that $\beta_1 = D_1(c) \neq 0$. Since $D_1(\beta_1^{-1} c + v - \beta_1^{-1} \gamma_{n-1} \frac{a^n}{n!}) = 0$ and $D_2(\beta_1^{-1} c + v - \beta_1^{-1} \gamma_{n-1} \frac{a^n}{n!}) =$

$\beta_1^{-1}\gamma_{n-1}\frac{a^{n-1}}{(n-1)!}-\beta_1^{-1}\gamma_{n-1}\frac{a^{n-1}}{(n-1)!}=0$ we have by Lemma 1 that $\beta_1^{-1}c+v-\beta_1^{-1}\gamma_{n-1}\frac{a^n}{n!}=\nu$ for some $\nu\in\mathbb{K}$. Subtracting the element $\nu D_1\in I'$ from the element cD_1+aD_2 we may assume that $\nu=0$. Then we obtain $c=-\beta_1 v+\gamma_{n-1}\frac{a^n}{n!}$. Denoting α_{n-1} by α, γ_{n-1} by γ and β_1 by β we obtain a basis of L of the form

$$\Big\{D_1, aD_1, \dots, \frac{a^{n-1}}{(n-1)!}D_1, (v-\alpha\frac{a^n}{n!})D_1, D_2, (-\beta v+\gamma\frac{a^n}{n!})D_1-aD_2\Big\}$$

(here $D_1(a)=0, D_1(v)=-1, D_2(a)=1, D_2(v)=0$). Now suppose that $\beta=D_1(c)=0$. Since $D_2(c)=\gamma\frac{a^{n-1}}{(n-1)!}$ we see that for the element $c_1=c-\gamma\frac{a^n}{n!}$ it holds $D_1(c)=\beta=0$, $D_2(c)=0$. So by Lemma 1 we obtain $c-\gamma\frac{a^n}{n!}=\nu_2$ for some $\nu_2\in\mathbb{K}$. Subtracting the element ν_2D_1 from cD_1+aD_2 we may assume that $\nu_2=0$. So we have that $c=\gamma\frac{a^n}{n!}$ i.e. the basis of L is of the same form as in case $\beta\neq 0$.

Now consider the product $[(v-\alpha a^n/n!)D_1, (\beta v+\gamma a^n/n!)D_1-aD_2]$. This product equals to $(-\alpha\beta+\gamma+n\alpha)D_1$ and belongs to I'. Hence $-\alpha\beta+\gamma+n\alpha=0$ and $\gamma=\alpha(\beta-n)$. We see that L is of type 3 of Lemma. □

Remark 3. There exist realizations for all types of Lie algebras from Lemma 8
1. $D_1=\frac{\partial}{\partial x}, D_2=\frac{\partial}{\partial y}$, $a=y, b=-x$.
2. $D_1=\frac{\partial}{\partial x}, D_2=\beta y\frac{\partial}{\partial y}-x\frac{\partial}{\partial x}$, $a=y, b=-x, a_i=y^{m_i},$.
3. $D_1=\frac{\partial}{\partial x}, D_2=\frac{\partial}{\partial y}$, $a=y, f=-x$.

The next three corollaries can be easily proved by using results of Lemmas 2, 7 and 8.

Corollary 1. *Let L be a finite dimensional nilpotent subalgebra of $\widetilde{W_2}(\mathbb{K})$. Then there exist elements $D_1, D_2\in L$ linearly independent over R such that L is one of the following algebras:*
(1) $L=\langle D_1, a_1D_1,\dots,a_nD_1\rangle$, *for some $a_i\in R$ such that $D_1(a_i)=0, i=1,\dots,n$.*
(2) $L=\langle D_1, D_2\rangle, [D_1,D_2]=0$.
(3) $L=\langle D_1, aD_1,\dots,(a^n/n!)D_1, D_2\rangle$ *for some $a\in R$ such that $D_1(a)=0, D_2(a)=1$,* $[D_1,D_2]=0$.

Corollary 2. *Let L be a finite dimensional solvable subalgebra of $\widetilde{W_2}(\mathbb{K})$. If L is nonabelian and decomposable into a direct sum of proper ideals, then $L=A\oplus B$, where A is a nonabelian ideal of dimension 2 and B is either a one-dimensional ideal or a two-dimensional nonabelian ideal of L.*

Corollary 3. *Let L be a finite dimensional solvable subalgebra of $\widetilde{W_2}(\mathbb{K})$. If L is nonabelian, then* $\dim L/L'\leq 2$.

