

ON LARGE COUPLING CONVERGENCE WITHIN TRACE IDEALS

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Dedicated to V. D. Koshmanenko on the occasion of his seventieth birthday

ABSTRACT. Let \mathcal{E} and \mathcal{P} be nonnegative quadratic forms such that $\mathcal{E} + b\mathcal{P}$ is closed and densely defined for every nonnegative real number b . Let H_b be the selfadjoint operator associated with $\mathcal{E} + b\mathcal{P}$. By Kato's monotone convergence theorem, there exists an operator L such that $(H_b + 1)^{-1}$ converges to L strongly, as b tends to infinity. We give a condition which is sufficient in order that the operators $(H_b + 1)^{-1}$ converge w.r.t. the trace norm with convergence rate $O(1/b)$. As an application we discuss trace norm resolvent convergence of Schrödinger operators with point interactions.

1. INTRODUCTION

We shall explain the goal and the content of the present note with the aid of an example.

Example 1.1. Let b be a positive real number (the so called coupling constant) and let $(a_n)_{n \in \mathbb{Z}}$ be a family of positive real numbers. Let δ_a be the Dirac measure with mass at a and

$$H_b = -\frac{d^2}{dx^2} + b \sum_{n \in \mathbb{Z}} a_n \delta_n.$$

By Kato's monotone convergence theorem, the nonnegative selfadjoint operators H_b in $L^2(\mathbb{R})$ converge in the strong resolvent sense to the Laplacian $-\frac{d^2}{dx^2}$ in $L^2(\mathbb{R})$ with Dirichlet boundary conditions at every point of \mathbb{Z} , as b tends to infinity. While it may be very difficult to analyze the operators H_b , it is trivial to derive the properties of the limit H_∞ . Thus if the coupling constant b is large, one may replace H_b by H_∞ and try to estimate the error one makes in this way. This leads to the following two questions:

1. In what sense do the operators H_b converge?
2. How fast?

The answer depends on the coefficients a_n . Let us consider three cases.

$$(1) \frac{a_n \rightarrow 0, |n| \rightarrow \infty}{\text{The operators } H_b \text{ do not converge in the norm resolvent sense, cf. [3].}}$$

$$(2) \frac{0 < \inf_{n \in \mathbb{Z}} a_n \leq \sup_{n \in \mathbb{Z}} a_n < \infty}{\text{The operators } H_b \text{ converge in the norm resolvent sense and the rate of convergence is of the order } O(1/b) \text{ (cf. [3]), but the resolvent differences do not belong to the trace class.}}$$

$$(3) \frac{\sum_{n \in \mathbb{Z}} \frac{1}{a_n} < \infty}{\text{There exists a finite constant } c \text{ such that}}$$

$$\| (H_b + 1)^{-1} - (H_\infty + 1)^{-1} \|_1 \leq \frac{c}{b}, \quad b > 0$$

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($\|\cdot\|_1$ denotes the trace norm and $\|\cdot\|_2$ the Hilbert-Schmidt norm). In particular, the absolutely continuous parts of H_b and H_∞ are unitarily equivalent. Since the spectrum of H_∞ is a discrete set, it follows in this special case that the absolutely continuous spectra of the operators H_b are empty, too.

The assertion about the trace norm in the third case has not been known before. In this note we shall present new general results on large coupling convergence and show that the mentioned assertion about the trace norm easily follows from these general results. Perturbations of Schrödinger operators by surface measures and by differential operators of the same order have been treated in [1] and [7], respectively. We expect that our general method can also be used in order to extend some of the results from [1] and [7].

