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SPECTRAL PROPERTIES OF STURM–LIOUVILLE EQUATIONS WITH SINGULAR ENERGY-DEPENDENT POTENTIALS

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Dedicated to Ya. V. Mykytyuk on occasion of his 60th birthday

Abstract. We study spectral properties of energy-dependent Sturm–Liouville equations, introduce the notion of norming constants and establish their interrelation with the spectra. One of the main tools is the linearization of the problem in a suitable Pontryagin space.

1. Introduction

The main aim of the present paper is to investigate spectral properties of Sturm–Liouville problems with energy-dependent potentials given by the differential equations

$$-y'' + qy + 2\lambda py = \lambda^2 y \tag{1.1}$$

on (0,1) and some boundary conditions. Here p is a real-valued function from $L_2(0,1)$, q is a real-valued distribution from the Sobolev space $W_2^{-1}(0,1)$, and $\lambda \in \mathbb{C}$ is a spectral parameter. (A detailed definition of the spectral problem of interest will be given in the next section).

The spectral equation (1.1) is of importance in classical and quantum mechanics. For example, such problems arise in solving the Klein–Gordon equations, which describe the motion of massless particles such as photons (see [13, 26]). Sturm–Liouville energy-dependent equations are also used for modeling vibrations of mechanical systems in viscous media (see [39]). Note that in such models the spectral parameter λ is related to the energy of the system, and this motivates the terminology "energy-dependent" used for the spectral problem of the form (1.1).

The equations under study were also considered on the line and discussed in the context of the inverse scattering theory (see, e.g. [12, 23, 32, 1, 14, 21, 36], and [9] for a more extensive reference list). Some of their spectral properties in this context were established in [24]. The spectral problem (1.1) on an interval with $p \in W_2^1[0,\pi]$ and $q \in L_2[0,\pi]$ and with general boundary conditions was also studied by I. Guseinov and I. Nabiev in [8, 25]. An interesting approach to the spectral analysis of problems under consideration suggested by P. Jonas [13] and H. Langer, B. Najman, and C. Tretter [18, 26, 19] uses the theory of Krein spaces (i.e. spaces with indefinite scalar products).

In the present paper, we consider (1.1) under minimal smoothness assumptions on the real-valued potentials p and q. As equation (1.1) contains terms depending on the spectral parameter λ and its square λ^2, the spectral problem of interest is better understood as that for the quadratic operator pencil related to (1.1). And indeed, some of the spectral properties of the Sturm–Liouville energy-dependent equations (1.1) are derived in this paper from the general spectral theory of polynomial operator pencils (see [22]) and

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some by the direct analysis of the corresponding quadratic operator pencil. We also prove equivalence of the spectral problem for (1.1) and that for its linearization $\mathcal{L}$. The operator $\mathcal{L}$ turns out to be self-adjoint in a suitably defined Pontryagin space, which provides some further properties of the operator pencil T.

We also introduce the notion of the norming constants for the problem (1.1). Norming constants are a significant tool in the inverse spectral theory (see e.g. [20, 28]). Therefore it is important to define these quantities in the most convenient way. For real and simple eigenvalues the definition of these quantities is analogous to that for the standard Sturm–Liouville operators. However, since the problem (1.1) can also have non-real and/or non-simple eigenvalues, our definition is more general. As in the classical Sturm–Liouville theory, the norming constants are related to the spectra of (1.1) under two different sets of boundary conditions; we derive the explicit formula in Section 5. We also find sufficient conditions for simplicity of the spectra. The obtained results have their important applications in the inverse problems of reconstruction of the potentials p and q from two spectra or from one spectrum and a set of norming constants (see [9, 30]).

The paper is organized as follows. In the next section we formulate the spectral problem under study as that for the corresponding operator pencil T and recall some notions from the operator pencil theory. In Section 3, we analyze the operator pencil T and obtain some of its spectral properties. We construct a linearization $\mathcal{L}$ of the spectral problem for T in Section 4. The operator $\mathcal{L}$ is considered in a specially defined Pontryagin space (i.e. in a space with indefinite inner product) and is shown to be self-adjoint therein. This gives more spectral properties of $\mathcal{L}$ and so of T. In Section 5, we introduce the notion of norming constants for the problem under study and derive some relations for these quantities. In Section 6, we obtain sufficient conditions for the spectra of the problems (1.1) under two types of boundary conditions to be real and simple.

Notations. Throughout the paper, $\rho(T)$, $\sigma(T)$ and $\sigma_{\mathrm{p}}(T)$ denote the resolvent set, the spectrum and the point spectrum of a linear operator or a quadratic operator pencil T. The superscript t will signify the transposition of vectors and matrices, e.g. $(c_1, c_2)^{\mathrm{t}}$ is the column vector $\binom{c_1}{c_2}$. The scalar product in $L_2(0,1)$ is denoted by $(\cdot,\cdot)_{L_2}$.

2. Preliminaries

Consider equation (1.1) subject to the Dirichlet boundary conditions

$$y(0) = y(1) = 0. \tag{2.1}$$

Notice that other separate boundary conditions can be treated similarly; in particular, in Sections 5 and 6 we shall consider (1.1) under the mixed conditions (5.9). We restrict our attention to (2.1) merely in order to enlighten the ideas and avoid unessential technicalities.

Equation (1.1) depends on the spectral parameter λ non-linearly. Thus to formulate the spectral problem of interest rigorously we should regard (1.1) as a spectral problem for an operator pencil. To start with, consider the differential expression

$$\ell(y) := -y'' + qy.$$

As q is a real-valued distribution from $W_2^{-1}(0,1)$, we need to explain how $\ell(y)$ is defined. The simplest and most convenient way uses the method of regularization by quasi-derivatives (see, e.g. [33, 34]) that proceeds as follows. Take a real-valued $r \in L_2(0,1)$ such that $q = r'$ in the distributional sense and for every absolutely continuous function y denote by $y^{[1]} := y' - ry$ its *quasi-derivative*. We then define ℓ as

$$\ell(y) = -\big(y^{[1]}\big)' - ry^{[1]} - r^2 y$$

on the domain

$$\operatorname{dom}\ell = \{y \in AC(0,1) \mid y^{[1]} \in AC(0,1),\ \ell(y) \in L_2(0,1)\}.$$

A direct verification shows that under this definition $\ell(y)$ coincides with $-y'' + qy$ in the distributional sense. Observe also that for every f from $L_2(0,1)$, every complex a, b and every x_0 from $[0,1]$ the equation $\ell(y) = \mu y + f$ possesses a unique solution satisfying the initial conditions $y(x_0) = a$ and $y^{[1]}(x_0) = b$ (see, e.g. [33]).

Denote by A the operator acting via

$$Ay := \ell(y)$$

on the domain

$$\operatorname{dom} A := \{y \in \operatorname{dom} \ell \mid y(0) = y(1) = 0\}.$$

For regular q, the operator A is a standard Sturm–Liouville operator with potential q and the Dirichlet boundary conditions. It was shown in [33, 34] that if $q \in W_2^{-1}(0,1)$ is real-valued, then the operator A is self-adjoint, bounded below and has a simple discrete spectrum.

Remark 2.1. Recall that an operator S is said to possess discrete spectrum if $\sigma(S)$ consists of isolated points, each of which is an eigenvalue of finite algebraic multiplicity. By Theorem III.6.29 of [15], S has discrete spectrum if its resolvent is compact for one (and then for all) $\lambda \in \rho(S)$.

Next we denote by B the operator of multiplication by the function $2p \in L_2(0,1)$, by I the identity operator, and define the quadratic operator pencil T as

$$T(\lambda) := \lambda^2 I - \lambda B - A, \quad \lambda \in \mathbb{C}, \tag{2.2}$$

on the λ-independent domain $\operatorname{dom} T := \operatorname{dom} A$. Then the spectral problem (1.1), (2.1) can be regarded as the spectral problem for the operator pencil T. Properties of the operators A and B guarantee that the pencil T is well defined; more exactly, the following statement holds true.

Proposition 2.2. *For every fixed $\lambda_0 \in \mathbb{C}$ the operator $T(\lambda_0)$ is closed on the domain* $\operatorname{dom} T$ *and has discrete spectrum.*

Proof. Since the domain of the operator A consists only of bounded functions, we have that $\operatorname{dom} B \supset \operatorname{dom} A$. This immediately implies that the operator $T(\lambda_0)$ is well defined for every $\lambda_0 \in \mathbb{C}$.

Let us fix $\lambda_0 \in \mathbb{C}$. Take an arbitrary $\mu \in \rho(A)$ and denote by φ_- and φ_+ solutions of the equation $\ell(y) = \mu y$ satisfying the boundary conditions $\varphi_-(0) = 0$, $\varphi_-^{[1]}(0) = 1$ and $\varphi_+(1) = 0$, $\varphi_+^{[1]}(1) = 1$. Then the Green function of $A - \mu I$, i.e. the kernel of the operator $(A - \mu I)^{-1}$ is equal to

$$k_0(x,s) := \begin{cases} \varphi_+(x)\varphi_-(s)/W, & \text{when } x > s, \\ \varphi_-(x)\varphi_+(s)/W, & \text{when } x \le s, \end{cases}$$

where $W = \varphi_+(x)\varphi_-^{[1]}(x) - \varphi_-(x)\varphi_+^{[1]}(x)$ is the Wronskian of the solutions φ_+ and φ_-. In particular, the Green function is continuous on the square $\Omega := [0,1] \times [0,1]$. It follows that the operator $(\lambda_0^2 I - \lambda_0 B)(A - \mu I)^{-1}$ is an integral one with the kernel k given by

$$k(x,s) = (\lambda_0^2 - 2\lambda_0 p(x)) k_0(x,s).$$

As k is square integrable on Ω, the operator $(\lambda_0^2 I - \lambda_0 B)(A - \mu I)^{-1}$ is of the Hilbert–Schmidt class and thus $\lambda_0^2 I - \lambda_0 B$ is A–compact (see [15, Ch. IV]). In view of Theorem IV.1.11 of [15] the operator $T(\lambda_0)$ is closed on $\operatorname{dom} A$. Moreover, Theorem IV.5.35 of [15] yields coincidence of the essential spectra of the operators A and $T(\lambda_0)$. As A has discrete spectrum, we get that $\sigma_{\mathrm{ess}} T(\lambda_0) = \sigma_{\mathrm{ess}}(A) = \emptyset$ and thus the spectrum of $T(\lambda_0)$ is discrete. $\square$

Let us now recall some notions of the spectral theory of operator pencils; see [22].

