Methods of Functional Analysis and Topology
Vol. 20 (2014), no. 1, pp. 68–91

# EIGENFUNCTION EXPANSIONS ASSOCIATED WITH AN OPERATOR DIFFERENTIAL EQUATION NON-LINEARLY DEPENDING ON A SPECTRAL PARAMETER

VOLODYMYR KHRABUSTOVSKYI

*Dedicated to my teacher Professor F. S. Rofe-Beketov on the occasion of his jubilee*

Abstract. For an operator differential equation that depends on a spectral parameter in the Nevanlinna manner we obtain expansions in eigenfunctions.

## Introduction

We consider, either on a finite or an infinite interval, the operator differential equation of arbitrary order

$$l_\lambda[y] = m[f], \quad t \in \bar{\mathcal{I}}, \quad \mathcal{I} = (a,b) \subseteq \mathbb{R}^1 \tag{1}$$

in the space of vector-functions with values in a separable Hilbert space $\mathcal{H}$, where

$$l_\lambda[y] = l[y] - \lambda m[y] - n_\lambda[y], \tag{2}$$

$l[y], m[y]$ are symmetric operator differential expression. The order of $l_\lambda[y]$ is equal to $r > 0$. For the expression $m[y]$ the subintegral quadratic form $m\{y,y\}$ of its Dirichlet integral $m[y,y] = \int_{\mathcal{I}} m\{y,y\}dt$ is nonnegative for $t \in \bar{\mathcal{I}}$. The leading coefficient of the expression $m[y]$ may not have the inverse one in $B(\mathcal{H})$ for any $t \in \bar{\mathcal{I}}$ and even it may vanish on some intervals. For the operator differential expression $n_\lambda[y]$ the form $n_\lambda\{y,y\}$ depends on $\lambda$ in the Nevanlinna manner for $t \in \bar{\mathcal{I}}$. Therefore the order $s \geq 0$ of $m[y]$ is even and $\leq r$.

In paper [28] in the Hilbert space $L^2_m(\mathcal{I})$ with metrics generated by the form $m[y,y]$, for equation (1)–(2) we constructed analogs $R(\lambda)$ of the generalized resolvents which in general are non-injective and which possess the following representation:

$$R(\lambda) = \int_{\mathbb{R}^1} \frac{dE_\mu}{\mu - \lambda}, \tag{3}$$

where $E_\mu$ is a generalized spectral family for which $E_\infty$ is less or equal to the identity operator.

This analogue is an integro-differential operator depending on the characteristic operator of the equation

$$l_\lambda[y] = -\frac{(\Im l_\lambda)[f]}{\Im\lambda}, \quad t \in \bar{\mathcal{I}}, \tag{4}$$

where $(\Im l_\lambda)[f] = \frac{1}{2i}(l[f] - l^*[f])$. This characteristic operator was defined in [28]. It is an analogue of the characteristic matrix of a scalar differential operator [40] (see also

2000 *Mathematics Subject Classification.* Primary 34B05, 34B07, 34L10.

*Key words and phrases.* Relation generated by pair of differential expressions one of which depends on spectral parameter in nonlinear manner, non-injective resolvent, generalized resolvent, Weyl type operator function and solution, eigenfunction expansion.

[32, p. 280]). The operator $R(\lambda)$ in the case $n_\lambda[y] \equiv 0$ is the generalized resolvent of the minimal relation corresponding to equation (1) (see the details in [26, 28]).

In this paper we calculate $E_\Delta$ and derive an inequality of Bessel type. In the case where the expression $n_\lambda[y]$ submits in a special way to the expression $m[y]$ we obtain inversion formulae and the Parseval equality. The general results obtained in the work are illustrated with the example of equation (1) with coefficients which are periodic on the semi-axes. We remark that in the case $n_\lambda[y] \equiv 0$ it follows from [22, 26] that if $\mathcal{I} = \mathbb{R}^1$, $r > s$ and $\dim \mathcal{H} < \infty$, then $E_\mu$ for equation (1) with periodic coefficients has no jumps. (For $r = s$ in the described case $E_\mu$ may have jump (see e.g. [26])). We show that in contrast to the case $n_\lambda[y] \equiv 0$ if $r = 1$, $\dim \mathcal{H} = 2$ then $E_\mu$ for equation (1) with periodic coefficients on the axis may have jump.

In the case $n_\lambda[y] \equiv 0$ the eigenfunction expansion results above are obtained in paper [26], which contains its comparison with the results that were obtained earlier for this case. For the case $n_\lambda[y] \equiv 0$ we also refer to papers [5, 6, 7, 15]. In contrast to [26] in [5, 6] $\dim\mathcal{H} = 1$, $0 \leq s < r$, the leading coefficient of the expression $m[y]$ does not vanish and the operator $m$ is uniformly positive. In [5] the elementary approach by [9] has been used to show the existence of eigenfunction expansions. In [6] another existence proof was given, based on the spectral theorem in a direct integral form. The results of [5, 6] can not be used (in contrast to [26]) if for example $m[y] = w(t)y$ where the weight $w(t)$ may vanish or $m[y] = (-1)^n y^{(2n)}$, $\mathcal{I} = \mathbb{R}^1$. However they can be used (in contrast to [26]) in some cases if $m\{y, y\} < 0$ for some $y(t)$ and $t$. In the case where $l$ and $m$ are partial differential operators, expansions in distribution solutions of the equation $(l - \lambda m)[y] = 0$ for formally self-adjoint $l$ and positive $m$ have been obtained in [15] if $m^{-1}l$ has a self-adjoint realization and in [7] without this assumption.

Eigenfunction expansions for differential operators and relations are considered in the monographs [9, 14, 2, 3, 4, 29, 30, 37, 38]. Let us notice that for infinite systems first eigenfunction expansion results are obtained in [34] for operator Sturm-Liouville equation. (Later it was done in [16] in another way). An expansion in eigenfunctions of an operator equation of highest order (analogous to scalar case [40]) was obtained in [8]).

Also for the case of the half-axis we obtain for equation (1) a generalization of the result from [39] on the expansion in solutions of a scalar Sturm-Liouville equation which satisfy, in the regular end-point, a boundary condition depending on a spectral parameter. To do this we introduce, for equation (1), Weyl type functions and solutions. Such solutions for operator equation of the first order containing a spectral parameter in Nevanlinna manner was constructed in [24]. In the case of finite canonical systems, a parametrization of Weyl functions for such an equation was obtained in [33] with the help of $\mathcal{J}$-theory. For the first order systems depending on a spectral parameter in a linear manner a parametrization of Weyl functions immediately in terms of boundary conditions was obtained in [1]. The method of [1] is based on the theory of the abstract Weyl function for symmetric linear relations (see [10, 12] and references therein).

A part of the results of this paper is contained in a preliminary form in the preprint [27].

We denote by ( . ) and $\|\cdot\|$ the scalar product and the norm in various spaces with special indices if it is necessary. For a differential operation $l$, we denote $\Re l = \frac{1}{2}(l + l^*)$, $\Im l = \frac{1}{2i}(l - l^*)$.

Let an interval $\Delta \subseteq \mathbb{R}^1$, $f(t)$ $(t \in \Delta)$ be a function with values in some Banach space $B$. The notation $f(t) \in C^k(\Delta, B)$, $k = 0, 1, \ldots$ (we omit the index $k$ if $k = 0$) means that, in any point of $\Delta$, $f(t)$ has continuous in the norm $\|\cdot\|_B$ derivatives of order up to and including $l$ that are taken in the norm $\|\cdot\|_B$; if $\Delta$ is either semi-open or closed interval then on its ends belonging to $\Delta$ there exist one-side continuous derivatives. The notation

$f(t) \in C_0^k(\Delta, B)$ means that $f(t) \in C^k(\Delta, B)$ and $f(t) = 0$ in the neighborhoods of the ends of $\Delta$.

## 1. Characteristic operator. Weyl type operator function and solution

In order to formulate the eigenfunction expansion results we present in this section several results from [28]. Comparing with [28] some of these results are given here in either a more general form (Propositions 1.1, 1.2) or a weaker form (Theorem 1.3). Lemmas 1.1, 1.2, Theorem 1.2 and Corollary 1.1 are new.

We consider an operator differential equation in a separable Hilbert space $\mathcal{H}_1$,

$$\frac{i}{2}\Big(\left(Q(t)\,x(t)\right)' + Q^*(t)\,x'(t)\Big) - H_\lambda(t)\,x(t) = W_\lambda(t)\,F(t), \quad t \in \bar{\mathcal{I}}, \tag{5}$$

where $Q(t)$, $[\Re Q(t)]^{-1}$, $H_\lambda(t) \in B(\mathcal{H}_1)$, $Q(t) \in C^1\left(\bar{\mathcal{I}}, B(\mathcal{H}_1)\right)$; the operator function $H_\lambda(t)$ is continuous in $t$ and is Nevanlinna's in $\lambda$. Namely, the following condition holds:

(**A**) There is a set $\mathcal{A} \supseteq \mathbb{C}\backslash\mathbb{R}^1$, every its point has a neighborhood independent of $t \in \bar{\mathcal{I}}$, in this neighborhood $H_\lambda(t)$ is analytic $\forall t \in \bar{\mathcal{I}}$; $\forall \lambda \in \mathcal{A}\, H_\lambda(t) = H^*_{\bar{\lambda}}(t) \in C\left(\bar{\mathcal{I}}, B(\mathcal{H}_1)\right)$; the weight $W_\lambda(t) = \Im H_\lambda(t)/\Im\lambda \geq 0\,(\Im\lambda \neq 0)$.

In view of [24] $\forall \mu \in \mathcal{A}\bigcap\mathbb{R}^1 :\ W_\mu(t) = \partial H_\lambda(t)/\partial\lambda|_{\lambda=\mu}$ is Bochner locally integrable in the uniform operator topology.

For convenience we suppose that $0 \in \bar{\mathcal{I}}$ and we denote $\Re Q(0) = G$.

Let $X_\lambda(t)$ be an operator solution of the homogeneous equation (5) satisfying the initial condition $X_\lambda(0) = I$, where $I$ is an identity operator in $\mathcal{H}_1$.

For any $\alpha$, $\beta \in \bar{\mathcal{I}}$, $\alpha \leq \beta$, we denote $\Delta_\lambda(\alpha, \beta) = \int_\alpha^\beta X^*_{\bar{\lambda}}(t)\,W_\lambda(t)\,X_\lambda(t)\,dt$, and $N = \{h \in \mathcal{H}_1\,|h \in Ker\Delta_\lambda(\alpha,\beta)\ \forall\alpha,\beta\}$, $P$ is the ortho-projection onto $N^\perp$. $N$ is independent of $\lambda \in \mathcal{A}$ [24].

For $x(t) \in \mathcal{H}_1$ we denote $U[x(t)] = ([\Re Q(t)]\,x(t), x(t))$.

**Definition 1.1.** [23, 24] An analytic operator-function $M(\lambda) = M^*\left(\bar{\lambda}\right) \in B(\mathcal{H}_1)$ of non-real $\lambda$ is called a characteristic operator of equation (5) on $\mathcal{I}$, if for $\Im\lambda \neq 0$ and for any $\mathcal{H}_1$ - valued vector-function $F(t) \in L^2_{W_\lambda}(\mathcal{I})$ with compact support the corresponding solution $x_\lambda(t)$ of equation (5) of the form

$$x_\lambda(t, F) = \mathcal{R}_\lambda F = \int_{\mathcal{I}} X_\lambda(t)\left\{M(\lambda) - \frac{1}{2}\operatorname{sgn}(s-t)\,(iG)^{-1}\right\}X^*_{\bar{\lambda}}(s)\,W_\lambda(s)\,F(s)\,ds \tag{6}$$

satisfies the condition

$$(\Im\lambda)\lim_{(\alpha,\beta)\uparrow\mathcal{I}}\left(U[x_\lambda(\beta, F)] - U[x_\lambda(\alpha, F)]\right) \leq 0, \quad \Im\lambda \neq 0. \tag{7}$$

Let us note that in [24], a characteristic operator was defined if $Q(t) = Q^*(t)$. Our case is equivalent to this one since equation (5) coincides with equation of (5) type with $\Re Q(t)$ instead of $Q(t)$ and with $H_\lambda(t) - \frac{1}{2}\Im Q'(t)$ instead of $H_\lambda(t)$.

Properties of the characteristic operator and sufficient conditions of the characteristic operators existence are obtained in [23, 24].

In the case $\dim\mathcal{H}_1 < \infty$, $Q(t) = \mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$, $-\infty < a = 0$, a description of characteristic operators was obtained in [33] (the results of [33] were specified and supplemented in [25]). In the case $\dim\mathcal{H}_1 = \infty$ and $\mathcal{I}$ is finite, a description of characteristic operators was obtained in [24]. These descriptions are obtained under the condition that

$$\exists\lambda_0 \in \mathcal{A}, \quad [\alpha,\beta] \subseteq \overline{\mathcal{I}}:\ \Delta_{\lambda_0}(\alpha,\beta) \gg 0. \tag{8}$$

**Definition 1.2.** [23, 24] Let $M(\lambda)$ be the characteristic operator of equation (5) on $\mathcal{I}$. We say that the corresponding condition (7) is separated for nonreal $\lambda = \mu_0$ if for any $\mathcal{H}_1$-valued vector function $f(t) \in L^2_{W_{\mu_0}(t)}(\mathcal{I})$ with compact support the following inequalities hold simultaneously for the solution $x_{\mu_0}(t)$ (6) of equation (5):

$$\lim_{\alpha \downarrow a} \Im \mu_0 U\left[x_{\mu_0}(\alpha)\right] \geq 0, \quad \lim_{\beta \uparrow b} \Im \mu_0 U\left[x_{\mu_0}(\beta)\right] \leq 0. \tag{9}$$

**Theorem 1.1.** [23, 24] *Let $P = I$, $M(\lambda)$ be the characteristic operator of equation (5), $\mathcal{P}(\lambda) = iM(\lambda)G + \frac{1}{2}I$, so that we have the following representation:*

$$M(\lambda) = \left(\mathcal{P}(\lambda) - \frac{1}{2}I\right)(iG)^{-1}. \tag{10}$$

*Then the condition (7) corresponding to $M(\lambda)$ is separated for $\lambda = \mu_0$ if and only if the operator $\mathcal{P}(\mu_0)$ is a projection, i.e.,*

$$\mathcal{P}(\mu_0) = \mathcal{P}^2(\mu_0).$$

**Definition 1.3.** [23, 24] If the operator-function $M(\lambda)$ of the form (10) is a characteristic operator of equation (5) on $\mathcal{I}$ and, moreover, $\mathcal{P}(\lambda) = \mathcal{P}^2(\lambda)$, then $\mathcal{P}(\lambda)$ is called a characteristic projection of equation (5) on $\mathcal{I}$.

