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SCHRÖDINGER OPERATORS WITH NON-SYMMETRIC ZERO-RANGE POTENTIALS

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Dedicated to Professor V. D. Koshmanenko on the occasion of his seventieth birthday

Abstract. Non-self-adjoint Schrödinger operators $A_{\mathbf{T}}$ which correspond to non-symmetric zero-range potentials are investigated. For a given $A_{\mathbf{T}}$, a description of non-real eigenvalues, spectral singularities and exceptional points are obtained; the possibility of interpretation of $A_{\mathbf{T}}$ as a self-adjoint operator in a Krein space is studied, the problem of similarity of $A_{\mathbf{T}}$ to a self-adjoint operator in a Hilbert space is solved.

1. Introduction

An important class of Schrödinger operators is formed by operators with singular perturbations. For example, this class contains Schrödinger operators with zero-range potentials or point interactions. These operators effectively simulate short range interactions and belong to the class of exactly solvable models. Numerous works are devoted to a study of singularly perturbed Schrödinger operators, in which a series of approaches to the construction and investigation of such operators are developed (see, e.g., [1, 2, 3, 12, 13] and references therein). These studies, in the majority of cases, deal with *symmetric singular perturbations* that lead to *self-adjoint* Schrödinger operators.

In the present paper we study *non-self-adjoint* Schrödinger operators which correspond to non-symmetric zero-range potentials.

Our work was inspirited in part by an intensive development of Pseudo-Hermitian ($\mathcal{PT}$-Symmetric) Quantum Mechanics PHQM (PTQM) during last decades [7, 8, 16]. The key point of PHQM/PTQM theories is the employing of non-self-adjoint operators with certain properties of symmetry for the description of experimentally observable data. Briefly speaking, in order to interpret a given non-self-adjoint operator A in a Hilbert space $\mathfrak{H}$ as a physically meaning Hamiltonian we have to check the reality of its spectrum and to prove the existence of a new inner product that ensures the (hidden) self-adjointness of A.

The paper is devoted to the implementation of this program for various classes ($\mathcal{PT}$-symmetric operators, δ- and δ'- potentials with complex couplings, see definitions in Examples II-IV of Sec. 2) of non-self-adjoint Schrödinger operators $A_{\mathbf{T}}$ corresponding to the Schrödinger type differential expression (2.1) with singular zero-range potential

$$a < \delta, \cdot > \delta(x) + b < \delta', \cdot > \delta(x) + c < \delta, \cdot > \delta'(x) + d < \delta', \cdot > \delta'(x),$$

where the parameters a, b, c, d are complex numbers. The matrix $\mathbf{T}$ is formed by these parameters and operators $A_{\mathbf{T}}$ are defined by Lemma 2.1.

In Sec. 2, the necessary and sufficient conditions for the existence of non-real eigenvalues, spectral singularities and exceptional points of $A_{\mathbf{T}}$ are given.

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Sec. 3 is devoted to an interpretation of $A_{\mathbf{T}}$ as a self-adjoint operator in a Krein space. Such kind of self-adjointness cannot be considered as completely satisfactory in PHQM/PTQM because it does not guarantee unitarity of the dynamics generated by $A_{\mathbf{T}}$. However, possible realization of $A_{\mathbf{T}}$ as self-adjoint with respect to some indefinite metrics (indefinite inner product) allows us to apply well-developed tools of the Krein spaces theory [6] to solving the problem of similarity of $A_{\mathbf{T}}$ to a self-adjoint operator in a Hilbert space. The similarity property means that $A_{\mathbf{T}}$ turns out to be a self-adjoint operator in a Hilbert space with respect to a suitably chosen inner product.

In Sec. 4, we solve the similarity problem for $A_{\mathbf{T}}$ with the use of a general criterion of similarity [18] and the Krein spaces methods.

The properties of $A_{\mathbf{T}}$ established in the paper illustrate a typical PHQM/PTQM evolution of spectral properties which can be obtained by changing entries of $\mathbf{T}$: complex eigenvalues $\to$ spectral singularities / exceptional points $\to$ similarity to a self-adjoint operator. For this reason, the operators $A_{\mathbf{T}}$ considered in the work can be used as exactly solvable models of PHQM/PTQM.

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of a linear operator A, respectively, while $A \upharpoonright \mathcal{D}$ stands for the restriction of A to the set $\mathcal{D}$. The resolvent set and the spectrum of an operator A are denoted as $\rho(A)$ and $\sigma(A)$, respectively.

2. Operator realizations and their simplest properties

A one-dimensional Schrödinger operator corresponding to a general zero-range potential at the point $x = 0$ can be defined by the heuristic expression

$$-\frac{d^2}{dx^2} + a < \delta, \cdot > \delta(x) + b < \delta', \cdot > \delta(x) + c < \delta, \cdot > \delta'(x) + d < \delta', \cdot > \delta'(x), \tag{2.1}$$

where δ and δ' are, respectively, the Dirac δ-function and its derivative (with support at 0) and a, b, c, d are complex numbers.

The expression (2.1) gives rise to the symmetric operator

$$A_{\text{sym}} = -\frac{d^2}{dx^2}, \quad \mathcal{D}(A_{\text{sym}}) = \{u(x) \in W_2^2(\mathbb{R}) \mid u(0) = u'(0) = 0\} \tag{2.2}$$

acting in $L_2(\mathbb{R})$ and, generally speaking, any proper extension A of A_{sym} (i.e., $A_{\text{sym}} \subset A \subset A^*_{\text{sym}}$) can be considered as an operator realization of (2.1) in $L_2(\mathbb{R})$.

In order to specify more exactly which a proper extension A of A_{sym} corresponds to (2.1) we will use an approach suggested in [3]. The idea consists in the construction of some regularization $\mathbb{A}_{\mathbf{r}}$ of (2.1) that is well defined as an operator from $\mathcal{D}(A^*_{\text{sym}}) = W_2^2(\mathbb{R}\backslash\{0\})$ to $W_2^{-2}(\mathbb{R})$. Then, the corresponding operator realization of (2.1) in $L_2(\mathbb{R})$ is determined as follows:

$$A = \mathbb{A}_{\mathbf{r}} \upharpoonright_{\mathcal{D}(A)}, \quad \mathcal{D}(A) = \{f \in \mathcal{D}(A^*_{\text{sym}}) \mid \mathbb{A}_{\mathbf{r}} f \in L_2(\mathbb{R})\}. \tag{2.3}$$

To obtain a regularization of (2.1) it suffices to extend the distributions δ and δ' onto $W_2^2(\mathbb{R}\backslash\{0\})$. The most reasonable way (based on preserving of initial homogeneity of δ and δ' with respect to scaling transformations, see, for details, [3], [9]) leads to the following definition:

$$< \delta_{\text{ex}}, f >= \frac{f(+0) + f(-0)}{2}, \quad < \delta'_{\text{ex}}, f >= -\frac{f'(+0) + f'(-0)}{2}$$

for all $f(x) \in W_2^2(\mathbb{R}\backslash\{0\})$. In this case, the regularization of (2.1) onto $W_2^2(\mathbb{R}\backslash\{0\})$ takes the form

$$\mathbb{A}_{\mathbf{r}} = -\frac{d^2}{dx^2} + a < \delta_{\text{ex}}, \cdot > \delta(x) + b < \delta'_{\text{ex}}, \cdot > \delta(x) + c < \delta_{\text{ex}}, \cdot > \delta'(x) + d < \delta'_{\text{ex}}, \cdot > \delta'(x),$$

where $-d^2/dx^2$ acts on $W_2^2(\mathbb{R}\backslash\{0\})$ in the distributional sense.

The definition (2.3) is not always easy to use. Repeating the proof of Theorem 1 in [5] we obtain an equivalent description of operators determined by (2.3).

Lemma 2.1. *Let A be determined by (2.3). Then A coincides with the restriction of $A^*_{\mathrm{sym}} = -d^2/dx^2$ onto the domain*

$$\mathcal{D}(A) = \{f(x) \in W_2^2(\mathbb{R}\backslash\{0\}) \ \mid \ \mathbf{T}\Gamma_0 f = \Gamma_1 f\}, \quad \mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.4}$$

where

$$\Gamma_0 f = \frac{1}{2}\begin{pmatrix} f(+0) + f(-0) \\ -f'(+0) - f'(-0) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} f'(+0) - f'(-0) \\ f(+0) - f(-0) \end{pmatrix}. \tag{2.5}$$

Remark 2.2. In what follows the notation $A_{\mathbf{T}}$ will be used for operator realizations of (2.1) defined by (2.4) and (2.5).

It is known that the continuous spectrum of an operator $A_{\mathbf{T}}$ coincides with $[0, \infty)$ and the point spectrum of $A_{\mathbf{T}}$ may appear only in $\mathbb{C}\backslash\mathbb{R}_+$.

Denote

$$\mathbf{det}\ \mathbf{T} = ad - bc. \tag{2.6}$$

Lemma 2.3. *An operator $A_{\mathbf{T}}$ has an eigenvalue $z = \tau^2$ if and only if the equation*

$$2d\tau^2 + i(\mathbf{det}\ \mathbf{T} - 4)\tau + 2a = 0 \tag{2.7}$$

has a solution $\tau \in \mathbb{C}_+ = \{\tau \in \mathbb{C} \ : \ \mathrm{Im}\ \tau > 0\}$.