An *operator pencil* T is an operator-valued function on $\mathbb{C}$. The *spectrum* of an operator pencil T is the set $\sigma(T)$ of all $\lambda \in \mathbb{C}$ such that $T(\lambda)$ is not boundedly invertible, i.e.

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid 0 \in \sigma(T(\lambda))\}.$$

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if $T(\lambda)y = 0$ for some non-zero function $y \in \operatorname{dom} T$, which is then the corresponding *eigenfunction*. The eigenvalues of T constitute its *point spectrum* $\sigma_{\mathrm{p}}(T)$, i.e.

$$\sigma_{\mathrm{p}}(T) = \{\lambda \in \mathbb{C} \mid 0 \in \sigma_{\mathrm{p}}(T(\lambda))\}.$$

The set

$$\rho(T) := \mathbb{C} \setminus \sigma(T)$$

is the *resolvent set* of an operator pencil T.

Vectors $y_1, \dots, y_{m-1}$ are said to be associated with an eigenvector y_0 corresponding to an eigenvalue λ if

$$\sum_{k=0}^{j} \frac{1}{k!} T^{(k)}(\lambda) y_{j-k} = 0, \quad j = 1, \dots, m-1.$$

Here $T^{(k)}$ denotes the k-th derivative of T with respect to λ. The number m is called the length of the chain $y_0, \dots, y_{m-1}$ of an eigen- and associated vectors. The maximal length of a chain starting with an eigenvector y_0 is called the *algebraic multiplicity* of an eigenvector y_0.

For an eigenvalue λ of T the dimension of the null-space of $T(\lambda)$ is called the *geometric multiplicity* of λ. The eigenvalue is said to be *geometrically simple* if its geometric multiplicity equals to one.

For the pencil T of (2.2) the operator $-T(\lambda_0)$ is a Sturm–Liouville operator with potential $q + 2\lambda_0 p - \lambda_0^2$ under the Dirichlet boundary conditions, whence the dimension of its null-space is at most one. Therefore all the eigenvalues of the pencil T under study are geometrically simple, and then the *algebraic multiplicity* of an eigenvalue is the algebraic multiplicity of the corresponding eigenvector. (If the eigenvalue λ is not geometrically simple, its algebraic multiplicity is the number of vectors in the corresponding canonical system, see [22, 16]). An eigenvalue is said to be *algebraically simple* (or just *simple*) if its algebraic multiplicity is one.

3. Spectral properties of the operator pencil

In this section we discuss some basic spectral properties of the operator pencil T. We start with the following lemmas.

Lemma 3.1. *The spectrum of the operator pencil T consists entirely of eigenvalues.*

Proof. By definition, $\lambda_0 \in \mathbb{C}$ belongs to the spectrum of the operator pencil T if and only if $0 \in \sigma(T(\lambda_0))$. Since $\sigma(T(\lambda_0)) = \sigma_{\mathrm{p}}(T(\lambda_0))$ (see Proposition 2.2), every λ_0 in the spectrum of T is its eigenvalue. $\square$

Lemma 3.2. *The resolvent set of the operator pencil T is not empty.*

Proof. As the operator A is lower semibounded, a number μ exists such that the operator $A + \mu^2 I$ is positive. Let us show that then the number $i\mu$ belongs to the resolvent set $\rho(T)$ of the operator pencil T. Suppose it does not; then, by the previous lemma, $i\mu$ is an eigenvalue of T and there exists a nonzero eigenfunction y. The equality $T(i\mu)y = 0$ yields

$$((A + \mu^2 I)y, y)_{L_2} + i\mu(By, y)_{L_2} = 0,$$

which contradicts positivity of $A + \mu^2 I$. Therefore $i\mu$ belongs to $\rho(T)$ and the lemma is proved. $\square$

Using these lemmas, we can prove discreteness of the spectrum of the operator pencil T.

Lemma 3.3. *The spectrum of the operator pencil T is a discrete subset of $\mathbb{C}$.*

Proof. Let us take some $\lambda_0 \in \rho(T)$ and rewrite $T(\lambda)$ as

$$T(\lambda) = T(\lambda_0) + (\lambda - \lambda_0)[2\lambda_0 I - B] + (\lambda - \lambda_0)^2 I.$$

Set $\widehat{B} := 2\lambda_0 I - B$, $\widehat{A} := T(\lambda_0)$, $\mu := \lambda - \lambda_0$ and consider the pencil $\widehat{T}(\mu) := T(\lambda)T^{-1}(\lambda_0)$, which can be written as

$$\widehat{T}(\mu) := I + \mu \widehat{B}\widehat{A}^{-1} + \mu^2 \widehat{A}^{-1}.$$

The arguments analogous to those used in the proof of Proposition 2.2 yield that the operator $\mu\widehat{B}\widehat{A}^{-1} + \mu^2\widehat{A}^{-1}$ belongs to the Hilbert–Schmidt class and so is compact. Then applying the Gohberg theorem on analytic operator-valued functions [7, Ch. I] to the pencil $I - S(\mu)$ with $S(\mu) := -(\mu\widehat{B}\widehat{A}^{-1} + \mu^2\widehat{A}^{-1})$, we obtain that for all $\mu \in \mathbb{C}$ except for possibly some isolated points the operator $\widehat{T}(\mu)$ is boundedly invertible, while these isolated points are eigenvalues of $\widehat{T}$ of finite algebraic multiplicity. This shows that the spectrum of $\widehat{T}$ is a discrete subset of $\mathbb{C}$.

Assume $\lambda \in \sigma(T)$, which by Lemma 3.1 means that $\lambda \in \sigma_{\mathrm{p}}(T)$, and let x be the corresponding eigenfunction. Then $y = T^{-1}(\lambda_0)x$ is an eigenfunction of $\widehat{T}$ corresponding to the eigenvalue $\mu = \lambda - \lambda_0$. Therefore,

$$\lambda \in \sigma(T) \Rightarrow \mu = \lambda - \lambda_0 \in \sigma(\widehat{T}).$$

Observe also that if $\lambda \in \rho(T)$, i.e. if $T(\lambda)$ is boundedly invertible, then the operator $T(\lambda_0)T^{-1}(\lambda)$ is closable, defined on the whole space $L_2(0,1)$, and thus bounded by the closed graph theorem [15, Theorem III.5.20]. Direct verification shows that it is the inverse operator of $\widehat{T}(\mu)$ with $\mu = \lambda - \lambda_0$. Therefore,

$$\lambda \in \rho(T) \Rightarrow \mu = \lambda - \lambda_0 \in \rho(\widehat{T}).$$

These two implications give the equivalence

$$\lambda \in \sigma(T) \Leftrightarrow \mu = \lambda - \lambda_0 \in \sigma(\widehat{T});$$

thus the spectrum of the operator pencil T is discrete in $\mathbb{C}$ along with the spectrum of $\widehat{T}$. □

We summarize the above considerations in the following theorem.

Theorem 3.4. *The spectrum of the operator pencil T of* (2.2) *is a discrete subset of $\mathbb{C}$ and consists of geometrically simple eigenvalues.*

Remark 3.5. Without loss of generality we may and shall assume further in this paper that 0 is not in $\sigma(T)$ or, equivalently, that the operator A is boundedly invertible. In view of the above theorem, we can always achieve this by shifting the spectral parameter by a real number.

As was noted in Section 2, every eigenvalue of T is geometrically simple. However, in general the spectrum of the operator pencil T is not necessarily real or algebraically simple as the following example demonstrates.

Example 3.6. *Consider the operator pencil*

$$T(\lambda) := \lambda^2 - 2\lambda\pi + \frac{d^2}{dx^2} + 5\pi^2 = (\lambda - \pi)^2 + 4\pi^2 + \frac{d^2}{dx^2},$$

i.e. the pencil T with $p \equiv \pi$ and $q = r' \equiv -5\pi^2$. Then $\lambda_{\pm 1} = (1 \pm i\sqrt{3})\pi$ are complex conjugate eigenvalues of this operator pencil, while $\lambda_2 = \pi$ is its eigenvalue of algebraic multiplicity at least 2, since $y_0 = \sin 2\pi x$ and $y_1 \equiv 0$ form the corresponding chain of eigen- and associated vectors.

4. Linearization and its properties

In this section we shall recast the spectral problem for the operator pencil T as a spectral problem for some linear operator $\mathcal{L}$ and show equivalence of these problems. Considering $\mathcal{L}$ in a specially defined Pontryagin space will then reveal some further spectral properties of the pencil T.