3. Nonsolvable subalgebras of $\widetilde{W_2}(\mathbb{K})$

Lemma 9. *If L is a finite dimensional semisimple subalgebra of the Lie algebra $\widetilde{W_2}(\mathbb{K})$, then L is isomorphic to $sl_2(\mathbb{K})$ or $sl_3(\mathbb{K})$, or $sl_2(\mathbb{K})\oplus sl_2(\mathbb{K})$.*

Proof. If L is of rank 1 (as a system of vectors) over R, then $L\simeq sl_2(\mathbb{K})$ by Lemma 2. So, we can assume that L is of rank 2 over R. Fix a Cartan subalgebra H of the algebra L, a basis π of the system Δ of roots which correspond to H and let Δ^+ be the set of positive roots relatively to the ordering on Δ. Consider the triangular decomposition

$$L=N_+ + H + N_-,\quad N_+=\oplus_{\alpha_i>0}L_{\alpha_i},\quad N_-=\oplus_{\alpha_i<0}L_{\alpha_i}$$

and the Borel subalgebra $B=H+N_+$ of L. If the subalgebra N_+ is abelian, then L is a direct sum $L=L_1\oplus\cdots\oplus L_k$ of ideals isomorphic to $sl_2(\mathbb{K})$ (see, for example [5]). Then B is a direct sum $B=B_1\oplus\cdots\oplus B_k$ of Borel subalgebras of $L_i\simeq sl_2(\mathbb{K})$ and using Corollary 2 we see that either $L=L_1\simeq sl_2(\mathbb{K})$ or $L=L_1\oplus L_2\simeq sl_2(\mathbb{K})\oplus sl_2(\mathbb{K})$.

Now, let the subalgebra N_+ be nonabelian. Since N_+ is nilpotent it is indecomposable into a direct sum of nonzero ideals by Corollary 1. But then the algebra L is also indecomposable into a direct sum of proper ideals and hence is simple. By Corollary 3 we have relations

$$\dim B/B' = \dim B/N = \dim H \leq 2.$$

Therefore, if N_+ is nonabelian, then $\dim H = 2$ and L is a simple Lie algebra of one of the types A_2, B_2 or G_2. First suppose that L is of type G_2. Then the subalgebra N_+ from its triangular decomposition has nonabelian derived subalgebra $[N_+, N_+]$. The latter is impossible (see Corollary 1) and hence L cannot be of type G_2.

Further, let us show that L is not of type B_2. Fix a Cartan subalgebra H of L and a basis $\{\alpha, \beta\}$ of the root system Δ. Then the subalgebra N_+ has the basis $\{e_\alpha,\ e_\beta,\ e_{\alpha+\beta}, e_{\alpha+2\beta}\}$. It follows from Corollary 1 that $e_{\alpha+\beta} = f \cdot e_{\alpha+2\beta}$ for some element $f \in R$. Consider the element σ_α of the Weyl group of the root system Δ acting by the rule $\sigma_\alpha(\gamma) = \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha$, where γ is an arbitrary root from Δ. Then $\{-\alpha, \beta+\alpha.\beta, \alpha+2\beta\}$ are positive roots relatively to the new basis $\{\sigma_\alpha(\alpha), \sigma_\alpha(\beta)\}$. The subalgebra $\langle e_{-\alpha}, e_{\beta+\alpha}, e_\beta, e_{\alpha+2\beta}\rangle$ is nilpotent and by Corollary 1 it holds $e_\beta = g \cdot e_{\alpha+2\beta}$ for some $g \in R$. Analogously one can show that $e_\alpha = h \cdot e_{\alpha+2\beta}$ for some $h \in R$. Three relations with coefficients f, g, h obtained above imply that all elements from the basis of N_+ are multiple to one of them and hence the subalgebra N_+ is abelian by Lemma 2. This is impossible and obtained contradiction shows that L is not of type B_2. Thus, L is of type A_2. □

Lemma 10. *Let L be a finite dimensional nonsolvable subalgebra of $\widetilde{W_2}(\mathbb{K})$ whose Levi factor is either of type A_2 or of type $A_1 \times A_1$. Then L is semisimple of type A_2 or of type $A_1 \times A_1$ respectively.*

