Methods of Functional Analysis and Topology
Vol. 20 (2014), no. 1, pp. 17–33

DELTA-TYPE SOLUTIONS FOR A SYSTEM OF INDUCTION EQUATIONS WITH DISCONTINUOUS VELOCITY FIELD

A. I. ESINA AND A. I. SHAFAREVICH

The article is dedicated to the 70th anniversary of brilliant mathematician V. D. Koshmanenko

Abstract. We study asymptotic solutions of a Cauchy problem for induction equations describing magnetic field in a well conducting fluid. We assume that the coefficient (the velocity field of the fluid) changes rapidly in a small vicinity of a two-dimensional surface. We prove that the weak limit of the solution has delta-type singularity on this surface; in the case of a perfectly conducting fluid, we describe several regularizations of the problem with discontinuous coefficients which allow to define generalized solutions.

1. Statement of the problem

A description of the magnetic field temporal evolution in a conductive fluid plays an important role in the study of strong fields of planets, stars and galaxies. In particular, a lot of papers was devoted to a detailed study of the effect of hydrodynamic dynamo– unlimited growth of the magnetic field at large times caused by irregular behavior of the trajectories of the smooth velocity field of the fluid (see, e.g. [8], [1]). From the mathematical point of view, it means there are eigenvalues with positive real part of the induction operator and the Cauchy problem‘s solutions, which grow exponentially as $t \to \infty$. We study an alternative effect, an instantaneous growth of the field due to the velocity field‘s jump (break) (cf. [4], [5]). It appears that the magnetic field has delta-type singularity on the surface of discontinuity. The correct generalized statement of problem exists, in particular, if passing through the surface only the amplitude changes rather than direction of the velocity field.

In the linear approximation, magnetic field in a conducting fluid satisfies the induction equation

$$\frac{\partial B}{\partial t} + (V,\nabla)B - (B,\nabla)V = \varepsilon^2\mu\triangle B, \quad (\nabla, V) = (\nabla, B) = 0. \tag{1.1}$$

Here, $x \in \mathbb{R}^n$, $n = 2, 3$, B is the magnetic field (a vector field in $\mathbb{R}^n$), $V(x)$ a given vector field (fluid velocity), the coefficient of resistance (the inverse of the magnetic Reynolds number) is written as $\varepsilon^2\mu$ for convenience when writing future asymptotic behavior of the solution (we will assume that $\varepsilon \to 0$). For the equation (1.1), consider the Cauchy problem

$$B|_{t=0} = B^0(x), \tag{1.2}$$

2000 *Mathematics Subject Classification.* 35Q, 46F.

Key words and phrases. Induction equation, Cauchy problem, generalized solutions.

The work is supported by the Grant of the Russian Government for the support of scientific research, carried out under the supervision of leading scientists in FGBOU VPO "Moscow State University", grant N 11G.34.31.0054, and by RFBR grants 11-01-00937-, 13-01-00664, grant for young scientists "My first grant" 12-01-31235.

where B_0 is a smooth compactly supported divergence-free vector field. We are interested in the case of a discontinuous velocity field V; it is well known that the divergence-free field can have only a tangential discontinuity on a smooth surface (on a curve in the two-dimensional case). We denote this surface by M and assume it to be compact and defined by the equation $\Phi(x) = 0$, where Φ is a smooth function and $\nabla\Phi|_M \neq 0$. As we will see later, the magnetic field B has, in general, a δ-type singularity on the surface M; thus a generalized formulation of the problem (1.1)–(1.2), in general, is not clear (even if we write the system in the divergent form we have to multiply the δ-function by a discontinuous one). Therefore, we regularize the problem as follows. We introduce the fast variable $y = \frac{\Phi(x)}{\varepsilon}$ and consider the "smoothened" velocity field $V(y,x)$: we assume that V is a smooth function, and

$$\lim_{y\to\pm\infty} V(x,y) = V_\pm(x), \quad \forall x. \tag{1.3}$$

Here, $V_\pm$ are smooth divergence-free vector fields, and convergence to the limit is assumed to be sufficiently fast (faster than any power of y).

In particular, we can consider the field of the form

$$V(x,y) = \frac{V_+(x) + V_-(x)}{2} + \beta(y)\frac{V_+(x) - V_-(x)}{2},$$

where $\beta(y)$ is a smooth function which tends to ± 1 as $y \to \pm\infty$.

We are particularly interested in the weak limit of B as $\varepsilon \to 0$ and its dependence on the method of regularization of the velocity field V. Note that, if the magnetic viscosity coefficient is not $O(\varepsilon^2)$, the delta-type singularity, in general, does not appear, — the viscosity prevents rapid changes of B. That is why this factor in the system (1.1) was chosen as $\varepsilon^2\mu$. We begin studying solutions of the Cauchy problem (1.1)–(1.2) in the case $\mu = 0$ (an ideally conducting fluid).

2. Regularization and generalized solutions in the case of an ideally conducting fluid

2.1. **Two-dimensional case.** First, we consider the two-dimensional case. Since the field V is divergence-free, trajectories of V are the level lines of a scalar function (the current function). A smooth closed curve M is a trajectory of the field; in a neighborhood of M, we can introduce the action-angle variables – I, φ (see, e.g. [1]). The field V has the form

$$V = \omega(y, I)\frac{\partial}{\partial\varphi},$$

where $\omega \to \omega_\pm$ for $y \to \pm\infty$. We can assume that the curve M is defined by the equation $I = 0$ and $y = I/\varepsilon$. We also assume that action-angle variables are defined globally (i.e. in the area containing the support of the initial field B^0) do not depend on y.

Theorem 1. *Let $\mu = 0$. Under the assumptions formulated above, the weak limit of the magnetic field does not depend on the form of regularization of V and has a delta-type singularity on the curve M.*

Proof. Let us write the equation (1.1) in the coordinates I, φ,

$$\begin{aligned}
&\frac{\partial B_I}{\partial t} + \omega\frac{\partial B_I}{\partial\varphi} = 0,\\
&\frac{\partial B_\varphi}{\partial t} + \omega\frac{\partial B_\varphi}{\partial\varphi} = \frac{1}{\varepsilon}B_I\frac{\partial\omega}{\partial y} + B_I\frac{\partial\omega}{\partial I},\\
&B_I|_{t=0} = B_I^0(I,\varphi),\\
&B_\varphi|_{t=0} = B_\varphi^0(I,\varphi).
\end{aligned}$$

Solution of this system has the form

$$B_I = B_I^0(I, \varphi - \omega t),$$
$$B_\varphi = t(\frac{1}{\varepsilon}\frac{\partial\omega}{\partial y} + \frac{\partial\omega}{\partial I})B_I^0(I, \varphi - \omega t) + B_\varphi^0(I, \varphi).$$

Let us compute the weak limit of the solution as $\varepsilon \to 0$. Due to the fact that

$$\omega \to \omega_\pm, \quad \frac{\partial\omega}{\partial I} \to \frac{\partial\omega_\pm}{\partial I}, \quad \frac{\partial\omega}{\partial y} \to 0$$

as $y \to \pm\infty$, we can deduce

$$\lim_{\varepsilon\to 0} B_I = B_I^0(I, \varphi - \omega_- t) + \theta(I)(B_I^0(I, \varphi - \omega_+ t) - B_I^0(I, \varphi - \omega_- t)),$$
$$\lim_{\varepsilon\to 0} B_\varphi = t\delta(I)\int_{-\infty}^{\infty} \frac{\partial\omega}{\partial y}(y, 0)B_I^0(0, \varphi - \omega(y, 0)t)\,dy$$
$$+ t\frac{\partial\omega_-}{\partial I}B_I^0(I, \varphi - \omega_- t) + B_\varphi^0(I, \varphi - \omega_- t) + \theta(I)\Big(t\frac{\partial\omega_+}{\partial I}B_I^0(I, \varphi - \omega_+ t)$$
$$+ B_\varphi^0(I, \varphi - \omega_+ t) - t\frac{\partial\omega_-}{\partial I}B_I^0(I, \varphi - \omega_- t) - B_\varphi^0(I, \varphi - \omega_- t)\Big).$$

Here, $\theta(I)$ is the Heaviside function. Evidently, only the factor of $\delta(I)$ can depend on the form of regularization of V; changing the integration variable $y \to t\omega y$ and calculating the integral, we find that this coefficient equals

$$-\int_{\varphi-\omega_-(0)t}^{\varphi-\omega_+(0)t} B_I^0(0, z)dz, \quad z = \varphi - \omega t, \tag{2.1}$$

and it doesn't depend on the form of regularization of V (i.e. depends only on $V_\pm$). □