4.1. **Linearization.** Setting $u_1 := y$ and $u_2 := \lambda y$, we recast the problem (1.1), (2.1) as the first order system

$$\begin{aligned} u_2 &= \lambda u_1, \\ Au_1 + Bu_2 &= \lambda u_2. \end{aligned} \tag{4.1}$$

The system (4.1) is the spectral problem for the operator

$$\mathcal{L}_0 := \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}. \tag{4.2}$$

Therefore the spectral properties of the operator pencil T should be closely related to those of the operator $\mathcal{L}_0$. It is natural to consider the latter in the so-called energy space $\mathcal{E}$, which we next define.

Recall that the operator A is supposed to be boundedly invertible (see Remark 3.5). Denote by H_α, $\alpha \in \mathbb{R}$, the scale of Hilbert spaces generated by the operator A. Thus the space H_0 coincides with $L_2(0,1)$, for $\alpha > 0$ the space H_α is the domain of the operator $|A|^\alpha$ endowed with the norm $\|x\|_\alpha := \||A|^\alpha x\|$, and for $\alpha < 0$ the space H_α is the completion of H_0 by the norm $\|\cdot\|_\alpha$. Since the operator A has compact resolvent, for every $\beta < \alpha$ the embedding $H_\alpha \hookrightarrow H_\beta$ is compact. Note that for any $\alpha < \theta$ the restriction of the operator A^α to H_θ is a homeomorphism between H_θ and $H_{\theta-\alpha}$. Similarly, for $\alpha > \theta$ the extension by continuity of the operator A^α as a mapping from H_θ to $H_{\theta-\alpha}$ is a homeomorphism.

Introduce the Hilbert space $(\mathcal{E}, (\cdot,\cdot)_\mathcal{E})$, where $\mathcal{E} := H_{1/2} \times H_0$ and the scalar product $(\cdot,\cdot)_\mathcal{E}$ is given by

$$(\mathbf{x},\mathbf{y})_\mathcal{E} = (|A|^{1/2}x_1, |A|^{1/2}y_1)_{L_2} + (x_2, y_2)_{L_2}$$

for every $\mathbf{x} = (x_1,x_2)^{\mathrm{t}}$ and $\mathbf{y} = (y_1,y_2)^{\mathrm{t}}$ in $\mathcal{E}$. Then the operator $\mathcal{L}_0$ of (4.2) is well defined on the domain

$$\operatorname{dom}\mathcal{L}_0 := \{(u_1,u_2)^{\mathrm{t}} \mid u_1 \in H_1;\ u_2 \in H_{1/2} \cap \operatorname{dom} B\}.$$

However, $\mathcal{L}_0$ is not closed on this domain. To describe its closure, we need the following auxiliary result.

Lemma 4.1. *The operator B extends by continuity to a compact mapping $\widetilde{B}$ from $H_{1/2}$ to $H_{-1/2}$.*

Proof. Using the arguments analogous to those in the proof of Proposition 2.2 one can show that the operator $BA^{-1} : H_0 \to H_0$ is compact. This yields compactness of B as a mapping from H_1 to H_0.

Observe that $H_1 \hookrightarrow H_0 \hookrightarrow H_{-1}$ is a rigged Hilbert space triple ([3, Ch. 1], [6, Ch. 1]). Denoting by $\langle\cdot,\cdot\rangle$ the pairing between H_{-1} and H_1, we get for $x \in H_1$ that

$$\begin{aligned} \|Bx\|_{-1} &= \sup_{y\in H_1 \,:\, \|y\|_1=1} |\langle Bx, y\rangle| = \sup_{y\in H_1 \,:\, \|y\|_1=1} |(Bx,y)_{L_2}| \\ &= \sup_{y\in H_1 \,:\, \|y\|_1=1} |(x,By)_{L_2}| \le \|x\|_0 \|B\|_{H_1\to H_0}. \end{aligned}$$

This implies that B extends by continuity to a bounded mapping from H_0 to H_{-1}. Now, using the interpolation theorem for compact operators [27], we obtain compactness of $\widetilde{B} : H_{1/2} \to H_{-1/2}$. □

Recall also that A extends by continuity to a homeomorphism $\widetilde{A} : H_{1/2} \to H_{-1/2}$.

Lemma 4.2. ([10]). *The operator $\mathcal{L}_0$ is closable in $\mathcal{E}$ and the closure $\mathcal{L}$ is given by the formulas*

$$\begin{gathered}\mathcal{L}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \widetilde{A}x_1 + \widetilde{B}x_2 \end{pmatrix}, \\ \operatorname{dom}\mathcal{L} = \left\{ (x_1, x_2)^{\mathrm{t}} \mid x_1, x_2 \in H_{1/2},\ \widetilde{A}x_1 + \widetilde{B}x_2 \in H_0 \right\}.\end{gathered} \tag{4.3}$$

We are going to show the coincidence of the spectra of T and $\mathcal{L}$. To start with, we prove that the spectrum of $\mathcal{L}$ is discrete.

Lemma 4.3. *The spectrum of the operator $\mathcal{L}$ is discrete.*

Proof. By Remark 2.1, to show discreteness of the spectrum of $\mathcal{L}$ it is enough to establish that its inverse $\mathcal{L}^{-1}$ is compact.

For $\mathbf{x} = (x_1, x_2)^{\mathrm{t}}$ from $\operatorname{dom}\mathcal{L}$ and $\mathbf{y} = (y_1, y_2)^{\mathrm{t}}$ from $\mathcal{E}$, the equation $\mathcal{L}\mathbf{x} = \mathbf{y}$ can be recast as the system

$$\begin{aligned} x_2 &= y_1, \\ \widetilde{A}x_1 + \widetilde{B}x_2 &= y_2. \end{aligned}$$

Since the operator $\widetilde{A}$ is a homeomorphism of $H_{1/2}$ and $H_{-1/2}$, we conclude that the operator $\mathcal{L}$ is boundedly invertible and its inverse $\mathcal{L}^{-1}$ is given by the matrix

$$\mathcal{L}^{-1} = \begin{pmatrix} -\widetilde{A}^{-1}\widetilde{B} & A^{-1} \\ I & 0 \end{pmatrix}$$

on $\mathcal{E}$. Next we prove that $\mathcal{L}^{-1} : \mathcal{E} \to \mathcal{E}$ is a compact operator. To do this, we show compactness of all entries of the corresponding matrix.

In view of Lemma 4.1, the operator $\widetilde{B} : H_{1/2} \to H_{-1/2}$ is compact. Since the operator $\widetilde{A}^{-1} : H_{-1/2} \to H_{1/2}$ is bounded, this yields compactness of $\widetilde{A}^{-1}\widetilde{B} : H_{1/2} \to H_{1/2}$.

Since the operator $A^{-1} : H_0 \to H_1$ is bounded and the embedding $H_1 \hookrightarrow H_{1/2}$ is compact, the operator $A^{-1} : H_0 \to H_{1/2}$ is compact as the composition of a bounded operator and a compact operator.

The lower-right entry I is an embedding of the space $H_{1/2}$ into H_0 and thus it is a compact operator.

These observations yield compactness of $\mathcal{L}^{-1}$ and complete the proof. □

For $\lambda \in \mathbb{C}$ we set

$$\widetilde{T}(\lambda) := \lambda^2 I - \lambda\widetilde{B} - \widetilde{A}. \tag{4.4}$$

and consider $\widetilde{T}(\lambda)$ as an operator from $H_{1/2}$ to $H_{-1/2}$.

Theorem 4.4. ([10], *see also* [38, 37]). *The spectrum of the operator $\mathcal{L}$ coincides with the spectrum $\sigma(\widetilde{T})$ of the operator pencil $\widetilde{T}$. For every nonzero $\lambda \in \rho(\widetilde{T})$ the following representation holds:*

$$(\mathcal{L} - \lambda\mathcal{I})^{-1} = \begin{pmatrix} -\lambda^{-1}(\widetilde{T}^{-1}(\lambda)\widetilde{A} + I) & -T^{-1}(\lambda) \\ -\widetilde{T}^{-1}(\lambda)\widetilde{A} & -\lambda T^{-1}(\lambda) \end{pmatrix}. \tag{4.5}$$

Now we can show coincidence of the spectra of the operator $\mathcal{L}$ and of the operator pencil T. In view of Lemmas 3.3 and 4.3, it is sufficient to show coincidence of the corresponding eigenvalues.

Theorem 4.5. *The eigenvalues of the operator pencil T coincide with those of the operator $\mathcal{L}$ counting geometric and algebraic multiplicities.*

Proof. Observe firstly that for every $\lambda_0 \in \mathbb{C}$ the operator $\widetilde{T}(\lambda_0)$ is an extension of $T(\lambda_0)$. Therefore if for some $\mu \in \rho(T(\lambda_0)) \cap \rho(\widetilde{T}(\lambda_0))$ one has $(\widetilde{T}(\lambda_0) - \mu)u \in H_0$, then u belongs to H_1, i.e. to $\operatorname{dom} T$. We shall use this remark in our further discussions.