Proof. Let $S = S(L)$ be the solvable radical of L. By Theorem of Levi-Maltsev $L = L_1 \leftthreetimes S$, where L_1 is a Levi factor of L. First suppose that L_1 is of type A_2. Let us fix a Cartan subalgebra H of L_1 and the root system Δ corresponding to H. Consider the triangular decomposition

$$L = N_- + H + N_+ \tag{7}$$

of L_1 relatively to H and Δ. Since the subalgebra N_+ is nonabelian (this follows from the multiplication law in algebras of type A_2) it contains by Corollary 1 elements D_1 and D_2, linearly independent over R such that $[D_1, D_2] = 0$. Consider S as an L_1-module and take the older vector $D \in S$ relatively to the decomposition (7). Then we have

$$[D_1, D] = 0, \quad [D_2, D] = 0. \tag{8}$$

If we write $D = aD_1 + bD_2$ for some $a, b \in R$, then from the previous relation we get

$$D_1(a) = 0, \quad D_1(b) = 0, \quad D_2(a) = 0 \quad \text{and} \quad D_2(b) = 0.$$

Lemma 1 yields now that $a, b \in \mathbb{K}$, i.e. $D \in L_1$. As $L_1 \cap S = 0$ we obtain $S = 0$ and therefore $L = L_1$ is a simple Lie algebra of type A_2.

Let now L_1 be of type $A_1 \times A_1$. Write $L_1 = G_1 \oplus G_2$, where $G_i \simeq sl_2(\mathbb{K})$ and fix Cartan subalgebras $H_1 \subset G_1, H_2 \subset G_2$. Consider any triangular decompositions

$$G_1 = N_{1+} + H_1 + N_{1-}, \quad G_2 = N_{2+} + H_2 + N_{2-}$$

relatively to H_1 and H_2. Take any nonzero element $D_1 \in N_{1+}$. Then at least one of the subalgebras N_{1-}, N_{2+}, N_{2-} contains a nonzero element D_2 such that D_1 and D_2 are linearly independent over R. Really, in other case $H_1 = [N_{1+}, N_{1-}]$ and $H_2 = [N_{2+}, N_{2-}]$ lie also in RD_1 and therefore $L = G_1 \oplus G_2 \subset RD_1$ which is impossible by Lemma 2. It is easily shown that the two-dimensional abelian subalgebra $N_+ = \langle D_1, D_2\rangle$ is a part of triangular decomposition $L = N_+ + H + N_-$ of L relatively to the Cartan subalgebra $H = H_1 \oplus H_2$. Choosing as above the older vector in S relatively to N_+ and repeating the considerations from the case $L_1 \simeq A_2$ we get $S = 0$, i.e. L is semisimple of type $A_1 \times A_1$. □

Lemma 11. *Let L be a nonsolvable finite dimensional subalgebra of $\widetilde{W}_2(\mathbb{K})$. Then L is isomorphic to one of the following algebras:*

(1) $sl_3(\mathbb{K})$.

(2) $sl_2(\mathbb{K})$ *or* $sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$.

(3) $sl_2(\mathbb{K}) \leftthreetimes V_m$, *where V_m is the irreducible module over $sl_2(\mathbb{K})$ of dimension $m+1$, $m = 0, 1, \dots$*

(4) $gl_2(\mathbb{K}) \leftthreetimes V_m$, *where V_m is the irreducible module over $gl_2(\mathbb{K})$ of dimension $m+1$, $m = 0, 1, \dots$ and nonzero central elements of $gl_2(\mathbb{K})$ act on V_m as nonzero scalars.*

Proof. Let S be the solvable radical of L and L_1 be a Levi factor of the algebra L. We can consider only the case $S \neq 0$ because of Lemma 9. It follows from Lemma 10 that $L_1 \simeq sl_2(\mathbb{K})$. Choose a Cartan subalgebra H of the algebra L_1 and a triangular decomposition $L_1 = N_+ + H + N_-$ of L_1.

