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ON THE DISCRETE SPECTRUM OF A LINEAR OPERATOR PENCIL ARISING IN TRANSPORT THEORY

P. A. COJUHARI

Dedicated to V. D. Koshmanenko on the occasion of his 70th birthday

Abstract. We study the problem of the finiteness of the discrete spectrum for linear operator pencils occurring in one-velocity transport theory. The results are obtained using direct methods of perturbation theory for linear operators. The proposed approach allowed to give a relatively quick proofs of the main results improving related results obtained previously by K. M. Case and C. G. Lekkerkerker.

1. Introduction

The time-independent linear transport equation in one dimensional slab configuration with anisotropic scattering has the following form:

$$\omega\omega_0 \frac{\partial}{\partial x} f(x,\omega) + f(x,\omega) - \int_{S^2} k(x,\omega,\omega^{'}) f(x,\omega^{'})\, d\omega^{'} = 0, \tag{1.1}$$

where f is the distribution function defined on $\Omega = \Delta \times S^2$ (the phase space), Δ is an open interval on the real axis $\mathbb{R}$, S^2 denotes the unit sphere in $\mathbb{R}^3$, ω_0 is a fixed unit vector (selected in the direction of increasing x), by $\omega\omega^{'}$ it is denoted the scalar product (defined on $\mathbb{R}^3$) of $\omega,\ \omega^{'} \in S^2$ [5].

We consider the situation of azimuthal symmetry that means that the distribution function is independent of the azimuth, in other words, the dependence on ω is only thru the variable $\mu = \omega\omega_0,\ \ -1 \le \mu \le 1$. In addition, we assume that the scattering kernel k is of the form

$$k(x,\omega,\omega^{'}) = g(\omega\omega^{'}), \quad x \in \Delta, \quad \omega,\ \omega^{'} \in S^2, \tag{1.2}$$

that is, k does not depend on the position variable x (the host medium is homogeneous) depending only on $\omega\omega^{'}$ (the rotational invariance property). The function g determined k as in (1.2) is also called the scattering function or, in other terminology especially in the theory of radiative transfer, the dispersion indicatrix [7].

Looking for a solution in the form $u(\mu)e^{-\lambda x}$, in our assumptions, from Eq. (1.1) it follows that

$$u(\mu) - \lambda\mu u(\mu) - \int_{S^2} g(\omega\omega^{'}) u(\mu^{'})\, d\omega^{'} = 0. \tag{1.3}$$

In transport theory (cf., for instance, [7]), the auxiliary equation (1.3) is called the characteristic equation of transport processes described, in our case, by Eq. (1.1). The main question is to determine λ for which the equation (1.3) has a non-trivial solution or, in more general setting, to investigate the nature of the spectrum of the operator pencil defined by the expression from the left side of Eq. (1.3).

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For our purposes Eq. (1.3) is considered in the space $L_2[-1,1]$, the scattering function g is assumed to be summable on $[-1,1]$, i. e., $g \in L_1[-1,1]$ and, it is required that

$$0 < g_0 \leq 1, \quad | g_j | < g_0 \quad (j = 1, 2, \ldots), \tag{1.4}$$

where

$$g_n = 2\pi \int_{-1}^{1} g(\mu) P_n(\mu)\, d\mu \quad (n = 0, 1, \ldots)$$

and $P_n(\mu)$ are the Legendre polynomials.

The main purpose of the present paper is to give conditions in order that the discrete part of the spectrum of the operator pencil (already mentioned) corresponding to the characteristic equation (1.3) be finite. The results are formulated in terms of the coefficients g_n, improve those obtained by K. M. Case [1] and C. G. Lekkerkerker [6]. In [1] Case proved the finiteness of the set of the discrete eigenvalues under the condition $\sum_{n=0}^{\infty} n^2 \mid g_n \mid < \infty$. Lekkerkerker [6] showed, however, that the assertion remains true under the following substantially weaker condition:

$$\sum_{n=1}^{\infty} (n \log n) \mid g_n \mid < \infty. \tag{1.5}$$

Notice [5] for historical remarks and perspectives of the problem, and other related results (see also references on the subject cited in [5]).

The present paper is a continuation of the author's work [3] in which similar results are established in terms of smoothness of the scattering function g (see Theorems 3.1 and 3.3 [3]). In fact, we treat the problem in the abstract framework developed in [3] (cf. Theorem 2.1 [3]). However, for the sake of convenience, we present the corresponding abstract results in suitable forms in which they are need in our concrete applications. The results thus presented in an abstract setting are important by themselves and useful for other applications. The abstract results are stated in Section 2, and their applications to transport theory are given in Section 3.