Assume that $\lambda_0 \in \mathbb{C}$ is an eigenvalue of T with the corresponding chain of eigen- and associated vectors $y_0, y_1, \dots, y_{m-1}$. By definition, this means that

$$(\lambda_0^2 - \lambda_0 B - A)y_k + (2\lambda_0 - B)y_{k-1} + y_{k-2} = 0$$

for $k = 0, \dots, m-1$ with y_{-1}, y_{-2} being zero. A direct verification shows that these equalities are equivalent to the following ones:

$$(\mathcal{L} - \lambda_0)\mathbf{y}_k = \mathbf{y}_{k-1}, \quad k = 0, \dots, m-1,$$

with $\mathbf{y}_k := (y_k, \lambda_0 y_k + y_{k-1})^{\mathrm{t}}$. In particular, λ_0 is an eigenvalue of $\mathcal{L}$ and the vectors $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{m-1}$ belong to the domain of $\mathcal{L}$ and so form a chain of eigen- and associated vectors of $\mathcal{L}$ corresponding to λ_0.

Next suppose that $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $\mathcal{L}$ with the corresponding chain of eigen- and associated vectors $\mathbf{y}_0, \dots, \mathbf{y}_{m-1}$ of length m. By definition, $(\mathcal{L} - \lambda_0)\mathbf{y}_k = \mathbf{y}_{k-1}$ (with $\mathbf{y}_{-1} := 0$) or, setting $\mathbf{y}_k := (y_{k,1}, y_{k,2})^{\mathrm{t}}$,

$$\begin{aligned} -\lambda_0 y_{k,1} + y_{k,2} &= y_{k-1,1}, \\ \widetilde{A} y_{k,1} + (\widetilde{B} - \lambda_0) y_{k,2} &= y_{k-1,2}. \end{aligned}$$

This gives that $y_{k,2} = \lambda_0 y_{k,1} + y_{k-1,1}$ and

$$(\lambda_0^2 - \lambda_0 \widetilde{B} - \widetilde{A}) y_{k,1} + (2\lambda_0 - \widetilde{B}) y_{k-1,1} + y_{k-2,1} = 0$$

for $k = 0, \dots, m-1$, with $\mathbf{y}_{-2} := 0$. Since $\mathbf{y}_k = (y_{k,1}, y_{k,2})^{\mathrm{t}} \in \operatorname{dom} \mathcal{L}$, we have that $y_{k,1}$ is from $H_{1/2}$. Thus the last equality yields that $y_{0,1}, y_{1,1}, \dots, y_{m-1,1}$ is a chain of eigen- and associated vectors of the operator pencil $\widetilde{T}$ corresponding to the eigenvalue λ_0.

Next we prove by induction that all the vectors $y_{0,1}, y_{1,1}, \dots, y_{m-1,1}$ belong to H_1 and, therefore, form a chain of eigen- and associated vectors of T corresponding to λ_0. Take $\mu \in \rho(T(\lambda_0)) \cap \rho(\widetilde{T}(\lambda_0))$ and observe that $(\widetilde{T}(\lambda_0) - \mu) y_{0,1} = -\mu y_{0,1} \in H_0$. In view of the remark made at the beginning of the proof, this yields that $y_{0,1}$ belongs to H_1 and so it is an eigenvector of T corresponding to λ_0. Now suppose that $y_{j,1}$ belongs to H_1 for every $j < k$; then

$$(\widetilde{T}(\lambda_0) - \mu) y_{k,1} = -\mu y_{k,1} - (2\lambda_0 - \widetilde{B}) y_{k-1,1} + y_{k-2,1} \in H_0.$$

By the same arguments we deduce that $y_{k,1}$ is from H_1; thus the chain of eigen- and associated vectors of $\mathcal{L}$ generates the corresponding chain for T.

The above reasonings show that there is a one-to-one correspondence between the chains of eigen- and associated vectors of T and $\mathcal{L}$ corresponding to the same eigenvalues, thus establishing the claim. $\square$

Coincidence of the eigenvalues of $\mathcal{L}$ and T can be proved in another way. Observe that Lemma 4.3 and Theorem 4.4 imply that the spectrum of $\widetilde{T}$ is discrete. But it is known (see [35]) that the discrete parts of the spectra of T and $\widetilde{T}$ coincide. In view of Proposition 3.3, this yields that $\sigma(T) = \sigma(\widetilde{T})$ and so $\sigma(T) = \sigma(\mathcal{L})$.

4.2. **The Pontryagin space properties of $\mathcal{L}$.** Now we show that the linearization $\mathcal{L}$ is self-adjoint in some Pontryagin space. Consider the operator $J = P_+ - P_-$, where P_+ and P_- are the orthogonal projectors onto the spectral subspaces of A corresponding to the positive and negative parts of the spectrum respectively. Set $\mathcal{J} := \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$ and define an inner product

$$[\mathbf{x}, \mathbf{y}] := (\mathcal{J}\mathbf{x}, \mathbf{y})_{\mathcal{E}} = (J|A|^{1/2} x_1, |A|^{1/2} y_1)_{L_2} + (x_2, y_2)_{L_2}$$

for every $\mathbf{x} = (x_1, x_2)^{\mathrm{t}}$ and $\mathbf{y} = (y_1, y_2)^{\mathrm{t}}$ from $\mathcal{E}$. The number of negative eigenvalues of the operator $\mathcal{J}$ equals that of J, which in turn is the number of negative eigenvalues of A. But the operator A is lower semibounded and the number of its negative eigenvalues is finite, say κ. Therefore the product $[\cdot,\cdot]$ is indefinite and the space $\Pi := (\mathcal{E}, [\cdot,\cdot])$ is a Pontryagin space of negative index κ. Note that the topology of Π coincides with that of $\mathcal{E}$.

Consider the operator $\mathcal{L}_0$ in the space Π. For every $\mathbf{x} = (x_1, x_2)^{\mathrm{t}}$ from $\operatorname{dom}\mathcal{L}_0$ we have

$$\begin{aligned}[\mathcal{L}_0\mathbf{x}, \mathbf{x}] &= (J|A|^{1/2}x_2, |A|^{1/2}x_1)_{L_2} + (Ax_1, x_2)_{L_2} + (Bx_2, x_2)_{L_2} \\ &= (Ax_1, x_2)_{L_2} + (x_2, Ax_1)_{L_2} + (Bx_2, x_2)_{L_2} \\ &= 2\mathrm{Re}(Ax_1, x_2)_{L_2} + (Bx_2, x_2)_{L_2}.\end{aligned}$$

Clearly, this shows that $[\mathcal{L}_0\mathbf{x}, \mathbf{x}]$ is real and thus the operator $\mathcal{L}_0$ is symmetric in Π. This together with the fact that the spectrum of $\mathcal{L}$ is discrete (see Lemma 4.3) gives the following proposition (see [11, Theorem II.9.1]).

Proposition 4.6. *The operator $\mathcal{L}$ is self-adjoint in the Pontryagin space Π.*

Using this result and properties of self-adjoint operators in Pontryagin spaces (see Proposition A.1) we obtain that the spectrum of the operator $\mathcal{L}$ is real with possible exception of at most κ pairs of complex-conjugate eigenvalues λ and $\bar{\lambda}$. The algebraic multiplicity of each eigenvalue of $\mathcal{L}$ can not exceed $2\kappa + 1$.

When the operator A is positive, Π is a Hilbert space (the negative index $\kappa = 0$). Then $\mathcal{L}$ is a self-adjoint operator in a Hilbert space and thus its spectrum is real and the algebraic and geometric multiplicities of the eigenvalues are equal. The eigenvalues of $\mathcal{L}$ are geometrically simple by Theorems 3.4 and 4.5; therefore, the spectrum of the operator $\mathcal{L}$ is then real and simple.

Summing up all these results and using the connection between the spectral problems for the operator pencil T and for the operator $\mathcal{L}$, we derive further spectral properties of T.

Theorem 4.7. *Let κ be the number of negative eigenvalues of the operator A. Then*

(i) *the spectrum of the operator pencil T is real with possible exception of at most κ pairs of complex-conjugate eigenvalues λ and $\bar{\lambda}$;*

(ii) *denote by $m(\lambda)$ the algebraic multiplicity of the eigenvalue λ; then*

$$\sum_{\mathrm{Im}\lambda>0} m(\lambda) + \sum_{\mathrm{Im}\lambda=0} \left[\frac{m(\lambda)}{2}\right] \le \kappa.$$

In particular, the algebraic multiplicity of every eigenvalue of T does not exceed $2\kappa+1$ and the number of non-simple real eigenvalues of T does not exceed κ.

Corollary 4.8. *If the operator A is positive, then the spectrum of the operator pencil T is real and simple.*

5. Norming constants

5.1. **Notion of norming constants.** In this section we introduce the notion of norming constants for the operator pencil T and establish some of their properties. In the case where the pencil T has only real and simple eigenvalues the norming constants were used in [9] to solve the inverse spectral problem of determining the potentials p and q of the pencil.

We say that a $k \times k$ matrix is upper (lower) anti-triangular if all its elements under (above) anti-diagonal are zero. Denote by $M^+[\gamma_1, \gamma_2, \dots, \gamma_k]$ (resp. by $M^-[\gamma_1, \gamma_2, \dots, \gamma_k]$)

a Hankel upper (resp. lower) anti-triangular matrices given by

$$M^{+}[\gamma_1,\gamma_2,\dots\gamma_k]=\begin{pmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_k\\ \gamma_2 & & \ddots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \gamma_k & 0 & \cdots & 0\end{pmatrix}$$

and

$$M^{-}[\gamma_1,\gamma_2,\dots,\gamma_k]=\begin{pmatrix}0 & \cdots & 0 & \gamma_1\\ \vdots & \ddots & \ddots & \gamma_2\\ 0 & \ddots & \ddots & \vdots\\ \gamma_1 & \gamma_2 & \cdots & \gamma_k\end{pmatrix}.$$

We say that the sequence $\gamma_1,\dots,\gamma_k$ is *associated* with the matrices $M^{\pm}[\gamma_1,\gamma_2,\dots,\gamma_k]$.