Proof. Let us denote by τ the square root of the energy parameter $z = \tau^2$ determined uniquely by the condition $\mathrm{Im}\ \tau > 0$ and consider the functions

$$h_{1\tau}(x) = \begin{cases} e^{i\tau x}, & x > 0 \\ e^{-i\tau x}, & x < 0 \end{cases}, \quad h_{2\tau}(x) = \begin{cases} -e^{i\tau x}, & x > 0 \\ e^{-i\tau x}, & x < 0 \end{cases} \tag{2.8}$$

that form a basis of $\ker(A^*_{\mathrm{sym}} - zI)$, where $z = \tau^2$ runs $\mathbb{C}\backslash\mathbb{R}_+$. It is clear that z belongs to the point spectrum of A if and only if there exists a function $f \in \ker(A^*_{\mathrm{sym}} - zI) \cap \mathcal{D}(A)$. Representing $f(x)$ in the form

$$f(x) = c_1 h_{1\tau}(x) + c_2 h_{2\tau}(x), \quad c_i \in \mathbb{C}$$

and substituting this expression into (2.4) we arrive at the conclusion that z is an eigenvalue of A if and only if the system of equations

$$(a - 2i\tau)c_1 + ib\tau c_2 = 0,$$
$$cc_1 + (id\tau + 2)c_2 = 0$$

has a nontrivial solution c_1, c_2. This is possible if the determinant of the coefficient matrix of the system is equal to zero, i.e., $2d\tau^2 + i(ad - bc - 4)\tau + 2a = 0$. Rewriting the obtained equation in the form (2.7) we complete the proof. □

Definition 2.4. *Let $A_{\mathbf{T}}$ be defined by (2.4), (2.5) and let the spectrum of $A_{\mathbf{T}}$ do not coincide with $\mathbb{C}$. We will say that the operator $A_{\mathbf{T}}$ has:*

- a nonzero spectral singularity $z = \tau^2$ *if the equation (2.7) has a solution $\tau \in \mathbb{R} \setminus \{0\}$;*
- a spectral singularity at point $z = 0$ *if (2.7) has a solution $\tau = 0$ with multiplicity* 2*;*
- a spectral singularity at point $z = \infty$ *if there are no solutions of (2.7) in $\mathbb{C}$.*

The non-self-adjoint operator $A_{\mathbf{T}}$ has an exceptional point $z = \tau^2$ *if the equation (2.7) has a solution $\tau \in \mathbb{C}_+$ with multiplicity* 2.

A spectral singularity (an exception point) z lies on the continuous spectrum (on the point spectrum) of $A_{\mathbf{T}}$ and it is a serious defect that rules out the operator as a viable candidate for a physical observable [11, 15].

Example I. *Symmetric potential.*
The singular potential

$$V = a < \delta, \cdot > \delta(x) + b < \delta', \cdot > \delta(x) + c < \delta, \cdot > \delta'(x) + d < \delta', \cdot > \delta'(x) \tag{2.9}$$

in (2.1) is symmetric (i.e., $V^* = V$) if and only if

$$a,\ d \in \mathbb{R}, \quad c = \overline{b}. \tag{2.10}$$

The corresponding operators $A_{\mathbf{T}}$ turn out to be self-adjoint operators in $L_2(\mathbb{R})$ with respect to the initial inner product

$$(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx. \tag{2.11}$$

Lemma 2.5. *The spectrum $\sigma(A_{\mathbf{T}})$ is real and it contains the continuous part $[0, \infty)$ and possibly, negative eigenvalues. There are no spectral singularities and exceptional points.*

Proof. An operator $A_{\mathbf{T}}$ is a finite dimensional extension of the symmetric operator A_{sym} determined by (2.2). This means that the continuous spectrum of $A_{\mathbf{T}}$ coincides with $[0, \infty)$.

Let $d \neq 0$. Then the solutions $\tau_{1,2}$ of (2.7) have the form

$$\tau_{1,2} = i\frac{4 - \mathbf{det\ T} \pm \sqrt{D}}{4d}, \tag{2.12}$$

where $\mathbf{det\ T}$ and $D = (4 - \mathbf{det\ T})^2 + 16ad$ are real numbers.

Taking (2.6) into account we rewrite

$$D = (4 - ad + bc)^2 + 16ad = (4 + ad - bc)^2 + 16bc = (4 + \mathbf{det\ T})^2 + 16bc. \tag{2.13}$$

Moreover, in view of (2.10), $bc = |b|^2$. Therefore $D = (4 + \mathbf{det\ T})^2 + 16|b|^2 \geq 0$. This means that the solutions $\tau_{1,2}$ determined by (2.12) *always belong to* $i\mathbb{R}$.

Similarly, if $d = 0$, equation (2.7) is reduced to $-i(|b|^2 + 4)\tau + 2a = 0$. The solution $\tau_1 = -2ai/(|b|^2 + 4)$ *belongs to* $i\mathbb{R}$.

The two cases above and Lemma 2.3 show that $A_{\mathbf{T}}$ may have negative eigenvalues $z = \tau^2$ and there are no spectral singularities and exceptional points of $A_{\mathbf{T}}$. □

Example II. $\mathcal{PT}$*-symmetric potential.*
Denote by $\mathcal{P}$ and $\mathcal{T}$ the operators of space parity and complex conjugation, respectively

$$\mathcal{P}f(x) = f(-x), \quad \mathcal{T}f(x) = \overline{f(x)}. \tag{2.14}$$

The potential V is called $\mathcal{PT}$-symmetric if $\mathcal{PT}V = V\mathcal{PT}$. Extending $\mathcal{P}$ onto $W_2^{-2}(\mathbb{R})$, one gets $\mathcal{P}\delta = \delta$ and $\mathcal{P}\delta' = -\delta'$. These relations and (2.9) imply that V is $\mathcal{PT}$-symmetric if and only if[1]

$$a,\ d \in \mathbb{R}, \quad b,\ c \in i\mathbb{R}. \tag{2.15}$$

The corresponding operators $A_{\mathbf{T}}$ turn out to be $\mathcal{PT}$-symmetric operators, i.e., the relation

$$\mathcal{PT}A_{\mathbf{T}} = A_{\mathbf{T}}\mathcal{PT} \tag{2.16}$$

holds on the domain $\mathcal{D}(A_{\mathbf{T}})$.

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[1] the cases of symmetric and $\mathcal{PT}$-symmetric potentials differs by conditions imposed on parameters b, c.

Remark 2.6. In what follows we will often use operator identities

$$XA = BX, \tag{2.17}$$

where A and B are (possible) unbounded operators in $L_2(\mathbb{R})$ and X is a bounded operator in $L_2(\mathbb{R})$. In that case, we *always assume that (2.17) holds on $\mathcal{D}(A)$*. This means that $X : \mathcal{D}(A) \to \mathcal{D}(B)$ and the identity $XAu = BXu$ holds for all $u \in \mathcal{D}(A)$. If A is bounded, then (2.17) should hold on the whole $L_2(\mathbb{R})$. In particular, relation (2.15) means that the operator $\mathcal{PT}$ maps $\mathcal{D}(A_{\mathbf{T}})$ onto $\mathcal{D}(A_{\mathbf{T}})$ and $\mathcal{PT}A_{\mathbf{T}}f = A_{\mathbf{T}}\mathcal{PT}f$ for all $f \in \mathcal{D}(A_{\mathbf{T}})$.

Comparing the condition of self-adjointness (2.10) and the condition of $\mathcal{PT}$-symmetry (2.16) we obtain that $\mathcal{PT}$-symmetric operators $A_{\mathbf{T}}$ are not self-adjoint with respect to the initial inner product (2.11) except the case $b = -c \in i\mathbb{R}$. Therefore, $\mathcal{PT}$-symmetric operators $\mathcal{D}(A_{\mathbf{T}})$ may have non-real eigenvalues. In particular, it may happen that the set of complex eigenvalues of $A_{\mathbf{T}}$ coincide with $\mathbb{C} \setminus \mathbb{R}_+$.

Lemma 2.7. *1. A $\mathcal{PT}$-symmetric operator $A_{\mathbf{T}}$ has* non-real eigenvalues *if and only if one of the following conditions are satisfied:*

$$(i) \quad D = (4 - \mathbf{det}\ \mathbf{T})^2 + 16ad < 0, \quad (4 - \mathbf{det}\ \mathbf{T})d > 0;$$
$$(ii) \quad \mathbf{det}\ \mathbf{T} = 4, \quad a = d = 0.$$

Condition (i) corresponds to the case where $A_{\mathbf{T}}$ has two non-real eigenvalues, which are conjugate to each other. Condition (ii) describes the situation where any point $z \in \mathbb{C}\backslash\mathbb{R}_+$ is an eigenvalue of $A_{\mathbf{T}}$. In that case the spectrum of $A_{\mathbf{T}}$ coincides with $\mathbb{C}$;

2. A $\mathcal{PT}$-symmetric operator $A_{\mathbf{T}}$ has:

- nonzero spectral singularity *if and only if*

$$(iii) \quad D < 0, \quad (4 - \mathbf{det}\ \mathbf{T})d = 0.$$

In that case, the positive number $z = -\frac{a}{d}$ is the spectral singularity of $A_{\mathbf{T}}$;

- spectral singularity at point $z = 0$ *if and only if*

$$(iv) \quad D = 0, \quad (4 - \mathbf{det}\ \mathbf{T})d = 0, \quad d \neq 0, \quad a = 0;$$

- spectral singularity at point $z = \infty$ *if and only if*

$$(v) \quad D = 0, \quad (4 - \mathbf{det}\ \mathbf{T})d = 0, \quad d = 0, \quad a \neq 0;$$

- exceptional point *if and only if*

$$(vi) \quad D = 0, \quad (4 - \mathbf{det}\ \mathbf{T})d > 0.$$

In that case, the negative number $z = \frac{a}{d}$ is the exceptional point of $A_{\mathbf{T}}$.