2. A TRACE NORM ESTIMATE

Let H be a nonnegative selfadjoint operator in the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ and let \mathcal{E} be the closed quadratic form in \mathcal{H} associated with H , i.e.

$$(2.1) \quad \begin{aligned} D(H) &= \{h \in D(\mathcal{E}) : \exists f \in \mathcal{H} \forall g \in D(\mathcal{E}) : \mathcal{E}(h, g) = (f, g)\}, \\ \mathcal{E}(h, g) &= (Hh, g), \quad g \in D(\mathcal{E}), \quad h \in D(H); \end{aligned}$$

above implicit definition of Hh is correct, since, by Kato's representation theorem, $D(H)$ is dense in $(D(\mathcal{E}), \mathcal{E}_1)$. Put

$$(2.2) \quad G := (H + 1)^{-1}, \quad \mathcal{E}_1(f, g) := \mathcal{E}(f, g) + (f, g), \quad f, g \in D(\mathcal{E}).$$

By (2.1) (with \mathcal{E} replaced by \mathcal{E}_1),

$$(2.3) \quad Gf = h, \quad \text{if, and only if} \quad \mathcal{E}_1(h, g) = (f, g), \quad g \in D(\mathcal{E}).$$

Let J be a closed operator from the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$ to the auxiliary Hilbert space $(\mathcal{H}_{\text{aux}}, (\cdot, \cdot)_{\text{aux}})$, such that the range $\text{ran}(J)$ of J is dense in \mathcal{H}_{aux} and the domain $D(JG)$ of JG is dense in \mathcal{H} . Since $\text{ran}(G) = D(H)$ is dense in $(D(\mathcal{E}), \mathcal{E}_1)$ and $D(JG)$ is dense in \mathcal{H} , the domain $D(J)$ of J is dense in $(D(\mathcal{E}), \mathcal{E}_1)$. Thus JJ^* is a nonnegative selfadjoint operator in the Hilbert space \mathcal{H}_{aux} . Since the range of J is dense in \mathcal{H}_{aux} , the kernel $\ker(J^*)$ of J^* and hence also the kernel of JJ^* is trivial. We put

$$(2.4) \quad \check{H} := (JJ^*)^{-1}.$$

Note that \check{H} is an invertible nonnegative selfadjoint operator in \mathcal{H}_{aux} .

For $b \in (0, \infty)$ let

$$\begin{aligned} D(\mathcal{E}^{bJ}) &:= D(J), \\ \mathcal{E}^{bJ}(f, g) &:= \mathcal{E}(f, g) + b(Jf, Jg)_{\text{aux}}, \quad f, g \in D(J). \end{aligned}$$

Along with \mathcal{E} and J also the quadratic form \mathcal{E}^{bJ} in \mathcal{H} is closed. Let H_b be the nonnegative selfadjoint operator associated with \mathcal{E}^{bJ} . By Kato's monotone convergence theorem, there exists a nonnegative selfadjoint operator L in \mathcal{H} such that

$$(2.5) \quad (H_b + 1)^{-1}f \longrightarrow Lf, \quad \text{as } b \longrightarrow \infty, \quad f \in \mathcal{H}.$$

In this section we shall discuss the following questions:

1. In which sense do the operators $(H_b + 1)^{-1}$ converge?
2. How fast?

In the next section we shall show that our general results cover the example in the introduction.

Let $h \in D(J^*)$. For every $f \in D(JG)$

$$(h, JGf) = \mathcal{E}_1(J^*h, Gf) = (J^*h, f).$$

Thus

$$(2.6) \quad D(J^*) \subset D((JG)^*) \quad \text{and} \quad J^*h = (JG)^*h, \quad h \in D(J^*).$$

Let $f \in D(JG)$. For every $g \in D(J)$

$$\begin{aligned} & \mathcal{E}_1^{bJ} \left(Gf - J^* \left(\frac{1}{b} + JJ^* \right)^{-1} JGf, g \right) \\ &= \mathcal{E}_1(Gf, g) + b(JGf, Jg)_{\text{aux}} - \mathcal{E}_1(bJ^*(1 + bJJ^*)^{-1} JGf, g) \\ & \quad - b(JJ^*(1 + bJJ^*)^{-1} JGf, Jg)_{\text{aux}} \\ &= (f, g) + b[(JGf, Jg)_{\text{aux}} - ((1 + bJJ^*)^{-1} JGf, Jg)_{\text{aux}} \\ & \quad - bJJ^*(1 + bJJ^*)^{-1} JGf, Jg)_{\text{aux}}] = (f, g). \end{aligned}$$