In this section we shall often work with infinite block-diagonal matrices with upper (lower) anti-triangular blocks of two types. The first type blocks are just upper (lower) anti-triangular Hankel matrices. The second type blocks have the form

$$\begin{pmatrix}0 & B_1\\ B_2 & 0\end{pmatrix}, \tag{5.1}$$

where B_1 is an upper (lower) anti-triangular Hankel matrix and B_2 is its complex conjugate. Denote the diagonal blocks of such an infinite matrix M by M_n, $n\in\mathbb{Z}$. To every block M_n of size m there is associated a number sequence of length m; these finite sequences together form an infinite sequence $(\gamma_k)_{k\in\mathbb{Z}}$ *associated* with M.

Now we list the eigenvalues λ_k, $k\in\mathbb{Z}$, of the operator pencil T so that

(i) each eigenvalue is repeated according to its multiplicity;
(ii) the real parts of eigenvalues do not decrease, i.e. $\mathrm{Re}\lambda_i\le\mathrm{Re}\lambda_j$ for $i<j$;
(iii) the moduli of the imaginary parts of the eigenvalues with equal real parts do not decrease, i.e. if $\mathrm{Re}\lambda_i=\mathrm{Re}\lambda_j$ for some $i<j$, then $|\mathrm{Im}\lambda_i|\le|\mathrm{Im}\lambda_j|$; if, in addition, $|\mathrm{Im}\lambda_i|=|\mathrm{Im}\lambda_j|$, then $\mathrm{Im}\lambda_i\ge\mathrm{Im}\lambda_j$.

This enumeration ensures that if some λ is an eigenvalue of T of multiplicity m, then there is $n\in\mathbb{Z}$ such that $\lambda=\lambda_n=\lambda_{n+1}=\dots=\lambda_{n+m-1}$. If, moreover, λ is non-real, then $\overline{\lambda}=\lambda_{n+m}=\lambda_{n+m+1}=\dots=\lambda_{n+2m-1}$.

Along with the eigenvalue sequence $(\lambda_k)_{k\in\mathbb{Z}}$ we introduce the sequence $(y_k)_{k\in\mathbb{Z}}$ of vectors from $\operatorname{dom}A$ such that if $\lambda_n=\lambda_{n+1}=\dots=\lambda_{n+m-1}$ is an eigenvalue of T of multiplicity m, then $y_n,y_{n+1},\dots,y_{n+m-1}$ is a chain of eigen- and associated vectors of T corresponding to λ_n defined as follows. Let $y(\,\cdot\,,\lambda)$ be the solution of (1.1) satisfying the initial conditions $y(0,\lambda)=0$ and $y^{[1]}(0,\lambda)=1$; then

$$y_{n+j}(x):=\frac{1}{j!}\frac{\partial^j y(x,\lambda)}{\partial\lambda^j}\bigg|_{\lambda=\lambda_n},\qquad j=0,1,\dots,m-1. \tag{5.2}$$

For complex conjugate eigenvalues $\lambda_n=\lambda_{n+1}=\dots=\lambda_{n+m-1}=\overline{\lambda_{n+m}}=\dots=\overline{\lambda_{n+2m-1}}$ the corresponding vectors obey the relations $y_{n+m+j}=\overline{y_{n+j}}$ for $j=0,\dots,m-1$.

Next we define the *norming constants* α_n, $n\in\mathbb{Z}$, for the operator pencil T as follows. For real eigenvalue $\lambda=\lambda_n=\lambda_{n+1}=\dots=\lambda_{n+m-1}$ of algebraic multiplicity m we put

$$\begin{aligned}\alpha_{n+j}&=(T'(\lambda)(\lambda y_{n+j}+y_{n+j-1}),y_{n+m-1})_{L_2}+(\lambda y_{n+j-1}+y_{n+j-2},y_{n+m-1})_{L_2}\\&\quad+(\lambda y_{n+j}+y_{n+j-1},y_{n+m-2})_{L_2},\quad j=0,\dots,m-1;\end{aligned} \tag{5.3}$$

in this formula y_{n-1} and y_{n-2} are assumed to be zero to simplify the expression. For nonreal eigenvalue $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1} = \overline{\lambda_{n+m}} = \cdots = \overline{\lambda_{n+2m-1}}$

$$\begin{aligned}
\alpha_{n+j} &= \big(T'(\lambda)(\lambda y_{n+j} + y_{n+j-1}), y_{n+2m-1}\big)_{L_2} \\
&\quad + (\lambda y_{n+j-1} + y_{n+j-2}, y_{n+2m-1})_{L_2} \\
&\quad + (\lambda y_{n+j} + y_{n+j-1}, y_{n+2m-2})_{L_2}, \quad j = 0, \ldots, m-1, \\
\alpha_{n+m+j} &= \overline{\alpha_{n+j}}, \quad j = 0, \ldots, m-1;
\end{aligned} \tag{5.4}$$

here we put $y_{n-1} = y_{n-2} = 0$ as well as in (5.3). Defining the norming constants for the operator pencil T in the described way is quite natural. Firstly, note that for real and simple eigenvalues so defined norming constants determine the type of eigenvalues (see [22]) as

$$\big(T'(\lambda_n)y_n, y_n\big)_{L_2} = \frac{\alpha_n}{\lambda_n}.$$

Secondly, if the potential p is identically zero, (1.1) is the spectral equation for the Sturm–Liouville operator A and the given definition of the norming constants for the operator pencil T coincides with the standard definition of the norming constants for A [5]. Further we shall see that so defined norming constants determine the Gramm matrix for the linearization $\mathcal{L}$.

By Theorem 4.5, the sequence $(\lambda_k)_{k\in\mathbb{Z}}$ is an eigenvalue sequence for the operator $\mathcal{L}$. Consider the sequence of vectors $(\mathbf{y}_k)_{k\in\mathbb{Z}}$ in $\mathcal{E}$, such that for the eigenvalue $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1}$ of multiplicity m the vectors $\mathbf{y}_k$, $k = n, \ldots, n+m-1$, are defined by the formulas $\mathbf{y}_n = (y_n, \lambda y_n)^{\mathrm{t}}$ and $\mathbf{y}_j = (y_j, \lambda y_j + y_{j-1})^{\mathrm{t}}$, $j = n+1, \ldots, n+m-1$. From the proof of Theorem 4.5 we know that $\mathbf{y}_n, \mathbf{y}_{n+1}, \ldots, \mathbf{y}_{n+m-1}$ is the chain of eigen- and associated vectors of $\mathcal{L}$ corresponding to the eigenvalue λ and so $(\mathbf{y}_n)_{n\in\mathbb{Z}}$ is the sequence of all eigen- and associated vectors of $\mathcal{L}$.

Put $g_{kl} := [\mathbf{y}_k, \mathbf{y}_l]$ and associate with the operator $\mathcal{L}$ the Gramm matrix $G = (g_{kl})$. Since the root subspaces corresponding to eigenvalues λ_k and λ_l are orthogonal in Π as soon as $\lambda_k \neq \overline{\lambda_l}$, we immediately see that the Gramm matrix G is of block-diagonal form. For real $\lambda = \lambda_n = \cdots = \lambda_{n+m-1}$ and $k, l = n, \ldots, n+m-1$, the equality

$$[(\mathcal{L} - \lambda\mathcal{I})\mathbf{y}_k, \mathbf{y}_l] = [\mathbf{y}_k, (\mathcal{L} - \lambda\mathcal{I})\mathbf{y}_l],$$

yields $g_{ij} = g_{kl}$ provided $i+j = k+l$ and the indices i, j, k, l are between n and $n+m-1$; moreover, $g_{kl} = 0$ if $k+l < 2n+m-1$. Next observe that

$$\begin{aligned}
g_{n+j,n+m-1} &= [\mathbf{y}_{n+j}, \mathbf{y}_{n+m-1}] = (Ay_{n+j,1}, y_{n+m-1,1})_{L_2} + (y_{n+j,2}, y_{n+m-1,2})_{L_2} \\
&= \big(T'(\lambda)(\lambda y_{n+j,1} + y_{n+j-1,1}), y_{n+m-1,1}\big)_{L_2} \\
&\quad + (\lambda y_{n+j-1,1} + y_{n+j-2,1}, y_{n+m-1,1})_{L_2} \\
&\quad + (\lambda y_{n+j,1} + y_{n+j-1,1}, y_{n+m-2,1})_{L_2} = \alpha_{n+j}, \quad j = 0, \ldots, m-1,
\end{aligned} \tag{5.5}$$

where we put $y_{n-1,1} = y_{n-2,1} = 0$ to simplify the expression. Therefore the block G_n of G corresponding to an eigenvalue λ of multiplicity m is a Hankel lower anti-triangular matrix $M^-[\alpha_n, \ldots, \alpha_{n+m-1}]$.