Proof. Let $A_{\mathbf{T}}$ be a $\mathcal{PT}$-symmetric operator. Then, the values of $\mathbf{det}\ \mathbf{T}$ and D are real (it follows from (2.15)).

Using Lemma 2.3 and (2.12), we arrive at the conclusion that condition (i) is necessary and sufficient for the existence of two non-real eigenvalues $z_{1,2} = \tau_{1,2}^2$ of $A_{\mathbf{T}}$, which are conjugate to each other.

The requirement that any point $z = \tau^2 \in \mathbb{C}\backslash\mathbb{R}_+$ is an eigenvalue of $A_{\mathbf{T}}$ is equivalent to the condition that any $\tau \in \mathbb{C}_+$ is a solution of (2.7). This is possible only in the case where the left-hand side of (2.7) vanishes. The latter is equivalent to the condition (ii).

It follows from (2.12) that a nonzero spectral singularity exists in the case where $D = (4 - \mathbf{det}\ \mathbf{T})^2 + 16ad < 0$ and $\mathbf{det}\ \mathbf{T} = 4$ that is equivalent to (iii). In that case, $\tau_{1,2} = \pm\sqrt{-\frac{a}{d}}$ and $z = \tau_{1,2}^2 = -\frac{a}{d}$.

The descriptions (iv) and (v) of spectral singularities at 0 and at ∞ are obvious due to (2.7).

By Definition 2.4 a point $z = \tau^2$ ($\tau \in \mathbb{C}_+$) is an exceptional point of $A_{\mathbf{T}}$ if its multiplicity is 2. Hence, $d \neq 0$ in (2.7) and τ is determined by (2.12) with $D = 0$.

Furthermore, the condition $\tau \in \mathbb{C}_+$ implies $(4 - \mathbf{det}\ \mathbf{T})d > 0$. Thus we show that condition (vi) corresponds to exceptional points. In that case

$$z = \tau^2 = -\frac{(4 - \mathbf{det}\ \mathbf{T})^2}{16d^2} = \frac{16ad}{16d^2} = \frac{a}{d} < 0$$

(since $D = 0$, $(4 - \mathbf{det}\ \mathbf{T}) \neq 0$ and hence, $ad < 0$). Lemma 2.7 is proved. □

Example III. *δ-potential with a complex coupling* [15].

Let $a \in \mathbb{C}$ and $b = c = d = 0$. Then (2.1) takes the form

$$-\frac{d^2}{dx^2} + a < \delta, \cdot > \delta(x), \quad a \in \mathbb{C}$$

and (2.4) gives rise to operators $A_{\mathbf{T}} \equiv A_a = -\frac{d^2}{dx^2}$ with domains of definition

$$\mathcal{D}(A_a) = \left\{ f(x) \in W_2^2(\mathbb{R}\backslash\{0\}) \ \middle|\ \begin{array}{c} f(0+) = f(0-)\ (\equiv f(0)) \\ f'(0+) - f'(0-) = af(0) \end{array} \right\}.$$

By virtue of Lemma 2.3 and Definition 2.4 we conclude:

- if $\mathsf{Re}\ a < 0$, then A_a has a unique eigenvalue $z = -a^2/4$, which is real $\iff$ $\mathsf{Im}\ a = 0$;
- if $\mathsf{Re}\ a \geq 0$, then the spectrum of A_a is real, continuous and it coincides with $[0, \infty)$;
- if $a \in i\mathbb{R} \setminus \{0\}$, then A_a has spectral singularity $z = \frac{|a|^2}{4}$;
- there are no exceptional points of A_a.

Example IV. *δ'-potential with a complex coupling.*

Let $d \in \mathbb{C}$ and $a = b = c = 0$. Then (2.1) takes the form

$$-\frac{d^2}{dx^2} + d < \delta', \cdot > \delta'(x), \quad d \in \mathbb{C}$$

and (2.4) gives rise to operators $A_{\mathbf{T}} \equiv A_d = -\frac{d^2}{dx^2}$ with domains of definition

$$\mathcal{D}(A_d) = \left\{ f(x) \in W_2^2(\mathbb{R}\backslash\{0\}) \ \middle|\ \begin{array}{c} f'(0+) = f'(0-)\ (\equiv f'(0)) \\ f(0+) - f(0-) = -df'(0) \end{array} \right\}.$$

By virtue of Lemma 2.3 and Definition 2.4:

- if $\mathsf{Re}\ d > 0$, then A_d has a unique eigenvalue $z = -4/d^2$, which is real $\iff$ $\mathsf{Im}\ d = 0$;
- if $\mathsf{Re}\ d \leq 0$, then the spectrum of A_d is real, continuous and it coincides with $[0, \infty)$;
- if $d \in i\mathbb{R} \setminus \{0\}$, then A_d has spectral singularity $z = \frac{4}{|d|^2}$;
- there are no exceptional points of A_d.

3. Interpretation as self-adjoint operators in Krein spaces

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$ and with fundamental symmetry J (i.e., $J = J^*$ and $J^2 = I$). The space $\mathfrak{H}$ endowed with the indefinite inner product (indefinite metric) $[f, g]_J := (Jf, g)$, $\forall f, g \in \mathfrak{H}$ is called a Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$.

The difference between the initial inner product $(\cdot, \cdot)$ and indefinite metric $[\cdot, \cdot]_J$ consists in the fact that, except the cases $J = \pm I$, the sign of the sesqulinear form $[f, f]_J$ is not determined (i.e, it is possible $[f, f]_J < 0$, $[f, f]_J = 0$, or $[f, f]_J > 0$ for various $f \neq 0$). The Hilbert space $\mathfrak{H}$ can be considered as a particular case of the Krein space $(\mathfrak{H}, [\cdot, \cdot]_J)$ with $J = I$.

A linear densely defined[2] operator A acting in $\mathfrak{H}$ is called self-adjoint in the Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$ if A is self-adjoint with respect to the indefinite metric $[\cdot,\cdot]_J$. This condition is equivalent to the relation

$$A^* = JAJ. \tag{3.1}$$

The spectrum of a self-adjoint operator in Krein space is symmetric with respect to the real axis. For an additional information about Krein spaces and operators acting therein we refer to [6].

We recall [16, 17] that a linear densely defined operator A acting in a Hilbert space $\mathfrak{H}$ is said to be *pseudo-Hermitian* if there exists a bounded and boundedly invertible self-adjoint operator $\eta : \mathfrak{H} \to \mathfrak{H}$ such that

$$A^* = \eta A \eta^{-1}. \tag{3.2}$$

Relation (3.2) means that A is self-adjoint with respect to the pseudo-metric $[\cdot,\cdot]_\eta = (\eta\cdot,\cdot)$.

It follows from (3.1) and (3.2) that self-adjoint operators in Krein spaces are pseudo-Hermitian. The inverse implication is also true. Indeed, let A be pseudo-Hermitian. Then (3.2) holds for some η. Denote

$$|\eta| = \sqrt{\eta^2}, \quad J = \eta|\eta|^{-1} \tag{3.3}$$

and consider the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_{|\eta|})$ endowed with new (equivalent to $(\cdot,\cdot)$) inner product $(\cdot,\cdot)_{|\eta|} = (|\eta|\cdot,\cdot)$. Then, the pseudo-metric $[\cdot,\cdot]_\eta$ coincides with the indefinite metric $[\cdot,\cdot]_{J|\eta|} = (J\cdot,\cdot)_{|\eta|}$ constructed with the use of fundamental symmetry $J = \eta|\eta|^{-1}$ and new inner product $(\cdot,\cdot)_{|\eta|}$, i.e.,

$$[\cdot,\cdot]_\eta = (\eta\cdot,\cdot) = (J|\eta|\cdot,\cdot) = (J\cdot,\cdot)_{|\eta|} = [\cdot,\cdot]_{J|\eta|}.$$

This means that A turns out to be a self-adjoint operator in the Krein space $(\mathfrak{H}, [\cdot,\cdot]_{J|\eta|})$.