Note, that in the two-dimensional case it is easy to obtain a regularization of the original problem which directly admits a generalized formulation. In fact, let $\omega(I)$ be a discontinuous function: $\omega = \omega_-(I) + \theta(I)(\omega_+(I) - \omega_-(I))$ ($\omega_{\pm(I)}$ are assumed to be smooth). We write the system in the "divergent" form,

$$\begin{aligned} &\frac{\partial B_I}{\partial t} + \omega\frac{\partial B_I}{\partial\varphi} = 0,\\ &\frac{\partial B_\varphi}{\partial t} = \frac{\partial}{\partial I}(B_I\omega),\\ &B_I|_{t=0} = B_I^0(I, \varphi),\\ &B_\varphi|_{t=0} = B_\varphi^0(I, \varphi). \end{aligned} \tag{2.2}$$

Theorem 2. *The generalized function*

$$\begin{aligned} B_I &= B_I^0(I, \varphi - \omega t),\\ B_\varphi &= \frac{\partial}{\partial I}\omega_-\int_0^t B_I^0(I, \varphi - \omega_-(I)t)\,dt\\ &+ \theta(I)\frac{\partial}{\partial I}\Big(\omega_+\int_0^t B_I^0(I, \varphi - \omega_+(I)t)\,dt - \omega_-\int_0^t B_I^0(I, \varphi - \omega_-(I)t)\,dt\Big)\\ &+ \delta(I)\int_0^t \big(\omega_+(0)B_I^0(0, \varphi - \omega_+(0)t) - \omega_-(0)B_I^0(0, \varphi - \omega_-(0)t)\big)\,dt + B_\varphi^0(I, \varphi) \end{aligned} \tag{2.3}$$

satisfies (2.2)

Proof. First, we solve the first equation of the system,

$$B_I = B_I^0(I, \varphi - \omega t).$$

Then we obtain a solution of the second one,

$$B_\varphi = \int_0^t \frac{\partial}{\partial I}(B_I \omega)dt + B_\varphi^0(I, \varphi).$$

Rewriting $\omega B_I^0(I, \varphi - \omega(I)t)$ in the form

$$\omega B_I^0(I, \varphi - \omega(I)t) = \omega_-(I) B_I^0(I, \varphi - \omega_-(I)t) \\ + \theta(I)\big(\omega_+ B_I^0(I, \varphi - \omega_+(I)t) - \omega_- B_I^0(I, \varphi - \omega_-(I)t)\big)$$

and differentiating the product of $\theta(I)$ and a smooth function, we obtain

$$B_\varphi = \frac{\partial}{\partial I}\omega_- \int_0^t B_I^0(I, \varphi - \omega_-(I)t)\, dt \\ + \theta(I)\frac{\partial}{\partial I}\Big(\omega_+ \int_0^t B_I^0(I, \varphi - \omega_+(I)t)\, dt - \omega_- \int_0^t B_I^0(I, \varphi - \omega_-(I)t)\, dt\Big) \\ + \delta(I) \int_0^t \big(\omega_+(0) B_I^0(0, \varphi - \omega_+(0)t) - \omega_-(0) B_I^0(0, \varphi - \omega_-(0)t)\big)\, dt + B_\varphi^0(I, \varphi).$$

□

Note that the obtained generalized solution coincides with the weak limit of the smooth solution of the "smoothened" problem (see Theorem 1). Compare, for example, the coefficients of the delta function. We denote by $F(z)$ the integral of $B_I^0(0, z)$. Obviously, the function (2.1) can be written as

$$F(\varphi - \omega_-(0)t) - F(\varphi - \omega_+(0)t);$$

it is clear that it equals the function

$$\int_0^t \big(\omega_+(0) B_I^0(0, \varphi - \omega_+(0)t) - \omega_-(0) B_I^0(0, \varphi - \omega_-(0)t)\big)\, dt,$$

determining the coefficient of the delta function in formula (2.3). Similarly one can check that all the other terms also coincide.

2.2. **Three-dimensional case.** Now let us consider the three-dimensional case. It turns out that in this case the weak limit of smooth solution of the problem is uniquely defined (i.e. independent of the method of regularization of V) if, when passing through the surface, the jump of the field V is "one-dimensional". In another words, we have the jump either of the length or of the direction of the vector V (but not of the both quantities). Let us formulate a precise result. We will assume that V is an Euler field (i.e., a stationary solution of the Euler equations) in general position. The latter means that the fields V and curl$V$ are linearly independent almost everywhere in the treated area of the three-dimensional space. In this case, the trajectories of V lie on two-dimensional surfaces, Bernoulli surfaces, which, if they are compact, are homeomorphic to tori. Moreover, in the area, foliated by these tori, one can introduce action — angle variables I, φ, $\varphi = (\varphi_1, \varphi_2)$, such that

$$V = \Big(\omega(I), \frac{\partial}{\partial \varphi}\Big) = \omega_1(I)\frac{\partial}{\partial \varphi_1} + \omega_2(I)\frac{\partial}{\partial \varphi_2}.$$

Here, the variable I numbers the tori and is proportional to the volume of the area bounded by the torus, $\varphi \mod 2\pi$ are angular coordinates on the tori. We again assume that these variables are defined globally on support of B^0 and are independent of y. We assume that the surface of the field discontinuity is given by $I = 0$ and represent the vector of frequencies ω in the form of

$$\omega(I) = \lambda\omega_0,$$

where λ is a scalar function, ω_0 is the unit vector

$$\omega_0 = (\cos\alpha, \sin\alpha).$$

The jump of the field when crossing the surface is the jump (or, in the smoothened problem, a rapid change) of the two-dimensional vector ω, the jump of its absolute value is the jump of λ, and the jump of the direction is the jump of the angle function α. Consider the regularized problem, let $y = I/\varepsilon$ and let $\omega \to \omega_\pm$ as $y \to \pm\infty$.

Theorem 3. *Let one of the two functions α, λ be independent of y. Then the weak limit of the smooth solution of the Cauchy problem is independent of the regularizing function (i.e. depends only on $\omega_\pm$).*

Proof. Let's write the equations in the action–angle coordinates,

$$\begin{cases} \dfrac{\partial B_I}{\partial t} + \Big(\omega, \dfrac{\partial B_I}{\partial \varphi}\Big) = 0, \\ \dfrac{\partial B_\varphi}{\partial t} + \Big(\omega, \dfrac{\partial}{\partial \varphi}\Big) B_\varphi = B_I\Big(\dfrac{\partial \omega}{\partial I} + \dfrac{1}{\varepsilon}\dfrac{\partial \omega}{\partial y}\Big). \end{cases} \tag{2.4}$$

A solution of the first equation has the form $B_I = B_I^0(I, \varphi - \omega t)$. The second (vector) equation we project to the unit vector ω_0 and to the orthogonal unit vector $n = (-\sin\alpha, \cos\alpha)$. We denote the corresponding components of the vector B_φ by B_ω and B_n. For these functions, we obtain the following equations:

$$\Big(\frac{\partial}{\partial t} + \Big(\omega, \frac{\partial}{\partial \varphi}\Big)\Big) B_\omega = B_I\Big(\omega_0, \Big(\frac{\partial}{\partial I} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\Big)\omega\Big),$$
$$\Big(\frac{\partial}{\partial t} + \Big(\omega, \frac{\partial}{\partial \varphi}\Big)\Big) B_n = B_I\Big(n, \Big(\frac{\partial}{\partial I} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\Big)\omega\Big),$$

Let ω be $\lambda\omega_0$; using the fact that the derivative of the unit vector ω_0 equals n, multiplied by the derivative of α, we obtain the following system:

$$\Big(\frac{\partial}{\partial t} + \Big(\omega, \frac{\partial}{\partial \varphi}\Big)\Big) B_\omega = B_I\Big(\frac{\partial}{\partial I} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\Big)\lambda,$$
$$\Big(\frac{\partial}{\partial t} + \Big(\omega, \frac{\partial}{\partial \varphi}\Big)\Big) B_n = \lambda B_I\Big(\frac{\partial}{\partial I} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\Big)\alpha.$$

A solution has the form

$$B_\omega = t\Big(\frac{\partial \lambda}{\partial I} + \frac{1}{\varepsilon}\frac{\partial \lambda}{\partial y}\Big) B_I^0(I, \varphi - \omega t) + B_\omega^0(I, \varphi - \omega t),$$
$$B_n = t\lambda\Big(\frac{\partial \alpha}{\partial I} + \frac{1}{\varepsilon}\frac{\partial \alpha}{\partial y}\Big) B_I^0(I, \varphi - \omega t) + B_n^0(I, \varphi - \omega t).$$

Now we use the fact that only one of the functions α, λ depends on y.