2. Preliminaries. Abstract framework

Let $\mathcal{H}$ be a Hilbert space. We denote by $\mathcal{B}\ (\mathcal{H})$ the set of all bounded operators defined on $\mathcal{H}$, $\mathcal{B}_\infty\ (\mathcal{H})$ $(\subset \mathcal{B}(\mathcal{H}))$ stands for the set of all compact operators in $\mathcal{H}$. The resolvent set, the spectrum and the discrete spectrum for an operator A are denoted by $\rho(A), \sigma(A)$ and $\sigma_d\ (A)$, respectively. In the case of a self-adjoint operator A, we say that an open interval Λ of the real axis is a gap in the spectrum of A (or, simply, a spectral gap of A) if $\Lambda \subset \rho\ (A)$. Also, we say that the spectrum of the operator A is finite on Λ if $\Lambda\ \cap\ \sigma\ (A)$ consists only of a finite number of eigenvalues of finite multiplicity. We will keep the same terminology for the general case of operator-valued functions or, in particular, operator pencils with which we are concerned in this paper.

Next, let A and B be bounded operators on the space $\mathcal{H}$, and consider the following linear operator pencil

$$L(\lambda) = B - \lambda A, \quad \lambda \in \mathbb{C}.$$

We are interested to find conditions under which the discrete spectrum (i.e. the set of eigenvalues lying outside of the essential spectrum) of the operator pencil $L(\lambda)$ is only finite. The restrictions on A and B which we made are given by the assumptions listed below. Although these restrictions stimulate typical situations in transport theory considered in our applications, it seems that they are natural and, in a sense, optimal for the posed problem.

(A1) i) A and B are self-adjoint operators in $\mathcal{H}$.

ii) The interval $\Lambda = (a,b)$ is a spectral gap of the (unperturbed) operator pencil $L_0(\lambda) = I - \lambda A$.

iii) $B = I + C$, where $C \in \mathcal{B}_\infty(\mathcal{H})$ and $\| C \| \leq 1$.

Under these assumptions the spectrum of $L(\lambda)$ is entirely situated on the real axis (cf. [3], Proposition 2.1), and, due to Weyl type theorems, the spectrum of $L(\lambda)$ on the interval Λ can be only discrete with its possible points of accumulation the edges of Λ. As has been shown in [3] (see Theorem 2.7 [3]), the finiteness of this discrete part of the spectrum is ensured by the assumptions in (A1) together with the following.

(A2) For an operator of finite rank K the operator $C - K$ admits a factorization of the form $C - K = S^*TS$ with $S \in \mathcal{B}(\mathcal{H})$, $T = T^*$ and $T \in \mathcal{B}_\infty(\mathcal{H})$) such that the operator-valued functions

$$Q_j(\lambda) = \lambda S A^j (L_0(\lambda))^{-1} S^* \quad (j = 0,1,2;\ \lambda \in \rho(L_0))$$

are uniformly bounded on Λ , i.e. there exists $c > 0$ such that

$$\| Q_j(\lambda) \| \leq c \quad (j = 0,1,2;\ \lambda \in \Lambda).$$

Below we give other formulations of this result more suitable for our purposes. To this end, denote by E the spectral measure associated with A, and put

$$| L_0(\lambda) | = \int | 1 - \lambda s | \, dE(s), \quad W_0(\lambda) = \int \mathrm{sgn}(1 - \lambda s) \, dE(s)$$

(the integration is taken over the spectrum of A).

Theorem 2.1. *Let A, B and $\Lambda = (a,b)$ (with $-\infty < a < b \leq \infty$) be as in (A1), and assume that a is not a characteristic number of A, i.e. $Ker(L_0(a)) = \{0\}$. Furthermore, let S and T denote bounded operators on $\mathcal{H}$ so that $C = S^*TS$, suppose that T is self-adjoint and*

$$| L_0(a) |^{-1/2} S^* P \in \mathcal{B}_\infty(\mathcal{H}), \qquad (2.1)$$

where P is an orthogonal projection such that $dim(I - P)\mathcal{H} < \infty$. Then the spectrum of the operator pencil $L(\lambda)$ on Λ is only discrete for which a is not an accumulation point.