By (2.3) (with \mathcal{E} replaced by \mathcal{E}_1^{bJ}) this implies that

$$(H_b + 1)^{-1} f = Gf - J^* \left(\frac{1}{b} + JJ^* \right)^{-1} JGf.$$

By (2.2), (2.4) and (2.6), this can be rewritten as

$$(2.7) \quad (H + 1)^{-1} f - (H_b + 1)^{-1} f = (JG)^* \left(\frac{1}{b} + \check{H}^{-1} \right)^{-1} JGf, \quad f \in D(JG).$$

Put

$$(2.8) \quad D_b := (H + 1)^{-1} - (H_b + 1)^{-1}, \quad D_\infty f := \lim_{b \rightarrow \infty} D_b f, \quad f \in \mathcal{H}.$$

Since D_b is a bounded selfadjoint operator in \mathcal{H} and $\|D_b\| \leq 2$ for every $b \in (0, \infty)$, the same holds true for D_∞ . Note that

$$(2.9) \quad (H_b + 1)^{-1} - L = D_\infty - D_b, \quad b \in (0, \infty).$$

Let $(E_{\check{H}}(\lambda))_{\lambda \in \mathbb{R}}$ be the spectral family of the selfadjoint operator \check{H} . Note that $E_{\check{H}}(\lambda) = 0$ for every nonpositive λ , since the operator \check{H} is nonnegative and invertible. Let $f \in D(JG)$. Let $0 < b < b' < \infty$. Then it follows from (2.7) and (2.8) that

$$\begin{aligned} ((D_{b'} - D_b)f, f) &= \left(\left[\left(\frac{1}{b'} + \check{H}^{-1} \right)^{-1} - \left(\frac{1}{b} + \check{H}^{-1} \right)^{-1} \right] JGf, JGf \right)_{\text{aux}} \\ &= \int \left(\frac{1}{\frac{1}{b'} + \frac{1}{\lambda}} - \frac{1}{\frac{1}{b} + \frac{1}{\lambda}} \right) d \| E_{\check{H}}(\lambda) JGf \|_{\text{aux}}^2. \end{aligned}$$

The integrands increase pointwise to $\frac{\lambda^2}{\lambda + b}$, as b' increases to infinity. By the monotone convergence theorem and since $D_\infty f = \lim_{b' \rightarrow \infty} D_{b'} f$, we get now that

$$(2.10) \quad ((D_\infty - D_b)f, f) = \int \frac{\lambda^2}{\lambda + b} d \| E_{\check{H}}(\lambda) JGf \|_{\text{aux}}^2, \quad f \in D(JG).$$

Since $D_\infty - D_b$ is bounded and selfadjoint, $D(JG)$ is dense in \mathcal{H} , and $E_{\check{H}}(\lambda) = 0$ for every $\lambda \leq 0$, it follows that the operator $D_\infty - D_b$ is nonnegative.

Since $D(JG)$ is dense in \mathcal{H} , we can choose an orthonormal basis $(e_k)_{k \in I}$ of \mathcal{H} such that $e_k \in D(JG)$ for every $k \in I$. By (2.10),

$$b \sum_{k \in I} ((D_\infty - D_b)e_k, e_k) = \sum_{k \in I} \int \frac{b\lambda^2}{\lambda + b} d \| E_{\check{H}}(\lambda) JGe_k \|_{\text{aux}}^2.$$

By the monotone convergence theorem, it follows that

$$(2.11) \quad b \sum_{k \in I} ((D_\infty - D_b)e_k, e_k) \uparrow \sum_{k \in I} \int \lambda^2 d \| E_{\check{H}}(\lambda) JGe_k \|_{\text{aux}}^2, \quad \text{as } b \uparrow \infty.$$

Since the operator $D_\infty - D_b$ is nonnegative, bounded and selfadjoint, it belongs to the trace class if, and only if, $\sum_{k \in I} ((D_\infty - D_b)e_k, e_k) < \infty$. If this is true, then

$$(2.12) \quad \|D_\infty - D_b\|_1 = \sum_{k \in I} ((D_\infty - D_b)e_k, e_k).$$

Moreover

$$(2.13) \quad \int \lambda^2 d \| E_{\check{H}}(\lambda) JGf \|_{\text{aux}}^2 = \| \check{H} JGf \|_{\text{aux}}^2,$$

if $JGf \in D(\check{H})$.