Properties of characteristic projections and sufficient conditions for their existence are obtained in [24]. Also [24] contains a description of characteristic projections and an abstract analogue of Theorem 1.1. Necessary and sufficient conditions for existence of a characteristic operator, which corresponds to such separated boundary conditions that the corresponding boundary condition in a regular point is self-adjoint, are obtained in [28] with a help of Theorem 1.1. In the case of self-adjoint boundary conditions an analogue of this result for regular differential operators in the space of vector-functions was proved in [35] (see also [37]). For finite canonical systems depending on the spectral parameter in a linear manner such an analogue was proved in [31]. These analogues were obtained in a different way comparing with the proof in [28].

From this point and till the end of Corollary 1.1 we suppose that $\mathcal{H}_1 = \mathcal{H}^{2n}$,

$$Q(t) = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix} = J/i, \tag{11}$$

where $I_n$ is the identity operator in $\mathcal{H}^n$, $\mathcal{I} = (0, b)$, $b \leq \infty$ and condition (8) holds. Let condition (7) be separated and $\mathcal{P}(\lambda)$ be a corresponding characteristic projection. In view of [24, p. 469] the Nevanlinna pair $\{-a(\lambda), b(\lambda)\}$, $a(\lambda), b(\lambda) \in B(\mathcal{H}^n)$ (see for example [11]) and Weyl function $m(\lambda) \in B(\mathcal{H}^n)$ of equation (5) on $\mathcal{I}$ [24] exist such that

$$\mathcal{P}(\lambda) = \begin{pmatrix} I_n \\ m(\lambda) \end{pmatrix} \left(b^*(\bar{\lambda}) - a^*(\bar{\lambda})\, m(\lambda)\right)^{-1} \left(a_2^*(\bar{\lambda}),\, -a_1^*(\bar{\lambda})\right), \tag{12}$$

$$I - \mathcal{P}(\lambda) = \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix} \left(b(\lambda) - m(\lambda)\, a(\lambda)\right)^{-1} \left(-m(\lambda), I_n\right), \tag{13}$$

$$\left(b^*(\bar{\lambda}) - a^*(\bar{\lambda})\, m(\lambda)\right)^{-1},\ \left(b(\lambda) - m(\lambda)\, a(\lambda)\right)^{-1} \in B(\mathcal{H}^n). \tag{14}$$

(Conversely [24] $\mathcal{P}(\lambda)$ (12) is a characteristic projection for any Nevanlinna pair $(-a(\lambda), b(\lambda))$ and any Weyl function $m(\lambda)$ of equation (5) on $\mathcal{I}$.)

Let also the domain $D \subseteq \mathbb{C}_+$ be such that $\forall \lambda \in D:\ 0 \in \rho\left(a(\lambda) - ib(\lambda)\right)$ (for example $D = \mathbb{C}_+$ if $\exists \lambda_\pm \in \mathbb{C}_\pm$ such that $a^*(\lambda_\pm)\, b(\lambda_\pm) = b^*(\lambda_\pm)\, a(\lambda_\pm)$). Let domain $D_1$ be symmetric to $D$ with respect to real axis. Then the operator $\mathcal{R}_\lambda F$ (6) for $\lambda \in D \bigcup D_1$ can be represented in the following form using the operator solution $U_\lambda(t) \in B\left(\mathcal{H}^n, \mathcal{H}^{2n}\right)$ of equation (5), ($F = 0$) satisfying an accumulative (or dissipative) initial condition and the operator solution $V_\lambda(t) \in B\left(\mathcal{H}^n, \mathcal{H}^{2n}\right)$ of Weyl type of the same equation. More precisely the following proposition holds.

**Proposition 1.1.** *Let $\lambda \in D \bigcup D_1$ and $\mathcal{H}_1$-valued $F(t) \in L^2_{W_\lambda}(\mathcal{I})$[1]. Then solution* (6), (10), (12) *of equation* (5) *is equal to*

$$\mathcal{R}_\lambda F = \int_0^t V_\lambda(t)\, U^*_{\bar\lambda}(s)\, W_\lambda(s)\, F(s)\, ds + \int_t^b U_\lambda(t)\, V^*_{\bar\lambda}(s)\, W_\lambda(s)\, F(s)\, ds, \tag{15}$$

*where the integrals converge strongly if the interval of integration is infinite. Here*

$$U_\lambda(t) = X_\lambda(t) \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix}, \quad V_\lambda(t) = X_\lambda(t) \begin{pmatrix} b(\lambda) \\ -a(\lambda) \end{pmatrix} K^{-1}(\lambda) + U_\lambda(t)\, m_{a,b}(\lambda), \tag{16}$$

*where*

$$K(\lambda) = a^*\left(\bar\lambda\right) a(\lambda) + b^*\left(\bar\lambda\right) b(\lambda), \quad K^{-1}(\lambda) \in B(\mathcal{H}^n), \tag{17}$$

$$m_{a,b}(\lambda) = m^*_{a,b}\left(\bar\lambda\right) = K^{-1}(\lambda)\left(a^*\left(\bar\lambda\right) + b^*\left(\bar\lambda\right) m(\lambda)\right)\left(b^*\left(\bar\lambda\right) - a^*\left(\bar\lambda\right) m(\lambda)\right)^{-1}, \tag{18}$$

$$\int_0^\beta V^*_\lambda(t)\, W_\lambda(t)\, V_\lambda(t)\, dt \le \frac{\left(b(\bar\lambda) - m^*(\lambda) a(\bar\lambda)\right)^{-1} (\Im m(\lambda))(b^*(\bar\lambda) - a^*(\bar\lambda) m(\lambda))^{-1}}{\Im\lambda}, \tag{19}$$

$\forall [0,\beta] \subseteq \bar{\mathcal{I}}$, *and therefore*

$$\forall h \in \mathcal{H}^n : \ V_\lambda(t)\, h \in L^2_{W_\lambda(t)}(\mathcal{I}). \tag{20}$$

*Moreover if $a(\lambda) = a\left(\bar\lambda\right)$, $b(\lambda) = b\left(\bar\lambda\right)$ as $\Im\lambda \ne 0$ then we can set $D = \mathbb{C}_+$ and*

$$\int_0^\beta V^*_\lambda(t)\, W_\lambda(t)\, V_\lambda(t)\, dt \le \frac{\Im m_{a,b}(\lambda)}{\Im\lambda}, \quad \Im\lambda \ne 0. \tag{21}$$

*Proof.* In view of (10), (12), (13) $\mathcal{R}_\lambda F$ has a representation (15) where

$$V_\lambda(t) = X_\lambda(t) \begin{pmatrix} I_n \\ m(\lambda) \end{pmatrix} \left(a^*_2\left(\bar\lambda\right) - a^*_1\left(\bar\lambda\right) m(\lambda)\right)^{-1}. \tag{22}$$

Due to Lemma 1.2 from [28] the integrals in (15) converge strongly if the interval of integration is infinite.

In view of [17, 24] and the fact that $\mathcal{P}^*\left(\bar\lambda\right) G \mathcal{P}(\lambda) = 0$ [24] one has

$$a(\lambda) = \mp i\,(u(\lambda) + I_n)\, S(\lambda), \quad b(\lambda) = (u(\lambda) - I_n)\, S(\lambda), \quad \lambda \in \mathbb{C}_\pm, \tag{23}$$

where $u(\lambda) = u^*\left(\bar\lambda\right) \in B(\mathcal{H}^n)$ is some contraction, $S(\lambda), S^{-1}(\lambda) \in B(\mathcal{H}^n)$; $u(\lambda)$, $S(\lambda)$ analytically depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}^1$.

In view of (23),

$$K(\lambda) = -4 S^*\left(\bar\lambda\right) u(\lambda)\, S(\lambda) \tag{24}$$

and so $K^{-1}(\lambda) \in B(\mathcal{H}^n)$, $\lambda \in D \bigcup D_1$ since $u^{-1}(\lambda) = -2iS(\lambda)\,(a(\lambda) - ib(\lambda))^{-1} \in B(\mathcal{H}^n)$ $\lambda \in D \cup D_1$.

Using (23), (24) it can be directly shown that the initial conditions in point $t = 0$ for solutions $V_\lambda(t)$ (16) and $V_\lambda$ (22) coincide and that $m_{a,b}(\lambda) = m^*_{a,b}\left(\bar\lambda\right)$. So (15)–(18) is proved.

In view of [24, p. 450] one has $\forall [0,\beta] \subseteq \bar{\mathcal{I}}$

$$\mathcal{P}^*(\lambda)\, \Delta_\lambda(0,\beta)\, \mathcal{P}(\lambda) \le \frac{1}{2\Im\lambda} \mathcal{P}^*(\lambda)\, G \mathcal{P}(\lambda). \tag{25}$$

Now inequality (19) (and therefore (20)) follows from (25) in view of (12), (17), (22).

If $a(\lambda) = a\left(\bar\lambda\right)$, $b(\lambda) = b\left(\bar\lambda\right)$ as $\Im\lambda \ne 0$, then the operator $u(\lambda)$ is unitary and independent of $\lambda$ (cf. [24]). Now in formulae (17), (18) and the right-hand-side of (19) we substitute $a\left(\bar\lambda\right)$, $b\left(\bar\lambda\right)$ by $a(\lambda)$, $b(\lambda)$ and by direct calculations with the help of (23)

---

[1] Norms $\|\cdot\|_{L^2_{W_\lambda}(\mathcal{I})}$ are equivalent for $\lambda \in \mathcal{A}$ [24].

we prove that the right hand sides of inequalities (19), (21) coincide. The proposition is proved. □

For an arbitrary Nevanlinna pair $\{-a(\lambda), b(\lambda)\}$, the Weyl solution $V_\lambda(t)$ (16) does not satisfy, in general, inequality (21) for $\lambda \in D\bigcup D_1$, and the corresponding Weyl function $m_{a,b}(\lambda)$ (18) does not satisfy the condition

$$\frac{\Im m_{a,b}(\lambda)}{\Im \lambda} \geq 0, \quad \lambda \in D\bigcup D_1. \tag{26}$$

But if we choose the pair $\{-a(\lambda), b(\lambda)\}$ "in a canonical way", then the corresponding Weyl solution $V_\lambda$ (16) satisfies (21).

Namely let $v(\lambda) \in B(\mathcal{H}^n)$ be a contraction analytically depending on $\lambda$ in a domain $D \subseteq \mathbb{C}_+$ and let $v^{-1}(\lambda) \in B(\mathcal{H}^n)$, $\lambda \in D$. Let us consider the following pair $\{a(\lambda), b(\lambda)\}$, where

$$a(\lambda) = -i\,(v(\lambda) + I_n), \quad b(\lambda) = v(\lambda) - I_n, \quad \lambda \in D. \tag{27}$$

Let us extend the pair $\{a(\lambda), b(\lambda)\}$ (27) to the domain $D_1$ which is symmetric to $D$ with the respect to the real axis in the following way:

$$v(\lambda) = \left(v^*\left(\bar{\lambda}\right)\right)^{-1}, \quad \lambda \in D_1 \tag{28}$$

(and therefore $v^*\left(\bar{\lambda}\right)$ is stretching as $\lambda \in D$). As a result we obtain a pair of (23) type with $D$ (respectively $D_1$) instead of $\mathbb{C}_+$ (respectively $\mathbb{C}_-$) and

$$u(\lambda) = \begin{cases} v(\lambda), & \lambda \in D \\ v^*(\bar{\lambda}), & \lambda \in D_1 \end{cases}, \quad S(\lambda) = \begin{cases} I_n, & \lambda \in D \\ -v(\lambda), & \lambda \in D_1 \end{cases}.$$

Therefore if $\lambda \in D\bigcup D_1$ then for the pair $\{a(\lambda), b(\lambda)\}$ (27), (28) the projections (12), (13) exist and therefore for the operator $M(\lambda)$ (10), (12) the condition (7) holds and is separated.

**Lemma 1.1.** *The operator Weyl function* $m_{a,b}(\lambda)$ (18) *corresponding to the pair* $\{a(\lambda), b(\lambda)\}$ (27), (28) *satisfies for any* $h \in \mathcal{H}^n$ *the identity*

$$\Im\left(m_{a,b}(\lambda)\,h, h\right) = \frac{1}{4}\left\|\sqrt{v\left(\bar{\lambda}\right)v^*\left(\bar{\lambda}\right) - I_n}\,(I_n - im(\lambda))\,g\right\|^2 + \Im\left(m(\lambda)\,g, g\right), \quad \lambda \in D, \tag{29}$$

*where* $g = \left(\left(I_n - v^*\left(\bar{\lambda}\right)\right) + i\left(I_n + v^*\left(\bar{\lambda}\right)m(\lambda)\right)\right)^{-1} h$, $(\dots)^{-1} \in B(\mathcal{H}^n)$.

*Proof.* The Weyl function $m_{a,b}(\lambda)$ (18), (27), (28) is equal to

$$\begin{aligned} m_{a,b}(\lambda) = &-\frac{1}{4}\left(i(I_n + v^*(\bar{\lambda})) - (I_n - v^*(\bar{\lambda}))m(\lambda)\right) \\ &\times\left(I_n - v^*(\bar{\lambda}) + i(I_n + v^*(\bar{\lambda}))m(\lambda)\right)^{-1}, \quad \lambda \in D, \end{aligned} \tag{30}$$

where $(\dots)^{-1} \in B(\mathcal{H}^n)$ in view of (14), (27), (28).