Proof. From

$$\big(| L_0(a) |^{-1/2} S^* P \big)^* \supset PS \, | L_0(a) |^{-1/2}$$

it follows that the densely defined operator $PS \, | L_0(a) |^{-1/2}$ is bounded and has a unique extension, namely $(| L_0(a) |^{-1/2} S^* P)^*$, on $\mathcal{H}$. Therefore, we can write

$$C = S^*TS = S^*PTPS + K = | L_0(a) |^{1/2} \, T_0 \, | L_0(a) |^{1/2} + K,$$

where

$$T_0 = | L_0(a) |^{-1/2} S^* P T \big(| L_0(a) |^{-1/2} S^* P \big)^*$$

and

$$K = S^*PT(I-P)S + S^*(I-P)TPS + S^*(I-P)T(I-P)S.$$

It is seen that the operator T_0 is self-adjoint and compact in $\mathcal{H}$, and that the operator-valued functions

$$\lambda \, | L_0(a) |^{1/2} A^j (L_0(\lambda))^{-1} \, | L_0(a) |^{1/2} \ \ (= \lambda A^j \, | L_0(a) | \, (L_0(\lambda))^{-1}) \quad (j = 0,1,2)$$

are uniformly bounded on an arbitrary, but fixed, subinterval $\Lambda' = (a, a') \subset \Lambda$. Therefore, the obtained factorization for $C - K$ (with K as above), namely

$$C - K = | L_0(a) |^{1/2} \, T_0 \, | L_0(a) |^{1/2},$$

is satisfactory for the assumption (A2) to be true. Thus Theorem 2.7 [3] can be applied in order to conclude the fact that the spectrum of $L(\lambda)$ on Λ' is finite. □

Corollary 2.2. *If A, B and $\Lambda = (a, b)$ (with $-\infty < a < b < \infty$) satisfy the conditions of Theorem 2.1 at both end points of Λ, then the spectrum of the operator pencil $L(\lambda)$ on Λ is finite.*

Next we make some useful remarks concerning the condition (2.1), most difficult in verifications. First of all we observe that if $\mid I - aA \mid^{-1/2} S^* \in \mathcal{B}_\infty(\mathcal{H})$ for a suitable S, then, due to of

$$S(I - aA)^{-1}S^* = S \mid I - aA \mid^{-1/2} W_0(a)^* \mid I - aA \mid^{-1/2} S^*,$$

it follows that $S(I - aA)^{-1}S^*$ has an extension $Q(a)$ on $\mathcal{H}$. In case $L_0(a) = I - aA$ is definite, i.e. either $L_0(a) \geq 0$ or $L_0(a) \leq 0$ (in the sense of quadratic forms) the converse assertion is also true. Let, for instance, $L_0(a) \geq 0$, and suppose that the operator $S(I - aA)^{-1}S^*$ is densely defined (let on the set $\mathcal{D}$) and has a compact extension $Q(a)$ on $\mathcal{H}$. Then, for any $u \in \mathcal{D}$, one has

$$\begin{aligned}\| (I - aA)^{-1/2}S^*u \|^2 &= \langle (I - aA)^{-1/2}S^*u,\ (I - aA)^{-1/2}S^*u\rangle \\ &= \langle S(I - aA)^{-1}S^*u, u\rangle \leq \| Q(a) \| \, \| u \|^2,\end{aligned}$$

so that the operator $(I - aA)^{-1/2}S^*$ being densely defined possesses also a bounded extension. However, in these conditions the domains of the operators $(I - aA)^{-1/2}S^*$ and S^* coincide (cf. Remark 4.2 in [2]). Thus the operator $(I - aA)^{-1/2}S^*$ is in fact bounded on $\mathcal{H}$ and, moreover,

$$Q(a) = \big((I - aA)^{-1/2}S^*\big)^*\big((I - aA)^{-1/2}S^*\big)$$

that implies $(I - aA)^{-1/2}S^* \in \mathcal{B}_\infty(\mathcal{H})$.

Remark 2.3. *Estimate formulae for the number of the eigenvalues given in [3] (see Theorem 2.8 [3]) can be adjusted to situations discussed above, in particular, to those of Corollary 2.2.*

3. Applications to transport theory

In this section we study the eigenvalue problem for the transport equation (1.1) under the restrictions mentioned in Section 1 (Introduction). As was mentioned the problem reduces to the investigation of the operator pencil defined by the characteristic equation (1.3). Under the prescribed conditions for the scattering function g the integral operator on the left side of (1.3) is compact in the space $L_2[-1, 1]$, g_n are its eigenvalues, and the corresponding eigenfunctions are the Legendre polynomials $P_n(\mu)$ [5]. Accordingly, using the orthogonality and recursion properties of the Legendre polynomials, E_q. (1.3) can be reduced equivalently to an equation of the form

$$u_n - \lambda(a_{n+1}u_{n+1} + a_n u_{n-1}) - g_n u_n = 0 \quad (n = 0, 1, \ldots;\ u_{-1} := 0), \tag{3.1}$$