Example II contd. It is known [14] that an arbitrary $\mathcal{PT}$-symmetric operator $A_{\mathbf{T}}$ can be interpreted as self-adjoint one in a suitable chosen Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_J)$. Using [14] we can specify the relevant indefinite metrics $[\cdot,\cdot]_J$. Denote

$$\mathcal{R}f(x) = \mathsf{sign}(x)f(x), \quad f \in L_2(\mathbb{R}). \tag{3.4}$$

The operator $\mathcal{R}$ is a fundamental symmetry which anti-commutes with $\mathcal{P}$: $\mathcal{PR} = -\mathcal{RP}$. It is easy to check that the operator $i\mathcal{PR}$ is also a fundamental symmetry and, moreover, any operator

$$J_{\vec{\alpha}} = \alpha_1\mathcal{P} + \alpha_2\mathcal{R} + \alpha_3 i\mathcal{PR}, \quad \alpha_j \in \mathbb{R}, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \tag{3.5}$$

turns out to be a fundamental symmetry in $L_2(\mathbb{R})$.

We consider also a subset of the set of fundamental symmetries $J_{\vec{\alpha}}$ by imposing an additional condition of $\mathcal{PT}$-symmetry: $\mathcal{PT}J_{\vec{\alpha}} = J_{\vec{\alpha}}\mathcal{PT}$.

The operator $J_{\vec{\alpha}}$ is $\mathcal{PT}$-symmetric if and only if $\alpha_2 = 0$. In that case the latter relation in (3.5) takes the form $\alpha_1^2 + \alpha_3^2 = 1$ and we may set $\alpha_1 = \cos\phi$ and $\alpha_3 = \sin\phi$. Then

$$J_{\vec{\alpha}} = (\cos\phi)\mathcal{P} + i(\sin\phi)\mathcal{PR} = \mathcal{P}(\cos\phi + i(\sin\phi)\mathcal{R}) = \mathcal{P}e^{i\phi\mathcal{R}}.$$

Thus, fundamental symmetries $J_{\vec{\alpha}}$ *with the additional property of* $\mathcal{PT}$*-symmetry coincide with fundamental symmetries* $\mathcal{P}_\phi = \mathcal{P}e^{i\phi\mathcal{R}}$, $\phi \in [0, 2\pi)$.

Consider the following collection of indefinite metrics on $L_2(\mathbb{R})$:

$$[\cdot,\cdot]_{J_{\vec{\alpha}}} = (J_{\vec{\alpha}}\cdot,\cdot), \quad [\cdot,\cdot]_{\mathcal{P}_\phi} = (\mathcal{P}_\phi\cdot,\cdot).$$

[2] with respect to the initial inner product $(\cdot,\cdot)$

Proposition 3.1. ([14]) *Every $\mathcal{PT}$-symmetric operator $A_{\mathbf{T}}$ can be interpreted as a self-adjoint operator in the Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_{\mathcal{P}_\phi})$, where the parameter ϕ is determined by the relation*

$$2(b-c)\cos\phi = i(4+\det\mathbf{T})\sin\phi. \tag{3.6}$$

Proposition 3.1 shows that the collection of Krein spaces $(L_2(\mathbb{R}), [\cdot,\cdot]_{\mathcal{P}_\phi})$ generated by $\mathcal{PT}$-symmetric fundamental symmetries $\mathcal{P}_\phi$ *is sufficient for the interpretation of $A_{\mathbf{T}}$ as a self-adjoint operator*. The possible interpretation of some $A_{\mathbf{T}}$ as a self-adjoint operator in a Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_{J_{\vec{\alpha}}})$, where $J_{\vec{\alpha}}$ *is not $\mathcal{PT}$-symmetric* (i.e. $J_{\vec{\alpha}} \neq \mathcal{P}_\phi$) immediately leads to specific spectral properties of $A_{\mathbf{T}}$.

Proposition 3.2. ([14]) *Let $A_{\mathbf{T}}$ be a non-self-adjoint $\mathcal{PT}$-symmetric operator. Then*

- *if $A_{\mathbf{T}}$ admits an interpretation as a self-adjoint operator in a Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_{J_{\vec{\alpha}}})$, where $J_{\vec{\alpha}}$ is not $\mathcal{PT}$-symmetric, then $\sigma(A_{\mathbf{T}}) = \mathbb{C}$;*
- *if $A_{\mathbf{T}}$ admits an interpretation as a self-adjoint operator in two different Krein spaces $(L_2(\mathbb{R}), [\cdot,\cdot]_{\mathcal{P}_{\phi_1}})$ and $(L_2(\mathbb{R}), [\cdot,\cdot]_{\mathcal{P}_{\phi_2}})$, where $\mathcal{P}_{\phi_1}$ and $\mathcal{P}_{\phi_2}$ are* linearly independent*, then the spectrum of $A_{\mathbf{T}}$ contains a pair of complex conjugated eigenvalues;*
- *if $A_{\mathbf{T}}$ has a real spectrum, then $A_{\mathbf{T}}$ has interpretation as self-adjoint operator for the* unique choice[3] *of the Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_{\mathcal{P}_\phi})$.*

Examples III. VI contd. It follows from (2.10) and (2.15) that operators A_a (A_d) with real a (real d) are self-adjoint in the initial Hilbert space $L_2(\mathbb{R})$. Moreover, taking [5] into account, we decide that these operators are self-adjoint in the Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_{\mathcal{P}})$, where

$$[f,g]_{\mathcal{P}} = (\mathcal{P}f, g) = \int_{\mathbb{R}} f(-x)\overline{g(x)}\,dx. \tag{3.7}$$

If a is non-real and $\mathsf{Re}\ a < 0$ (if d is non-real and $\mathsf{Re}\ d > 0$), then A_a (A_d) *cannot be interpreted as pseudo-Hermitian operator*, or that is equivalent, *cannot be interpreted as a self-adjoint operator in a Krein space.* Indeed, if we assume that such an interpretation is possible, then the spectrum of A_a (of A_d) must be symmetric with respect to the real axis that contradicts to the fact that the spectrum of A_a (of A_d) contains *a unique complex eigenvalue $z = -a^2/4$ $(d = -4/d^2)$*.

If $a \in i\mathbb{R}\setminus\{0\}$, $(d \in i\mathbb{R}\setminus\{0\})$ the operator A_a (A_d) has a spectral singularity. Hence, A_a (A_d) cannot be interpreted as a self-adjoint in a Hilbert space (see Theorem 4.6). The problem of interpretation of A_a (A_d) as a self-adjoint operator in a Krein space is still open.

If $\mathsf{Re}\ a > 0$ ($\mathsf{Re}\ d < 0$), then the operator A_a (A_d) turns out to be self-adjoint in $L_2(\mathbb{R})$ for a certain choice of inner product equivalent to the initial one $(\cdot,\cdot)$ (it follows from Corollary 4.11).

4. Similarity to self-adjoint operators

An operator A acting in a Hilbert space $\mathfrak{H}$ is called *similar* to a self-adjoint operator H if there exists a bounded and boundedly invertible operator Z such that

$$A = Z^{-1}HZ. \tag{4.1}$$

The similarity of A to a self-adjoint operator means that A *turns out to be self-adjoint for a certain choice of inner product of $\mathfrak{H}$, which is equivalent to the initial inner product* $(\cdot,\cdot)$. Indeed, let (4.1) hold. By analogy with (3.3) we denote

$$|Z| = \sqrt{Z^*Z}, \quad U = Z|Z|^{-1} \tag{4.2}$$

[3] up to linearly dependent fundamental symmetries $\mathcal{P}_\phi$

and rewrite (4.1) as follows:

$$H = ZAZ^{-1} = U|Z|A|Z|^{-1}U^{-1} = UKU^{-1}, \quad K = |Z|A|Z|^{-1}.$$

The operator U is unitary but, in general, U is not self-adjoint.[4] Taking into account that H is self-adjoint, we obtain

$$H^* = (UKU^{-1})^* = UK^*U^{-1} = H = UKU^{-1}.$$

Therefore, $K^* = K$. Then

$$(|Z|A|Z|^{-1})^* = |Z|^{-1}A^*|Z| = |Z|A|Z|^{-1}$$

or, that is equivalent $A^*|Z|^2 = |Z|^2A$. The obtained relation allows us to prove the self-adjointness of A in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_{|Z|^2})$ endowed with new (equivalent to $(\cdot,\cdot)$) inner product $(\cdot,\cdot)_{|Z|^2} = (|Z|^2\cdot,\cdot)$. Indeed,

$$(Af,g)_{|Z|^2} = (|Z|^2Af,g) = (f,A^*|Z|^2g) = (f,|Z|^2Ag) = (f,Ag)_{|Z|^2}, \quad f,g \in \mathcal{D}(A).$$

Thus A is self-adjoint in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot)_{|Z|^2})$.

If A is a self-adjoint operator in Krein space, then similarity of A to a self-adjoint operator in a Hilbert space admits an equivalent characterization. Indeed, a characteristic property of a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$ is the possibility of its decomposition onto the direct sum of maximal uniformly positive $\mathfrak{L}_+$ and maximal uniformly negative $\mathfrak{L}_-$ subspaces, which are orthogonal with respect to the indefinite metric $[\cdot,\cdot]_J$:

$$\mathfrak{H} = \mathfrak{L}_+[\dot{+}]\mathfrak{L}_- \tag{4.3}$$

(here $[\dot{+}]$ means the orthogonality with respect to the indefinite metric $[\cdot,\cdot]_J$).