Now identity (29) follows from (30) by a direct calculation. □

In view of the fact that $m_{a,b}\left(\bar{\lambda}\right) = m^*_{a,b}(\lambda)$, inequality (19), Lemma 1.1, condition (8) and formula (22) the following theorem is valid.

**Theorem 1.2.** *The solution* $V_\lambda(t)$ (16)–(18), (27), (28) *satisfies inequality* (21) *for* $\lambda \in D\bigcup D_1$ *(and therefore* $m_{a,b}(\lambda)$ (18), (27), (28) *satisfies inequality* (26) *with* " $\gg$ " *replaced with* " $\geq$ "*).*

**Lemma 1.2.** *Let $v(\lambda) \in B(\mathcal{H}^n)$ be a contraction analytically depending on $\lambda \in \mathbb{C}_+$. Let limit points of the set $S = \{\lambda \in \mathbb{C}_+ : v^{-1}(\lambda) \notin B(\mathcal{H}^n)\}$ that belong to $\mathbb{C}_+$ be isolated. Let $D = \mathbb{C}_+ \backslash S$. Let us consider the operator-function $m_{ab}(\lambda)$ (18), (27), (28) as $\lambda \in D \bigcup D_1$. Then the points of the set $S$ are removable singular points of this function. If $m_{a,b}(\lambda)$ is extended to the set $S$ in a proper way then we obtain the Nevanlinna operator-function $m_{a.b}(\lambda) = m^*_{a,b}(\bar{\lambda})$.*

*Proof.* Let $\lambda_0 \in S$ not be a limit point of $S$. Then $\lambda_0$ is a removable singular point of the scalar function $(m_{a,b}(\lambda) f, f)\, \forall f \in \mathcal{H}^n$ in view of (29). Hence $\exists\, m_0 \in B(\mathcal{H}^n)$ : $\lim\limits_{\lambda \to \lambda_0} (m_{a,b}(\lambda) f, f) = (m_0 f, f)\, \forall f \in \mathcal{H}^n$ in view of principle of uniform boundedness [18, p. 164], [19, p. 322]. If we define $m_{a,b}(\lambda)$ in the point $\lambda_0$ as $m_{a,b}(\lambda_0) = m_0$ then we obtain an operator-function which is analytic in the point $\lambda_0$ in view of [19, p. 195]. Analyticity of $m_{a,b}(\lambda)$ in the limit points of $S$ belonging to $\mathbb{C}_+$ is proved analogously. ☐

**Corollary 1.1.** *Let the construction $v(\lambda) \in B(\mathcal{H}^n)$ satisfy condition of Lemma 1.2. Then the corresponding solution $V_\lambda(t)$ (16)–(18), (27), (28) satisfies the inequality (21) ($V_\lambda(t) \overset{\text{def}}{=}$ (22), $\lambda \notin D \cup D_1$).*

For the construction of solutions of Weyl type and descriptions of Weyl function in various situation see [1, 24] and references in [1].

We consider in the separable Hilbert space $\mathcal{H}$ differential expression $l_\lambda[y]$ of order $r > 0$ with coefficients from $B(\mathcal{H})$. This expression is presented in the divergent form, namely

$$l_\lambda[y] = \sum_{k=0}^{r} i^k l_k(\lambda)[y], \tag{31}$$

where $l_{2j}(\lambda) = D^j p_j(t,\lambda) D^j$, $l_{2j-1}(\lambda) = \frac{1}{2} D^{j-1} \{D q_j(t,\lambda) + s_j(t,\lambda) D\} D^{j-1}$, $D = \frac{d}{dt}$.

Let $-l_\lambda$ depend on $\lambda$ in Nevanlinna manner. Namely, from now on the following condition holds:

(**B**) The set $\mathcal{B} \supseteq \mathbb{C} \setminus \mathbb{R}^1$ exists, every its point has a neighborhood independent on $t \in \bar{\mathcal{I}}$, in this neighborhood coefficients $p_j = p_j(t,\lambda)$, $q_j = q_j(t,\lambda)$, $s = s_j(t,\lambda)$ of the expression $l_\lambda$ are analytic $\forall t \in \bar{\mathcal{I}}$; $\forall \lambda \in \mathcal{B}$, $p_j(t,\lambda)$, $q_j(t,\lambda)$, $s_j(t,\lambda) \in C^j(\bar{\mathcal{I}}, B(\mathcal{H}))$ and

$$\begin{gathered} p_n^{-1}(t,\lambda) \in B(\mathcal{H})\ (r = 2n), \\ (q_{n+1}(t,\lambda) + s_{n+1}(t,\lambda))^{-1} \in B(\mathcal{H})\ (r = 2n+1), \quad t \in \bar{\mathcal{I}}; \end{gathered} \tag{32}$$

these coefficients satisfy the following conditions:

$$p_j(t,\lambda) = p_j^*(t,\bar{\lambda}), \quad q_j(t,\lambda) = s_j^*(t,\bar{\lambda}), \quad \lambda \in \mathcal{B}\ (\Longleftrightarrow\ l_\lambda = l^*_{\bar{\lambda}},\ \lambda \in \mathcal{B}), \tag{33}$$

$$\begin{gathered} \forall h_0, \dots, h_{\left[\frac{r+1}{2}\right]} \in \mathcal{H} : \\ \frac{\Im\Big( \sum\limits_{j=0}^{[r/2]} (p_j(t,\lambda) h_j, h_j) + \frac{i}{2} \sum\limits_{j=1}^{\left[\frac{r+1}{2}\right]} \{(s_j(t,\lambda) h_j, h_{j-1}) - (q_j(t,\lambda) h_{j-1}, h_j)\} \Big)}{\Im\lambda} \le 0, \\ t \in \bar{\mathcal{I}}, \quad \Im\lambda \ne 0. \end{gathered} \tag{34}$$

Therefore the order of expression $\Im l_\lambda$ is even and therefore if $r = 2n+1$ is odd, then $q_{m+1}$, $s_{m+1}$ are independent on $\lambda$ and $s_{n+1} = q^*_{n+1}$.

Condition (34) is equivalent to the condition: $(\Im l_\lambda)\{f,f\}/\Im\lambda \le 0$, $t \in \bar{\mathcal{I}}$, $\Im\lambda \ne 0$. Here for differential expression $L[y] = \sum\limits_{k=0}^{R} i^k L_k[y]$ with sufficiently smooth coefficients from $B(\mathcal{H})$, where $L_{2j} = D^j P_j\,(t)\,D^j$, $L_{2j-1} = \frac{1}{2} D^{j-1}\left\{DQ_j\,(t) + S_j\,(t)\;D\right\} D^{j-1}$, we denote by

$$
L\{f,g\} = \sum_{j=0}^{[R/2]} \Big(P_j\,(t)\,f^{(j)}\,(t)\,,\,g^{(j)}\,(t)\Big) + \frac{i}{2}\sum_{j=1}^{\left[\frac{R+1}{2}\right]} \Big(S_j\,(t)\,f^{(j)}\,(t)\,,\,g^{(j-1)}\,(t)\Big) - \Big(Q_j\,(t)\,f^{(j-1)}\,(t)\,, g^{(j)}\,(t)\Big) \tag{35}
$$

the bilinear form which corresponds to subintegral expression of the Dirichlet integral for expression $L[y]$.

Let $m[y]$ be the same as $l_\lambda[y]$ differential expression of even order $s \le r$ with operator coefficients $\tilde p_j\,(t) = \tilde p_j^*\,(t)\,,\ \tilde q_j\,(t)\,,\ \tilde s_j\,(t) = \tilde q_j^*\,(t) \in C^j(\bar{\mathcal{I}}, B(\mathcal{H}))$ that are independent on $\lambda$. Let

$$
\begin{gathered}
\forall h_0,\,\ldots,\,h_{\left[\frac{r+1}{2}\right]} \in \mathcal{H}:\ 0 \le \sum_{j=0}^{s/2} (\tilde p_j\,(t)\,h_j, h_j) + \Im \sum_{j=1}^{s/2} (\tilde q_j\,(t)\,h_{j-1},\,h_j) \\
\le -\frac{\Im\Big(\sum\limits_{j=0}^{[r/2]} (p_j\,(t,\lambda)\,h_j, h_j) + \frac{i}{2}\sum\limits_{j=1}^{\left[\frac{r+1}{2}\right]} ((s_j\,(t,\lambda)\,h_j, h_{j-1}) - (q_j\,(t,\lambda)\,h_{j-1}, h_j))\Big)}{\Im\lambda}, \\
t \in \bar{\mathcal{I}}, \quad \Im\lambda \ne 0.
\end{gathered} \tag{36}
$$

Condition (36) is equivalent to the condition: $0 \le m\,\{f,f\} \le -(\Im l_\lambda)\,\{f,f\}\,/\Im\lambda$, $t \in \bar{\mathcal{I}}$, $\Im\lambda \ne 0$.

In the case of even $r = 2n \ge s$ we denote

$$
Q\,(t, l_\lambda) = J/i, \qquad S\,(t, l_\lambda) = Q\,(t, l_\lambda)\,, \tag{37}
$$

$$
H\,(t,\,l_\lambda) = \|h_{\alpha\beta}\|^2_{\alpha,\,\beta=1}\,, \quad h_{\alpha\beta} \in B\,(\mathcal{H}^n)\,, \tag{38}
$$

where $h_{11}$ is a three diagonal operator matrix whose elements under the main diagonal are equal to $\big(\frac{i}{2}q_1,\,\ldots,\,\frac{i}{2}q_{n-1}\big)$, the elements over the main diagonal are equal to $\big(-\frac{i}{2}s_1,\ \ldots,\ -\frac{i}{2}s_{n-1}\big)$, the elements on the main diagonal are equal to $\big(-p_0,\ \ldots,\ -p_{n-2},\ \frac{1}{4}s_n p_n^{-1} q_n - p_{n-1}\big)$; $h_{12}$ is an operator matrix with the identity operators $I_1$ under the main diagonal, the elements on the main diagonal are equal to $\big(0,\ \ldots,\ 0,\ -\frac{i}{2}s_n p_n^{-1}\big)$, all the rest elements are equal to zero; $h_{21}$ is an operator matrix with identity operators $I_1$ over the main diagonal, the elements on the main diagonal are equal to $\big(0,\ \ldots,\ 0,\ \frac{i}{2}p_n^{-1}q_n\big)$, all the rest elements are equal to zero; $h_{22} = \operatorname{diag}\big(0,\ \ldots,\ 0,\ p_n^{-1}\big)$.

Also in this case we denote [2]

$$
W\,(t,\,l_\lambda,\,m) = C^{*-1}\,(t, l_\lambda)\left\{\|m_{\alpha\beta}\|^2_{\alpha,\,\beta=1}\right\} C^{-1}\,(t, l_\lambda)\,, \quad m_{\alpha\beta} \in B\,(\mathcal{H}^n)\,, \tag{39}
$$

where $m_{11}$ is a three diagonal operator matrix whose elements under the main diagonal are equal to $\big(-\frac{i}{2}\tilde q_1,\,\ldots,\,-\frac{i}{2}\tilde q_{n-1}\big)$, the elements over the main diagonal are equal to $\big(\frac{i}{2}\tilde s_1,\,\ldots,\,\frac{i}{2}\tilde s_{n-1}\big)$, the elements on the main diagonal are equal to $(\tilde p_0,\,\ldots,\,\tilde p_{n-1})$; $m_{12} = \operatorname{diag}\big(0,\,\ldots,\,0,\,\frac{i}{2}\tilde s_n\big)$, $m_{21} = \operatorname{diag}\big(0,\,\ldots,\,0,\,-\frac{i}{2}\tilde q_n\big)$, $m_{22} = \operatorname{diag}(0,\,\ldots,\,0,\,\tilde p_n)$.

---

[2] $W\,(t, l_\lambda, m)$ is given for the case $s = 2n$ . If $s < 2n$ one have to set the corresponding elements of the operator matrices $m_{\alpha\beta}$ to be equal to zero. In particular if $s < 2n$ then $m_{12} = m_{21} = m_{22} = 0$ and therefore $W\,(t, l_\lambda, m) = \operatorname{diag}\,(m_{11}, 0)$ in view of (14) from [28].

The operator matrix $C\,(t,l_\lambda)$ is defined by the condition

$$(40)\qquad \begin{aligned} &C\,(t,l_\lambda)\,\mathrm{col}\left\{f\,(t)\,,\,f'\,(t)\,,\,\ldots,\,f^{(n-1)}\,(t)\,,\,f^{(2n-1)}\,(t)\,,\,\ldots,\,f^{(n)}\,(t)\right\}\\ &=\mathrm{col}\,\left\{f^{[0]}\,(t|l_\lambda)\,,\,f^{[1]}\,(t|l_\lambda)\,,\,\ldots,\,f^{[n-1]}\,(t|l_\lambda)\,,\,f^{[2n-1]}\,(t\,|l_\lambda\,)\,,\,\ldots,\,f^{[n]}\,(t\,|l_\lambda\,)\right\},\end{aligned}$$

where $f^{[k]}\,(t\,|L\,)$ are quasi-derivatives of vector-function $f\,(t)$ that correspond to differential expression $L[y]$; $C^{-1}(t,l_\lambda)\in B(\mathcal{H}^r)$, $t\in\bar{\mathcal{I}}$, $\lambda\in\mathcal{B}$ in view of (14) from [28].