considered in the space $l_2(\mathbb{Z}_+)$ of square summable sequences $u = (u_n)$, $u_n \in \mathbb{C}$ ($n = 0, 1, \ldots$), where

$$a_n = \frac{n}{\sqrt{4n^2 - 1}} \quad (n = 0, 1, \ldots). \tag{3.2}$$

Next, A denote the operator defined on $l_2(\mathbb{Z}_+)$ by

$$(Au)_n = a_{n+1}u_{n+1} + a_n u_{n-1} \quad (n = 0, 1, \ldots;\ u_{-1} := 0), \tag{3.3}$$

and C the multiplication operator by g_n (also defined on $l_2(\mathbb{Z}_+)$)

$$(Cu)_n = g_n u_n \quad (n = 0, 1, \ldots). \tag{3.4}$$

Eq. (3.1) can be written into the following compact form:

$$(I - \lambda A - C)u = 0, \quad u \in l_2(\mathbb{Z}_+),$$

and, in this way, the problem is reduced to the study of the eigenvalues of the operator pencil

$$L(\lambda) = I - \lambda A - C$$

in the space $l_2(\mathbb{Z}_+)$, where A and C are defined as in (3.3) and (3.4), respectively.

In fact, A is an operator generated in $l_2(\mathbb{Z}_+)$ by a Jacobi matrix with the null main diagonal and with its elements (3.2) on secondary diagonals. The operator A is self-adjoint, $\sigma(A) = [-1, 1]$ and $\sigma_d(A) = \emptyset$. So, in the spectrum of the unperturbed operator pencil $L_0(\lambda) = I - \lambda A$ there is a spectral gap $\Lambda = (-1, 1)$, and our aim is to find conditions under which the spectrum of $L(\lambda)$ on Λ is finite. The arguments will based on the abstract results discussed in Section 2. We let $\gamma_n = | g_n |^{1/2}$ and define $S \in \mathcal{B}(l_2(\mathbb{Z}_+))$ by

$$(Su)_n = \gamma_n u_n \quad (n = 0, 1, \ldots).$$

In view of symmetric nature of the problem (the discrete spectrum, and therefore the whole spectrum, of the operator pencil $L(\lambda)$ is situated symmetrically with respect to the origin $\lambda = 0$) it is enough to study, for instance, the eigenvalues in the interval $(0, 1)$. To this end, consider the operator

$$Q = (I - A)^{-1/2} S \quad (\text{clearly, } I - A \geq 0).$$

The operator Q has a matrix representation $Q = [q_{kn}]$ (hereafter, we identify Q with its matrix) with respect to the standard basis in $l_2(\mathbb{Z}_+)$ given by

$$q_{nk} = (k + 1/2)^{1/2} (n + 1/2)^{1/2} \gamma_n \int_{-1}^{1} (1 - \mu)^{-1/2} P_n(\mu) P_k(\mu) \, d\mu \quad (k, n = 0, 1, \ldots),$$

which can be obtained by formal calculations. Operator Q, in general, is not defined on the whole space $l_2(\mathbb{Z}_+)$. To avoid such an inconvenience we will consider the operator Q be defined on all elements $u = (u_n) \in l_2(\mathbb{Z}_+)$ such that

$$\sum_{n=0}^{\infty} (n + 1/2)^{1/2} \; \gamma_n u_n = 0. \tag{3.5}$$

In the sequel, it will be assumed that $\gamma = (\gamma_n)$ converges rapidly to zero so that the sequence $(n^{1/2}\gamma_n)$ is an element of $l_2(\mathbb{Z}_+)$, i.e.

$$\sum_{n=0}^{\infty} n \mid g_n \mid < \infty. \tag{3.6}$$

Then, instead of Q it is in fact considered the operator $G = (I - A)^{-1/2} SP$, P being the orthogonal projection onto the subspace of all $u = (u_n) \in l_2(\mathbb{Z}_+)$ satisfying (3.5). Obviously, in this case, $\dim(I - P) = I$. Moreover, the operator Q_0 having the representation $Q_0 = [q^0_{kn}]$ with

$$q^0_{kn} = \sqrt{2}(k + 1/2)^{-1/2} (n + 1/2)^{1/2} \gamma_n \quad (k, n = 0, 1, \ldots)$$

vanishes on $Pl_2(\mathbb{Z}_+)$. Taking into account this fact and the fact that

$$q^0_{kn} = (k + 1/2)^{1/2} (n + 1/2)^{1/2} \gamma_n \int_{-1}^{1} (1 - \mu)^{-1/2} P_k(\mu) \, d\mu,$$

since

$$\int_{-1}^{1} (1 - \mu)^{-1/2} P_k(\mu) \, d\mu = \sqrt{2}(k + 1/2)^{-1} \quad (k = 0, 1, \ldots)$$