Case 1. $\dim S = 1$ or $\dim S = 2$. If $\dim S = 1$, then $L = L_1 \oplus S$ is a sum of two ideals and $L \simeq sl_2(\mathbb{K}) \oplus V_0$, where V_0 is a one-dimensional module over $sl_2(\mathbb{K})$. The algebra L is of type 4 with $m = 0$. Suppose that $\dim S = 2$. If S is a nonabelian ideal of L, then L is a direct sum of ideals $L = L_1 \oplus S$. Since $S = \langle w \rangle \leftthreetimes \langle v_0 \rangle$ for some elements $w, v_0 \in S$, then $L \simeq gl_2(\mathbb{K}) \leftthreetimes \langle v_0 \rangle$ is of type (5) with $m = 0$ because $L_1 \oplus \langle w \rangle \simeq gl_2(\mathbb{K})$. Let S be abelian. Suppose that S is a reducible module. Then $S = S_1 \oplus S_2$ is a direct sum of L_1-modules of dimension 1 over $\mathbb{K}$. Take the Borel subalgebra $B = H + N_+$ of L_1. Then the subalgebra $B \oplus S_1 \oplus S_2$ of L is solvable of dimension 4. But such an algebra does not exist by Lemmas 7 and 8. This contradiction shows that S is irreducible and $L \simeq sl_2(\mathbb{K}) \leftthreetimes V_1$, where V_1 is of dimension 2 over $\mathbb{K}$. The algebra L is of type 4. Further, we will assume that $\dim S \geq 3$.

Case 2. S is abelian (of dimension ≥ 3). Let us show that S is an irreducible module over L_1. Assume to the contrary that S is reducible. If S is a sum of one-dimensional submodules over L_1, then $L = L_1 \oplus S$ is a direct sum of ideals. Its subalgebra $B + S$ is solvable, nonabelian and decomposable into direct sum of subalgebras $B \oplus S$. The latter is impossible by Corollary 2. So we can assume $S = S_1 \oplus S_2$ where S_1, S_2 are L_1-submodules, $\dim S_1 \geq 2$ and S_1 is irreducible (note that S_1 and S_2 are ideals of L because S is abelian). Let $D_2 \in N_+$ be a nonzero element. Then the subalgebra $M = \langle D_2 \rangle + S$ is nonabelian, nilpotent and $\dim M/[M, M] \leq 2$ by Corollary 3. On the other hand, since $[M, M] = [D_2, S_1] \oplus [D_2, S_2]$, $\dim S_i/[D_2, S_i] \geq 1$, $i = 1, 2$ (because $\operatorname{ad} D_2$ acts nilpotently on S_i) we have

$$\dim M/[M, M] = \dim\langle D_2 \rangle + \dim S_1/[D_2, S_1] + \dim S_2/[D_2, S_2] \geq 3.$$

The latter contradicts to Corollary 3 and hence S is a simple L_1-module. It is obvious that L is of type 4. Note that the subalgebra $M = \langle D_2 \rangle + S$ is of the form

$$\langle D_2, D_1, aD_1, \dots, \frac{a^k}{k!} D_1 \rangle, \quad [D_2, D_1] = 0, \quad D_1(a) = 0, \quad D_2(a) = 1.$$

Case 3. S is a nilpotent (nonabelian) ideal. Then by Corollary 1 there exist elements $D_1, D_2 \in S$ such that

$$S = \langle D_2, D_1, aD_1, \dots, \quad (a^k/k!)D_1 \rangle, \quad [D_2, D_1] = 0,$$
$$D_1(a) = 0, \quad D_2(a) = 1, \quad \dim S \geq 3.$$

Therefore $\langle D_1 \rangle = S^{k-1}$ and $\langle D_1 \rangle$ is an ideal of L. Using Lemma 3 we see that $RD_1 \cap L$ is an ideal of L and therefore $L_1 \leftthreetimes \langle D_1, aD_1, \dots, \frac{a^k}{k!} D_1 \rangle$ is a subalgebra of L. This subalgebra has the abelian decomposable ideal $\langle D_1, aD_1, \dots, \frac{a^k}{k!} D_1 \rangle$. This is impossible by the Case 1 and therefore the Case 3 is impossible.