The quasi-derivatives corresponding to $l_\lambda[y]$ are equal (cf. [36]) to

$$(41)\qquad y^{[j]}\,(t\,|l_\lambda\,)=y^{(j)}\,(t)\,,\quad j=0,\,\ldots,\,\left[\frac{r}{2}\right]-1,$$

$$(42)\qquad y^{[n]}\,(t\,|l_\lambda\,)=\begin{cases} p_n y^{(n)}-\frac{i}{2}q_n y^{(n-1)}, & r=2n\\ -\frac{i}{2}q_{n+1}y^{(n)}, & r=2n+1\end{cases},$$

$$(43)\qquad \begin{aligned} y^{[r-j]}\,(t\,|l_\lambda\,)&=-Dy^{[r-j-1]}\,(t\,|l_\lambda\,)+p_j y^{(j)}+\frac{i}{2}\left[s_{j+1}y^{(j+1)}-q_j y^{(j-1)}\right],\\ j&=0,\,\ldots,\,\left[\frac{r-1}{2}\right],\quad q_0\equiv 0.\end{aligned}$$

Then $l_\lambda\,[y]=y^{[r]}\,(t\,|l_\lambda\,)$. The quasi-derivatives $y^{[k]}\,(t\,|m)$ corresponding to $m[y]$ are defined in the same way with even $s$ instead of $r$ and $\tilde{p}_j,\tilde{q}_j,\tilde{s}_j$ instead of $p_j,q_j,s_j$.

In the case of odd $r=2n+1>s$ we denote

$$(44)\qquad Q\,(t,l_\lambda)=\begin{cases} J/i\oplus q_{n+1}\\ q_1\end{cases},\quad S\,(t,l_\lambda)=\begin{cases} J/i\oplus s_{n+1}, & n>0\\ s_1, & n=0\end{cases},$$

$$(45)\qquad H\,(t,\,l_\lambda)=\begin{cases} \|h_{\alpha\,\beta}\|^2_{\alpha,\,\beta=1}\,, & n>0\\ p_0, & n=0\end{cases},$$

where $B\,(\mathcal{H}^n)\ni h_{11}$ is a three-diagonal operator matrix whose elements under the main diagonal are equal to $\left(\frac{i}{2}q_1,\,\ldots,\,\frac{i}{2}q_{n-1}\right)$, the elements over the main diagonal are equal to $\left(-\frac{i}{2}s_1,\,\ldots,\,-\frac{i}{2}s_{n-1}\right)$, the elements on the main diagonal are equal to $(-p_0,\,\ldots,\,-p_{n-1})$, all the rest elements are equal to zero. $B\,\left(\mathcal{H}^{n+1},\,\mathcal{H}^n\right)\ni h_{12}$ is an operator matrix whose elements with numbers $j$, $j-1$ are equal to $I_1$, $j=2,\,\ldots,\,n$, the element with number $n$, $n+1$ is equal to $\frac{1}{2}s_n$, all the rest elements are equal to zero. $B\,\left(\mathcal{H}^n,\,\mathcal{H}^{n+1}\right)\ni h_{21}$ is an operator matrix whose elements with numbers $j-1$, $j$ are equal to $I_1$, $j=2,\,\ldots,\,n$, the element with number $n+1$, $n$ is equal to $\frac{1}{2}q_n$, all the rest elements are equal to zero. $B\,\left(\mathcal{H}^{n+1}\right)\ni h_{22}$ is an operator matrix whose last row is equal to $(0,\,\ldots,\,0,\,-iI_1,\,-p_n)$, last column is equal to $\mathrm{col}\,(0,\,\ldots,\,0,\,iI_1,\,-p_n)$, all the rest elements are equal to zero.

Also in this case we denote [3]

$$(46)\qquad W\,(t,\,l_\lambda,\,m)=\|m_{\alpha\beta}\|^2_{\alpha,\,\beta=1}\,,$$

where $m_{11}$ is defined in the same way as $m_{11}$ (39). $B\,\left(\mathcal{H}^{n+1},\,\mathcal{H}^n\right)\ni m_{12}$ is an operator matrix whose element with number $n$, $n+1$ is equal to $-\frac{1}{2}\tilde{s}_n$, all the rest elements are equal to zero. $B\,\left(\mathcal{H}^n,\,\mathcal{H}^{n+1}\right)\ni m_{21}$ is an operator matrix whose element with number $n+1$, $n$ is equal to $-\frac{1}{2}\tilde{q}_n$, all the rest elements are equal to zero. $B\,\left(\mathcal{H}^{n+1}\right)\ni m_{22}=\mathrm{diag}\,(0,\,\ldots,\,0,\,\tilde{p}_n)$.

Obviously in view of (33), (36) for $H\,(t,l_\lambda)$ (38), (45) and $W\,(t,l_\lambda,m)$ (39), (46) one has

$$(47)\qquad H^*\,(t,l_\lambda)=H\,(t,l_{\bar{\lambda}})\,,\quad W^*\,(t,l_\lambda,m)=W\,(t,l_\lambda,\,m)\,,\quad t\in\bar{\mathcal{I}},\quad \lambda\in\mathcal{B}.$$

[3] See the previous footnote.

For sufficiently smooth vector-function $f(t)$ we denote

$$\begin{aligned}(48)\quad &\mathcal{H}^r \ni F(t,\, l_\lambda, m)\\ &= \begin{cases} \left(\sum\limits_{j=0}^{s/2} \oplus f^{(j)}(t)\right) \oplus 0 \oplus \cdots \oplus 0, & r = 2n, \quad r = 2n+1 > 1,\ s < 2n \\ \left(\sum\limits_{j=0}^{n-1} \oplus f^{(j)}(t)\right) \oplus 0 \oplus \cdots \oplus 0 \oplus \left(-if^{(n)}(t)\right), & r = 2n+1 > 1,\ s = 2n \\ f(t), & r = 1 \\ \left(\sum\limits_{j=0}^{n-1} \oplus f^{(j)}(t)\right) \oplus \left(\sum\limits_{j=1}^{n} \oplus f^{[r-j]}(t\,|l_\lambda)\right), & r = s = 2n. \end{cases}\end{aligned}$$

**Theorem 1.3.** [28] *Equation* (1) *is equivalent to the following first order system:*

$$(49)\quad \frac{i}{2}\Big(\left(Q(t,l_\lambda)\,\vec{y}(t)\right)' + Q^*(t,l_\lambda)\,\vec{y}px'(t)\Big) - H(t,l_\lambda)\,\vec{y}(t) = W(t,l_{\bar\lambda},m)\,F(t,l_{\bar\lambda},m),$$

*where* $Q(t,l_\lambda)$, $H(t,l_\lambda)$ *are defined by* (37), (38), (44), (45) *and* $W(t,l_{\bar\lambda},m)$, $F(t,l_{\bar\lambda},m)$ *are defined by* (39), (46), (48) *with* $l_{\bar\lambda}$ *instead of* $l_\lambda$. *Namely if* $y(t)$ *is a solution of equation* (1) *then*

$$\begin{aligned}(50)\quad &\vec{y}(t) = \vec{y}(t,\, l_\lambda,\, m,\, f)\\ &= \begin{cases} \Big(\sum\limits_{j=0}^{n-1} \oplus y^{(j)}(t)\Big) \oplus \Big(\sum\limits_{j=1}^{n} \oplus \left(y^{[r-j]}(t\,|l_\lambda) - f^{[s-j]}(t\,|m)\right)\Big), & r = 2n \\ \Big(\sum\limits_{j=0}^{n-1} \oplus y^{(j)}(t)\Big) \oplus \Big(\sum\limits_{j=1}^{n} \oplus \left(y^{[r-j]}(t\,|l_\lambda) - f^{[s-j]}(t\,|m)\right)\Big) \oplus \left(-iy^{(n)}(t)\right), & r = 2n+1 > 1 \\ \qquad \left(\text{here } f^{[k]}(t\,|m) \equiv 0 \text{ as } k < \frac{s}{2}\right) & \\ y(t), & r = 1 \end{cases}\end{aligned}$$

*is a solution of* (49). *Any solution of equation* (49) *is equal to* (50), *where* $y(t)$ *is some solution of equation* (1).

Let us notice that different vector-functions $f(t)$ can generate different right-hand-sides of equation (49) but only unique right-hand-side of equation (1).

Due to Lemma 1.1 and Theorem 1.2 from [28] we have $\frac{\Im H(t,l_\lambda)}{\Im\lambda} = W\left(t, l_\lambda, -\frac{\Im l_\lambda}{\Im\lambda}\right) \geq 0$, $t \in \bar{\mathcal{I}}$, $\Im\lambda \neq 0$ and therefore $H(t,l_\lambda)$ satisfies condition **(A)** with $\mathcal{A} = \mathcal{B}$. Therefore $\forall\mu \in \mathcal{B} \cap \mathbb{R}^1$ $W(t,l_\mu,-\frac{\Im l_\mu}{\Im\mu}) = \frac{\partial H(t,l_\lambda)}{\partial\lambda}\Big|_{\lambda=\mu}$ is Bochner locally integrable in uniform operator topology. Here in view of (38), (45) $\forall\mu \in \mathcal{B}\bigcap\mathbb{R}^1$ $\exists\frac{\Im l_\mu}{\Im\mu} \overset{def}{=} \frac{\Im l_{\mu+i0}}{\Im(\mu+i0)} = \frac{\partial l_\lambda}{\partial\lambda}\big|_{\lambda=\mu}$, where the coefficients $\frac{\partial p_j(t,\mu)}{\partial\lambda}$, $\frac{\partial q_j(t,\mu)}{\partial\lambda}$, $\frac{\partial s_j(t,\mu)}{\partial\lambda}$ of expression $\partial l_\mu/\partial\mu$ are Bochner locally integrable in the uniform operator topology.

Also in view of Theorem 1.2 and Lemma 1.1 from [28] one has

$$(51)\qquad 0 \leq W(t,l_\lambda,m) \leq W\left(t, l_\lambda, -\frac{\Im l_\lambda}{\Im\lambda}\right) = \frac{\Im H(t,l_\lambda)}{\Im\lambda}, \quad t \in \bar{\mathcal{I}}, \quad \Im\lambda \neq 0.$$

Let us consider in $\mathcal{H}_1 = \mathcal{H}^r$ the equation

$$(52)\quad \frac{i}{2}\Big(\left(Q(t,l_\lambda)\,\vec{y}(t)\right)' + Q^*(t,l_\lambda)\,\vec{y}'(t)\Big) - H(t,l_\lambda)\,\vec{y}(t) = W\left(t, l_\lambda, -\frac{\Im l_\lambda}{\Im\lambda}\right) F(t).$$

This equation is an equation of (5) type due to (47), (51). Equation (4) is equivalent to equation (52) with $F(t) = F\left(t, l_{\bar\lambda}, -\frac{\Im l_\lambda}{\Im\lambda}\right)$ due to Theorem 1.3 since $W(t,l_\lambda,-\Im l_\lambda/\Im\lambda) = W(t,l_{\bar\lambda},-\Im l_\lambda/\Im\lambda)$ in view of (33), (47), (51).

**Definition 1.4.** [28] Every characteristic operator of equation (52) corresponding to the equation (4) is said to be a characteristic operator of equation (4) on $\mathcal{I}$.

In some cases we will suppose additionally that

$$\begin{gathered}\exists\lambda_0 \in \mathcal{B};\ \alpha,\beta \in \bar{\mathcal{I}},\ 0 \in [\alpha,\beta]\,, \text{ the number } \delta > 0 : \\ -\int_\alpha^\beta \left(\frac{\Im l_{\lambda_0}}{\Im \lambda_0}\right)\{y\,(t,\lambda_0)\,, y\,(t,\lambda_0)\}\; dt \geq \delta \left\| P\vec{y}\,(0, l_{\lambda_0}, m, 0)\right\|^2\end{gathered} \tag{53}$$

for any solution $y\,(t,\lambda_0)$ of (1) as $\lambda = \lambda_0,\ f = 0$, where $P \in B(\mathcal{H}^r)$ is the orthoprojection onto subspace $N^\perp$ which corresponds to equation (52). In view of Theorem 1.2 from [28] this condition is equivalent to the fact that for the equation (52)

$$\begin{gathered}\exists\lambda_0 \in \mathcal{A} = \mathcal{B};\ \alpha,\beta \in \bar{\mathcal{I}},\ 0 \in [\alpha,\beta]\,, \text{ the number } \delta > 0 : \\ (\Delta_{\lambda_0}\,(\alpha,\beta)\,g,g) \geq \delta \left\|Pg\right\|^2, \quad g \in \mathcal{H}^r.\end{gathered}$$

Therefore in view of [24] the fulfillment of (53) implies its fulfillment with $\delta\,(\lambda) > 0$ instead of $\delta$ for all $\lambda \in \mathcal{B}$.

Let us notice that in view of (36) $l_\lambda$ can be a represented in form (2) where

$$l = \Re l_i, \quad n_\lambda = l_\lambda - l - \lambda m; \quad \Im n_\lambda \{f,f\}\,/\Im\lambda \geq 0, \quad t \in \bar{\mathcal{I}}, \quad \Im\lambda \neq 0. \tag{54}$$

From now on we suppose that $l_\lambda$ has a representation (2), (54) and therefore the order of $n_\lambda$ is even.

We consider pre-Hilbert spaces $\overset{\circ}{H}$ and $H$ of vector-functions $y\,(t) \in C_0^s\left(\bar{\mathcal{I}},\mathcal{H}\right)$ and $y\,(t) \in C^s\left(\bar{\mathcal{I}},\mathcal{H}\right)$, $m\,[y\,(t)\,,\, y\,(t)] < \infty$ correspondingly with a scalar product

$$(f\,(t)\,,\, g\,(t))_m = m\,[f\,(t)\,,\, g\,(t)]\,,$$

where

$$m\,[f,\, g] = \int\limits_{\mathcal{I}} m\,\{f,\, g\}\, dt. \tag{55}$$

Here $m\,\{f,\, g\}$ is defined by (35) with expression $m[y]$ from condition (36) instead of $L[y]$. Namely,

$$\begin{aligned} m\,\{f,\, g\} = &\sum_{j=0}^{s/2} (\tilde{p}_j\,(t)\, f^{(j)}(t), g^{(j)}(t)) \\ &+ \frac{i}{2}\sum_{j=1}^{s/2}\Big((\tilde{q}_j^*\,(t)\, f^{(j)}(t),\, g^{(j-1)}(t)) - (\tilde{q}_j\,(t)\, f^{(j-1)}(t),\, g^{(j)}(t))\Big)\,.\end{aligned}$$

By $\overset{\circ}{L}{}^2_m\,(\mathcal{I})$ and $L^2_m\,(\mathcal{I})$ we denote the completions of spaces $\overset{\circ}{H}$ and $H$ in the norm $\|\bullet\|_m = \sqrt{(\bullet,\,\bullet)_m}$ correspondingly. By $\overset{\circ}{P}$ we denote the orthoprojection in $L^2_m\,(\mathcal{I})$ onto $\overset{\circ}{L}{}^2_m\,(\mathcal{I})$.