(see [4], 7.255 (3)), the matrix representation of Q on elements of the subspace $P(l_2(\mathbb{Z}_+))$ is given by

$$g_{kn} = (k+1/2)^{1/2}(n+1/2)^{1/2}\gamma_n \int_{-1}^{1} (1-\mu)^{-1/2} P_k(\mu)(P_n(\mu)-1)\, d\mu \qquad (k,n = 0,1,\ldots). \tag{3.7}$$

In case this matrix defines a compact operator in $l_2(\mathbb{Z}_+)$ by Theorem 2.1 we can conclude that the spectrum of the operator pencil $L(\lambda)$ in the interval $(0,1)$, and therefore on $\Lambda = (-1,1)$, is finite. We formulate this result in the following.

Theorem 3.1. *Suppose the scattering function $g, g \in L_1[-1,1]$, is such that the conditions (1.4) and (3.6) are satisfied. If the operator $G = [g_{kn}]$, where g_{kn} are given by (3.7), is compact in the space $l_2(\mathbb{Z}_+)$, then the spectrum of the operator pencil $L(\lambda)$ on the interval $(-1,1)$ is finite.*

Remark 3.2. *The condition $G = [g_{kn}] \in \mathcal{B}_\infty(l_2(\mathbb{Z}_+))$, in Theorem 3.1, is equivalent to the fact that the integral operator with the kernel*

$$h(\mu,\nu) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \gamma_n \frac{P_n(\mu)-1}{\sqrt{1-\mu}} P_n(\nu) \quad (-1 \le \mu,\nu \le 1)$$

is compact in the space $L_2[-1,1]$.

It turns out that under the Lekkerkerker's condition (1.5) the operator G defined as in Theorem 3.1 belongs to the class of Hilbert-Schmidt operators. This fact is settled by the following theorem.

Theorem 3.3. *Let G be defined as in Theorem 3.1. If the condition (1.5) is satisfied, then G is an operator of Hilbert-Schmidt class. Moreover, for the Hilbert-Schmidt norm there holds*

$$\| G \|_2^2 = \sum_{n=0}^{\infty} \frac{2n+1}{2} H_n \mid g_n \mid$$

in which $H_n := \sum_{j=1}^{n} \frac{1}{j}$.

Proof. Taking into account (3.7) and that $\gamma_n^2 = \mid g_n \mid$, we have

$$\| G \|_2^2 = \sum_{n=0}^{\infty} \frac{2n+1}{2} \mid g_n \mid \sum_{k=0}^{\infty} \frac{2k+1}{2} \left| \int_{-1}^{1} \frac{P_n(\mu)-1}{\sqrt{1-\mu}} P_k(\mu)\, d\mu \right|^2$$

and, by Parseval equality, we get

$$\| G \|_2^2 = \sum_{n=0}^{\infty} \frac{2n+1}{2} \mid g_n \mid \int_{-1}^{1} \frac{(P_n(\mu)-1)^2}{1-\mu}\, d\mu. \tag{3.8}$$

The integral on the right side of (3.8) can be evaluated as follows. By Christoffel-Darboux formula (cf. [4], 8.915) it can be obtained

$$\frac{P_n(x)-1}{x-1} = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=0}^{j-1} (2k+1) P_k(x) = \sum_{k=0}^{n-1} (2k+1) \left(\sum_{j=k+1}^{n} \frac{1}{j} \right) P_k(x).$$

Thus

$$\int_{-1}^{1} \frac{(P_n(\mu)-1)^2}{1-\mu}\, d\mu = \int_{-1}^{1} \frac{P_n(\mu)-1}{1-\mu} P_n(\mu)\, d\mu + \int_{-1}^{1} \frac{P_n(\mu)-1}{1-\mu}\, d\mu$$
$$= \sum_{k=0}^{n-1} (2k+1) \left(\sum_{j=k+1}^{n} \frac{1}{j} \right) \int_{-1}^{1} P_k(x)\, dx = H_n,$$

and the assertion follows. $\square$

Remark 3.4. *Under Lekkerkerker's condition (1.5) the operator* $PS(I-A)^{-1}SP$ *admits an extension* G^*G *on* $l_2(\mathbb{Z}_+)$ *representing an nuclear operator. Note that for the finiteness of eigenvalues of* $L(\lambda)$ *on* Λ *it is enough to require only the compactness of this extension (cf. the remarks made in Section 2).*

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Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Cracow, Poland

E-mail address: cojuhari@agh.edu.pl

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