1. Let α be independent of y; let us compute the weak limit of the solution

$$\begin{aligned} \lim_{\varepsilon\to 0} B_\omega = \delta(I) \int_{-\infty}^{\infty} t\frac{\partial \lambda}{\partial y} B_I^0(0, \varphi - t\lambda\omega_0(0))\, dy \\ + B_\omega^0(I, \varphi - \omega_- t) + \theta(I)\big(B_\omega^0(I, \varphi - \omega_+ t) - B_\omega^0(I, \varphi - \omega_- t)\big) \\ + t\frac{\partial \lambda_-}{\partial I} B_I^0(I, \varphi - \omega_- t) + \theta(I)\Big(t\frac{\partial \lambda_+}{\partial I} B_I^0(I, \varphi - \omega_+ t) - t\frac{\partial \lambda_-}{\partial I} B_I^0(I, \varphi - \omega_- t)\Big). \end{aligned}$$

Clearly, only the coefficient of the delta function can depend on the way of regularization of V. Changing the variable of integration $y \to t\lambda(y)$, this coefficient can be rewritten as

$$\int_{\lambda_- t}^{\lambda_+ t} B_I^0(0, \varphi - \omega_0(0)z)\, dz, \quad z = \lambda t,$$

which does not depend on the form of the function λ (i.e. on the way of regularization of V).

The weak limit of B_n can be calculated similarly,

$$\begin{aligned}\lim_{\varepsilon\to 0} B_n &= B_n^0(I, \varphi - \omega_- t) \\ &\quad + \theta(I)\big(B_n^0(I, \varphi - \omega_+ t) - B_n^0(I, \varphi - \omega_- t)\big) + t\lambda_- \frac{\partial \alpha_-}{\partial I} B_I^0(I, \varphi - \omega_- t) \\ &\quad + \theta(I)\Big(t\lambda_+ \frac{\partial \alpha_+}{\partial I} B_I^0(I, \varphi - \omega_+ t) - t\lambda_- \frac{\partial \alpha_-}{\partial I} B_I^0(I, \varphi - \omega_- t)\Big).\end{aligned}$$

This limit is also independent of the regularizing function $\lambda(y)$.

2. Now let λ be independent of y. Let us compute the weak limit of the vector $B_\varphi = B_\omega \omega_0 + B_n n$,

$$\begin{aligned}\lim_{\varepsilon\to 0} B_\varphi &= \delta(I) \int_{-\infty}^{\infty} t\lambda \frac{\partial \alpha}{\partial y} B_I^0(0, \varphi - t\lambda\omega_0(0, \alpha)) n(0, \alpha)\, dy \\ &\quad + n_-\Big(B_n^0(I, \varphi - \omega_- t) + t\lambda_- \frac{\partial \alpha_-}{\partial I} B_I^0(I, \varphi - \omega_- t)\Big) \\ &\quad + \theta(I)\Bigg(n_+\Big(B_n^0(I, \varphi - \omega_+ t) + t\lambda_+ \frac{\partial \alpha_+}{\partial I} B_I^0(I, \varphi - \omega_+ t)\Big) \\ &\quad - n_-(B_n^0(I, \varphi - \omega_- t)) + t\lambda_- \frac{\partial \alpha_-}{\partial I} B_I^0(I, \varphi - \omega_- t)\Bigg) \\ &\quad + \omega_0^-\Big(B_\omega^0(I, \varphi - \omega_- t) + t\frac{\partial \lambda_-}{\partial I} B_I^0(I, \varphi - \omega_- t)\Big) \\ &\quad + \theta(I)\Bigg(\omega_0^+\Big(B_\omega^0(I, \varphi - \omega_+ t) + t\frac{\partial \lambda_+}{\partial I} B_I^0(I, \varphi - \omega_+ t)\Big) \\ &\quad - \omega_0^-(B_\omega^0(I, \varphi - \omega_- t)) + t\frac{\partial \lambda_-}{\partial I} B_I^0(I, \varphi - \omega_- t)\Bigg),\end{aligned}$$

where $\omega_0^\pm = \omega_0(0, \alpha_\pm)$, $n_\pm = n(0, \alpha_\pm)$. It is clear that only the factor of the delta function can depend on the way of regularization (i.e. on the function α). Changing the variable of integration, we present this form factor in the form

$$t\lambda(0) \int_{\alpha_-}^{\alpha_+} B_I^0(o, \varphi - t\lambda(0)\omega_0(\alpha)) n(0, \alpha)\, d\alpha,$$

which does not depend on α. □

Remark 1. Generally speaking, there are other ways to regularize the velocity field; one of them, for example, is indicated at the beginning of the paper. The methods of regularization, discussed above (the “smoothening” is applied either to the direction or to the absolute value of V) seem to be quite natural. In particular, as we will prove later, if α is independent of y, the weak limit of the solution coincides with the generalized solution of the special regularized problem.

Similarly to the two-dimensional case, the invariance of the weak limit with respect to the method of smoothing the velocity field is associated with the possibility of the direct generalized formulation of the original problem. First we consider the case when the absolute value (but not the direction) of the field is discontinuous. The weak formulation is based on the two simple ideas.

1. If V is smooth, the system of induction equations can be written in the "divergence" form

$$\frac{\partial B}{\partial t} = \operatorname{curl}(V \times B). \tag{2.5}$$

2. From the formulas for the solutions obtained in the previous theorem, it follows that the delta-type singularity occurs only in the components of the magnetic field parallel to V. However, in the right-hand side of (2.5) this component is obviously not included, — this system can be rewritten as

$$\frac{\partial B}{\partial t} = \operatorname{curl}(V \times B_\perp), \tag{2.6}$$

where $B_\perp$ denotes the projection of B to the plane orthogonal to V (note that, under our assumptions, this plane depends smoothly on x). In what follows, by the generalized formulation of the Cauchy problem with discontinuous field V we mean (2.6). As before, we assume that V is an Euler field in a general position without a jump of the direction, i.e. that in the action–angle variables

$$V = \lambda(I)(\omega_0(I), \frac{\partial}{\partial\varphi}),$$

where $\omega_0(I)$ is a smooth unit vector function, and the scalar function λ is discontinuous at $I = 0$,

$$\lambda = \lambda_-(I) + \theta(I)(\lambda_+(I) - \lambda_-(I)),$$

where $\lambda_\pm$ are smooth functions.

Theorem 4. *The generalized function*

$$\begin{aligned}
B_I &= B_I^0(I, \varphi - \lambda\omega_0^- t) + \theta(I)(B_I^0(I, \varphi - \lambda\omega_0^+ t) - B_I^0(I, \varphi - \lambda\omega_0^- t)),\\
B_n &= t\frac{\partial\alpha_-}{\partial I}\lambda_-(I)B_I^0(I, \varphi - \omega_- t) + t\theta(I)\Big(\frac{\partial\alpha_+}{\partial I}\lambda_+(I)B_I^0(I, \varphi - \omega_+ t)\\
&\quad - \frac{\partial\alpha_-}{\partial I}\lambda_-(I)B_I^0(I, \varphi - \omega_- t)\Big)\\
&\quad + B_n^0(I, \varphi - \omega_- t) + \theta(I)\big(B_n^0(I, \varphi - \omega_+ t) - B_n^0(I, \varphi - \omega_- t)\big),\\
B_\omega &= \int_0^t \Big(\lambda(I)\Big(n, \frac{\partial}{\partial\varphi}\Big)B_n + \frac{\partial}{\partial I}\big(\lambda_- B_I(I, \varphi - \omega_- t)\big)\Big)\,dt\\
&\quad + \delta(I)\int_0^t \big(\lambda_+ B_I^0(0, \varphi - \omega_+ t) - \lambda_- B_I^0(0, \varphi - \omega_- t)\big)\,dt\\
&\quad + \theta(I)\int_0^t \frac{\partial}{\partial I}\big(\lambda_+ B_I^0(0, \varphi - \omega_+ t) - \lambda_- B_I^0(0, \varphi - \omega_- t)\big)\,dt.
\end{aligned}$$

satisfies (2.6) *with the initial conditions* (1.2). *This function coincides with the weak limit of the "smoothened" problem.*