**Theorem 1.4.** [28] *Let $M\,(\lambda)$ be a characteristic operator of equation* (4)*, for which the condition* (53) *with $P = I_r$ holds if $\mathcal{I}$ is infinite. Let $\Im\lambda \neq 0$, $f\,(t) \in H$ and*

$$\begin{aligned} &\operatorname{col}\{y_j\,(t,\lambda,f)\} \\ &= \int_{\mathcal{I}} X_\lambda\,(t)\left\{M\,(\lambda) - \frac{1}{2}\operatorname{sgn}(s-t)\,(iG)^{-1}\right\}X^*_{\bar\lambda}\,(s)\, W\,(s, l_{\bar\lambda}, m)\, F\,(s, l_{\bar\lambda}, m)\; ds, \\ &\hspace{26em} y_j \in \mathcal{H},\end{aligned} \tag{56}$$

*be a solution of equation* (49)*, that corresponds to equation* (1)*, where $X_\lambda\,(t)$ is the operator solution of homogeneous equation* (49) *such that $X_\lambda\,(0) = I_r$; $G = \Re Q\,(0, l_\lambda)$*

*(if $\mathcal{I}$ is infinite integral* (56) *converges strongly). Then the first component of vector function* (56) *is a solution of equation* (1)*. It defines densely defined in $L^2_m(\mathcal{I})$ integro-differential operator*

$$R(\lambda) f = y_1(t,\lambda,f), \quad f \in H, \tag{57}$$

*which has the following properties after closing:*

*1°*

$$R^*(\lambda) = R(\bar{\lambda}), \quad \Im\lambda \neq 0; \tag{58}$$

*2°*

$$R(\lambda) \quad \text{is holomorphic on} \quad \mathbb{C}\setminus\mathbb{R}^1; \tag{59}$$

*3°*

$$\|R(\lambda) f\|^2_{L^2_m(\mathcal{I})} \leq \frac{\Im (R(\lambda) f, f)_{L^2_m(\mathcal{I})}}{\Im\lambda}, \quad \Im\lambda \neq 0, \quad f \in L^2_m(\mathcal{I}). \tag{60}$$

Let us notice that the definition of the operator $R(\lambda)$ is correct. Indeed if $f(t) \in H$, $m[f,f] = 0$, then $R(\lambda) f \equiv 0$ since $W(t, l_{\bar{\lambda}}, m) F(t, l_{\bar{\lambda}}, m) \equiv 0$ due to (51) and Theorem 1.2 from [28].

Also let us notice that if $L^2_m(\mathcal{I}) = \overset{\circ}{L}{}^2_m(\mathcal{I})$ then Theorem 1.4 is valid with $f(t) \in \overset{\circ}{H}$ instead of $f(t) \in H$ and without condition (53) with $P = I_r$ if $\mathcal{I}$ is infinite.

The resolvent $R(\lambda)$ can be represented in another forms (see Remarks 3.1, 3.2 from [28]). The following proposition is the generalization of Remark 3.2 from [28].

**Proposition 1.2.** *Let $r = 2n$, $\mathcal{I} = (0,b)$, $b \leq \infty$, condition* (53) *hold with $P = I_r$. (Therefore for equation* (52) *condition* (8) *holds.) Let for characteristic operator $M(\lambda)$ of equation* (4) *condition* (7) *be separated. (Therefore $M(\lambda)$ has representation* (10) *where characteristic projection $\mathcal{P}(\lambda)$ can be represented in the form* (12)*,* (13) *with the help of some Nevanlinna pair $\{-a(\lambda), b(\lambda)\}$ and some Weyl function $m(\lambda)$ of equation* (52)*; this equation with $F(t) = 0$ has an operator solutions $U_\lambda(t)$, $V_\lambda(t)$* (16)*-*(18)*). Let domains $D, D_1$ be the same as in Proposition 1.1. Then $R(\lambda)f$* (57) *for $\lambda \in D \bigcup D_1$ can be represented in the form*

$$R(\lambda) f = \int_0^t \sum_{j=1}^n v_j(t,\lambda) \sum_{k=0}^{s/2} \left(u_j^{(k)}(s,\bar{\lambda})\right)^* \mathrm{m}_k[f(s)]\, ds$$
$$+ \int_t^b \sum_{j=1}^n u_j(t,\lambda) \sum_{k=0}^{s/2} \left(v_j^{(k)}(s,\bar{\lambda})\right)^* \mathrm{m}_k[f(s)]\, ds,$$

*where the integrals converge strongly if the interval of integration is infinite. Here $u_j(t,\lambda)$, $v_j(t,\lambda) \in B(\mathcal{H})$ are operator solutions of equation* (1) *as $f = 0$, such that*

$$\begin{aligned} &(u_1(t,\lambda), \dots u_n(t,\lambda)) = [X_\lambda(t)]_1 \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix}, \\ &(v_1(t,\lambda), \dots, v_n(t,\lambda)) \\ &\quad = [X_\lambda(t)]_1 \begin{pmatrix} b(\lambda) \\ -a(\lambda) \end{pmatrix} K^{-1}(\lambda) + (u_1(t,\lambda), \dots, u_n(t,\lambda))\, m_{a,b}(\lambda), \end{aligned} \tag{61}$$

*$K(\lambda)$, $m_{a,b}(\lambda)$ see* (17)*,* (18)*,*

$$\begin{aligned} \mathrm{m}_k[f(s)] = \tilde{p}_k(s) f^{(k)}(s) + \frac{i}{2}\left(\tilde{q}^*_{k+1}(s) f^{(k+1)}(s) - \tilde{q}_k(s) f^{(k-1)}(s)\right) \\ \left(\tilde{q}_0 \equiv 0, \quad \tilde{q}_{\frac{s}{2}+1} \equiv 0\right), \end{aligned} \tag{62}$$

$$\| \left(v_1\left(t,\lambda\right),\dots,v_n\left(t,\lambda\right)\right) h\|_m^2 \le \frac{\Im(m(\lambda)g,g)}{\Im\lambda}, \quad \Im\lambda \neq 0,$$

*where* $g=(b^*(\bar\lambda)-a^*(\bar\lambda)m(\lambda))^{-1}h$, $h\in\mathcal{H}^n$ *and therefore*

$$\left(v_1\left(t,\lambda\right),\dots,v_n\left(t,\lambda\right)\right) h \in L_m^2\left(\mathcal{I}\right) \quad \forall h\in\mathcal{H}^n.$$

*Moreover if* $a\left(\lambda\right)=a\left(\bar\lambda\right)$, $b\left(\lambda\right)=b\left(\bar\lambda\right)$ *as* $\Im\lambda\neq 0$ *then we can set* $D=\mathbb{C}_+$ *and*

$$\|\left(v_1\left(t,\lambda\right),\dots,v_n\left(t,\lambda\right)\right) h\|_m^2 \le \frac{\Im\left(m_{a,b}\left(\lambda\right)h,h\right)}{\Im\lambda}, \quad \Im\lambda\neq 0. \tag{63}$$

*Let the contructed* $v(\lambda)\in B(\mathcal{H}^n)$ *satisfy the conditions of Lemma 1.2 and domains* $D$, $D_1$ *be the same as in Lemma 1.2. Then corresponding solution* $\left(v_1\left(t,\lambda\right),\dots,v_n\left(t,\lambda\right)\right)$ (61)*,* (17)*),* (18)*,* (27)*,* (28) *satisfies inequality* (63) *(*$\left(v_1\left(t,\lambda\right),\dots,v_n\left(t,\lambda\right)\right)\overset{\text{def}}{=}[V_\lambda(t)]_1$, $\lambda\notin D\cup D_1$*, where* $[V_\lambda(t)]_1\in B(\mathcal{H}^n,\mathcal{H})$ *is an analogue of* $[X_\lambda(t)]_1$ *for* $V_\lambda(t)$ (22)*).*

*Proof.* The proof follows from Proposition 1.1, Corollary 1.1, Theorem 1.4 and also Theorem 1.2 from [28]. □

Comparison of Theorem 1.4, Propositions 1.2 (in less complete form) with results for various particular cases see in [28].

## 2. Eigenfunction expansions

It is known [13] that (58)–(60) implies (3), where $E_\mu\in B\left(L_m^2\left(\mathcal{I}\right)\right)$, $E_\mu=E_{\mu-0}$,

$$0\le E_{\mu_1}\le E_{\mu_2}\le \mathbf{I}, \quad \mu_1<\mu_2; \quad E_{-\infty}=0. \tag{64}$$

Here $\mathbf{I}$ is the identity operator in $L_m^2\left(\mathcal{I}\right)$. We denote $E_{\alpha\beta}=\frac{1}{2}\left[E_{\beta+0}+E_\beta-E_{\alpha+0}-E_\alpha\right]$.

**Theorem 2.1.** *Let* $M\left(\lambda\right)$ *be the characteristic operator of equation* (4) *(and therefore by* [24, p. 162] $\Im PM\left(\lambda\right)P/\Im\lambda\ge 0$ *as* $\Im\lambda\neq 0$*) and* $\sigma\left(\mu\right)=w-\lim\limits_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_0^\mu \Im PM\left(\mu+i\varepsilon\right)Pd\mu$ *be the spectral operator-function that corresponds to* $PM\left(\lambda\right)P$.

*Let the condition* (53) *with* $P=I_r$ *hold if* $\mathcal{I}$ *is infinite. Let* $E_\mu$ *be generalized spectral family* (64) *corresponding by* (3) *to the resolvent* $R\left(\lambda\right)$ *from Theorem 1.4 which is constructed with the help of characteristic operator* $M\left(\lambda\right)$*. Let* $\mathcal{B}^1=\mathcal{B}\cap\mathbb{R}^1$*. Then for any* $[\alpha,\beta]\subset\mathcal{B}^1$ *the equalities*

$$\begin{aligned} \overset{\circ}{P}E_{\alpha,\beta}f\left(t\right)&=\overset{\circ}{P}\int_\alpha^\beta\left[X_\mu\left(t\right)\right]_1 d\sigma\left(\mu\right)\varphi\left(\mu,f\right), \quad \text{if } f\left(t\right)\in\overset{\circ}{H},\ \mathcal{I} \text{ is infinite},\\ E_{\alpha,\beta}f\left(t\right)&=\int_\alpha^\beta\left[X_\mu\left(t\right)\right]_1 d\sigma\left(\mu\right)\varphi\left(\mu,f\right), \quad \text{if } f\left(t\right)\in H,\ \mathcal{I} \text{ is finite}\end{aligned} \tag{65}$$

*are valid in* $L_m^2\left(\mathcal{I}\right)$*, where* $\left[X_\lambda\left(t\right)\right]_1\in B\left(\mathcal{H}^r,\mathcal{H}\right)$ *is the first row of the operator solution* $X_\lambda\left(t\right)$ *of homogeneous equation* (49) *which is written in the matrix form and such that* $X_\lambda\left(0\right)=I_r$,

$$\varphi\left(\mu,f\right)=\begin{cases}\displaystyle\int_{\mathcal{I}}\left(\left[X_\mu\left(t\right)\right]_1\right)^* m\left[f\right]dt, & \text{if } f\left(t\right)\in\overset{\circ}{H},\\ \displaystyle\int_{\mathcal{I}}\left(\left[X_\mu\left(t\right)\right]_1\right)^* W\left(t,l_\mu,m\right)F\left(t,l_\mu,m\right)dt, & \text{if } f\left(t\right)\in H, \mathcal{I} \text{ is finite}\\ & \text{or } f(t)\in\overset{\circ}{H},\end{cases} \tag{66}$$

$\mu\in[\alpha,\beta]$.

*Moreover, if vector-function $f(t)$ satisfy the following conditions:*

$$\begin{aligned} &\overset{\circ}{P} E_\infty f = f, \quad \overset{\circ}{P}\int_{\mathbb{R}^1\setminus\mathcal{B}^1} dE_\mu f = 0, \qquad \text{if } f\in \overset{\circ}{H},\ \mathcal{I} \text{ is infinite,} \\ &E_\infty f = f, \quad \int_{\mathbb{R}^1\setminus\mathcal{B}^1} dE_\mu f = 0, \quad \text{if } f\in H,\ \mathcal{I} \text{ is finite} \end{aligned} \tag{67}$$

*then the inversion formulae in $L_m^2(\mathcal{I})$*

$$\begin{aligned} f(t) &= \overset{\circ}{P}\int_{\mathcal{B}^1} [X_\mu(t)]_1\, d\sigma(\mu)\,\varphi(\mu,f), \qquad \text{if } f(t)\in \overset{\circ}{H},\ \mathcal{I} \text{ is infinite,} \\ f(t) &= \int_{\mathcal{B}^1} [X_\mu(t)]_1\, d\sigma(\mu)\,\varphi(\mu,f), \qquad \text{if } f(t)\in H,\ \mathcal{I} \text{ is finite} \end{aligned} \tag{68}$$

*and Parceval's equality*

$$m[f,g] = \int_{\mathcal{B}^1} (d\sigma(\mu)\,\varphi(\mu,f),\varphi(\mu,g)) \tag{69}$$

*are valid, where $g(t)\in \overset{\circ}{H}$ if $\mathcal{I}$ is infinite or $g(t)\in H$ if $\mathcal{I}$ is finite.*

*In general case for $f(t), g(t)\in \overset{\circ}{H}$ if $\mathcal{I}$ is infinite or $f(t), g(t)\in H$ if $\mathcal{I}$ is finite, the inequality of Bessel type*

$$m[f(t),g(t)] \le \int_{\mathcal{B}^1} (d\sigma(\mu)\,\varphi(\mu,f)\,,\,\varphi(\mu,g)) \tag{70}$$

*is valid.*

Let us notice that $\mathcal{B}^1 = \cup_k (a_k,b_k)$, $(a_j,b_j)\cap(a_k,b_k)=\varnothing$, $k\ne j$ since $\mathcal{B}^1$ is an open set. In (68) $\overset{\circ}{P}\int_{\mathcal{B}^1} = \sum_k \lim\limits_{\alpha_k\downarrow a_k,\beta_k\uparrow b_k} \overset{\circ}{P}\int_{\alpha_k}^{\beta_k}$. In (68)–(70) we understand $\int_{\mathcal{B}^1}$ similarly.