Case 4. S is solvable (nonnilpotent). The L_1-submodule $S' = [S, S]$ is nilpotent, therefore S' is abelian by the previous case and S' is an irreducible L_1-module by Cases 1 and 2. Since $\dim S/S' \leq 2$ by Corollary 3 we have a direct decomposition $S = S' \oplus S_2$ of L_1-submodules with $\dim S_2 \leq 2$. First suppose that $\dim S_2 = 2$. Let us show that S_2 is an irreducible L_1-module. Indeed, in other case $S_2 \subseteq C_S(L_1)$ and the centralizer $C_S(L_1)$ a submodule of the L-module S. Because of previous cases we can assume that $\dim S' \geq 2$ and hence S' is an irreducible L_1-module. Then obviously $C_S(L_1) = S_2$. Since $C_S(L_1) = S_2$ is a subalgebra of L the sum $S_2 + L_1$ is a subalgebra of L. The latter is impossible because the subalgebra $S_2 + L_1$ does not exist by the Case 1. This contradiction shows that S_2 is an irreducible L_1-module.

Choose any nonzero elements $D_2 \in N_+$ and $h \in H$ and take standard bases $\{e_0, e_1\} \subset S_2$ and $\{f_0, f_1, \ldots, f_m\} \subset S'$ of the L_1-modules S_2 and S' respectively (recall that $L_1 \simeq sl_2(\mathbb{K})$). Then the linear operator $\operatorname{ad} h$ has eigenvalues $1, -1$ on S_2. If the eigenvalues of $\operatorname{ad} h$ on S' are even, then the elements $[e_i, f_j]$ are eigenvectors for $\operatorname{ad} h$ with odd eigenvalues. Since $[e_i, f_j] \in S'$ we see that $[e_i, f_j] = 0$. Let now the eigenvalues of $\operatorname{ad} h$ on S' be odd. Then $[e_i, f_j]$ are eigenvectors for $\operatorname{ad} h$ with even eigenvalues, so $[e_i, f_j] = 0$, $i = 0, 1$, $j = 0, 1, \ldots, m$. As S' is abelian the latter means that $S' \subset Z(S)$. This is impossible because of our assumption on S and therefore $\dim S/S' = 1$. Hence $\dim S_2 = 1$. The subalgebra $S_2 + L_1$ is obviously isomorphic to $gl_2(\mathbb{K})$ and S' is an irreducible $S_2 + L_1$-module. Since S_2 lies in the center of $S_2 + L_1$ and S is nonabelian we see that each nonzero element of S_2 acts on S' as multiplication by a nonzero scalar. We get a Lie algebra of type 5 from this Lemma. □

Remark 4. For each type of Lie algebras from this Lemma one can easily point out its realization

(1) $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), y(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})\rangle \simeq sl_3(\mathbb{K})$;

(2) $\langle \frac{\partial}{\partial x}, -x^2\frac{\partial}{\partial x}, -2x\frac{\partial}{\partial x}\rangle \simeq sl_2(\mathbb{K})$ and $\langle \frac{\partial}{\partial x}, -x^2\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, -y^2\frac{\partial}{\partial y}, -2y\frac{\partial}{\partial y}\rangle \simeq sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$;

(3) $\langle x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, x^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), x^{m-1}y(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \ldots, y^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})\rangle \simeq sl_2(\mathbb{K}) \leftthreetimes V_m$;

(4) $\langle x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, x^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), x^{m-1}y(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}), \ldots, y^m(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})\rangle \simeq gl_2(\mathbb{K}) \leftthreetimes V_m$.

We give a description of finite dimensional subalgebras of the Lie algebra $\widetilde{W}_2(\mathbb{K})$ up to isomorphism as Lie algebras. In fact we give more information about such Lie algebras (up to choice of basis $\{D_1, D_2\}$ of the two-dimensional vector space $\widetilde{W}_2(\mathbb{K})$ over the field $R = \mathbb{K}(x, y)$). In order to clarify the structure of described subalgebras of $\widetilde{W}_2(\mathbb{K})$ we formulate the main Theorem in terms of generators and relations.

Theorem 1. *Let L be a nonzero finite dimensional subalgebra of the Lie algebra $\widetilde{W}_2(\mathbb{K})$. Then the algebra L belongs to one of the following types:*

(1) $L = \langle e_1, \ldots, e_n\rangle$, *where* $[e_i, e_j] = 0$, $i, j = 1, \ldots, n$.

(2) $L = \langle e_1, \ldots, e_n, f\rangle$, *where* $[e_i, e_j] = 0, [f, e_i] = e_i$, $i = 1, \ldots, n$.