Proof. Let us turn to the action–angle coordinates I, $\varphi = (\varphi_1, \varphi_2)$. The system can be rewritten as follows:

$$\begin{aligned}
&\frac{\partial B_I}{\partial t} + \Big(\omega, \frac{\partial}{\partial \varphi}\Big) B_I = 0,\\
&\frac{\partial B_\varphi}{\partial t} + \Big(\omega, \frac{\partial}{\partial \varphi}\Big) B_\varphi - B_I \frac{\partial \omega}{\partial I} = 0,\\
&\lambda(I) \int_0^t \Big(n, \frac{\partial}{\partial \varphi}\Big) B_n \, dt = \lambda(I) \frac{\partial \alpha}{\partial I} \int_0^t B_I^0\big(I, \varphi - \lambda\omega_0 t\big) t \, dt.
\end{aligned}$$

Using the equality $(\nabla, B) = 0$, the equation for B_φ can be rewritten as

$$\frac{\partial B_\varphi}{\partial t} + \Big(\omega, \frac{\partial}{\partial \varphi}\Big) B_\varphi - \omega\Big(\frac{\partial}{\partial \varphi}, B_\varphi\Big) = \frac{\partial}{\partial I}\big(\omega B_I\big).$$

We expand this component of the magnetic field in the following components:

$$B_\varphi = B_\omega \omega_0 + B_n n,$$

where $\omega = (\omega_1, \omega_2) = \lambda(I)\omega_0$, $\lambda(I)$ is a discontinuous function, $\omega_0(I)$ is a unit vector, depending smoothly on I. The right side of the equation can be written as follows:

$$\frac{\partial}{\partial I}\big(B_I \lambda \omega_0\big) = \omega_0 \frac{\partial}{\partial I}\big(B_I \lambda\big) + B_I \lambda \frac{\partial \omega_0}{\partial I}.$$

We project the equation on smooth unit vectors ω_0 and n; taking into account that $\frac{\partial \omega_0}{\partial I} = n(I) \frac{\partial \alpha}{\partial I}$, we obtain

$$\begin{aligned}
&\frac{\partial B_I}{\partial t} + \lambda\Big(\omega_0, \frac{\partial}{\partial \varphi}\Big) B_I = 0,\\
&\frac{\partial B_n}{\partial t} + \lambda\Big(\omega_0, \frac{\partial}{\partial \varphi}\Big) B_n = \frac{\partial \alpha}{\partial I} \lambda(I) B_I,\\
&\frac{\partial B_\omega}{\partial t} = \lambda(I)\Big(n, \frac{\partial}{\partial \varphi}\Big) B_n + \frac{\partial}{\partial I}\big(\lambda B_I\big),\\
&B_I|_{t=0} = B_I^0(I, \varphi), \quad B_\omega|_{t=0} = B_\omega^0(I, \varphi), \quad B_n|_{t=0} = B_n^0(I, \varphi).
\end{aligned}$$

Solutions of the first two equations have the form

$$\begin{aligned}
B_I &= B_I^0(I, \varphi - \lambda\omega_0 t),\\
B_n &= t\lambda \frac{\partial \alpha}{\partial I} B_I^0(I, \varphi - \lambda\omega_0 t).
\end{aligned}$$

We note that these functions have simple discontinuities on the surface M. Expressing them through the Heaviside function and substituting to the equation for B_ω, we finally obtain

$$\begin{aligned}
B_I &= B_I^0\big(I, \varphi - \lambda\omega_0^- t\big) + \theta(I)\big(B_I^0(I, \varphi - \lambda\omega_0^+ t) - B_I^0(I, \varphi - \lambda\omega_0^- t)\big),\\
B_n &= t \frac{\partial \alpha_-}{\partial I} \lambda_-(I) B_I^0(I, \varphi - \omega_- t)\\
&\quad + t\theta(I)\Big(\frac{\partial \alpha_+}{\partial I} \lambda_+(I) B_I^0(I, \varphi - \omega_+ t) - \frac{\partial \alpha_-}{\partial I} \lambda_-(I) B_I^0(I, \varphi - \omega_- t)\Big)\\
&\quad + B_n^0(I, \varphi - \omega_- t) + \theta(I)\big(B_n^0(I, \varphi - \omega_+ t) - B_n^0(I, \varphi - \omega_- t)\big),\\
B_\omega &= \int_0^t \Big(\lambda(I)\Big(n, \frac{\partial}{\partial \varphi}\Big) B_n + \frac{\partial}{\partial I}\big(\lambda_- B_I(I, \varphi - \omega_- t)\big)\Big) \, dt\\
&\quad + \delta(I) \int_0^t \big(\lambda_+ B_I^0(0, \varphi - \omega_+ t) - \lambda_- B_I^0(0, \varphi - \omega_- t)\big) \, dt\\
&\quad + \theta(I) \int_0^t \frac{\partial}{\partial I}\Big(\lambda_+ B_I^0(0, \varphi - \omega_+ t) - \lambda_- B_I^0(0, \varphi - \omega_- t)\Big) \, dt.
\end{aligned}$$

Thus, on the surface M, the component B_ω of the magnetic field has a delta-type singularity. Let us compare the coefficients of the delta function of the weak limit of the smooth problem and the generalized solution. The factor of the delta function in the generalized solution has the form

$$\begin{aligned}\int_0^t &\big(\lambda_+ B_I^0(0,\varphi-\omega_+ t) - \lambda_- B_I^0(0,\varphi-\omega_- t)\big)\,dt\\ &= \int_0^{\lambda_+ t} B_I^0(0,\varphi-\omega_+ t)\,d(\lambda_+ t) - \int_0^{\lambda_- t} B_I^0(0,\varphi-\omega_- t)\,d(\lambda_- t)\\ &= \int_{\lambda_- t}^{\lambda_+ t} B_I^0(0,\varphi-\omega t)\,d(\lambda t).\end{aligned}$$

This function coincides with the corresponding coefficient in the weak limit of the solution of the smooth problem. □

Now let α and n be discontinuous functions, and let the function $\lambda(I)$ be smooth. In this case the original system can also be rewritten in such a way that generalized solutions can be defined.

Consider the initial system of equations in the action-angle variables and suppose that V is smooth. This system has the form

$$\begin{aligned}\frac{\partial B_I}{\partial t} + \Big(\omega, \frac{\partial}{\partial\varphi}\Big)B_I &= 0,\\ \frac{\partial B_\varphi}{\partial t} + \Big(\omega, \frac{\partial}{\partial\varphi}\Big)B_\varphi &= B_I\frac{\partial\omega}{\partial I}.\end{aligned}$$

As before, we set $\omega = \lambda\omega_0$, $\omega_0 = (\cos\alpha, \sin\alpha)$.

The solution of the first equation with the initial condition $B_I|_{t=0} = B_I^0(I,\varphi)$ has the form

$$B_I = B_I^0(I,\varphi-\omega t).$$

Substituting this function into the equation for B_φ we obtain

$$\frac{\partial B_\varphi}{\partial t} + \Big(\omega, \frac{\partial}{\partial\varphi}\Big)B_\varphi = B_I^0(I,\varphi-\omega t)\frac{\partial\lambda}{\partial I}\omega_0 + B_I^0(I,\varphi-\omega t)\lambda\frac{\partial\alpha}{\partial I}n(\alpha). \tag{2.7}$$