*Proof.* Let for definiteness $r=s=2n$, $\mathcal{I}$ is infinite (for another cases the proof becomes simpler). Let the vector-functions $f(t), g(t)\in \overset{\circ}{H}$, $\lambda=\mu+i\varepsilon$, $G(t,l_\lambda,m)$ be defined by (48) with $g(t)$ instead of $f(t)$. In view of the Stieltjes inversion formula, we have

$$\begin{aligned} (E_{\alpha,\beta}f,g)_{L_m^2(\mathcal{I})} &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\alpha^\beta \left(\left[y_1(t,\lambda,f)-y_1\left(t,\bar\lambda,f\right)\right],g\right)_m d\mu \\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\alpha^\beta \Big[ (\vec{y}_1(t,l_\lambda,m,f), G(t,l_\lambda,m))_{L^2_{W(t,l_\lambda,m)}(\mathcal{I})} \\ &\quad - (\vec{y}_1(t,l_{\bar\lambda},m,f), G(t,l_{\bar\lambda},m))_{L^2_{W(t,l_{\bar\lambda},m)}(\mathcal{I})} \\ &\quad + 2i\int_{\mathcal{I}} \Big(\left(\Im p_n^{-1}(t,\lambda)\right) f^{[n]}(t\,|m)\,, g^{[n]}(t\,|m)\Big)\,dt\Big] d\mu \\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_\alpha^\beta \Big[\Big(M(\lambda)\int_{\mathcal{I}} X_{\bar\lambda}^*(t)\,W(t,l_{\bar\lambda},m)\,F(t,l_{\bar\lambda},m)\,dt, \\ &\qquad\qquad \int_{\mathcal{I}} X_\lambda^*(t)\,W(t,l_\lambda,m)\,G(t,l_\lambda,m)\,dt\Big) \end{aligned} \tag{71}$$

$$- \Big(M^* (\lambda) \int_{\mathcal{I}} X^*_{\lambda} (t) \, W (t, l_{\lambda}, m) \, F (t, l_{\lambda}, m) \, dt,$$
$$\int_{\mathcal{I}} X^*_{\bar\lambda} (t) \, W (t, l_{\bar\lambda}, m) \, G (t, l_{\bar\lambda}, m) \, dt\Big)\Big] d\mu$$
$$= \int_{\alpha}^{\beta} \Big(d\sigma (\mu) \int_{\mathcal{I}} X^*_{\mu} (t) \, W (t, l_{\mu}) \, F (t, l_{\mu}, m) \, dt,$$
$$\int_{\mathcal{I}} X^*_{\mu} (t) \, W (t, l_{\mu}) \, G (t, l_{\mu}, m) \, dt\Big),$$

where $\vec{y}_1(t, l_\lambda, m, f)$ is defined by (50) with $y_1(t, \lambda, f)$ (57) instead of $y(t)$; the second equality is a corollary of formula (40) from [28], the next to last is a corollary of (56) and the last one follows from the well-known generalization of the Stieltjes inversion formula [40, p. 803], [8, p. 952]. (In the case of finite $\mathcal{I}$ we have to substitute in (71) $M(\lambda)$ by $PM(\lambda)P$ and then when passing to the next to the last equality in (71) we have to use the remark after the proof of Lemma 2.1 from [28].) But for $\lambda \in \mathcal{B}$

$$\int_{\mathcal{I}} X^*_{\bar\lambda} (t) \, W (t, l_{\bar\lambda}, m) \, F (t, l_{\bar\lambda}) \, dt = \int_{\mathcal{I}} ([X_{\bar\lambda} (t)]_1)^* \, m \, [f] \, dt, \tag{72}$$

because in view of Theorem 2.1 from [28]

$$\forall h \in \mathcal{H}^r : (\int_{\mathcal{I}} X^*_{\bar\lambda} (t) \, W (t, l_{\bar\lambda}, m) \, F (t, l_{\bar\lambda}) \, dt, h)$$
$$= \int_{\mathcal{I}} (W (t, l_{\bar\lambda}, m) \, F (t, l_{\bar\lambda}) \, , X_{\bar\lambda} (t) \, h) \, dt = (\int_{\mathcal{I}} ([X_{\bar\lambda}]_1)^* \, m \, [f] \, , h) \, dt.$$

Due to (71), (72), (66)

$$(E_{\alpha,\beta} f, g)_{L^2_m(\mathcal{I})} = \int_{\alpha}^{\beta} (d\sigma (\mu) \, \varphi (\mu, f) \, , \varphi (\mu, g)) \, . \tag{73}$$

The equality (69) and inequality (70) are the corollaries of (73).

Representing $\varphi (\mu, g)$ in (73) by the second variant of (66), changing in (73) the order of integration and replacing $\int_\alpha^\beta$ by integral sum and using Theorem 2.1 from [28] we obtain that

$$(E_{\alpha,\beta} f, g)_{L^2_m(\mathcal{I})} = (\int_{\alpha}^{\beta} [X_{\mu} (t)]_1 \, d\sigma (\mu) \, \varphi (\mu, f) \, , g (t))_{L^2_m(\mathcal{I})}$$
$$= (\overset{\circ}{P} \int_{\alpha}^{\beta} [X_{\mu} (t)]_1 \, d\sigma (\mu) \, \varphi (\mu, f) \, , g (t))_{L^2_m(\mathcal{I})}$$

since $g(t) \in \overset{\circ}{H}$ and (65) is proved. Equalities (68) are the corollary of (65), (67). Theorem 2.1 is proved. □

Let us notice that if $L^2_m(\mathcal{I}) = \overset{\circ}{L}{}^2_m(\mathcal{I})$ then Theorem 2.1 is valid without condition (53) with $P = I_r$ if $\mathcal{I}$ is infinite.

Formulae (65), (68), (69) are similar to corresponding formulas for scalar differential operators from [40] (operator case see [8]). For such operators the formulas that corresponds to (65), (68), (69) are represented in [14, p. 251, 255], [32, p. 516] in another form. Let us represent for example inversion formula (68), in the form analogues to [14, 32].

**Proposition 2.1.** *Let all conditions of Theorem 2.1 hold. Let us represent spectral operator-function $\sigma (\mu)$ in matrix form: $\sigma (\mu) = \|\sigma_{ij} (\mu)\|^r_{i,j=1}$, $\sigma_{ij} (\mu) \in B (\mathcal{H})$. Then* *the following inversion formulae in* $L_m^2(\mathcal{I})$

$$f(t) = \overset{\circ}{P} \int_{\mathcal{B}^1} \sum_{i,j=1}^{r} x_i(t,\mu)\, d\sigma_{ij}(\mu) \int_{\mathcal{I}} x_j^*(s,\mu)\, m[f(s)]\, ds, \tag{74}$$

$$f(t) = \overset{\circ}{P} \int_{\mathcal{B}^1} \sum_{i,j=1}^{r} x_i(t,\mu)\, d\sigma_{ij}(\mu) \int_{\mathcal{I}} \sum_{k=0}^{s/2} \left(x_j^{(k)}(s,\mu)\right)^* \mathrm{m}_k[f(s)]\, ds \tag{75}$$

*are valid if* $\mathcal{I}$ *is infinite, vector function* $f(t) \in \overset{\circ}{H}$ *satisfies* (67). *Let* $\mathcal{I}$ *is finite. Then* $\overset{\circ}{P}$ *in* (74)–(75) *disappears, and* (74) *(respectively,* (75)*) is valid for vector functions* $f(t) \in \overset{\circ}{H}$ *(respectively,* $f(t) \in H$*) satisfying* (67). *In formulae* (74), (75)*:* $x_i(t,\mu) \in B(\mathcal{H})$, $(x_1(t,\mu), \ldots, x_r(t,\mu)) = [X_\mu(t)]_1$, $\mathrm{m}_k[f(s)]$ *see* (62).

The proof of this proposition is carried out in the same way as the proof of (68) taking into account the proof of Remark 3.1 from [28].

Further we present several statements (Theorem 2.2, Proposition 2.3) which allow to check the fulfillment of conditions (67) of Theorem 2.1 in various situations.

It is known (see for example [21, 22] or [28, Ex. 3.2]) that even in the case $n_\lambda[y] \equiv 0$ in (1), (2) there is such $E_\mu$ satisfying (3), (56)–(60), (64) that $E_\infty \neq \mathbf{I}$.

On the other hand if $n_\lambda[y] \equiv 0$ then $R(\lambda)$ is a generalized resolvent of relation $\mathcal{L}_0$ and $\forall f \in D(\mathcal{L}_0)\, E_\infty f = f$ in view of [20, 22]. Here $\mathcal{L}_0$ is the minimal relation generated in $L_m^2(\mathcal{I})$ by the pair of expressions $l[y]$ and $m[y]$; in particular $\mathcal{L}_0 \supset \{\{y(t), f(t)\} :\ y(t) \in C_0^r(\mathcal{I}), f(t) \in \overset{\circ}{H}, l[y] = m[f]\}$ (see [26, 28]).

Let expression $n_\lambda$ in representation (2), (54) have a divergent form with coefficients $\tilde{\tilde{p}}_j = \tilde{\tilde{p}}_j(t,\lambda), \tilde{\tilde{q}}_j = \tilde{\tilde{q}}_j(t,\lambda), \tilde{\tilde{s}}_j = \tilde{\tilde{s}}_j(t,\lambda)$.

We denote $m(t)$ three-diagonal $(n+1)\times(n+1)$ operator matrix, whose elements under main diagonal are equal to $\left(-\frac{i}{2}\tilde{q}_1, \ldots, -\frac{i}{2}\tilde{q}_n\right)$, the elements over the main diagonal are equal to $\left(\frac{i}{2}\tilde{s}_1, \ldots, \frac{i}{2}\tilde{s}_n\right)$, the elements on the main diagonal are equal to $(\tilde{p}_0, \ldots, \tilde{p}_n)$, where $\tilde{p}_j, \tilde{q}_j, \tilde{s}_j = \tilde{q}_j^*$ are the coefficients of expressions $m$. (Here either $2n$ or $2n+1$ is equal to the order $r$ of $l_\lambda$). If order of $n_\lambda$ is less or equal to $2n$, we denote $n(t,\lambda)$ the analogues $(n+1)\times(n+1)$ operator matrix with $\tilde{\tilde{p}}_j, \tilde{\tilde{q}}_j, \tilde{\tilde{s}}_j$ instead of $\tilde{p}_j, \tilde{q}_j, \tilde{s}_j$. If order $m$ or order $n_\lambda$ is less than $2n$, we set the correspondent elements of $m(t)$ or $n(t,\lambda)$ be equal to zero.

**Theorem 2.2.** *Let in* (1), (2) *the order of the expression* $n_\lambda[y]$ *is less or equal to the order of the expression* $(l-\lambda m)[y]$ *(and therefore in view of* (54) *the order of* $l-\lambda m$ *is equal to* $r$*; so* $Q(t,l_\lambda) = Q(t,l-\lambda m)$*). Let* $y = R_\lambda f$, $f \in H$ *be the generalized resolvent of the minimal relation* $\mathcal{L}_0$ *generated in* $L_m^2(\mathcal{I})$ *by the pair of expressions* $l[y]$, $m[y]$ *and let* $y$ *satisfy equation* (1). *Let* $y_1 = R(\lambda) f$, $f \in H$ *be the operator* (56), (57) *from Theorem 1.4.*

*Let the following conditions hold for* $\tau > 0$ *large enough:*

*1°.*

$$\lim_{\alpha\downarrow a,\, \beta\uparrow b} \frac{([\Re Q(t,l_\lambda)](\vec{y}_1(t,l_\lambda,m,f) - \vec{y}(t,l{-}\lambda m,m,f)), (\vec{y}_1(t,l_\lambda,m,f)) - \vec{y}(t,l{-}\lambda m,m,f))}{\Im\lambda}\Bigg|_\alpha^\beta \leq 0, \quad \lambda = i\tau. \tag{76}$$

*2°.*

$$\Im n(\mathrm{t},\lambda) \leq c(t,\tau)\, m(t), \quad t \in \bar{\mathcal{I}}, \quad \lambda = i\tau, \tag{77}$$

*where the scalar function* $c(t,\tau)$ *satisfies the following condition:*

$$\sup_{t\in\bar{\mathcal{I}}} c(t,\tau) = o(\tau), \quad \tau \to +\infty. \tag{78}$$

*Then for generalized spectral family* $E_\mu$ (64) *corresponding by* (3) *to the resolvent* $R(\lambda)$ (56)–(57) *from Theorem 1.4 and for generalized spectral family* $\mathcal{E}_\mu$ *corresponding to the generalized resolvent* $R_\lambda$ *one has* $E_\infty = \mathcal{E}_\infty$.

Let us notice that in view of (77) the coefficient at the highest derivative in the expression $l - \lambda m$ has inverse from $B(\mathcal{H})$ if $t \in \bar{\mathcal{I}}$, $\Im\lambda \neq 0$.