Let now $\lambda = \lambda_n = \cdots = \lambda_{n+m-1} = \overline{\lambda_{n+m}} = \cdots = \overline{\lambda_{n+2m-1}}$. The root subspaces for the eigenvalues λ and $\overline{\lambda}$ are neutral and skewly-linked. Therefore the block of G corresponding to these two root spaces is of the form (5.1) with $m \times m$ matrices B_1 and B_2. Observe that $\mathbf{y}_{n+m+j} = \overline{\mathbf{y}_{n+j}}$; therefore B_2 is complex conjugate of B_1 and

$$g_{k-1,l+m} = [(\mathcal{L} - \lambda\mathcal{I})\mathbf{y}_k, \overline{\mathbf{y}_l}] = [\mathbf{y}_k, \overline{(\mathcal{L} - \lambda\mathcal{I})\mathbf{y}_l}] = g_{k,m+l-1}.$$

The arguments similar to those in the case of real λ yield that $g_{n+j,n+2m-1} = \alpha_{n+j}$ for $j = 0, \ldots, m-1$. It follows that B_1 is a Hankel lower anti-triangular matrix $M^-[\alpha_n, \ldots, \alpha_{n+m-1}]$.

We thus see that the sequence (α_k) of norming constants of the pencil T is associated with the block diagonal Gramm matrix of the system of eigen- and associated vectors $(\mathbf{y}_k)$ of the operator $\mathcal{L}$. This gives that having the Gramm matrix G, we automatically have the sequence (α_k). Having the sequence of norming constants (α_k) and the sequence of the eigenvalues (λ_k) of T we can construct the corresponding matrix G.

The following observation will be used in Section 6.

Lemma 5.1. *Assume that T has only real and simple eigenvalues. Then all the norming constants of T are positive if and only if the operator A is positive.*

Proof. In view of (5.5), under assumption of the lemma the norming constant α_n of T is equal to the element g_{nn} of the Gramm matrix G of $\mathcal{L}$. Therefore α_n are the Pontryagin space norms of the corresponding eigenvectors of $\mathcal{L}$.

Sufficiency. Obviously, if A is positive, then the space Π is a Hilbert space and so $\mathcal{L}$ is a self-adjoint operator in a Hilbert space and all norming constants α_n are positive as the Hilbert space norms of eigenvectors.

Necessity. Suppose that A is not positive. Then Π is a Pontryagin space of finite negativity index $\kappa > 0$, and so by the Pontryagin theorem (see e.g. [4]) there exists a maximal non-positive subspace of Π of dimension κ invariant under $\mathcal{L}$. Therefore $\mathcal{L}$ possesses an eigenvector in this subspace, and the norming constant α_n corresponding to this eigenvector is non-positive. Thus not all norming constants of T are positive, and the proof is complete. □

5.2. **Relations for norming constants.** Next we compute the residues of $(\mathcal{L} - z\mathcal{I})^{-1}$ at an eigenvalue λ in two different ways. Equating the results, we shall obtain some relations for norming constants α_k.

Observe firstly that for N sufficiently large the eigenvalues λ_n with $|n| > N$ are simple. Therefore the corresponding blocks of the matrix G are nonzero scalars equal to the corresponding norming constants tending to 1 as $|n| \to \infty$ [29]. The other blocks of G are nondegenerate lower anti-triangular matrices of two types described at the beginning of Subsection 5.1. Thus G is a block-diagonal matrix whose diagonal blocks G_n are invertible with $\sup \|G_n^{-1}\| < \infty$. As a result G is boundedly invertible; we denote its inverse by D. Next note that the inverse of a lower anti-triangular Hankel matrix is an upper anti-triangular Hankel matrix. Hence D as well as G has a block-diagonal structure but with upper anti-triangular Hankel matrices in blocks. Associate with D the sequence $(\delta_k)_{k\in\mathbb{Z}}$ as explained at the beginning of Subsection 5.1.

Consider firstly a real eigenvalue $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1}$ and observe that the residue of the resolvent of $\mathcal{L}$ at λ is the minus Riesz projector onto the root subspace (cf. e.g. [15, Ch. 1]), so that

$$\operatorname*{res}_{z=\lambda} (\mathcal{L} - z\mathcal{I})^{-1} = -\sum_{k,l=n}^{n+m-1} d_{kl}[\cdot, \mathbf{y}_l]\mathbf{y}_k = -\sum_{j=0}^{m-1}\sum_{k=0}^{j} \delta_{n+j}[\cdot, \mathbf{y}_{j+n-k}]\mathbf{y}_{n+k}. \tag{5.6}$$

Next using the representation (4.5) we obtain

$$\begin{aligned}\operatorname*{res}_{z=\lambda} (\mathcal{L} - z\mathcal{I})^{-1} &= -\operatorname*{res}_{z=\lambda} \begin{pmatrix} z^{-1}(\widetilde{T}^{-1}(z)\widetilde{A} + I) & T^{-1}(z) \\ \widetilde{T}^{-1}(z)\widetilde{A} & zT^{-1}(z) \end{pmatrix} \\ &= -\operatorname*{res}_{z=\lambda} \begin{pmatrix} z^{-1}\widetilde{T}^{-1}(z) & T^{-1}(z) \\ \widetilde{T}^{-1}(z) & zT^{-1}(z) \end{pmatrix} \begin{pmatrix} \widetilde{A} & 0 \\ 0 & I \end{pmatrix}.\end{aligned}$$

By Green's Formula

$$T(z)^{-1}f(x) = \frac{1}{W(z)}\left[\varphi_-(x,z)\int_x^1 f(t)\varphi_+(t,z)\,dt + \varphi_+(x,z)\int_0^x f(t)\varphi_-(t,z)\,dt\right],$$

where $\varphi_-(\cdot,z)$ is a solution of the equation $\ell(y)=(z^2-2zp)y$ satisfying the initial conditions $y(0)=0$, $y^{[1]}(0)=1$, $\varphi_+(\cdot,z)$ is a solution of the same equation satisfying the conditions $y(1)=0$, $y^{[1]}(1)=1$ and $W(z)=\varphi_-(x,z)\varphi_+^{[1]}(x,z)-\varphi_+(x,z)\varphi_-^{[1]}(x,z)$ is the Wronskian of the solutions $\varphi_-(\cdot,z)$ and $\varphi_+(\cdot,z)$. Set $s(z):=\varphi_-(1,z)$ and $c(z):=\varphi_-^{[1]}(1,z)$. Since the Wronskian W does not depend on x, we get that $W(z)=\varphi_-(1,z)=s(z)$. Next, note that for an eigenvalue λ of (1.1), (2.1) the functions $\varphi_+(x,\lambda)$ and $\varphi_-(x,\lambda)$ are related as follows

$$\varphi_+(x,\lambda)=\frac{\varphi_+^{[1]}(1,\lambda)}{\varphi_-^{[1]}(1,\lambda)}\varphi_-(x,\lambda)=\frac{1}{c(\lambda)}\varphi_-(x,\lambda).$$

Taking these remarks into account, we compute

$$\begin{aligned}\operatorname*{res}_{z=\lambda} z^{-1}\widetilde{T}^{-1}(z)f(x)&=\operatorname*{res}_{z=\lambda}\frac{\varphi_-(x,z)}{zs(z)c(z)}\int_0^1 f(t)\varphi_-(t,z)\,dt\\&=\sum_{j=0}^{m-1}\sum_{k=0}^{j}\eta_{n+j}(f,y_{j+n-k})_{L_2}y_{n+k}(x)\end{aligned}$$

with

$$\eta_{n+j}=\frac{1}{(m-1-j)!}\frac{\partial^{m-1-j}}{\partial z^{m-1-j}}\left[\frac{(z-\lambda)^m}{zs(z)c(z)}\right]\Bigg|_{z=\lambda} \tag{5.7}$$

for $j=0,\dots,m-1$. Analogously we obtain

$$\begin{aligned}\operatorname*{res}_{z=\lambda}\widetilde{T}^{-1}(z)f(x)&=\operatorname*{res}_{z=\lambda}\frac{z\varphi_-(x,z)}{zs(z)c(z)}\int_0^1 f(t)\varphi_-(t,z)\,dt\\&=\sum_{j=0}^{m-1}\sum_{k=0}^{j}\eta_{n+j}(f,\lambda y_{j+n-k}+y_{j+n-k-1})_{L_2}y_{n+k}(x)\\&=\sum_{j=0}^{m-1}\sum_{k=0}^{j}\eta_{n+j}(f,y_{j+n-k})_{L_2}(\lambda y_{n+k}(x)+y_{n+k-1}(x))\\\operatorname*{res}_{z=\lambda} z\widetilde{T}^{-1}(z)f(x)&=\operatorname*{res}_{z=\lambda}\frac{z\varphi_-(x,z)}{zs(z)c(z)}\int_0^1 f(t)z\varphi_-(t,z)\,dt\\&=\sum_{j=0}^{m-1}\sum_{k=0}^{j}\eta_{n+j}(f,\lambda y_{j+n-k}+y_{j+n-k-1})_{L_2}(\lambda y_{n+k}(x)+y_{n+k-1}(x)).\end{aligned}$$

This gives that

$$-\operatorname*{res}_{z=\lambda}\begin{pmatrix} z^{-1}\widetilde{T}^{-1}(z) & T(z)^{-1}\\ \widetilde{T}^{-1}(z) & zT(z)^{-1}\end{pmatrix}=-\sum_{j=0}^{m-1}\sum_{k=0}^{j}\eta_{n+j}(\cdot,\mathbf{y}_{j+n-k})_{L_2}\mathbf{y}_{n+k},$$

and so

$$\operatorname*{res}_{z=\lambda}(\mathcal{L}-z\mathcal{I})^{-1}=-\sum_{j=0}^{m-1}\sum_{k=0}^{j}\eta_{n+j}[\cdot,\mathbf{y}_{j+n-k}]\mathbf{y}_{n+k}, \tag{5.8}$$

where η_j are determined by (5.7). Now taking into account linear independence of $\mathbf{y}_j$ (see Proposition A.1) we obtain that

$$\delta_j=\eta_j,\quad j=n,\dots,n+m-1.$$

A similar result holds for nonreal eigenvalue $\lambda = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m-1} = \overline{\lambda_{n+m}} = \overline{\lambda_{n+m+1}} = \cdots = \overline{\lambda_{n+2m-1}}$. Namely, on the one hand,

$$\operatorname*{res}_{z=\lambda}(\mathcal{L} - z\mathcal{I})^{-1} = -\sum_{j=0}^{m-1}\sum_{k=0}^{j}\delta_{n+j}[\cdot, \overline{\mathbf{y}_{j+n-k}}]\mathbf{y}_{n+k},$$

where δ_j are the elements of the sequence associated with $D = G^{-1}$. On the other hand,

$$\operatorname*{res}_{z=\lambda}(\mathcal{L} - z\mathcal{I})^{-1} = -\sum_{j=0}^{m-1}\sum_{k=0}^{j}\eta_{n+j}[\cdot, \overline{\mathbf{y}_{j+n-k}}]\mathbf{y}_{n+k},$$

where η_j, $j = n, \dots, n+m-1$, are defined by (5.7). Equating both results and taking into account linear independence of $\mathbf{y}_j$, we obtain that $\delta_j = \eta_j$ and thus are defined by (5.7).