We represent the solution as a sum of $B_\varphi = B_1 + B_2$, where each of the vectors B_j satisfies the equation with the right-hand side equal to one of the two components in (2.7). Note that the field B_1 is directed along ω_0, and B_2 is orthogonal to this vector. For B_1, B_2 we obtain the equations

$$\begin{cases}\dfrac{\partial B_1}{\partial t} + \Big(\omega, \dfrac{\partial}{\partial\varphi}\Big)B_1 = B_I^0(I,\varphi-\omega t)\dfrac{\partial\lambda}{\partial I}\omega_0,\\[2mm] \dfrac{\partial B_2}{\partial t} + \Big(\omega, \dfrac{\partial}{\partial\varphi}\Big)B_2 = B_I^0(I,\varphi-\omega t)\lambda\dfrac{\partial\alpha}{\partial I}n(\alpha),\end{cases} \tag{2.8}$$

$$B_1|_{t=0} = B_1^0(I,\varphi) \quad B_2|_{t=0} = B_2^0(I,\varphi).$$

It is easy to see that, if α has a simple discontinuity on M, the first equation in (2.8) has generalized solution which also has a simple discontinuity. The second equation can be transformed as follows. We denote $\int_{\alpha_0}^{\alpha} n(\alpha)B_I^0(I,\varphi-\lambda\omega_0(\alpha)t)\,d\alpha = w(\alpha,I)$, then the second equation can be rewritten as:

$$\frac{\partial B_2}{\partial t} + \Big(\omega, \frac{\partial}{\partial\varphi}\Big)B_2 = \lambda(I)\Big(\frac{\partial}{\partial I}\big(w(\alpha(I),I)\big) - \frac{\partial w}{\partial I}\Big).$$

Thus, if the field V is smooth, the solution of (2.8) has the form

$$B_1 = tB_I^0(I,\varphi-\omega t)\frac{\partial\lambda}{\partial I}\omega_0 + B_1^0(I,\varphi-\omega t),$$

$$B_2 = t\lambda(I)\Big(\frac{\partial}{\partial I}\big(w(\alpha(I),I)\big) - \frac{\partial w}{\partial I}\Big) + B_1^0(I,\varphi-\omega t),$$

where $B_1^0 = (B_\varphi^0,\omega_0)\omega_0$, $B_2^0 = (B_\varphi^0,n)n$. It is easy to see that these functions are well-defined generalized functions in the case of discontinuous α; we call them (along with a discontinuous function B_I) a generalized solution of (2.7).

Theorem 5. *The generalized solution of* (2.7) *coincides with the weak limit as* $\varepsilon \to 0$ *of the smooth regularized problem.*

Proof. Note that the function w and its derivatives have the following form:

$$\frac{\partial w}{\partial I} = \int_{\alpha_0}^{\alpha(I)} \frac{\partial B_I^0}{\partial I}(I,\varphi-\lambda\omega_0 t)\,d\alpha - t\int_{\alpha_0}^{\alpha(I)} n(\alpha)\Big(\frac{\partial B_I^0}{\partial\varphi},\omega_0(\alpha)\Big)\frac{\partial\lambda}{\partial I}\,d\alpha,$$

$$w = \int_{\alpha_0}^{\alpha_-(I)} n(\alpha)B_I^0(I,\varphi-\lambda\omega_0 t)\,d\alpha + \theta(I)\int_{\alpha_-(I)}^{\alpha_+(I)} n(\alpha)B_I^0(I,\varphi-\lambda\omega_0 t)\,d\alpha.$$

It is clear that the generalized solution may differ from the weak limit of the smooth problem only in the term containing the delta function. We write down the coefficient of $\delta(I)$ in the expression for B_φ,

$$\begin{aligned} &t\lambda\delta(I)\int_{\alpha_-}^{\alpha_+} n(0,\alpha)B_I^0(0,\varphi-\lambda\omega_0(0,\alpha)t)\,d\alpha \\ &= \delta(I)\int_{-\infty}^{\infty} t\lambda\frac{\partial\alpha}{\partial y} n(0,\alpha)B_I^0(0,\varphi-\lambda\omega_0(0,\alpha)t)\,dy. \end{aligned}$$

This function coincides with the corresponding weak limit in the smooth problem. □

3. Highly conducting fluid: asymptotic solutions of the Cauchy problem

In what follows we describe the asymptotic behavior of solutions to the Cauchy problem (1.1)–(1.2) with $\mu > 0$ and $\varepsilon \to 0$ (low resistance). We assume that the initial field is smooth, divergence-free, compactly supported and independent of ε. First, we describe the formal asymptotic solutions of this problem, i.e., we impose a formal series satisfying the equation and the initial conditions. Then we present the justification of the asymptotics, that is, prove that the partial sums of this series differ from the exact solution by a function, sufficiently fast decreasing as $\varepsilon \to 0$. At the second stage, we need certain estimates for the resolving operator of the Cauchy problem, which we prove separately.

3.1. The formal asymptotics.

Let $V(x,y)$ ($x \in \mathrm{R}^3, y \in \mathrm{R}$) be a smooth vector function satisfying the equations (1.3) and uniformly bounded together with all its derivatives. Let $\Phi(x)$ be a smooth scalar function; we assume that a smooth two-dimensional surface $M : \Phi(x) = 0$ is compact, $(V,\nabla\Phi)|_M = 0$ and $|\nabla\Phi|^2 = 1$ in the neighborhood of M. We also assume that $\Phi < 0$ in the domain, bounded by M and $|\nabla\Phi| \geq \mathrm{Const} > 0$ everywhere in $\mathbb{R}^3$. We construct an asymptotic solution of the equation (1.1) in the form

$$B\Big(x,\frac{\Phi(x)}{\varepsilon},t,\varepsilon\Big) = \frac{1}{\varepsilon}B_{-1}\Big(x,\frac{\Phi(x)}{\varepsilon},t\Big) + \sum_{k=0}^{\infty}\varepsilon^k B_k\Big(x,\frac{\Phi(x)}{\varepsilon},t\Big) \tag{3.1}$$

We assume that all the fields $B_k(y,x,t)$ are smooth function of all their arguments, $B_{-1} \to 0$ as $|y| \to \infty$, $B_k \to B_k^\pm$ as $y \to \pm\infty$ faster than any power of y. Thus, when $\varepsilon \to 0$, the first term of the series converges weakly to the delta-function on M, and other terms — to functions with a simple jump on M.

Fix a number T, independent of ε.

Theorem 6. *For $t \in [0,T]$ there exist such smooth vector fields $B_k(x,y,t)$, that the partial sums of* (3.1)

$$B^N\Big(x, \frac{\Phi(x)}{\varepsilon}, t, \varepsilon\Big) = \frac{1}{\varepsilon} B_{-1}\Big(x, \frac{\Phi(x)}{\varepsilon}, t\Big) + \sum_{k=0}^{N} \varepsilon^k B_k\Big(x, \frac{\Phi(x)}{\varepsilon}, t\Big)$$

satisfy the equations

$$\frac{\partial B^N}{\partial t} + (V,\nabla)B^N - (B^N,\nabla)V - \frac{1}{\varepsilon}(B^N,\nabla\Phi)\frac{\partial V}{\partial y} = \mu\varepsilon^2 \triangle B^N + O(\varepsilon^N). \tag{3.2}$$

Here the symbol $O(\varepsilon^N)$ denotes the estimate from above in the $C(\mathbb{R}^3)$-norm.

Proof. We introduce the following notations:

$$u(x,y,t) = B_{-1}(x,y,t)|_M,$$
$$f(x,y,t) = (B_0,\nabla\Phi)|_M,$$
$$w(x,y) \equiv y\frac{\partial}{\partial\Phi}|_M(V,\nabla\Phi),$$
$$v = V(x,y)|_M.$$

We substitute (3.1) into the equation (1.3) and equate the functions multiplied by the same powers of ε. For ε^{-2} we obtain

$$(V,\nabla\Phi)\frac{\partial B_{-1}}{\partial y} - (B_{-1},\nabla\Phi)\frac{\partial V}{\partial y} = 0.$$

Note that the left hand side of this equation decreases rapidly as $|y| \to \infty$, so, by the well-known estimate ([3])

$$\begin{aligned} F\Big(x, \frac{\phi(x)}{\varepsilon}\Big) &= F(x,y)|_{x\in M, y=\Phi/\varepsilon} + \Phi\Big(\frac{\partial}{\partial\Phi}F(x,y)\Big)\Big|_{x\in M, y=\Phi/\varepsilon} + \cdots \\ &= F(x,y)|_{x\in M, y=\Phi/\varepsilon} + \varepsilon y\Big(\frac{\partial}{\partial\Phi}F(x,y)\Big)\Big|_{x\in M, y=\Phi/\varepsilon} + \cdots \\ &= F(x,y)|_{x\in M, y=\Phi/\varepsilon} + O(\varepsilon) \end{aligned}$$

the equation can be mod$O(\varepsilon)$ restricted to the surface M. Note that in fact we apply a Taylor expansion with respect to the distance from the surface M. As $(V,\nabla\Phi) = 0$ on the surface M, the resulting equation implies that $(B_{-1},\nabla\Phi)|_M = 0$. Below, we continue functions decreasing in y and defined on M, to a neighborhood of the surface (and, further, to the whole space). We adopt the following convention: the functions and the fields F will be continued in a neighborhood of M in such way that they do not depend on Φ (i.e, as solutions of the equation $\nabla_{\nabla\Phi}F = 0$). Thus, we put $(B_{-1},\nabla\Phi) = 0$ in the neighborhood of M. This means that the main term in the expansion of B is tangent to the surface.