*Proof.* Let $f(t) \in H$, $y_1 = R(\lambda) f$, $y = R_\lambda f$. Then $z = y_1 - y$ satisfies the following equation:

$$l[z] - \lambda m[z] = n_\lambda [y_1]. \tag{79}$$

Applying to the equation (79) the Green formula [28, Theorem 1.3] one has

$$\int_\alpha^\beta \Im(n_\lambda\{y_1, z\})\, dt + \int_\alpha^\beta m\{z, z\}\, dt = \frac{1}{2}\frac{\Re(Q(t, l_\lambda)\vec{z}, \vec{z})}{\Im\lambda}\bigg|_\alpha^\beta,$$

where $\vec{z} = \vec{z}(t, l - \lambda m, n_\lambda, y_1) = \vec{y}_1(t, l_\lambda, m, f) - \vec{y}(t, l - \lambda m, m, f)$ in view Lemma 1.2 from [28] and of (50). Hence for $\tau > 0$ large enough

$$\begin{aligned} m[z, z] &\leq -\int_{\mathcal{I}} \Im(n_\lambda[y_1, z])\, dt/\tau \\ &\leq \int_{\mathcal{I}} \left|\left(n(\mathrm{t},\lambda)\operatorname{col}\left\{y_1, y_1', \ldots, y_1^{(n)}\right\}, \operatorname{col}\left\{z, z', \ldots, z^{(n)}\right\}\right)\right| dt\Big/\tau, \quad \lambda = i\tau \end{aligned} \tag{80}$$

in view of (76). But due to the inequality of the Cauchy type for dissipative operators [41, p. 199] and (77), (78): subintegral function in the last integral in (80) is less or equal to $(m\{z, z\})^{1/2}(m\{y_1, y_1\})^{1/2} o(1)$ with $\lambda = i\tau$, $\tau \to +\infty$. Therefore $\|z\|_m \leq o(1/\tau)\|f\|_m$ since $\|R_\lambda\| \leq 1/|\Im\lambda|$. Hence

$$\|R(\lambda) - R_\lambda\| \leq o(1/\tau), \quad \lambda = i\tau, \quad \tau \to +\infty.$$

To complete the proof of the theorem it remains to prove the following

**Lemma 2.1.** *Let* $R_k(\lambda) = \int_{\mathbb{R}^1} \frac{dE_\mu^k}{\mu - \lambda}$, $k = 1, 2$, *where* $E_\mu^k$ *are the generalized spectral families the type* (64) *in Hilbert space* $\mathbf{H}$. *If* $\|R_1(\lambda) - R_2(\lambda)\| \leq o(1/\tau)$, $\lambda = i\tau$, $\tau \to +\infty$, *then* $E_\infty^1 = E_\infty^2$.

*Proof.* Let $f \in \mathbf{H}$ $\sigma(\mu) = \left(\left(E_\mu^1 - E_\mu^2\right) f, f\right)$. One has

$$\begin{aligned} &|([R_1(\lambda) - R_2(\lambda)] f, f)| \\ &= \frac{1}{\tau}\left| -\int_\Delta d\sigma(\mu) + \int_\Delta \frac{\mu d\sigma(\mu)}{\mu - \lambda} + \int_{R^1\setminus\Delta} \frac{\lambda d\sigma(\mu)}{\mu - \lambda}\right| \leq o(1/\tau)\|f\|^2, \\ &\qquad \lambda = i\tau, \quad \tau \to +\infty. \end{aligned}$$

Therefore

$$\left| -\int_\Delta d\sigma(\mu) + \int_\Delta \frac{\mu d\sigma(\mu)}{\mu - \lambda} + \int_{R^1\setminus\Delta} \frac{\lambda d\sigma(\mu)}{\mu - \lambda}\right| \leq o(1), \quad \lambda = i\tau, \quad \tau \to +\infty. \tag{81}$$

For an arbitrarily small $\varepsilon > 0$ we choose such finite interval $\Delta(\varepsilon)$ that for any finite interval $\Delta \supseteq \Delta(\varepsilon)$ : $\left|\int_{R^1\setminus\Delta} \frac{\lambda d\sigma(\mu)}{\mu - \lambda}\right| < \frac{\varepsilon}{2}$, $\lambda = i\tau$. But for any finite interval $\Delta \supseteq \Delta(\varepsilon)$ $\exists N = N(\Delta) : \forall\tau > N : \left|\int_\Delta \frac{\mu d\sigma(\mu)}{\mu - \lambda}\right| < \frac{\varepsilon}{2}$, $\lambda = i\tau$. Therefore $\forall\varepsilon > 0, \Delta \supseteq \Delta(\varepsilon)$ : $\left|\int_\Delta d\sigma(\mu)\right| < \varepsilon$ in view of (81). Hence $\forall f \in \mathbf{H}$: $\left(E_\infty^1 f, f\right) = \left(E_\infty^2 f, f\right) \Rightarrow E_\infty^1 = E_\infty^2$.

Lemma 2.1 and Theorem 2.2 are proved. □

**Corollary 2.1.** *Let the conditions of Theorems 2.1, 2.2 hold. Then for generalized spectral family $E_\mu$ from Theorem 2.1 $\forall f(t) \in D(\mathcal{L}_0) : E_\infty f = f$.*

**Remark 2.1.** *If $L^2_m(\mathcal{I}) = \overset{\circ}{L}{}^2_m(\mathcal{I})$, then it is sufficient to verify condition* (76) *in Theorem 2.2 for $f \in \overset{\circ}{H}$.*

In regular case condition (76) in Theorem 2.2 holds if $R_\lambda f$ and $R(\lambda)f$ satisfy "the same" boundary conditions. More precisely, the following statement holds.

**Proposition 2.2.** *Let interval $\mathcal{I}$ be finite. Let the order of expression $n_\lambda[y]$ be less or equal to the order of expression $(l-\lambda m)[y]$ and the coefficient $p(t,\lambda)$ of $l-\lambda m$ at the highest derivative has the inverse from $B(\mathcal{H})$ for $\Im\lambda \neq 0$, $t \in \bar{\mathcal{I}}$. Let for equation* (1), (2) *with $n_\lambda[y] \equiv 0$ there exist $\lambda_0 \in \mathbb{C}$, $\alpha, \beta \in \bar{\mathcal{I}}$, number $\delta > 0$ such that $0 \in [\alpha,\beta]$, $p^{-1}(t,\lambda_0) \in B(\mathcal{H})$ for $t \in [\alpha,\beta]$ and condition* (53) *holds with $P = I_r$ and these $\lambda_0$, $\alpha$, $\beta$.*

*Then for equation* (1)–(2) *condition* (53) *with $P = I_r$ and these $\alpha, \beta$ also holds for any $\lambda_0 \in \mathcal{B}$ with $\delta(\lambda_0) > 0$ instead of $\delta$.*

*For an arbitrary $f(t) \in H$ the boundary value problem which is obtained by adding to equation* (1)–(2) *with $n_\lambda[y] \equiv 0$ (respectively,* (1)–(2)*) boundary conditions*

(82) $$\exists h = h(\lambda, f) \in \mathcal{H}^r : \ \vec{y}(a, l-\lambda m, m, f) = \mathcal{M}_\lambda h, \quad \vec{y}(b, l-\lambda m, m, f) = \mathcal{N}_\lambda h$$

(83) *(respectively, $\exists h_1 = h_1(\lambda, f) \in \mathcal{H}^r$: $\vec{y}(a, l_\lambda, m, f) = \mathcal{M}_\lambda h_1$, $\vec{y}(b, l_\lambda, m, f) = \mathcal{N}_\lambda h_1$),*

*has the unique solution $y = R_\lambda f$ (respectively, $y_1 = R(\lambda) f$) in $C^r(\bar{\mathcal{I}}, \mathcal{H})$ as $\Im\lambda \neq 0$. Here $\vec{y}(t, l_\lambda, m, f)$ see* (50)*; the operator-functions $\mathcal{M}_\lambda, \mathcal{N}_\lambda \in B(\mathcal{H}^r)$ depend analytically on the non-real $\lambda$,*

$$\mathcal{M}^*_{\bar\lambda}\left[\Re Q(a, l_\lambda)\right]\mathcal{M}_\lambda = \mathcal{N}^*_{\bar\lambda}\left[\Re Q(b, l_\lambda)\right]\mathcal{N}_\lambda, \quad \Im\lambda \neq 0,$$

*where $Q(t, l_\lambda)$ is the coefficient of equation* (49) *corresponding by Theorem 1.3 to equation* (1),

$$\|\mathcal{M}_\lambda h\| + \|\mathcal{N}_\lambda h\| > 0, \quad 0 \neq h \in \mathcal{H}^r, \quad \Im\lambda \neq 0,$$

*the lineal $\{\mathcal{M}_\lambda h \oplus \mathcal{N}_\lambda h \,|h \in \mathcal{H}^r\} \subset \mathcal{H}^{2r}$ is a maximal $\mathcal{Q}$-nonnegative subspace if $\Im\lambda \neq 0$, where $\mathcal{Q} = (\Im\lambda)\operatorname{diag}(\Re Q(a, l_\lambda), -\Re Q(b, l_\lambda))$ (and therefore*

$$\Im\lambda\left(\mathcal{N}^*_\lambda\left[\Re Q(b, l_\lambda)\right]\mathcal{N}_\lambda - \mathcal{M}^*_\lambda\left[\Re Q(a, l_\lambda)\right]\mathcal{M}_\lambda\right) \leq 0, \quad \Im\lambda \neq 0).$$

*Operator $R_\lambda f$ (respectively, $R(\lambda)f$) is a generalized resolvent of $\mathcal{L}_0$ (respectively, a resolvent of* (57) *type). This resolvent is constructed by applying of Theorem 1.4 with the characteristic operator*

(84) $$M(\lambda) = -\frac{1}{2}\left(X_\lambda^{-1}(a)\mathcal{M}_\lambda + X_\lambda^{-1}(b)\mathcal{N}_\lambda\right)\left(X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda\right)^{-1}(iG)^{-1},$$

*to equation* (1), (2) *with $n_\lambda[y] \equiv 0$ (respectively,* (1)–(2)*). Here $\left(X_\lambda^{-1}(a)\mathcal{M}_\lambda - X_\lambda^{-1}(b)\mathcal{N}_\lambda\right)^{-1} \in B(\mathcal{H}^r)$, $\Im\lambda \neq 0$, $X_\lambda(t)$ is an operator solution from Theorem 1.4 which corresponds to equation $(l-\lambda m)[y] = 0$ (respectively, $l_\lambda[y] = 0$).*

*The resolvents $y = R_\lambda f$ and $y_1 = R(\lambda) f$ satisfy condition* (76) *of Theorem 2.2.*

Let us notice that if $\mathcal{I}$ is finite and condition (53) holds with $P = I_r$ then any characteristic operator of equation (4) has representation (84) in view of [24]. Also we notice that if $\mathcal{I}$ is finite, $n_\lambda[y] \equiv 0$ and condition (53) holds then any generalized resolvent of $\mathcal{L}_0$ can be constructed as an operator $R(\lambda)$ from Theorem 1.4 in view of [28].

*Proof.* In view of Theorems 3.2, 3.3 from [28] it is sufficient to prove only proposition about condition (53).

Let for definiteness order $l$ = order $m$ = order $n_\lambda = 2n$.

Let for equation (1)–(2) with $n_\lambda[y] \equiv 0$ condition (53) hold with $P = I_r$ and $\lambda_0, \alpha, \beta$ as in formulation of proposition, but for equation (1), (2) that is not true for any $\lambda_0 \in \mathcal{B}$.

Then in view of [24, 26] the solutions $y_k(t)$ of equation (1)–(2) with $f(t)=0$, $\lambda=i$ exist for which

$$\int_\alpha^\beta (m+\Im n_i)\{y_k,y_k\}\,dt\to 0,\quad \vec{y}_k(0,l_i,m,0)=f_k,\quad \|f_k\|=1, \tag{85}$$

where $i\Im n_i=n_i$ in view of (54). Hence in view of Theorem 1.2 from [28])

$$\int_\alpha^\beta (W_i(t,l+im,n_i)\,Y_k(t,l+im,n_i),Y_k(t,l+im,n_i))\,dt=\int_\alpha^\beta n_i\{y_k,y_k\}\,dt\to 0, \tag{86}$$

where $Y_k(t,l+im,n_i)$ is defined by (48) with $y_k(t)$, $l+im$, $n_i$ instead of $f(t)$, $l_\lambda$, $m$ respectively.

On the other hand

$$X_i(t)\,f_k=\tilde{X}_i(t)\,f_k+\tilde{X}_i(t)\int_o^t \tilde{X}_i^{-1}(s)\,J^{-1}W(s,l+im,n_i)\,Y_k(s,l+im,n_i)\,ds \tag{87}$$

in view of Theorem 1.3 and the fact that $\vec{y}_k(t,l-im,n_i,y_k)=\vec{y}_k(t,l_i,m,0)$, where $\tilde{X}_\lambda(t)$ is an analogue of $X_\lambda(t)$ for the case $n_\lambda[y]\equiv 0$.

Comparing (86), (87) we see that

$$\left\|X_i(t)\,f_k-\tilde{X}_i(t)\,f_k\right\|\to 0 \tag{88}$$

uniformly in $t\in[\alpha,\beta]$.