The relation between δ_j and η_j together with (5.7) will give the formula determining δ_j via two spectra of the problem (1.1) under two types of boundary conditions (see Theorem 5.3).

5.3. **The case of mixed boundary conditions.** Let us now consider the problem (1.1) with the so-called mixed boundary conditions

$$y(0) = y^{[1]}(1) = 0 \tag{5.9}$$

and denote by A_M the operator acting via

$$A_M y := \ell(y)$$

on the domain

$$\operatorname{dom} A_M := \{y \in \operatorname{dom}\ell \mid y(0) = y^{[1]}(1) = 0\}.$$

Define the operator pencil T_M by (2.2) with A_M instead of A. Then the spectral problem (1.1), (5.9) can be regarded as that for T_M.

We can study the pencil T_M in the same way as T. Moreover, by means of the operator A_M we can construct an energy space $\mathcal{E}_M$, the corresponding Pontryagin space Π_M and consider the corresponding linearization $\mathcal{L}_M$ therein as it was done for T. Clearly, all the results of Sections 3 and 4 concerning the pencil T hold for T_M.

For the operator pencil T_M we can also define the norming constants β_n by an analogue of (5.3) with T_M instead of T and (y_n) being eigen- and associated vectors for T_M defined via (5.2), in which λ is an eigenvalue of T_M. The norming constants β_n enjoy similar properties as α_n do; in particular, the following holds.

Lemma 5.2. *Let T_M have only real and simple eigenvalues. Then all the norming constants of T_M are positive if and only if the operator A_M is positive.*

For the pencil T_M and the corresponding linearization $\mathcal{L}_M$ we can define the sequences (δ_n^M) and (η_n^M) in the same way as (δ_n) and (η_n) were defined for T and obtain analogous relations.

5.4. **Determining norming constants from two spectra.** Note that the function $s(z)$ is a characteristic function for the problem (1.1), (2.1) and $c(z)$ is that for the problem (1.1), (5.9). This means that some $\lambda \in \mathbb{C}$ is an eigenvalue of the pencil T (resp. T_M) of algebraic multiplicity m if and only if it is a zero of $s(z)$ (resp. of $c(z)$) of order m. The functions $s(z)$ and $c(z)$ are of exponential type one and are determined uniquely by their zeros, i.e. by the eigenvalues λ_n of T and μ_n of T_M, by means of a canonical product (see [40]). Namely (see details in [29]), there exist constants s_0 and c_0 such that

$$s(z) = s_0 \text{V.p.}\prod_{k=-\infty}^{\infty}\left(1 - \frac{z}{\lambda_k}\right), \quad c(z) = c_0 \text{V.p.}\prod_{k=-\infty}^{\infty}\left(1 - \frac{z}{\mu_k}\right).$$

Therefore we have the following theorem.

Theorem 5.3. *The spectra* (λ_n) *and* (μ_n) *of the operator pencils* T *and* T_M *determine the corresponding norming constants* (α_n) *and* (β_n) *uniquely. Namely, the elements of the sequence* (δ_n) *associated with the matrix* $D = G^{-1}$ *are determined by the formula*

$$\delta_{n+j} = \frac{1}{(m-1-j)!}\frac{\partial^{m-1-j}}{\partial z^{m-1-j}}\left[\frac{(z-\lambda)^m}{zs(z)c(z)}\right]\bigg|_{z=\lambda}, \quad j = 0,\dots,m-1$$

for an eigenvalue $\lambda = \lambda_n = \cdots = \lambda_{n+m-1}$ *of* T *of algebraic multiplicity* m*. Analogously,*

$$\delta^M_{n+j} = \frac{1}{(m-1-j)!}\frac{\partial^{m-1-j}}{\partial z^{m-1-j}}\left[\frac{(z-\mu)^m}{zs(z)c(z)}\right]\bigg|_{z=\mu}, \quad j = 0,\dots,m-1$$

for an eigenvalue $\mu = \mu_n = \cdots = \mu_{n+m-1}$ *of* T_M *of algebraic multiplicity* m*.*

Corollary 5.4. *Assume that* λ_n *is a real and simple eigenvalue of* T*; then*

$$\alpha_n = \lambda_n \dot{s}(\lambda_n)c(\lambda_n) = -s_0c_0 \mathrm{V.p.}\prod_{\substack{k=-\infty\\k\neq n}}^{\infty}\left(1-\frac{\lambda_n}{\lambda_k}\right)\mathrm{V.p.}\prod_{k=-\infty}^{\infty}\left(1-\frac{\lambda_n}{\mu_k}\right).$$

Analogously, if μ_n *is a real and simple eigenvalue of* T_M*, then*

$$\beta_n = \mu_n s(\mu_n)\dot{c}(\mu_n) = -s_0c_0 \mathrm{V.p.}\prod_{k=-\infty}^{\infty}\left(1-\frac{\mu_n}{\lambda_k}\right)\mathrm{V.p.}\prod_{\substack{k=-\infty\\k\neq n}}^{\infty}\left(1-\frac{\mu_n}{\mu_k}\right).$$

Indeed, if λ_n is a real and simple eigenvalue of T, then $\delta_n = 1/\alpha_n$. On the other hand, in view of Theorem 5.3,

$$\delta_n = \frac{1}{\lambda_n \dot{s}(\lambda_n)c(\lambda_n)},$$

giving the formula for α_n. The result for β_n corresponding to a real and simple eigenvalue μ_n of T_M is derived analogously.

6. The case of real and simple eigenvalues

In this section we shall establish some conditions which guarantee that the spectra of T and T_M are real and simple.

Note firstly that it is more natural to label the eigenvalues λ_n of the pencil T by the index set $\mathbb{Z}^* := \mathbb{Z}\setminus\{0\}$ due to asymptotics of λ_n (see [29]). We say that the spectra of the operator pencils T and T_M *almost interlace* if they consist only of real and simple eigenvalues, which can be labeled in increasing order as λ_n, $n\in\mathbb{Z}^*$, and μ_n, $n\in\mathbb{Z}$, respectively so that they satisfy the condition

$$\mu_k < \lambda_k < \mu_{k+1} \quad \text{for every} \quad k\in\mathbb{Z}^*. \tag{6.1}$$

Theorem 6.1. *The following statements are equivalent:*

(i) *the spectra of* T *and* T_M *almost interlace;*
(ii) *a real number* μ_* *exists such that the operator* $T_M(\mu_*)$ *is negative.*

Proof. ((i) ⇒ (ii)) To start with, we additionally assume that $0\in(\mu_0,\mu_1)$ and then prove that (ii) holds with $\mu_* = 0$. Indeed,

$$\frac{\beta_{n+1}}{\beta_n} = -\frac{\mu_{n+1}}{\mu_n}\prod_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{\lambda_k-\mu_{n+1}}{\lambda_k-\mu_n}\prod_{\substack{k=-\infty\\k\neq n,n+1}}^{\infty}\frac{\mu_k-\mu_{n+1}}{\mu_k-\mu_n}.$$

A straightforward verification shows that if (i) holds, then this ratio is positive for every $n\in\mathbb{Z}$. Therefore, all the norming constants β_n are of the same sign. Recall that $\beta_n = [\mathbf{y}_n,\mathbf{y}_n]_{\mathcal{E}_M}$, where $\mathbf{y}_n$ is an eigenvector of $\mathcal{L}_M$ corresponding to μ_n; therefore all but at most finitely many β_n must be positive. As a result all β_n are positive and the claim follows from Lemma 5.2.

If 0 does not belong to (μ_0, μ_1), then we take any point μ_* from this interval and shift the spectral parameter of T and T_M by μ_* to obtain the pencils

$$\widehat{T}(\lambda) = T(\lambda + \mu_*) = \lambda^2 I - 2\lambda \widehat{B} - \widehat{A}, \tag{6.2}$$

$$\widehat{T}_M(\lambda) = T_M(\lambda + \mu_*) = \lambda^2 I - 2\lambda \widehat{B} - \widehat{A}_M \tag{6.3}$$

with $\widehat{B} := B - 2\mu_* I$, $\widehat{A} := -T(\mu_*)$ and $\widehat{A}_M := -T_M(\mu_*)$. Clearly, the spectra of $\widehat{T}$ and $\widehat{T}_M$ almost interlace with $0 \in (\mu_0, \mu_1)$. In view of the first part of this proof the operator $\widehat{A}_M$ is positive. Therefore, the operator $T_M(\mu_*)$ with this μ_* is negative.