For the functions containing ε^{-1} we obtain

$$\begin{aligned} &y\frac{\partial B_{-1}}{\partial y}\Big|_M\Big[\frac{\partial}{\partial\Phi}(V,\nabla\Phi)\Big]\Big|_M + \frac{\partial B_{-1}}{\partial t} + (V,\nabla)B_{-1} \\ &\quad - (B_{-1},\nabla)V + (V,\nabla\Phi)\frac{\partial B_0}{\partial y} - (B_0,\nabla\Phi)\frac{\partial V}{\partial y} = \mu^2(\nabla\Phi)^2\frac{\partial^2 B_{-1}}{\partial y^2}. \end{aligned}$$

The first term in the left-hand side of this equation is obtained from the second term of the Taylor expansion for $(B_{-1},\nabla\Phi)\frac{\partial V}{\partial y}$ with respect to Φ. All the terms in the latter equation decrease rapidly as $|y| \to \infty$, so mod$O(\varepsilon)$ we can restrict the equation to M. Since $(\nabla\Phi)^2 = 1$, $B_{-1}|_M = u$ and $V|_M = v$, we have

$$\frac{\partial u}{\partial t} + \{v,u\} - f\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial y} = \mu^2\frac{\partial^2 u}{\partial y^2}. \tag{3.3}$$

Here $\{,\}$ denotes the commutator of vector fields on M. Note that this equation contains the function $f = (B_0, \nabla\Phi)|_M$ from the next term of the asymptotic expansion. In order to obtain a closed system of equations, consider the terms containing ε^0. Equating the corresponding functions we obtain

$$\begin{aligned}
&\frac{\partial B_0}{\partial t} - (B_0,\nabla)V + (V,\nabla)B_0 + (V,\nabla\Phi)\frac{\partial B_1}{\partial y} - (B_1,\nabla\Phi)\frac{\partial V}{\partial y} \\
&\quad + \frac{1}{2}y^2\frac{\partial B_{-1}}{\partial y}\frac{\partial^2}{\partial\Phi^2}(V,\nabla\Phi)\Big|_M + y\frac{\partial}{\partial\Phi}(V,\nabla\Phi)\Big|_M\frac{\partial B_0}{\partial y}\Big|_M \\
&\quad - (B_{-1},\nabla)y\frac{\partial V}{\partial y}\Big|_M - y\frac{\partial}{\partial\Phi}(B_0,\nabla\Phi)\Big|_M\frac{\partial V}{\partial y}\Big|_M + y(B_0,\nabla\Phi)\Big|_M\frac{\partial}{\partial\Phi}\frac{\partial V}{\partial y}\Big|_M \\
&\quad = 2\mu(\nabla\Phi,\nabla)\frac{\partial B_{-1}}{\partial y} + \mu\triangle\Phi\frac{\partial B_{-1}}{\partial y} + \mu(\nabla\Phi)^2\frac{\partial^2 B_0}{\partial y^2}.
\end{aligned}$$

Here we took into account the corresponding terms of the Taylor expansion with respect to Φ. It is easy to see that the left-hand side of the corresponding equality, in general, does not decrease as $y \to \pm\infty$. Let us turn here to the limit as $y \to \pm\infty$; since $B_k \to B_k^{\pm}$ and $V \to V^{\pm}$, we get the equation

$$\frac{\partial B_0^{\pm}}{\partial t} + (V^{\pm},\nabla)B_0^{\pm} - (B_0^{\pm},\nabla)V^{\pm} = 0.$$

Let $B_0^{\pm}$ be solutions of these equations in the corresponding domains, bounded by M; we also claim that

$$B_0^{\pm}|_{t=0} = B^0$$

in the corresponding domain (the existence and uniqueness of solution for this Cauchy problem is evident). Now entire multiplier of ε^0 vanishes as $y \to \pm\infty$, and hence, it can be mod$O(\varepsilon)$ restricted to M. Multiplying the resulting equality by the vector $\nabla\Phi|_M$, after direct calculations we obtain

$$\frac{\partial f}{\partial t} + (v,\nabla)f + w\frac{\partial f}{\partial y} - f\frac{\partial w}{\partial y} - (u,\nabla)w = \mu\frac{\partial^2 f}{\partial y^2}. \tag{3.4}$$

Thus, we got a coupled system of equations for the normal to the surface M component of the field B_0 and (tangent to M) field B_{-1}. Evidently, the initial conditions for this system have the form

$$u|_{t=0} = 0, \quad f|_{t=0} = (B^0,\nabla\Phi)|_M. \tag{3.5}$$

Let u, f be the solution of this problem. These functions with respect to the variables x are defined on the surface M; note that $u \to 0$ and $f \to (B_0^{\pm},\nabla\Phi)|_M$ as $y \to \pm\infty$. Now we define the corresponding functions in the whole space. We define B_{-1} according to the rule $\nabla_{\nabla\Phi}B_{-1} = 0$, $B_{-1}|_M = u$ (note that it is sufficient to define this function in the neighborhood of M). In order to define $(B_0,\nabla\Phi)$, we represent this function in the form

$$(B_0,\nabla\Phi) = (B_0^-,\nabla\Phi) + \eta(y)(B_0^+,\nabla\Phi) + f_0,$$

where $\eta(y) = \frac{1}{2}(1+\tanh y)$, $\nabla_{\nabla_\Phi} f_0 = 0$. Note that $f_0 \to 0$ as $|y| \to \infty$.

We now write down the equations for the projection of B_0 to the tangent plane to the surface M. After direct calculations, we obtain the equation of the form

$$\frac{\partial b_0}{\partial t} + \{v, b_0\} - (B_1,\nabla\Phi)|_M\frac{\partial v}{\partial y} + w\frac{\partial b_0}{\partial y} = \mu^2\frac{\partial^2 b_0}{\partial y^2} + F_0,$$

where b_0 is the tangent component of $B_0|_M$ and F_0 is already defined. This equation contains unknown function $w_1 = (B_1,\nabla\Phi)|_M$; in order to obtain the close system we consider the next approximation (terms, containing ε^1). Analogous considerations lead to the equation of the form

$$\frac{\partial f_1}{\partial t} + (v,\nabla)f_1 + w\frac{\partial f_1}{\partial y} - f_1\frac{\partial w}{\partial y} - (b_0,\nabla)w = \mu\frac{\partial^2 f_1}{\partial y^2} + G_0,$$

where $f_1 = (B_1, \nabla\Phi)|_M$ and G_0 is already defined. Using the same procedure we finally kill arbitrary term, appearing after the substitution of (3.1) to the equations (1.1). □

Remark 2. Described asymptotic procedure computes from the coupled systems the pairs B_k^τ, B_{k+1}^n, where B^τ and B^n denote tangent and normal to M components of the vector field.

Remark 3. For the leading terms of asymptotic solution we have the system (3.3)–(3.4) with initial conditions (3.5).

3.2. **Estimates for the Green's function of the equation** (1.1)**.** In order to justify the formal asymptotics obtained above, we have to obtain certain estimates for the resolving operator of the Cauchy problem (1.1)–(1.2). These estimates were proved by S. Smirnov in his bachelor thesis [6]; in the close case of periodic function $V(y)$ the analogous estimates were obtained in [5].

Lemma 1. *Suppose that as $|x| \to \infty$, all derivatives $V(x,y)$ and $\Phi(x)$ uniformly in y, t, and ε converge to constants faster then any power of $|x|$. Consider the scalar parabolic operator $L_0 = \frac{\partial}{\partial t} + (V,\nabla) - \varepsilon^2\triangle$. The Green's function $G(x,\xi,t,\tau)$ for the equation $L_0u = g$ satisfies the following estimates:*

$$|D^m G| \le A\frac{1}{(\varepsilon^2(t-\tau))^{3/2}}\frac{1}{(t-\tau)^{|m|/2}}\exp\Big(-\lambda\frac{|x-\xi|^2}{\varepsilon^2(t-\tau)}\Big), \tag{3.6}$$

where $m = (m_1, m_2, m_3)$, $|m| = m_1 + m_2 + m_3$, $D^m = \varepsilon^{|m|}\frac{\partial^{|m|}}{\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}}$, A does not depend on ε.