Lemma 2.1. ([5, Theorem 1.1]). *Let P be the orthogonal projection onto the orthogonal complement of the kernel of J , where 'orthogonal' refers to the Hilbert space $(D(\mathcal{E}), \mathcal{E}_1)$. Let $\check{\mathcal{E}}_1$ be the closed quadratic form in \mathcal{H}_{aux} associated with \check{H} . Then*

$$(2.14) \quad D(\check{\mathcal{E}}_1) = \text{ran}(J)$$

and

$$(2.15) \quad \check{\mathcal{E}}_1(Jf, Jg) = \mathcal{E}_1(Pf, Pg), \quad f, g \in D(J).$$

Corollary 2.1. *The operator $\check{H}^{1/2}JG$ is bounded,*

$$(2.16) \quad D(\check{H}^{1/2}JG) = D(JG),$$

and

$$(2.17) \quad \|\check{H}^{1/2}JG\| \leq \|G\|^{1/2} \leq 1.$$

Proof. (2.16) follows immediately from (2.14), since $D(\check{H}^{1/2}) = D(\check{\mathcal{E}}_1)$. By (2.15),

$$\begin{aligned} \|\check{H}^{1/2}JGf\|_{\text{aux}}^2 &= \check{\mathcal{E}}_1(JGf, JGf) = \mathcal{E}_1(PGf, PGf) \\ &\leq \mathcal{E}_1(Gf, Gf) = (f, Gf) \leq \|G\| \|f\|^2, \quad f \in D(JG). \quad \square \end{aligned}$$

Theorem 2.1. *Suppose that $D(\check{H}^{1/2}) = \mathcal{H}_{\text{aux}}$ and that $\check{H}^{1/2}\overline{\check{H}^{1/2}JG}$ is a Hilbert-Schmidt operator. Then $(H_b + 1)^{-1} - L$ is a trace class operator for every $b > 0$ and*

$$(2.18) \quad b \| (H_b + 1)^{-1} - L \|_1 \uparrow \| \check{H}^{1/2}\overline{\check{H}^{1/2}JG} \|_2^2, \quad \text{as } b \uparrow \infty.$$

Proof. Let $(e_k)_{k \in I}$ be any orthonormal basis of \mathcal{H} such that $e_k \in D(JG)$ for every $k \in I$. By (2.9) and (2.11)–(2.13),

$$b \| (H_b + 1)^{-1} - L \|_1 \uparrow \sum_{k \in I} \| \check{H}JGe_k \|_{\text{aux}}^2 = \| \check{H}^{1/2}\overline{\check{H}^{1/2}JG} \|_2^2, \quad \text{as } b \uparrow \infty. \quad \square$$

Corollary 2.2. *Suppose that $D(\check{H}^{1/2}) = \mathcal{H}_{\text{aux}}$ and that $\check{H}^{1/2}$ is a Hilbert-Schmidt operator. Then $(H_b + 1)^{-1} - L$ is a trace class operator for every $b > 0$ and*

$$(2.19) \quad \sup_{b > 0} b \| (H_b + 1)^{-1} - L \|_1 \leq \|G\| \| \check{H}^{1/2} \|_2^2.$$

Proof. (2.19) follows from (2.17) and (2.18). □

Remark 2.1. a) Theorem 2.1 extends Theorem 2.3 in [4].

b) If $\mathcal{H}_{\text{aux}} \neq \{0\}$, then there exists a c with $0 < c \leq \infty$ such that

$$b \| (H_b + 1)^{-1} - L \| \uparrow c, \quad \text{as } b \uparrow \infty$$

(cf. [2]) and hence it is not possible that the numbers $\| (H_b + 1)^{-1} - L \|$ converge to 0 faster than c/b . Thus Theorem 2.1 gives a condition that is sufficient for trace norm convergence with maximal rate of convergence.