The pair of subspaces $\mathfrak{L}_\pm$ in the decomposition (4.3) is not determined uniquely.

Let A be an operator in $\mathfrak{H}$. We say that the decomposition (4.3) *is invariant with respect to A* if

$$\mathcal{D}(A) = \mathcal{D}_+[\dot{+}]\mathcal{D}_-, \quad \mathcal{D}_\pm = \mathcal{D}(A) \cap \mathfrak{L}_\pm$$

and $A = A_+[\dot{+}]A_-$, where the operators $A_\pm = A\restriction_{\mathcal{D}_\pm}$ acts in the subspaces $\mathfrak{L}_\pm$, respectively.

Proposition 4.1. ([4]) *A pseudo-Hermitian operator A is similar to a self-adjoint operator if and only if there exists decomposition (4.3) of the Krein space*[5] $(\mathfrak{H}, [\cdot,\cdot]_{J|\eta|})$ *which is invariant with respect to A.*

The decomposition (4.3) can be easily characterized with the use of the following operator $\mathcal{C}$:

$$\mathcal{C}f = \mathcal{C}(f_+ + f_-) = f_+ - f_-, \quad f = f_+ + f_-, \quad f_\pm \in \mathfrak{L}_\pm \tag{4.4}$$

(since $\mathfrak{L}_+ = (I+\mathcal{C})\mathfrak{H}$ and $\mathfrak{L}_- = (I-\mathcal{C})\mathfrak{H}$). Therefore, *the invariance of a given decomposition (4.3) with respect to a linear operator A is equivalent to the relation* $A\mathcal{C} = \mathcal{C}A$.

Assume additionally that A is self-adjoint in a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$, then the operators $A_\pm$ in the decomposition $A = A_+[\dot{+}]A_-$ are self-adjoint in the Hilbert spaces $\mathfrak{L}_\pm$ endowed with the inner products $\pm[\cdot,\cdot]_J$, respectively. Therefore, A is self-adjoint in the Hilbert space $\mathfrak{H}$ with the inner product

$$(f,g)_1 = [f_+,g_+]_J - [f_-,g_-]_J, \quad f = f_+ + f_-, \quad g = g_+ + g_-, \quad f_\pm \in \mathfrak{L}_\pm, \quad g_\pm \in \mathfrak{L}_\pm.$$

Taking the definition of $\mathcal{C}$ into account, we get $(\cdot,\cdot)_1 = [\mathcal{C}\cdot,\cdot]_J$. Moreover, it is known (see, for example, [10]) that every operator $\mathcal{C}$ defined by (4.4) has the form $\mathcal{C} = Je^Q$, where Q is a *bounded self-adjoint operator in* $\mathfrak{H}$ *which anticommutes with* J: $QJ = -JQ$. Therefore,

$$(\cdot,\cdot)_1 = [\mathcal{C}\cdot,\cdot]_J = (JJe^Q\cdot,\cdot) = (e^Q\cdot,\cdot)$$

---

[4] this is a difference with the operator J in (3.3).

[5] see Sec. 3 for the definition of the Krein space $(\mathfrak{H}, [\cdot,\cdot]_{J|\eta|})$.

and, finally we conclude that A is self-adjoint in the Hilbert space $(\mathfrak{H}, (e^Q\cdot,\cdot))$.

Proposition 4.2. *A pseudo-Hermitian operator A is similar to a self-adjoint operator if and only there exists an operator $\mathcal{C} = Je^Q$ such that $J = \eta|\eta|^{-1}$, the operator Q satisfies the relations*

$$Q^*|\eta| = |\eta|Q, \quad -Q^*\eta = \eta Q \tag{4.5}$$

and $A\mathcal{C} = \mathcal{C}A$. In that case, the operator A turns out to be self-adjoint in the Hilbert space $\mathfrak{H}$ endowed with inner product $(|\eta|e^Q\cdot,\cdot)$.

Proof. According to Proposition 4.1 the similarity of A to a self-adjoint operator is equivalent to the existence of decomposition (4.3) of the Krein space $(\mathfrak{H}, [\cdot,\cdot]_{J|\eta|})$ that is invariant with respect to A. This condition is equivalent to the relation $A\mathcal{C} = \mathcal{C}A$, where $\mathcal{C} = Je^Q$ corresponds to the mentioned decomposition of $(\mathfrak{H}, [\cdot,\cdot]_{J|\eta|})$. Taking (3.3) into account, we conclude that $J = \eta|\eta|^{-1}$. Then the relation $QJ = -JQ$ and the condition of self-adjointness of Q with respect to the inner product $(\cdot,\cdot)_{|\eta|}$ take the form

$$Q\eta|\eta|^{-1} = -\eta|\eta|^{-1}Q, \quad Q^*|\eta| = |\eta|Q$$

that is equivalent to (4.5).

The operator A is self-adjoint in the Krein space $(\mathfrak{H}, [\cdot,\cdot]_{J|\eta|})$ and it commutes with operator $\mathcal{C} = Je^Q$. In that case, as was established above, the operator A is self-adjoint with respect to the inner product $[\mathcal{C}\cdot,\cdot]_\eta = (e^Q\cdot,\cdot)_{|\eta|} = (|\eta|e^Q\cdot,\cdot)$. The proof is completed. □

For the case, where A cannot be interpreted as self-adjoint operator in Krein space, the following general integral-resolvent criterion of similarity can be used:

Lemma 4.3. ([18]) *A closed densely defined operator A acting in $\mathfrak{H}$ is similar to a self-adjoint one if and only if the spectrum of A is real and there exists a constant M such that*

$$\begin{aligned} &\sup_{\varepsilon>0}\varepsilon\int_{-\infty}^{\infty}\|(A-zI)^{-1}g\|^2 d\xi \le M\|g\|^2, \\ &\sup_{\varepsilon>0}\varepsilon\int_{-\infty}^{\infty}\|(A^*-zI)^{-1}g\|^2 d\xi \le M\|g\|^2, \quad \forall g\in\mathfrak{H}, \end{aligned} \tag{4.6}$$

where the integrals are taken along the line $z = \xi + i\varepsilon$ ($\varepsilon > 0$ is fixed) of upper half-plane $\mathbb{C}_+$.

In order to apply Lemma 4.3 to Examples II-IV, we need an explicit form of the resolvent $(A_{\mathbf{T}} - zI)^{-1}$. Repeating the proof of Lemma 2 in [5], we obtain

Lemma 4.4. *Let $A_{\mathbf{T}}$ be defined by (2.4), (2.5) and let $A_0 = -d^2/dx^2$, $\mathcal{D}(A_0) = W_2^2(\mathbb{R})$ be the free Schrödinger operator in $L_2(\mathbb{R})$. Then, for all $g_\pm \in L_2(\mathbb{R}_\pm)$ and for all $z = \tau^2$ from the resolvent set of $A_{\mathbf{T}}$,*

$$[(A_{\mathbf{T}} - zI)^{-1} - (A_0 - zI)^{-1}]g_\pm = c_{1\pm}(\tau)h_{1\tau} + c_{2\pm}(\tau)h_{2\tau}, \tag{4.7}$$

where $h_{j\tau}(x)$ are defined by (2.8) and

$$c_{1\pm}(\tau) = \frac{iF_\pm(\tau)}{\tau}\left(-1 + \frac{2d\tau^2 - 2i\tau(2\pm b)}{p(\tau)}\right),$$

$$c_{2\pm}(\tau) = \pm\frac{iF_\pm(\tau)}{\tau}\left(-1 + \frac{-2i\tau(2\mp c) + 2a}{p(\tau)}\right),$$

where $F_\pm(\tau) = \frac{1}{2}\int_{\mathbb{R}} e^{\pm i\tau s}g_\pm(s)\,ds$ and $p_{\mathbf{T}}(\tau) = 2d\tau^2 + i(\mathbf{det}\ \mathbf{T} - 4)\tau + 2a$.

It is known that the resolvent of an arbitrary self-adjoint operator H satisfies the inequality $\|(H-zI)^{-1}\| \leq \frac{1}{|\mathsf{Im}\ z|}$ for all $z \in \mathbb{C}\setminus\mathbb{R}$. If A is similar to a self-adjoint operator H (i.e., (4.1) holds), then the inequality above takes the form

$$\|(A-zI)^{-1}\| \leq \frac{C}{|\mathsf{Im}\ z|}, \quad C=\|Z^{-1}\|\|Z\|, \quad z\in\mathbb{C}\setminus\mathbb{R}. \tag{4.8}$$

Lemma 4.5. *If an operator $A_{\mathbf{T}}$ is similar to a self-adjoint operator in $L_2(\mathbb{R})$, then the functions*

$$\Phi_{\pm}(\tau)=\frac{(\mathsf{Re}\ \tau)^2}{|\tau|^2}\cdot\frac{|2d\tau^2+i\tau(\mathbf{det}\ \mathbf{T}\mp 2c)|^2+|i\tau(\mathbf{det}\ \mathbf{T}\pm 2b)+2a|^2}{|p_{\mathbf{T}}(\tau)|^2} \tag{4.9}$$

are uniformly bounded on $\mathbb{C}_{++}=\{\tau\in\mathbb{C}_+ : \mathsf{Re}\ \tau>0\}$ (i.e., there exists $K>0$ such that $\Phi_{\pm}(\tau)<K$ for all $\tau\in\mathbb{C}_{++}$).