Proof. We use the Levy method (parametrix method, see [2]). We introduce the function

$$Z(x,t,\xi,\tau) = \frac{1}{(\varepsilon^2(t-\tau))^{3/2}}\exp\Big(-\lambda\frac{|x-\xi|^2}{\varepsilon^2(t-\tau)}\Big).$$

This is the fundamental solution of the operator $\frac{\partial}{\partial t} - \varepsilon^2\triangle$. We regard this operator as a "first approximation" to the operator L_0. The function Z will be regarded as " the leading part" of the fundamental solution G_0 of the operator L_0; namely, we construct G_0 in the form

$$G_0(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + \int_\tau^t\int_{\mathrm{R}^3} Z(x,t,\eta,\sigma)\Psi(\eta,\sigma,\xi,\tau)\,d\eta\,d\sigma. \tag{3.7}$$

From the equation $L_0G_0 = 0$ it follows that $\Psi(x,t,\xi,\tau)$ satisfies the Volterra equation with a singular kernel $LZ(x,t,\eta,\sigma)$

$$\Psi(x,t,\xi,\tau) = LZ(x,t,\xi,\tau) + \int_\tau^t\int_{\mathrm{R}^3} LZ(x,t,\eta,\sigma)\Psi(\eta,\sigma,\xi,\tau)\,d\eta\,d\sigma. \tag{3.8}$$

The solution of this equation is (see [2])

$$\Psi(x,t,\xi,\tau) = \Sigma_{\nu=1}^\infty (LZ)_\nu(x,t,\xi,\tau), \tag{3.9}$$

where $(LZ)_1 = LZ = (\frac{\partial}{\partial t} + (V,\nabla) - \varepsilon^2\triangle)Z$,

$$(LZ)_{\nu+1} = \int_\tau^t\int_{\mathrm{R}^3}[LZ(x,t,\eta,\sigma)](LZ)_\nu(\eta,\sigma,\xi,\tau)\,d\eta\,d\sigma.$$

Let us obtain estimates for the operator $(LZ)_\nu$

$$
\begin{aligned}
|(LZ)_1(x,t,\xi,\tau)| &= |LZ(x,t,\xi,\tau)| = |(V(x,y),\nabla)Z(x,t,\xi,\tau)| \\
&\le A_1 \frac{|x-\xi|}{(t-\tau)^{5/2}\varepsilon^5} \exp\Big(\frac{-|x-\xi|^2}{4\varepsilon^2(t-\tau)}\Big) \\
&\le A_1 \frac{1}{(t-\tau)^2\varepsilon^4} \exp\Big(-\lambda\frac{|x-\xi|^2}{4\varepsilon^2(t-\tau)}\Big).
\end{aligned}
\tag{3.10}
$$

In [2] the convergence of (3.9) for an arbitrary parabolic operator is proved. To obtain an estimate on the Green's function and its derivatives one has to differentiate the equation (3.7)

$$
D^m G_0(x,t,\xi,\tau) = D^m Z(x,t,\xi,\tau) + \int_\tau^t \int_{\mathrm{R}^3} (D^m Z(x,t,\eta,\sigma))\Psi(\eta,\sigma,\xi,\tau)\,d\eta\,d\sigma.
$$

Only the first term is important for the estimate. We have

$$
\begin{aligned}
\frac{\partial}{\partial x_i} Z(x,t,\xi,\tau) &= \frac{x_i-\xi_i}{2^{3/2+2|m|}\pi^{3/2}(t-\tau)^{3/2+|m|}\varepsilon^{3+2|m|}} \exp\Big(-\lambda\frac{|x-\xi|^2}{4\varepsilon^2|t-\tau|}\Big), \\
D^m Z(x,t,\xi,\tau) &\le \frac{C}{2^{3/2+|m|}\pi^{3/2}(t-\tau)^{3/2+|m|/2}\varepsilon^3} \exp\Big(-\lambda\frac{|x-\xi|^2}{4\varepsilon^2|t-\tau|}\Big).
\end{aligned}
$$

□

.

Remark 4. The proof of this lemma differs from that presented in [2] by the explicit assessment of the dependence of G_0 on ε.

Theorem 7. *The Green matrix of the Cauchy problem for the equation* (1.1) *for all* t *satisfies the estimates:* $|D^m G_{ij}| \le \frac{C}{\varepsilon}\frac{1}{\varepsilon^3 t^{1/2}}\frac{1}{t^{|m|/2}}\exp(-\frac{|x-z|^2}{\varepsilon^2 t})$.

Proof. Consider a column T of the matrix G and decompose it into components parallel and orthogonal to $\nabla\Phi$: $T = a\nabla\Phi + w$, $(w,\nabla\Phi) = 0$. This column satisfies the equation:

$$
L_0 T - \frac{1}{\varepsilon}\frac{\partial V}{\partial y}(\nabla\Phi, T) - \frac{\partial V}{\partial x} T = 0.
$$

For the further proof of the theorem we prove the following lemma.

Lemma 2. *The scalar function* a *and the vector* w *are related by the following equations:*

$$
\begin{cases}
L_0 a = \Lambda a + \varepsilon(M,w), \\
L_0 w = \dfrac{1}{\varepsilon} a \dfrac{\partial V}{\partial y} + Pw + \varepsilon Q a,
\end{cases}
\tag{3.11}
$$

where Λ, *the elements of the vector* Q *and matrix* P *are polynomials of degree 1 in the operators* $\varepsilon\frac{\partial}{\partial x_j}$, *whose coefficients are smooth functions of* x, y, t, ε *and* $(M,w) = \varepsilon(\nabla\Phi, \triangle w)$.

Proof. Substituting the expression for column T to the equations (1.1) we obtain

$$
L_0 a\nabla\Phi - \frac{1}{\varepsilon}\frac{\partial V}{\partial y} a - a\frac{\partial V}{\partial x}\nabla\Phi + L_0 w - \frac{\partial V}{\partial x} w = 0.
\tag{3.12}
$$

The projection of this equation on the direction given by the vector $\nabla\Phi$ has the form

$$
(\nabla\Phi, L_0 a\nabla\Phi) - \frac{1}{\varepsilon}\Big(\nabla\Phi, \frac{\partial V}{\partial y}\Big)a - a\Big(\nabla\Phi, \frac{\partial V}{\partial x}\nabla\Phi\Big) + (\nabla\Phi, L_0 w) - \Big(\nabla\Phi, \frac{\partial V}{\partial x} w\Big) = 0.
$$

Here $(\nabla\Phi, \frac{\partial V}{\partial y}) = 0$, $(\nabla\Phi, \frac{\partial w}{\partial t}) = 0$,

$$L_0 a\nabla\Phi = \Big(\frac{\partial a}{\partial t} + (V,\nabla)a - \varepsilon^2\triangle a\Big)\nabla\Phi + a(V,\nabla)\nabla\Phi + 2\varepsilon^2(\nabla a,\nabla)\nabla\Phi - \varepsilon^3\nabla^3\Phi a,$$

$$(\nabla\Phi,(V,\nabla)w) - \Big(\nabla\Phi, \frac{\partial V}{\partial x}w\Big) = (V,\nabla)(w,\nabla\Phi) - (w,(V,\nabla)\nabla\Phi) - \Big(w, \frac{\partial V^*}{\partial x}\nabla\Phi\Big)$$
$$= -\Big(w,(V,\nabla)\nabla\Phi + \frac{\partial V^*}{\partial x}\nabla\Phi\Big) = -(w,\nabla(V,\nabla\Phi)) = 0.$$

All the terms containing a and not included to the L_0a, are summarized in the Λa. Thus, we obtained the first equation in the system. In order to prove the second one, let us project (3.12) to plane, orthogonal to $\nabla\Phi$ (the tangent plane to the surface $M_c : \Phi = c.$). We have

$$\Big(L_0 a\nabla\Phi - a\frac{\partial V}{\partial x}\nabla\Phi - \frac{\partial V}{\partial x}w\Big) - \Big(\Big(L_0 a\nabla\Phi - a\frac{\partial V}{\partial x}\nabla\Phi - \frac{\partial V}{\partial x}w\Big),\nabla\Phi\Big)\nabla\Phi$$
$$-\frac{1}{\varepsilon}\frac{\partial V}{\partial y}a + \Pi|_{T_pM}(L_0 w) = 0.$$