3. POINT INTERACTIONS

Let \mathbb{D} be the classical Dirichlet form in $L^2(\mathbb{R})$, i.e.

$$\begin{aligned} D(\mathbb{D}) &:= H^1(\mathbb{R}), \\ \mathbb{D}(f, g) &:= \int \bar{f}'g' dx, \quad f, g \in D(\mathbb{D}). \end{aligned}$$

Let $\|\cdot\|$ be the norm in $L^2(\mathbb{R})$ and $-\Delta$ the selfadjoint operator in $L^2(\mathbb{R})$ associated with \mathbb{D} . Put $G := (-\Delta + 1)^{-1}$ and

$$\|u\|_{H^2} := \| -u'' + u \|, \quad u \in D(-\Delta).$$

Let $(a_n)_{n \in \mathbb{Z}}$ be a family in $(0, \infty)$ and $\mu := \sum_{n \in \mathbb{Z}} a_n \delta_n$. By Sobolev's embedding theorem, every $f \in D(\mathbb{D})$ has a unique continuous representative \tilde{f} . We define the operator J_μ from $(D(\mathbb{D}), \mathbb{D}_1)$ to $L^2(\mathbb{R}, \mu)$ as follows

$$D(J_\mu) : \left\{ f \in D(\mathbb{D}) : \int |\tilde{f}|^2 d\mu < \infty \right\},$$

$$J_\mu f := \tilde{f} \text{ } \mu\text{-a.e.}, \quad f \in D(J_\mu).$$

Let (f_n) be a sequence in $D(J_\mu)$ such that (f_n) converges to f in $(D(\mathbb{D}), \mathbb{D}_1)$ and (\tilde{f}_n) to g in $L^2(\mathbb{R}, \mu)$. Then, by Sobolev's inequality, the sequence (\tilde{f}_n) converges uniformly to \tilde{f} . Thus $\tilde{f} = g$ μ -a.e. and hence $f \in D(J_\mu)$. Thus the operator J_μ is closed. The space $C_0^\infty(\mathbb{R})$ of smooth functions with compact support is contained in the domain of J_μ and it is dense in $(D(-\Delta), \|\cdot\|_{H^2})$ and G is a unitary mapping from $L^2(\mathbb{R})$ to $(D(-\Delta), \|\cdot\|_{H^2})$. Thus $D(J_\mu G)$ is dense in $L^2(\mathbb{R})$.

Let H_∞ be the Laplace operator in $L^2(\mathbb{R})$ with Dirichlet boundary conditions at every point of \mathbb{Z} . For every $b \in (0, \infty)$ let $-\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_n$ be the selfadjoint operator associated with the closed quadratic form \mathbb{D}^{b, J_μ} . By Kato's monotone convergence theorem, the operators $-\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_n$ converge in the strong resolvent sense to the operator H_∞ .

By the preceding considerations, the general hypothesis at the beginning of the previous section is satisfied, if we put

- $\mathcal{H} = L^2(\mathbb{R})$,
- $H = -\Delta$, $\mathcal{E} = \mathbb{D}$,
- $\mathcal{H}_{\text{aux}} = L^2(\mathbb{R}, \mu)$,
- $J = J_\mu$,
- $H_b = -\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_n$,
- $L = (H_\infty + 1)^{-1}$.

Put $\check{H}_\mu := (J_\mu J_\mu^*)^{-1}$ and let $\check{\mathbb{D}}_1$ be the closed quadratic form in $L^2(\mathbb{R}, \mu)$ associated with \check{H}_μ .