Proof. Let $A_{\mathbf{T}}$ be similar to self-adjoint. Since A_0 is self-adjoint, the inequalities (4.8) hold for $A_{\mathbf{T}}$ and for A_0. Therefore, for all $g\in L_2(\mathbb{R})$ and $z=\tau^2\in\mathbb{C}_+$,

$$\|[(A_{\mathbf{T}}-zI)^{-1}-(A_0-zI)^{-1}]g\|^2 \leq \frac{M}{(\mathsf{Im}\ z)^2}\|g\|^2, \tag{4.10}$$

where M is a constant independent of g and z. In particular, the inequality (4.10) holds if we put $g=g_+$ or $g=g_-$, where

$$g_+(x)=\begin{cases} e^{-i\overline{\tau}x}, & x>0\\ 0, & x<0\end{cases}, \quad g_-(x)=\begin{cases} 0, & x>0\\ e^{i\overline{\tau}x}, & x<0\end{cases}, \quad \tau\in\mathbb{C}_{++}.$$

In these cases, using (4.7) and taking into account that: the functions $h_{j\tau}$ in (4.7) are orthogonal in $L_2(\mathbb{R})$,

$$\|g_{\pm}\|^2=\frac{1}{2(\mathsf{Im}\ \tau)}, \quad \|h_{j\tau}\|^2=\frac{1}{\mathsf{Im}\ \tau}, \quad F_{\pm}(\tau)=\frac{1}{4(\mathsf{Im}\ \tau)}, \tag{4.11}$$

and $(\mathsf{Im}\ z)^2=4(\mathsf{Im}\ z)^2(\mathsf{Re}\ \tau)^2$ we can rewrite (4.10) as follows

$$\Phi_{\pm}(\tau)=\frac{(\mathsf{Re}\ \tau)^2}{|\tau|^2}M_{\pm}(\tau)\leq 2M, \quad \forall\tau\in\mathbb{C}',$$

where

$$M_{\pm}(\tau)=\left|1-\frac{2d\tau^2-2i\tau(2\pm b)}{p_{\mathbf{T}}(\tau)}\right|^2+\left|1-\frac{-2i\tau(2\mp c)+2a}{p_{\mathbf{T}}(\tau)}\right|^2.$$

Finally, remembering that $p_{\mathbf{T}}(\tau)=2d\tau^2+i\tau(\mathbf{det}\ \mathbf{T}-4)+2a$ we rewrite $M_{\pm}(\cdot)$ as

$$M_{\pm}(\tau)=\frac{|2d\tau^2+i\tau(\mathbf{det}\ \mathbf{T}\mp 2c)|^2+|i\tau(\mathbf{det}\ \mathbf{T}\pm 2b)+2a|^2}{|p_{\mathbf{T}}(\tau)|^2} \tag{4.12}$$

that gives (4.9). Lemma 4.5 is proved. □

The proof of Lemma 4.8 is close to the part of the proof of Theorem 4 in [5], where the particular case of operators $A_{\mathbf{T}}$ was considered.

Theorem 4.6. *Let $A\in\{A_{\mathbf{T}},A_a,A_d\}$ be an operator considered in Examples II-IV. If the spectrum of A contains the spectral singularity (the exceptional point), then A cannot be similar to a self-adjoint operator.*

Proof. Assume that $A=A_{\mathbf{T}}$ is a $\mathcal{PT}$-symmetric operator from Example II. It follows from the proof of Lemma 2.7 that $A_{\mathbf{T}}$ has a nonzero spectral singularity if the positive number $\tau=\sqrt{-\frac{a}{d}}$ is the root of $p_{\mathbf{T}}(\tau)$; and $A_{\mathbf{T}}$ has an exceptional point if the imaginary number $\tau=i\sqrt{-\frac{a}{d}}\in\mathbb{C}_+$ is the root of $p_{\mathbf{T}}(\tau)$ with multiplicity 2.

Let us suppose that $A_{\mathbf{T}}$ is similar to self-adjoint. Then, by virtue of Lemma 4.5, the functions $\Phi_{\pm}(\cdot)$ have to be uniformly bounded on $\mathbb{C}_{++}$. This is impossible since $\Phi_{\pm}(\tau)$ tend to infinity in neighborhood of τ.

Consider now the case of spectral singularity at point 0. Then, in view of relations (iv) of Lemma 2.7,

$$\Phi_{\pm}(\tau) = \frac{(\mathsf{Re}\ \tau)^2}{|\tau|^2} \cdot \frac{|d\tau + i(2 \mp c)|^2 + |i(2 \pm b)|^2}{d^2|\tau|^2}.$$

Here $|bc| \neq 0$ because $4 = \mathbf{det}\ \mathbf{T} = -bc$. Hence, at least one of functions $\Phi_{\pm}(\cdot)$ tends to infinity when $\tau \to 0$.

Finally, if $A_{\mathbf{T}}$ has spectral singularity at ∞, then relations (v) of Lemma 2.7 hold and

$$\Phi_{\pm}(\tau) = \frac{(\mathsf{Re}\ \tau)^2}{|\tau|^2} \cdot \frac{|i\tau(2 \mp c)|^2 + |i\tau(2 \pm b) + a|^2}{|a|^2}.$$

It follows from relations (v) that $4 = \mathbf{det}\ \mathbf{T} = -bc$. Hence, $|bc| \neq 0$ and at least one of functions $\Phi_{\pm}(\cdot)$ tends to infinity when $\tau \to \infty$.

Summing the cases above we conclude that $A_{\mathbf{T}}$ cannot be similar to a self-adjoint operator.

The cases $A = A_a$ and $A = A_d$ can be considered similarly (it suffices to consider the case of spectral singularity only). Theorem 4.6 is proved. □

The functions $M_{\pm}(\tau)$ in (4.12) corresponds to the operator $A_{\mathbf{T}}$ defined by (2.4), (2.5). The adjoint operator $A^*_{\mathbf{T}}$ coincides with $A_{\overline{\mathbf{T}}^t}$. Hence, the following functions:

$$M'_{\pm}(\tau) = \frac{|2\overline{d}\tau^2 + i\tau(\mathbf{det}\ \overline{\mathbf{T}} \mp 2\overline{b})|^2 + |i\tau(\mathbf{det}\ \overline{\mathbf{T}} \pm 2\overline{c}) + 2\overline{a}|^2}{|p_{\overline{\mathbf{T}}}(\tau)|^2} \tag{4.13}$$

correspond to $A^*_{\mathbf{T}}$.

Theorem 4.7. *Let $A_{\mathbf{T}}$ be an operator defined by (2.4), (2.5) with real spectrum. If the functions $M_{\pm}(\tau), M'_{\pm}(\tau)$ are uniformly bounded in $\mathbb{C}_{++} = \{\tau \in \mathbb{C}_+ : \mathsf{Re}\ \tau > 0\}$, then $A_{\mathbf{T}}$ is similar to self-adjoint.*

Proof. The operator A_0 satisfies relations (4.6) as a self-adjoint operator. Hence, the inequalities

$$\begin{aligned} &\sup_{\varepsilon>0}\varepsilon \int_{-\infty}^{\infty} \|[(A_{\mathbf{T}} - zI)^{-1} - (A_0 - zI)^{-1}]g\|^2 d\xi \leq M\|g\|^2, \\ &\sup_{\varepsilon>0}\varepsilon \int_{-\infty}^{\infty} \|[(A^*_{\mathbf{T}} - zI)^{-1} - (A_0 - zI)^{-1}]g\|^2 d\xi \leq M\|g\|^2, \quad \forall g \in L_2(\mathbb{R}), \end{aligned} \tag{4.14}$$

are necessarily and sufficient condition for the similarity of $A_{\mathbf{T}}$ to a self-adjoint operator.

Let $g = g_+$ be an arbitrary function from $L_2(\mathbb{R}_+)$. Using Lemma 4.4 and the relation $\|h_{j\tau}(x)\|^2 = \frac{1}{\mathsf{Im}\ \tau}$ (see (4.11)), we get

$$\|[(A_{\mathbf{T}} - zI)^{-1} - (A_0 - zI)^{-1}]g_+\|^2 = \frac{|F_+(\tau)|^2}{|\tau|^2(\mathsf{Im}\ \tau)} M_+(\tau), \tag{4.15}$$

where $F_+(\tau)$ is the Fourier transform of g_+ and $z = \tau^2$ $(\tau \in \mathbb{C}_{++})$.