(3) $L = \langle e_0, \ldots, e_n, f\rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0, \ldots, n$, $[f, e_0] = \lambda e_0$, $[f, e_i] = \lambda e_i + e_{i-1}$, $i = 1, \ldots, n$, $\lambda = 0$ *or* $\lambda = 1$.

(4) $L = \langle e_0, \ldots, e_n, f\rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0, \ldots, n$, $[f, e_i] = (1 + \beta m_i)e_i$, $i = 0, \ldots, n$, $m_i \in \mathbb{Z}$, $\beta \in \mathbb{K}^\star$ *and* $m_i \neq m_j$ *provided that* $i \neq j$.

(5) $L = \langle e_0, \ldots, e_n, f, g\rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0, \ldots, n$, $[f, e_0] = 0$, $[f, e_i] = e_{i-1}$, $i = 1, \ldots, n$, $[g, e_i] = (i - \beta)e_i$, $i = 0, \ldots, n$, $[g, f] = f - \gamma e_n$, $\beta, \gamma \in \mathbb{K}$.

(6) $L = \langle e_0, \ldots, e_n, f, g\rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0, \ldots, n$, $[f, e_i] = e_i$, $i = 0, \ldots, n$, $[g, e_0] = 0$, $[g, e_i] = e_{i-1}$, $i = 1, \ldots, n$, $[f, g] = 0$.

(7) $L = \langle e_0, \dots, e_n, f, g\rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0, \dots, n$, $[f, e_i] = e_i$, $i = 0, \dots, n$, $[g, e_i] = (1 + \beta m_i)e_i$, $i = 0, \dots, n$, $[g, f] = 0$, $\beta \in \mathbb{K}^\star$, $m_i \in \mathbb{Z}$, *and* $m_i \neq m_j$ *if* $i \neq j$.

(8) $L = \langle e_0, \dots, e_n, f, g, h\rangle$, *where* $[e_i, e_j] = 0$, $i, j = 0, \dots, n$, $[f, e_0] = 0$, $[f, e_i] = e_{i-1}$, $i = 1, \dots, n$, $[g, e_i] = e_i$, $i = 0, \dots, n$, $[g, f] = \alpha e_n$, $[h, e_i] = -(\beta + i)e_i$, $[h, f] = f - \gamma e_n$, $[h, g] = 0, \alpha, \beta \in \mathbb{K}, \gamma = \alpha(\beta - n)$.

(9) $L \simeq sl_2(\mathbb{K})$, *or* $L \simeq sl_2(\mathbb{K}) \oplus sl_2(\mathbb{K})$.

(10) $L \simeq sl_3(\mathbb{K})$.

(11) $sl_2(\mathbb{K}) \leftthreetimes V_m$, *where* V_m *is the irreducible module over* $sl_2(\mathbb{K})$ *of dimension* $m+1$, $m = 0, 1, \dots$

(12) $gl_2(\mathbb{K}) \leftthreetimes V_m$, *where* V_m *is the irreducible module over* $gl_2(\mathbb{K})$ *of dimension* $m+1$, $m = 0, 1, \dots$ *and nonzero central elements of* $gl_2(\mathbb{K})$ *act on* V_m *as nonzero scalars.*

Proof. Let L be a finite dimensional solvable subalgebra of the Lie algebra $\widetilde{W}_2(\mathbb{K})$. If L is of rank 1 over R, then L is of type 1 or 2 by Lemma 2. Let L be of rank 2 over R. If L possesses an abelian ideal I of rank 1 over R which is maximal with this property, then L is of type 3, 4 or 5 by Lemma 7 (we denote $e_i = a_i D_1$ in type 4 and $e_i = (a^i/i!)D_1$ for types 3 and 5). Let the ideal I be nonabelian. Then by Lemma 8 L is one of types 6, 7 or 8 (as above we denote $e_i = a_i D_1$ in type 7 and $e_i = (a^i/i!)D_1$ for types 6 and 8, $f = bD_1$ for types 6 and 7 and $f = D_2, g = (v - \alpha(a^n/n!))D_1$ for type 8 of this Theorem). Further, let L be nonsolvable. If L is semisimple, then L is one of types 9 or 10 by Lemma 9. Finally, if solvable radical of L is nonzero, then L is either of type 11 or of type 12 by Lemma 11. □

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