((ii) $\Rightarrow$ (i)) Let the operator $T_M(\mu_*)$ be negative. Consider the operator pencil $\widehat{T}_M$ of (6.3) obtained from T_M by the shift of the spectral parameter by μ_*. Then the operator $\widehat{A}_M = -T_M(\mu_*)$ is positive and the minimax principal (see, e.g. [31]) implies that $\widehat{A} = -T(\mu_*)$ is also positive. By Corollary 4.8, the spectra of T and T_M are real and simple. The eigenvalues λ_n and μ_n can be enumerated so that (see [29])

$$\lambda_n = \pi n + p_0 + \tilde{\lambda}_n \quad \text{and} \quad \mu_n = \pi\left(n - \tfrac{1}{2}\right) + p_0 + \tilde{\mu}_n \tag{6.4}$$

with $p_0 := \int_0^1 p(x)\,dx$ and ℓ_2-sequences $(\tilde{\mu}_n), (\tilde{\lambda}_n)$.

Next we define the norming constants $\widehat{\beta}_j$, $j \in \mathbb{Z}$, for $\widehat{T}_M$. In view of Lemma 5.2, all these norming constants are positive and by Corollary 5.4 they are determined by the formula

$$\widehat{\beta}_n = -\widehat{s}_0 \widehat{c}_0 \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\lambda_k - \mu_n}{\lambda_k - \mu_*} \prod_{\substack{k=-\infty \\ k \neq n}}^{\infty} \frac{\mu_k - \mu_n}{\mu_k - \mu_*},$$

where $\widehat{s}_0$, $\widehat{c}_0$ are some constants. Therefore, the expression

$$\frac{\widehat{\beta}_{n+1}}{\widehat{\beta}_n} = -\frac{\mu_{n+1} - \mu_*}{\mu_n - \mu_*} \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\lambda_k - \mu_{n+1}}{\lambda_k - \mu_n} \prod_{\substack{k=-\infty \\ k \neq n, n+1}}^{\infty} \frac{\mu_k - \mu_{n+1}}{\mu_k - \mu_n}$$

is positive. This yields that if $\mu_n < \mu_* < \mu_{n+1}$, then there is an even number of λ_k between μ_n and μ_{n+1} and otherwise there is an odd number of λ_k between μ_n and μ_{n+1}. But then the asymptotics (6.4) of μ_n and λ_n implies that the number of elements of (λ_k) between μ_n and μ_{n+1} can not exceed 1 and that there is no λ_k between μ_0 and μ_1. Hence (λ_n) and (μ_n) almost interlace and $\mu_* \in (\mu_0, \mu_1)$, thus completing the proof. ☐

From the proof of the first implication in the above theorem we immediately obtain the following corollaries, that were used to solve the inverse spectral problem for the pencils T and T_M in [30].

Corollary 6.2. *If the spectra $(\lambda_n)_{n\in\mathbb{Z}^*}$ of T and $(\mu_n)_{n\in\mathbb{Z}}$ of T_M almost interlace, then for every number μ_* from the interval (μ_0, μ_1) the operator $T_M(\mu_*)$ is negative.*

Corollary 6.3. *If for some $\mu_* \in \mathbb{R}$ the operator $T_M(\mu_*)$ is negative, then the spectra $(\lambda_n)_{n\in\mathbb{Z}^*}$ of T and $(\mu_n)_{n\in\mathbb{Z}}$ of T_M almost interlace with $\mu_* \in (\mu_0, \mu_1)$. Moreover, for every μ from (μ_0, μ_1) the operator $T_M(\mu)$ is negative.*

Appendix A. Basics of Pontryagin spaces theory

In this appendix we recall some facts from the Pontryagin space theory, which we use in the paper. The details of the theory, more spectral properties of self-adjoint operators in Pontryagin spaces and the proofs of the propositions given here can be found in [4, 17, 2].

A linear space Π is called an *inner product space* if there is a complex-valued function $[\cdot,\cdot]$ defined on $\Pi \times \Pi$ so that the conditions

$$[\alpha_1 u_1 + \alpha_2 u_2, v] = \alpha_1[u_1, v] + \alpha_2[u_2, v],$$
$$[u, v] = \overline{[v, u]}$$

hold for every $\alpha_1, \alpha_2 \in \mathbb{C}$ and $u_1, u_2, u, v \in \Pi$. The function $[\cdot,\cdot]$ is then called an *inner product.* An inner product space $(\Pi, [\cdot,\cdot])$ is a *Pontryagin space* of *negative index* κ if Π can be written as

$$\Pi = \Pi_+ \,[\dot{+}]\, \Pi_-, \tag{A.1}$$

where $[\dot{+}]$ denotes the direct $[\cdot,\cdot]$-orthogonal sum, $(\Pi_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces and the component Π_- is of finite dimension κ.

An element $x \in \Pi$ is said to be *positive* (resp. *negative, non-positive, non-negative, neutral*) if $[x, x] > 0$ (resp. $[x, x] < 0$, $[x, x] \le 0$, $[x, x] \ge 0$, $[x, x] = 0$). A subspace $\mathcal{M}$ of P is called *positive* (resp. *negative, non-positive, non-negative, neutral*) if all its non-zero vectors are positive (resp. negative, non-positive, non-negative, neutral)

In Pontryagin space of negative index κ the dimension of any non-positive subspace can not exceed κ. Moreover, a non-positive subspace of Pontryagin space is maximal (i.e. such that it is not properly included in any other non-positive subspace) if and only if it is of dimension κ.

Pontryagin spaces often arise from Hilbert spaces in the following way. Suppose we have a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ and a bounded self-adjoint operator G in $\mathcal{H}$ with $0 \in \rho(G)$ which has exactly κ negative eigenvalues counted according to their multiplicities. Then with an inner product

$$[x, y] := (Gx, y), \quad x, y \in \mathcal{H},$$

the space $(\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space of negative index κ for which the decomposition (A.1) can be given with Π_+ and Π_- being the spectral subspaces of G corresponding to the positive and negative spectrum of G respectively.

Consider a Pontryagin space $\Pi := \big(\Pi, [\cdot,\cdot]\big)$ and a closed operator $\mathcal{A}$ densely defined on Π. An *adjoint* $\mathcal{A}^{[*]}$ of $\mathcal{A}$ in Π is defined on the domain

$$\operatorname{dom} \mathcal{A}^{[*]} := \{y \in \Pi \mid [\mathcal{A}\cdot, y] \text{ is a continuous linear functional on} \operatorname{dom} \mathcal{A}\}$$

by the relation

$$[\mathcal{A}x, y] = [x, \mathcal{A}^{[*]}y], \quad x \in \operatorname{dom} \mathcal{A}, \quad y \in \operatorname{dom} \mathcal{A}^{[*]}.$$

The operator $\mathcal{A}$ is *symmetric* if $\mathcal{A} \subset \mathcal{A}^{[*]}$ and *self-adjoint* if $\mathcal{A} = \mathcal{A}^{[*]}$. In contrast to the case of Hilbert space, the spectrum of self-adjoint operator in Pontryagin space is not necessarily real, but it is always symmetric with respect to the real axis.

If for some eigenvalue λ of a self-adjoint operator in a Pontryagin space all eigenvectors are positive (resp. negative) then λ is called of *positive* (resp. *negative*) *type.*

Proposition A.1. *Assume $\mathcal{A}$ is a self-adjoint operator in a Pontryagin space Π. Then*

(i) *the spectrum of $\mathcal{A}$ is real with possible exception of at most κ pairs of eigenvalues λ and $\overline{\lambda}$ of finite algebraic multiplicities.*

(ii) *if the spectrum of the operator $\mathcal{A}$ is discrete, then the set of all eigenvectors and the corresponding associated vectors of $\mathcal{A}$ forms a basis in Π.*

Denote by $\mathcal{M}_\lambda(\mathcal{A})$ the root space of the operator $\mathcal{A}$ corresponding to an eigenvalue λ.

Proposition A.2. *Suppose $\mathcal{A}$ is a self-adjoint operator in a Pontryagin space. Then*

(1) *for eigenvalues λ and μ of $\mathcal{A}$ such that $\lambda \neq \overline{\mu}$ the root spaces $\mathcal{M}_\lambda(\mathcal{A})$ and $\mathcal{M}_\mu(\mathcal{A})$ are $[\cdot,\cdot]$-orthogonal;*

(2) *the linear span of all the algebraic root spaces corresponding to the eigenvalues of $\mathcal{A}$ in the upper (or lower) half plane is a neutral subspace of Π;*

(3) *the root spaces* $\mathcal{M}_\lambda(\mathcal{A})$ *and* $\mathcal{M}_{\bar\lambda}(\mathcal{A})$ *corresponding to complex conjugate eigenvalues* λ *and* $\bar\lambda$ *are isomorphic; moreover, they have the same Jordan structure;*

(4) *denote by* $m(\lambda)$ *the algebraic multiplicity of the eigenvalue* λ*; then*

$$\sum_{\operatorname{Im}\lambda>0} m(\lambda) + \sum_{\operatorname{Im}\lambda=0} \left[\frac{m(\lambda)}{2}\right] \le \kappa.$$

In particular, the length of every chain of eigen- and associated vectors does not exceed $2\kappa+1$ *and the number of non-simple real eigenvalues does not exceed* κ*.*

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