Since $M_+(\tau)$ is uniformly bounded on $\mathbb{C}_{++}$, there is a constant $K_1 > 0$ such that $|M_+(\tau)| \leq K_1$. Then

$$\varepsilon \int_{-\infty}^{\infty} \|[(A_{\mathbf{T}} - zI)^{-1} - (A_0 - zI)^{-1}]g_+\|^2 d\xi \leq K_1 \int_{-\infty}^{\infty} \frac{\varepsilon |F_+(\tau)|^2}{|\tau|^2(\mathsf{Im}\ \tau)} d\xi. \tag{4.16}$$

Let us consider an auxiliary self-adjoint operator $\tilde{A}_{\mathbf{T}}$ with $b = c = 0$, $a = 1$, and $d = 4$. Then $\mathbf{det}\ \mathbf{T} = 4$ and

$$\tilde{M}_+(\tau) = \frac{|\gamma^2 + i\gamma|^2 + |i\gamma + 1|^2}{|\gamma^2 + 1|^2} = \frac{|\gamma|^2}{|\gamma - i|^2} + \frac{1}{|\gamma + i|^2}, \quad \gamma = 2\tau \in \mathbb{C}_{++}.$$

The obtained expression leads to the conclusion that $\tilde{M}_+(\tau) \geq \frac{1}{4}$ for all $\tau \in \mathbb{C}_{++}$. Taking this inequality into account and using (4.14) and (4.15) for the pair of self-adjoint operators $\tilde{A}_{\mathbf{T}}$, A_0, we obtain

$$\begin{aligned}\frac{1}{4}\int_{-\infty}^{\infty} \frac{\varepsilon |F_+(\tau)|^2}{|\tau|^2 (\mathsf{Im}\ \tau)}\, d\xi &\leq \int_{-\infty}^{\infty} \frac{\varepsilon |F_+(\tau)|^2}{|\tau|^2 (\mathsf{Im}\ \tau)} \tilde{M}_+(\tau)\, d\xi \\ &= \varepsilon \int_{-\infty}^{\infty} \|[(\tilde{A}_{\mathbf{T}} - zI)^{-1} - (A_0 - zI)^{-1}] g_+\|^2 d\xi < M\|g_+\|^2,\end{aligned}$$

where M is a constant independent of $\varepsilon > 0$ and g_+.

Combining the obtained evaluation with (4.16), we obtain

$$\varepsilon \int_{-\infty}^{\infty} \|[(A_{\mathbf{T}} - zI)^{-1} - (A_0 - zI)^{-1}] g_+\|^2 d\xi < 4K_1 M \|g_+\|^2,$$

where $4K_1M$ does not depend on $\varepsilon > 0$ and g_+.

Considering similarly the case $g = g_-(x) \in L_2(\mathbb{R}_-)$ (here the uniformly boundedness of $M_-(\tau)$ has to be used) and, consequently, the case of operator $A^*_{\mathbf{T}}$, we arrive at the conclusion that (4.14) hold for all functions from $L_2(\mathbb{R})$. Hence, A is similar to a self-adjoint operator. Theorem 4.7 is proved. □

Corollary 4.8. *Let $A_{\mathbf{T}}$ satisfy conditions of Theorem 4.7. If, in addition, $A_{\mathbf{T}}$ can be interpreted as a self-adjoint operator in a Krein space, then the property of $M_\pm(\tau)$ to be uniformly bounded in $\mathbb{C}_{++}$ implies the similarity of $A_{\mathbf{T}}$ to a self-adjoint operator.*

Proof. If $A_{\mathbf{T}}$ can be interpreted as self-adjoint in a Krein space, then, for a certain choice of fundamental symmetry J, the equality (3.1) holds for $A_{\mathbf{T}}$ and $A^*_{\mathbf{T}}$. In that case, the first and the second inequalities in (4.6) are equivalent. Obviously, the same remains true for the inequalities (4.14). Thus, for the similarity of $A_{\mathbf{T}}$ to a self-adjoint operator it suffices to establish the first inequality in (4.14). The latter is ensured by uniformly boundedness property of $M_\pm(\tau)$ in $\mathbb{C}_{++}$ (see the proof of Theorem 4.7). □

Example II contd.

Corollary 4.9. *Let $A_{\mathbf{T}}$ be a $\mathcal{PT}$-symmetric operator considered in Example II. If one of the following conditions is satisfied, then $A_{\mathbf{T}}$ is similar to a self-adjoint operator:*

(i) $D = (4 - \mathbf{det}\ \mathbf{T})^2 + 16ad < 0, \quad (4 - \mathbf{det}\ \mathbf{T})d < 0$;

(ii) $D = 0, \quad (4 - \mathbf{det}\ \mathbf{T})d < 0$.

Proof. Every condition (i), (ii) guarantees that $d \neq 0$ and the roots $\tau_{1,2}$ of the polynomial $p_{\mathbf{T}}(\tau)$ (see (2.12)) belong to $\mathbb{C}_-$. Then the functions $M_\pm(\tau)$ (see (4.12)) are uniformly bounded in $\mathbb{C}_{++}$. By Proposition 3.1, $A_{\mathbf{T}}$ can be realized as self-adjoint in a Krein space. Hence, we can apply Corollary 4.8 that completes the proof. □

The conditions of Corollary 4.9 ensure the uniformly boundedness of $M_\pm(\tau)$. This property is *sufficient* for the similarity of $A_{\mathbf{T}}$ to a self-adjoint operator. If $D > 0$ the corresponding roots $\tau_{1,2}$ in (2.12) lie on imaginary axes $i\mathbb{R}$ and may happen that at least one of them (let, for definiteness, τ_1) belongs to $\mathbb{C}_+$. In that case the functions $M_\pm(\tau)$ may tend to infinity as $\tau \to \tau_1$. However, as we show below, the corresponding operator $A_{\mathbf{T}}$ remains similar to a self-adjoint operator.

Theorem 4.10. *Let $A_{\mathbf{T}}$ be a $\mathcal{PT}$-symmetric operator considered in Example II and let $D = (4 - \mathbf{det}\ \mathbf{T})^2 + 16ad > 0$. Then $A_{\mathbf{T}}$ is similar to a self-adjoint operator.*

Proof. By virtue of Proposition 3.1, $A_{\mathbf{T}}$ is self-adjoint in the Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_{\mathcal{P}_\phi})$. Using Proposition 4.2 with $\eta = J = \mathcal{P}_\phi$ we conclude that the similarity of $A_{\mathbf{T}}$ to a self-adjoint operator in a Hilbert space is equivalent to the existence of an operator $\mathcal{C} = \mathcal{P}_\phi e^Q$ which satisfies the following conditions:

$$Q^* = Q, \quad \mathcal{P}_\phi Q = -Q\mathcal{P}_\phi, \quad A\mathcal{C} = \mathcal{C}A. \tag{4.17}$$

Let $Q = \chi i\mathcal{R}\mathcal{P}_\phi$, where $\chi \in \mathbb{R}$ and $\mathcal{R}$ be defined by (3.4). The fundamental symmetry $\mathcal{R}$ anti-commutes with $\mathcal{P}$ and hence, $\mathcal{R}$ anti-commutes with the fundamental symmetry $\mathcal{P}_\phi = \mathcal{P}e^{i\phi\mathcal{R}}$. This means that Q satisfies the first two conditions of (4.17). The third condition is equivalent to the relation

$$A^*_{\mathbf{T}} e^Q = e^Q A_{\mathbf{T}} \tag{4.18}$$

since $A_{\mathbf{T}}\mathcal{P}_\phi = \mathcal{P}_\phi A^*_{\mathbf{T}}$ and $\mathcal{C} = \mathcal{P}_\phi e^Q$.

The operator $i\mathcal{R}\mathcal{P}_\phi$ is a fundamental symmetry in $L_2(\mathbb{R})$ because $\mathcal{R}$ anti-commutes with $\mathcal{P}_\phi$. This property allows us to rewrite e^Q as

$$e^Q = e^{\chi i\mathcal{R}\mathcal{P}_\phi} = [\cosh\chi]I + [\sinh\chi]i\mathcal{R}\mathcal{P}_\phi\,. \tag{4.19}$$

The obtained expression shows that e^Q commutes with the symmetric operator $A_{\rm sym}$ defined by (2.2) and commutes with the adjoint operator $A^*_{\rm sym}$. Hence, (4.18) holds if $e^Q : \mathcal{D}(A_{\mathbf{T}}) \to \mathcal{D}(A^*_{\mathbf{T}})$. The latter relation is equivalent to the following implication:

$$\text{if} \quad \mathbf{T}\Gamma_0 f = \Gamma_1 f, \quad \text{then} \quad \overline{\mathbf{T}}^{\rm t}\Gamma_0 e^{\chi i\mathcal{R}\mathcal{P}_\phi} f = \Gamma_1 e^{\chi i\mathcal{R}\mathcal{P}_\phi} f, \tag{4.20}$$

where Γ_j are boundary operators from Lemma 2.1 and f is an arbitrary element of $\mathcal{D}(A_{\mathbf{T}})$.

Thus *if (4.20) holds for a certain $\chi \in \mathbb{R}$, then $A_{\mathbf{T}}$ is similar to a self-adjoint operator in a Hilbert space.*

Denote

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using (2.5), (2.14), (3.4), and (4.19) it is easy to verify that

$$\Gamma_0 e^Q f = \cosh\chi\Gamma_0 f + \sinh\chi[-\frac{i}{2}\cos\phi\sigma_1\Gamma_1 f + \sin\phi\sigma_3\Gamma_0 f],$$

and

$$\Gamma_1 e^Q f = \cosh\chi\Gamma_1 f + \sinh\chi[2i\cos\phi\sigma_1\Gamma_0 f + \sin\phi\sigma_3\Gamma_1 f]$$

for all $f \in \mathcal{D}(A^*_{\rm sym}) = W^2_2(\mathbb{R}\backslash\{0\})$.