The non-zero coefficients multiplied by a form the following expression:

$$(V,\nabla)\nabla\Phi - \frac{\partial V}{\partial x}\nabla\Phi - \Big(\Big((V,\nabla)\nabla\Phi - \frac{\partial V}{\partial x}\nabla\Phi\Big),\nabla\Phi\Big)\nabla\Phi.$$

The j th component of this vector is

$$V_i\frac{\partial^2\Phi}{\partial x_i\partial x_j} - \frac{\partial V_j}{\partial x_i}\frac{\partial\Phi}{\partial x_i} - V_i\frac{\partial^2\Phi}{\partial x_i\partial x_l}\frac{\partial\Phi}{\partial x_l}\frac{\partial\Phi}{\partial x_j} + \frac{\partial V_l}{\partial x_i}\frac{\partial\Phi}{\partial x_l}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}.$$

Note that $(V,\nabla)\nabla\Phi = \nabla(V,\nabla\Phi) - \frac{\partial V^*}{\partial x}\nabla\Phi = -\frac{\partial V_i}{\partial x_j}\frac{\partial\Phi}{\partial x_i}$, so the coefficient with a is

$$-\Big(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\Big)\frac{\partial\Phi}{\partial x_i} + \Big(\frac{\partial V_l}{\partial x_i} + \frac{\partial V_i}{\partial x_l}\Big)\frac{\partial\Phi}{\partial x_l}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}.$$

For the terms with $l = j$ we have

$$-\Big(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\Big)\frac{\partial\Phi}{\partial x_i} + \Big(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\Big)\frac{\partial\Phi}{\partial x_i}\Big(\frac{\partial\Phi}{\partial x_j}\Big)^2$$

as $|\nabla\Phi|^2 = 1$. Since the fluid is incompressible $(\nabla, V) = \frac{\partial V_i}{\partial x_i} = 0$, the terms with $l = i$ are: $\Big(\frac{\partial V_i}{\partial x_i} + \frac{\partial V_i}{\partial x_i}\Big)\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}$. The last term in $l \neq i \neq j$: $\Big(\frac{\partial V_i}{\partial x_l} + \frac{\partial V_l}{\partial x_i}\Big)\frac{\partial\Phi}{\partial x_l}\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}$. This expression does not depend on the choice of an orthonormal system of coordinates x_i, so we can choose a system in which one axis is directed along the $\nabla\Phi$, and the other two in the tangent plane. Conditions $l \neq i \neq j$ imply that at least one of the axes lies in the tangent plane to M_c; for this component we have $\frac{\partial\Phi}{\partial x_m} = 0$, so this expression vanishes. Carefully collecting all the terms, we get the second equation of the system. □

We continue the proof of the theorem. We look for solution of the obtained system in the form

$$a = \sum_{k=0}^{\infty}\varepsilon^k a_k, \quad w = \sum_{k=-1}^{\infty}\varepsilon^k w_k.$$

Equating terms with the respective powers of ε, we obtain

$$(L_0 - \Lambda)a_k = (\nabla\Phi, \triangle w_{k-2}),$$
$$(L_0 - P)w_k = a_{k+1}\frac{\partial V}{\partial y} + Qa_{k-1}.$$

For a_0 we have

$$(L_0 - \Lambda)a_0 = 0,$$

so

$$a_0 \leq A_0 \frac{1}{\varepsilon^3 t^{3/2}} \exp\Big(- \lambda_a \frac{|x-\xi|^2}{\varepsilon^2 t}\Big)$$

(see Lemma 1).

Now consider the estimate for w_{-1}

$$(L_0 - P) w_{-1} = a_0 \frac{\partial V}{\partial y},$$

$$w_{-1} \leq \int_\tau^t \int_{\mathrm{R}^3} B_0 \frac{1}{\varepsilon^3 (t-\sigma)^{3/2}} \Big| \frac{\partial V}{\partial y} \Big| \exp\Big(- \lambda \Big(\frac{|x-\eta|^2}{\varepsilon^2 (t-\sigma)} + \frac{|\eta-\xi|^2}{\varepsilon^2 \sigma}\Big)\Big) d\eta\, d\sigma,$$

where $\lambda = \max(\lambda_a, \lambda_w)$. To estimate the integral above we make the substitution

$$W_i = \Big(\frac{1}{\varepsilon^2} \frac{t-\sigma}{t-\tau}\Big)^{1/2} \frac{\eta_i - \xi_i}{2(\tau-\sigma)^{1/2}} + \Big(\frac{1}{\varepsilon^2} \frac{\tau-\sigma}{t-\tau}\Big)^{1/2} \frac{\xi_i - x_i}{2(\tau-\sigma)^{1/2}}.$$

Now the integral can be estimated from above $w_{-1} \leq W \frac{1}{\varepsilon^3 t^{1/2}} \exp(\lambda \frac{|x-\xi|^2}{\varepsilon^2 t})$.

The same can be done for the functions a_k and w_k. As a result, we obtain the following inequalities:

$$|a_k| \leq \frac{A_k}{\varepsilon^3} t^{-3/2+2k} \exp\Big(- \lambda \frac{|x-\xi|^2}{\varepsilon^2 t}\Big),$$

$$|w_k| \leq \frac{W_k}{\varepsilon^3} t^{-3/2+2k} \exp\Big(- \lambda \frac{|x-\xi|^2}{\varepsilon^2 t}\Big).$$

After summation of the series we get the following estimates :

$$|a| \leq \frac{A}{\varepsilon^3} t^{-3/2} \exp\Big(- \lambda \frac{|x-\xi|^2}{\varepsilon^2 t}\Big),$$

$$|w| \leq \frac{W}{\varepsilon^4} t^{-1/2} \exp\Big(- \lambda \frac{|x-\xi|^2}{\varepsilon^2 t}\Big).$$

Thus for all t the estimate $|T| \leq \frac{C}{\varepsilon^4} t^{-1/2} \exp(-\lambda \frac{|x-\xi|^2}{\varepsilon^2 t})$ holds. The same arguments can be used in order to obtain the estimates of the derivatives of the Green's matrix. □

Remark 5. The proof of the theorem is essentially based on the Jordan structure of the system (3.11).

3.3. **Justification of the formal asymptotics.** Now we can prove that the formal asymptotics constructed above is close to the exact solution of the problem (1.1)–(1.2). Namely, the following assertion follows immediately from the theorems 6 and 7.

Theorem 8. *For* $t \in [0, t_0]$ *the solution* $B(x,t,\varepsilon)$ *of the Cauchy problem of the equation* (1.1)–(1.2) *has the form*

$$B(x,t,\varepsilon) = \sum_{k=-1}^{N} \varepsilon^k B_k\Big(\frac{\Phi(x)}{\varepsilon} x, t\Big) + O(\varepsilon^{N+1}). \tag{3.13}$$

Remark 6. The weak limit of the solution as $\varepsilon \to 0$ has a δ-type singularity on the surface M. The corresponding vector is tangent to the surface.

Remark 7. The leading term of asymptotics is computed from the system (3.3)–(3.5).

References

1. V. I. Arnold, B. A. Khesin, *Topological Methods in Hydrodynamics*, Springer-Verlag, New York, 1998.
2. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.
3. V. P. Maslov, *The Complex WKB Method for Nonlinear Equations*, Birkhäuser, Basel, 1994.
4. V. P. Maslov, G. A. Omel'yanov, *Geometric Asymptotics for Nonlinear PDE*, AMS, Providence, RI, 2001.
5. A. I. Shafarevich, *Behavior of magnetic field in a conductiong fluid with rapidly varying velocity field*, Doklady Akademii Nauk **360** (1998), no. 1, 31–33.
6. S. A. Smirnov, *Bachelor thesis*, Moscow Institute for Physics and Technology, 2010.
7. M. M. Vishik, *Magnetic field generation by the motion of a highly conducting fluid*, Geophys. Astrophys. Fluid Dynam. **48** (1989), 151–167.
8. Ya. B. Zel'dovich, A. A. Ruzmaikin, D. D. Sokolov, *Magnetic Fields in Astrophysics*, Gordon and Breach, New York, 1983.

A. Yu. Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia
E-mail address: `esina_anna@list.ru`

M. V. Lomonosov Moscow State University, Moscow, Russia
E-mail address: `shafarev@yahoo.com`

Received 10/10/2013