Lemma 3.1. *Let $k = 2 \frac{e-1}{e+1}$ and $j = \frac{2e}{e^2-1}$. Then*

$$(3.1) \quad D(\check{\mathbb{D}}_1) = \left\{ f \in L^2(\mathbb{R}, \mu) : \sum_{n \in \mathbb{Z}} |f(n)|^2 < \infty \right\},$$

$$\check{\mathbb{D}}_1(f, f) = k \sum_{n \in \mathbb{Z}} |f(n)|^2 + j \sum_{n \in \mathbb{Z}} |f(n+1) - f(n)|^2, \quad f \in D(\check{\mathbb{D}}_1).$$

Proof. Let P be the orthogonal projection onto the orthogonal complement of the kernel of J_μ ('orthogonal' refers to the Hilbert space $(D(\mathbb{D}), \mathbb{D}_1)$). By Lemma 2.1,

$$D(\check{\mathbb{D}}_1) = \text{ran} J_\mu,$$

$$\check{\mathbb{D}}_1(J_\mu f, J_\mu f) = \mathbb{D}_1(Pf, Pf), \quad f \in D(\mathbb{D}).$$

Put

$$u(x) := \begin{cases} \frac{e^2}{e^2-1} e^{-|x|} - \frac{1}{e^2-1} e^{|x|}, & |x| \leq 1, \\ 0, & |x| > 1 \end{cases}$$

and $u_n := u(\cdot - n)$ for every $n \in \mathbb{Z}$. Integrating by parts, we get that

$$(3.2) \quad \mathbb{D}_1(u_n, f) = 0, \quad f \in \ker(J_\mu), \quad n \in \mathbb{Z}.$$

An elementary computation yields

$$(3.3) \quad \mathbb{D}_1(u_n, u_m) = \begin{cases} 2 \frac{e^2 + 1}{e^2 - 1}, & n = m, \\ -\frac{2e}{e^2 - 1}, & |n - m| = 1, \\ 0, & |n - m| > 1. \end{cases}$$

Let $f \in D(\mathbb{D})$. By Sobolev's inequality, $\sum_{n \in \mathbb{Z}} |\tilde{f}(n)|^2 < \infty$. By (3.2) and (3.3), this implies that the series $\sum_{n \in \mathbb{Z}} \tilde{f}(n)u_n$ converges in the orthogonal complement of the kernel of J_μ . Since $f - \sum_{n \in \mathbb{Z}} \tilde{f}(n)u_n \in \ker(J_\mu)$, this implies that

$$(3.4) \quad Pf = \sum_{n \in \mathbb{Z}} \tilde{f}(n)u_n, \quad f \in D(\mathbb{D}).$$

By (3.3), it follows that

$$\begin{aligned} \check{\mathbb{D}}_1(J_\mu f, J_\mu f) &= \mathbb{D}_1\left(\sum_{n \in \mathbb{Z}} \tilde{f}(n)u_n, \sum_{m \in \mathbb{Z}} \tilde{f}(m)u_m\right) \\ &= \sum_{n, m \in \mathbb{Z}} \overline{\tilde{f}(n)} \tilde{f}(m) \mathbb{D}_1(u_n, u_m) \\ &= \sum_{n \in \mathbb{Z}} 2 \frac{e^2 + 1}{e^2 - 1} |\tilde{f}(n)|^2 \\ &\quad - \sum_{n \in \mathbb{Z}} \frac{2e}{e^2 - 1} \left(\overline{\tilde{f}(n)} \tilde{f}(n+1) + \overline{\tilde{f}(n+1)} \tilde{f}(n) \right), \end{aligned}$$

and, since $k + 2j = 2 \frac{e^2 + 1}{e^2 - 1}$ and $-j = -\frac{2e}{e^2 - 1}$, the lemma is proved. \square

Remark 3.1. $\check{\mathbb{D}}_1$ is the so called trace of the Dirichlet form \mathbb{D}_1 w.r.t. the measure μ . General results on how to compute the trace of a Dirichlet form can be found in [6]. In the special situation of the lemma it is, however, easier to determine the form $\check{\mathbb{D}}_1$ directly.

Theorem 3.1. *Let $\sum_{n \in \mathbb{Z}} \frac{1}{a_n} < \infty$. Then there exists a positive real number c such that*

$$(3.5) \quad b \left\| \left(-\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_n + 1 \right)^{-1} - (H_\infty + 1)^{-1} \right\|_1 \uparrow c, \quad \text{as } b \uparrow \infty,$$

and $c \leq \sum_{m \in \mathbb{Z}} \frac{k + 2j}{a_m}$ (as before $k = 2 \frac{e - 1}{e + 1}$ and $j = \frac{2e}{e^2 - 1}$).