Substituting these expressions into (4.20), we obtain that (4.20) is equivalent to the matrix equality

$$\cosh\chi(\mathbf{T} - \overline{\mathbf{T}}^{\rm t}) = -\sinh\chi(\frac{i}{2}\cos\phi[\overline{\mathbf{T}}^{\rm t}\sigma_1\mathbf{T} + 4\sigma_1] + \sin\phi[\sigma_3\mathbf{T} - \overline{\mathbf{T}}^{\rm t}\sigma_3]). \tag{4.21}$$

Since $A_{\mathbf{T}}$ is $\mathcal{PT}$-symmetric, the entries of $\mathbf{T}$ satisfy (2.15) and we can set $b = ix$, $c = iy$, where x, y are arbitrary real numbers.

Simple calculations give

$$\mathbf{T} - \overline{\mathbf{T}}^{\rm t} = i(x+y)\sigma_1, \quad \overline{\mathbf{T}}^{\rm t}\sigma_1\mathbf{T} + 4\sigma_1 = (\mathbf{det}\ \mathbf{T} + 4)\sigma_1, \quad \sigma_3\mathbf{T} - \overline{\mathbf{T}}^{\rm t}\sigma_3 = i(x-y)\sigma_1.$$

Hence, the matrix relation (4.21) can be reduced to the equality

$$x + y = -\frac{\sinh\chi}{\cosh\chi}\left[\frac{1}{2}(\mathbf{det}\ \mathbf{T} + 4)\cos\phi + (x-y)\sin\phi\right],$$

which, obviously, has a solution χ if and only if

$$\left[\frac{1}{2}(\mathbf{det}\ \mathbf{T}+4)\cos\phi+(x-y)\sin\phi\right]^2>(x+y)^2. \tag{4.22}$$

Using the identity $2(x-y)\cos\phi=(\mathbf{det}\ \mathbf{T}+4)\sin\phi$, which follows directly from (3.6), and making elementary transformations, we reduce (4.22) to the inequality

$$(\mathbf{det}\ \mathbf{T}+4)^2-16xy>0. \tag{4.23}$$

To complete the proof it is sufficient to observe that the inequality (4.23) coincides with the condition $D>0$ (since (2.13) and $b=ix, c=iy$). Theorem 4.10 is proved. □

Summing the results above we obtain the following relationship between properties of $A_{\mathbf{T}}$ and the parameters $D=(4-\mathbf{det}\ \mathbf{T})^2+16ad$, $K=(4-\mathbf{det}\ \mathbf{T})d$.

	$K>0$	$K=0$	$K<0$
$D>0$	similarity	similarity	similarity
$D=0$	exceptional point	spectral singularity at 0 spectral singularity at ∞ $\sigma(A_{\mathbf{T}})=\mathbb{C}$	similarity
$D<0$	pair of complex eigenvalues	nonzero spectral singularity	similarity

Examples III. VI contd.

Corollary 4.11. *Let A_a (A_d) be an operator considered in Example III (IV). If* Re $a>0$ *(*Re $d<0$*), then A_a (A_d) is similar to a self-adjoint operator.*

Proof. Let Re $a>0$. Then the spectrum of A_a is real. The adjoint A_a^* coincides with $A_{\overline{a}}$. The functions $M_\pm(\tau), M'_\pm(\tau)$ have the form

$$M_+(\tau)=M_-(\tau)=\frac{|a|^2}{|-2i\tau+a|^2},\quad M'_+(\tau)=M'_-(\tau)=\frac{|\overline{a}|^2}{|-2i\tau+\overline{a}|^2}.$$

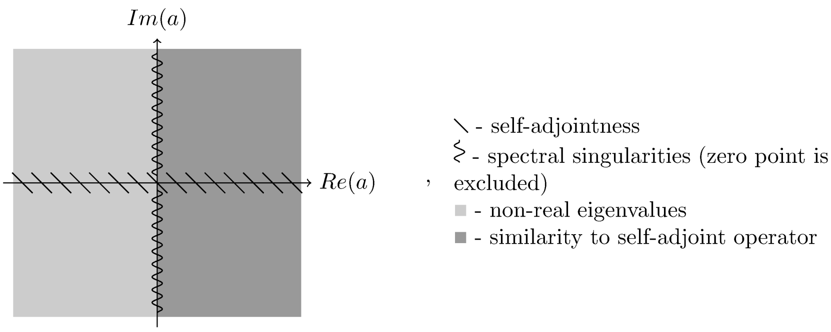
If Re $a>0$, then the roots $\tau_1=-\frac{ia}{2}$, $\tau_2=-\frac{i\overline{a}}{2}$ of the denominators belong to $\mathbb{C}_-$. In these cases, $M_\pm(\tau), M'_\pm(\tau)$ are uniformly bounded in $\mathbb{C}_{++}$. By Theorem 4.7, the operator A_a is similar to self-adjoint.

The operators A_d have real spectrum when Re $d<0$ and $A_d^*=A_{\overline{d}}$. The functions

$$M_+(\tau)=M_-(\tau)=\frac{|d\tau|^2}{|d\tau-2i|^2},\quad M'_+(\tau)=M'_-(\tau)=\frac{|\overline{d}\tau|^2}{|\overline{d}\tau-2i|^2}.$$

are uniformly bounded in $\mathbb{C}_{++}$. Using again Theorem 4.7 we complete the proof. □

The following picture illustrates the change of properties of A_a (complex eigenvalue → spectral singularity → similarity to a self-adjoint operator):



References

1. S. Albeverio, M. Dudkin, A. Konstantinov, and V. Koshmanenko, *On the point spectrum of $\mathcal{H}_{-2}$-singular perturbations*, Math. Nachr. **12** (2006), no. 280, 20–27.
2. S. Albeverio, A. Konstantinov, and V. Koshmanenko, *The Aronszajn-Donoghue theory for rank one perturbations of the $\mathcal{H}_{-2}$-class*, Integr. Equ. Oper. Theory **50** (2004), no. 1, 1–8.
3. S. Albeverio and P. Kurasov, *Singular Perturbations of Differential Operators*, Solvable Schrödinger Type Operators, London Mathematical Society Lecture Note Series, vol. 271, Cambridge University Press, Cambridge, 2000.
4. S. Albeverio and S. Kuzhel, *Pseudo-Hermiticity and theory of singular perturbations*, Letters Math. Phys. **67** (2004), no. 3, 223–238.
5. S. Albeverio and S. Kuzhel, *One-dimensional Schrödinger operators with $\mathcal{P}$-symmetric zero-range potentials*, J. Phys. A. **38** (2005), 4975–4988.
6. T. Ya. Azizov and I. S. Iokhvidov, *Linear Operators in Spaces with an Indefinite Metric*, John Wiley & Sons, Chichester, 1989.
7. C. M. Bender, *Introduction to PT-symmetric quantum theory*, Contemp. Phys. **46** (2005), 277–292.
8. C. M. Bender, *Making sense of non-Hermitian Hamiltonians*, Rep. Prog. Phys. **70** (2007), 947–1018.
9. S. Hassi and S. Kuzhel, *On J-self-adjoint operators with stable C-symmetries*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **143** (2013), no. 1, 141–167.
10. A. Grod, S. Kuzhel, and V. Sudilovskaja, *On operators of transition in Krein spaces*, Opuscula Mathematica **31** (2011), no. 1, 49–59.
11. U. Günther, I. Rotter, and B. Samsonov, *Projective Hilbert space structures at exceptional points*, J. Phys. A. **40** (2007), 8815–8833.
12. W. Karwowski, V. Koshmanenko, and S. Ota, *Schrödinger operator perturbed by operators related to null sets*, Positivity **2** (1998), no. 1, 77–99.
13. V. Koshmanenko, *Singular Quadratic Forms in Perturbation Theory*, Mathematics and Its Applications, vol. 474, Kluwer Academic Publishers, Dordrecht, 1999.
14. S. O. Kuzhel, O. M. Patsiuk, *Interpretation of $\mathcal{PT}$-symmetric operators like self-adjoint ones in Krein spaces*, Collection of research works of Institute of Mathematics of National Academy of Sciences of Ukraine **8** (2011) no. 1, 111–127. (Ukrianian)
15. A. Mostafazadeh, *Delta-function potential with a complex coupling*, J. Phys. A: Math. Gen. **39** (2006), 13495–13506.
16. A. Mostafazadeh, *Pseudo-Hermitian Representation of Quantum Mechanics*, Int. J. Geom. Meth. Mod. Phys. **7** (2010), 1191–1306.
17. A. Mostafazadeh, *Pseudo-Hermiticity versus $\mathcal{PT}$-symmetry: the necessary condition for the reality of the spectrum of a non-Hermitian Hamiltonian*, J. Math. Phys. **43** (2002), 205–214.
18. S. N. Naboko, *Conditions for similarity to unitary and selfadjoint operators*, Funktsional. Anal. i Prilozhen. **18** (1984), no. 1, 16–27. (Russian)

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