Proof. Put for every $m \in \mathbb{Z}$

$$e_m(n) := \begin{cases} \frac{1}{\sqrt{a_m}}, & n = m, \\ 0, & n \neq m. \end{cases}$$

$(e_m)_{m \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}, \mu)$ and hence

$$\| \check{H}_\mu^{1/2} \|_2^2 = \sum_{m \in \mathbb{Z}} \| \check{H}_\mu^{1/2} e_m \|_{L^2(\mathbb{R}, \mu)}^2 = \sum_{m \in \mathbb{Z}} \check{\mathbb{D}}_1(e_m, e_m) = \sum_{m \in \mathbb{Z}} \frac{k + 2j}{a_m} < \infty;$$

the last step follows from Lemma 3.1. The Theorem follows now from Theorem 2.1 and Corollary 2.2. \square

The proof of the assertions in the introduction is completed by the following example.

Example 3.1. Suppose that $S := \sup_{n \in \mathbb{Z}} a_n < \infty$. Let $b \in (0, \infty)$. Then the operator $(-\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_n + 1)^{-1} - (H_\infty + 1)^{-1}$ does not belong to the trace class.

Proof. Choose any $u \in C_0^\infty(\mathbb{R})$ such that $u(0) \neq 0$ and $u(x) = 0$, if $|x| \geq 1/2$. Put $v_n := u(\cdot - n)/\|u\|_{H^2}$ and $e_n := -v_n'' + v_n$. Then $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal system in $L^2(\mathbb{R})$, $Ge_n = v_n$ for every $n \in \mathbb{Z}$, $v_n(m) = 0$, if $n \neq m$, and there exists a positive real number c such that $v_n(n) = c$ for every $n \in \mathbb{Z}$. Thus

$$(3.6) \quad \|J_\mu Ge_n\|_{L^2(\mathbb{R}, \mu)} \leq c\sqrt{S}, \quad n \in \mathbb{Z}.$$

For notational brevity put $H_b := -\Delta + b \sum_{n \in \mathbb{Z}} a_n \delta_n$ and for every $n \in \mathbb{Z}$ denote by μ_n the measure with distribution function

$$d\mu_n(\lambda) = d\|E_{\check{H}_\mu}(\lambda)J_\mu Ge_n\|_{L^2(\mathbb{R}, \mu)}^2, \quad \lambda \in \mathbb{R}.$$

By (3.6) and since \check{H}_μ is an invertible nonnegative selfadjoint operator,

$$(3.7) \quad \mu_n(\mathbb{R}) = \mu_n((0, \infty)) \leq c^2 S.$$

By Lemma 3.1,

$$(3.8) \quad \int \lambda d\mu_n(\lambda) = \|\check{H}_\mu^{1/2} J_\mu Ge_n\|_{L^2(\mathbb{R}, \mu)}^2 \geq kc^2 > 0.$$

Put $a := k/(2S)$ and $\alpha := \min(1/2, a/(2b))$. Then for every $n \in \mathbb{Z}$

$$\begin{aligned} (((H_b + 1)^{-1} - (H_\infty + 1)^{-1})e_n, e_n) &= \int \frac{\lambda^2}{b + \lambda} d\mu_n(\lambda) \geq \int_{[a, \infty)} \alpha \lambda d\mu_n(\lambda) \\ &\geq \alpha \left(kc^2 - \int_{(0, a)} \lambda d\mu_n(\lambda) \right) \geq \alpha (kc^2 - ac^2 S) > 0; \end{aligned}$$

in the first step we have used (2.9) and (2.10), in the second and third step the fact that $\mu_n((-\infty, 0]) = 0$, in the third step (3.8), and in the fourth step (3.7). Thus

$$\sum_{n \in \mathbb{Z}} (((H_b + 1)^{-1} - (H_\infty + 1)^{-1})e_n, e_n) = \infty$$

and the nonnegative selfadjoint operator $(H_b + 1)^{-1} - (H_\infty + 1)^{-1}$ does not belong to the trace class. \square

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