In view of (85) subsequence $y_{k_q}$ exist such that

$$m\left\{y_{k_q},y_{k_q}\right\}\overset{a.a.}{\to}0,\quad n_i\left\{y_{k_q},y_{k_q}\right\}\overset{a.a.}{\to}0. \tag{89}$$

Due to second proposition in (89) and the arguments as in the proof of Proposition 3.1 from [28] one has

$$y_{k_q}^{[j]}(t\,|n_i)\overset{a.a.}{\to}0,\quad j=n,\dots,2n. \tag{90}$$

Let us denote $\tilde{y}_{k_q}(t)=\tilde{X}_i(t)\,f_{k_q}$. In view of Theorem 1.3 and (88)

$$\left\|y_{k_q}^{(j)}(t)-\tilde{y}_{k_q}^{(j)}(t)\right\|\to 0,\quad j=1,\dots,n-1, \tag{91}$$

$$\begin{aligned}&\Big\|\,(p_n(t)-i\tilde{p}_n(t))\left[y_{k_q}^{(n)}(t)-\tilde{y}_{k_q}^{(n)}(t)\right]\\ &\quad-\frac{i}{2}(q_n(t)-i\tilde{q}_n(t))\left[y_{k_q}^{(n-1)}(t)-\tilde{y}_{k_q}^{(n-1)}(t)\right]-y_{k_q}^{[n]}(t\,|n_i)\,\Big\|\\ &\quad=\left\|y_{k_q}^{[n]}(t|l_i)-\tilde{y}_{k_q}^{[n]}(t|l-im)\right\|\to 0\end{aligned} \tag{92}$$

uniformly in $t\in[\alpha,\beta]$. Comparing (89), (90), (92) and using $(p_n(t)-i\tilde{p}_n(t))^{-1}\in B(\mathcal{H})$ we have

$$\left(\tilde{p}_n(t)\,\tilde{y}_{k_q}^{(n)}(t),\tilde{y}_{k_q}^{(n)}(t)\right)\overset{a.a.}{\to}0. \tag{93}$$

In view of (91), (93), (89)

$$m\left\{\tilde{y}_{k_q},\tilde{y}_{k_q}\right\}\overset{a.a.}{\to}0,\quad \vec{y}_{k_q}(0,l-im,m,0)=f_{k_q},$$

that contradicts to the condition (53) with $P=I_r$ for equation (1), (2) with $n_\lambda[y]\equiv 0$. Hence for equation (1), (2) condition (53) with $P=I_r$ holds for $\lambda_0=i$ and therefore for any $\lambda_0\in\mathcal{B}$. Proposition 2.2 is proved. □

If the set $\mathbb{R}^1\backslash\mathcal{B}^1$ has no finite limit points then to verify the condition $\overset{\circ}{P}\int\dots=0$ or $\int\dots=0$ in (67) we can use the following proposition which is a corollary of Lemma from [39, p. 789].

**Proposition 2.3.** *Let* $R(\lambda) = \int_{\mathbb{R}^1} \frac{dE_\mu}{\mu-\lambda}$*, where* $E_\mu$ *is generalized spectral family in Hilbert space* $\mathbf{H}$*;* $g \in \mathbf{H}$*. If* $\sigma$ *is not a point of continuity of* $E_\mu g$*, then* $\exists c(\sigma, g) > 0$*:* $\|R(\sigma + i\tau) g\| \sim \frac{c(\sigma,g)}{|\tau|}$*,* $\tau \to 0$*.*

*Proof.* Let $\Delta$ be a jump of $E_\mu$ in the point $\sigma$. Then $\Delta g \neq 0$,

$$R(\sigma + i\tau) g = i\frac{\Delta}{\tau} g + \int_{\mathbb{R}^1} \frac{d\tilde{E}_\mu g}{\mu - \lambda} \quad \text{where} \quad \tilde{E}_\mu = \begin{cases} E_\mu, & \mu \le \sigma \\ E_\mu - \Delta, & \mu > \sigma \end{cases}. \tag{94}$$

Since the second term in (94) is $o\left(1/|\tau|\right)$ in view of [39, p. 789][4] proposition in proved. □

In the next theorem $\mathcal{I} = \mathbb{R}^1$ and condition (53) hold with $P = I_r$ both on the negative semi-axis $\mathbb{R}^1_-$ (i.e. as $\mathcal{I} = \mathbb{R}^1_-$) and on the positive semi-axis $\mathbb{R}^1_+$ (i.e. as $\mathcal{I} = \mathbb{R}^1_+$).

**Theorem 2.3.** *Let* $\mathcal{I} = \mathbb{R}^1$*, the coefficient of the expression* $l_\lambda[y]$ (2) *be periodic on each of the semi-axes* $\mathbb{R}^1_-$ *and* $\mathbb{R}^1_+$ *with periods* $T_- > 0$ *and* $T_+ > 0$ *correspondingly. Then the spectrums of the monodromy operators* $X_\lambda(\pm T_\pm)$ *(*$X_\lambda(t)$ *is from Theorem 1.4) do not intersect the unit circle as* $\Im\lambda \neq 0$*, the characteristic operator* $M(\lambda)$ *of the equation* (4) *is unique and equal to*

$$M(\lambda) = \left(\mathcal{P}(\lambda) - \frac{1}{2} I_r\right)(iG)^{-1} \quad (\Im\lambda \neq 0), \tag{95}$$

*where the projection* $\mathcal{P}(\lambda) = P_+(\lambda)(P_+(\lambda) + P_-(\lambda))^{-1}$, $P_\pm(\lambda)$ *are Riesz projections of the monodromy operators* $X_\lambda(\pm T_\pm)$ *that correspond to their spectrums lying inside the unit circle,* $(P_+(\lambda) + P_-(\lambda))^{-1} \in B(\mathcal{H}^r)$ *as* $\Im\lambda \neq 0$*.*

*Also let* $\dim \mathcal{H} < \infty$*, a finite interval* $\Delta \subseteq \mathcal{B}^1$*. Then in Theorem 2.1* $d\sigma(\mu) = d\sigma_{ac}(\mu) + d\sigma_d(\mu), \mu \in \Delta$*. Here* $\sigma_{ac}(\mu) \in AC(\Delta)$ *and, for* $\mu \in \Delta$*,*

$$\sigma'_{ac}(\mu) = \frac{1}{2\pi} G^{-1}\left(Q^*_-(\mu) G Q_-(\mu) - Q^*_+(\mu) G Q_+(\mu)\right) G^{-1},$$

*where the projections* $Q_\pm(\mu) = q_\pm(\mu)(P_+(\mu) + P_-(\mu))^{-1}$, $q_\pm(\mu)$ *are Riesz projections of the monodromy matrices* $X_\mu(\pm T_\pm)$ *corresponding to the multiplicators belonging to the unit circle and such that they are shifted inside the unit circle as* $\mu$ *is shifted to the upper half plane,* $P_\pm(\mu) = P_\pm(\mu + i0)$*;* $\sigma_d(\mu)$ *is a step function.*

Let us notice that the sets on which $q_\pm(\mu), P_\pm(\mu), (P_+(\mu) + P_-(\mu))^{-1}$ are not infinitely differentiable do not have finite limit points $\in \mathcal{B}^1$ as well as the set of points of increase of $\sigma_d(\mu)$.

*Proof.* The proof of Theorem 2.3 is similar to that on in the case $n_\lambda[y] \equiv 0$ [26]. □

The following examples demonstrate effects that are the results of appearance in $l_\lambda$ (2) of perturbation $n_\lambda$ depending nonlinearly on $\lambda$.

In Examples 2.1, 2.2 nonlinear in $\lambda$ perturbation does not change the type of the spectrum. In this examples $\dim \mathcal{H} = 1$, $m[y] = -y'' + y$. $L^2_m(\mathcal{I}) = \overset{\circ}{L}{}^2_m(\mathcal{I}) = W^{1,2}_2(\mathbb{R}^1)$. In Example 2.3 such perturbation implies an appearance of spectral gap with "eigenvalue" in this gap.

*Example* 2.1. Let

$$l_\lambda[y] = iy' - \lambda(-y'' + y) - \left(-\frac{h}{\lambda} y\right), \quad h \ge 0.$$

[4] Lemma from [39, p. 789] is proved for families $E_\mu$ with $E_\infty$ =identity operator. But analysis of its proof shows that it is valid in general case.

Here $\mathcal{B}=\mathbb{C}\setminus 0$, $E_0=E_{+0}$, spectral matrix $\sigma(\mu)\in AC_{\mathrm{loc}}$,
$\sigma'(\mu)=\frac{1}{2\pi}\begin{pmatrix}\frac{2}{\sqrt{4h+1-4\mu^2}} & 0\\ 0 & \frac12\sqrt{4h+1-4\mu^2}\end{pmatrix}$, as $|\mu|<\sqrt{h+1/4}$,
$\sigma'(\mu)=0$, as $|\mu|>\sqrt{h+1/4}$.

In Example 2.1 nonlinear in $\lambda$ perturbation change edges of spectral band.

*Example* 2.2. Let

$$l_\lambda[y]=y^{(IV)}-\lambda(-y''+y)-\left(-\frac{h}{\lambda}y\right),\quad h\geq 0.$$

Here $\mathcal{B}=\begin{cases}\mathbb{C}\setminus\{0\}, & h\neq 0\\ \mathbb{C}, & h=0\end{cases}$, $E_0=E_{+0}$, spectral matrix $\sigma(\mu)\in AC_{\mathrm{loc}}$,

$$\sigma'(\mu)=\begin{cases}\frac{1}{2\pi}\sqrt{\frac{\lambda+\sqrt{D}}{D}}\begin{pmatrix}\frac{2}{\lambda+\sqrt{D}} & 0 & 0 & -1\\ 0 & 1 & \frac{-\lambda+\sqrt{D}}{2} & 0\\ 0 & \frac{-\lambda+\sqrt{D}}{2} & \left(\frac{\lambda-\sqrt{D}}{2}\right)^2 & 0\\ -1 & 0 & 0 & \frac{\lambda+\sqrt{D}}{2}\end{pmatrix}, \\ \qquad \text{as} \quad -\sqrt{h}<\mu<0,\quad \mu>\sqrt{h} \\ \frac{1}{2\pi}\cdot\frac{1}{\sqrt{\lambda-2\sqrt{q}}}\begin{pmatrix}\frac{1}{\sqrt{q}} & 0 & 0 & -1\\ 0 & 1 & -\sqrt{q} & 0\\ 0 & -\sqrt{q} & \sqrt{q}\left(\lambda-\sqrt{q}\right) & 0\\ -1 & 0 & 0 & \lambda-\sqrt{q}\end{pmatrix}, \\ \qquad \text{as} \quad \mu^*<\mu<\sqrt{h}\end{cases},$$

where $D=\mu^2-4q$, $q=h/\mu-\mu$, $\mu^*=\mu^*(h)$ – nonnegative root of equation $D=0$. $\sigma'(\mu)=0$, as $\mu\notin\left[-\sqrt{h},0\right]\bigcup[\mu^*,\infty)$.

In Example 2.2 nonlinear in $\lambda$ perturbation implies an appearance of additional spectral band $[-h,0]$, variation of edge of semi-infinite spectral band and appearance of interval $\left(\mu^*,\sqrt{h}\right)$ of fourfold spectrum.

*Example* 2.3. Let $\dim\mathcal{H}=2$,

$$l_\lambda[y]=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}y'-\lambda y-\begin{pmatrix}-h/\lambda & 0\\ 0 & 0\end{pmatrix}y,\quad h\geq 0.$$

Here $\mathcal{B}=\begin{cases}\mathbb{C}\setminus\{0\}, & h\neq 0\\ \mathbb{C}, & h=0\end{cases}$, spectral matrix $\sigma(\mu)=\sigma_{\mathrm{ac}}(\mu)+\sigma_{\mathrm{d}}(\mu)$, $\sigma_{\mathrm{ac}}(\mu)\in AC_{\mathrm{loc}}$, $\sigma'_{\mathrm{ac}}(\mu)\neq 0$, as $|\mu|>\sqrt{h}$, $\sigma'_{\mathrm{ac}}(\mu)=0$, as $|\mu|<\sqrt{h}$, step-function $\sigma_{\mathrm{d}}(\mu)$ has only one jump $\begin{pmatrix}0 & 0\\ 0 & \sqrt{h}\big/2\end{pmatrix}$ in point $\mu=0$ (inside of spectral gap). In this point

$$(E_{+0}-E_0)f=\begin{pmatrix}0 & 0\\ 0 & \sqrt{h}\big/2\end{pmatrix}\int_{-\infty}^{\infty}e^{-\sqrt{h}|t-s|}f(s)\,ds,\quad f(t)\in L^2\left(\mathbb{R}^1\right).$$

Let us explain that in Examples 2.1, 2.2: 1) Spectral matrices are locally absolutely continuous in view of Theorem 3.6 and estimates of the type $\|M(\lambda)\|\sim\frac{c}{|\lambda|^\alpha}\ (\lambda\to i0)$, $\alpha<$

1 for corresponding characteristic operators (cf. [20]) that follows from (95); 2) Equalities $E_0 = E_{+0}$ follow from Proposition 2.3, equality $L_m^2(\mathbb{R}^1) = \overset{\circ}{L}{}_m^2(\mathbb{R}^1)$ and estimates of the type $\|R(i\tau)g\|_m \le \frac{c(g)}{|\tau|^\beta}$, $(\tau \to 0)$, $\beta < 1$, $g \in \overset{\circ}{\mathrm{H}}$ that follows from Theorems 1.4, 2.3 and Floquet Theorem.

Let us notice that in view of Floquet theorem conditions of Theorem 2.2 ((76) with account of Remark 2.1) hold for all Examples 2.1–2.3.

The following theorem is a generalization of results from [39] on the expansion in solutions of scalar Sturm-Liouville equation which satisfy in regular end point the boundary condition depending on spectral parameter.

**Theorem 2.4.** *Let $r = 2n$, $\mathcal{I} = (0, \infty)$, condition (53) with $P = \mathcal{I}_{2n}$ hold. Let contraction $v(\lambda) \in B(\mathcal{H}^n)$ satisfy the conditions of Lemma 1.2. Let $v(\lambda)$ analytically depend in $\lambda$ in any points of $\mathcal{B}^1 = \mathbb{R}^1 \bigcap \mathcal{B}$ and be unitary in this points.*

*Let $R(\lambda)$ (57) correspond to characteristic operator $M(\lambda)$ (10), (12) of equation (4), where characteristic projection (12) corresponds to some Weyl function $m(\lambda)$ of equation (52) and to pair (27), (28) which is constructed with the help of this $v(\lambda)$. Let the generalized spectral family $E_\mu$ correspond to $R(\lambda)$ by (3).*

*Let $m_{a,b}(\lambda)$ be Nevanlinna operator-function from Lemma 1.2 corresponding by (18), (27), (28) to these $v(\lambda)$ and $m(\lambda)$. Let $\sigma_{a,b}(\mu) = w - \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_0^\mu \Im m_{a,b}(\mu + i\varepsilon)\, d\mu$ be the spectral operator-function that corresponds to $m_{a,b}(\lambda)$.*

*Then every proposition of Theorem 2.1 is valid with $\sigma_{a,b}(\mu)$ instead of $\sigma(\mu)$, $(u_1(t,\lambda), \dots, u_n(t,\lambda))$ instead of $[X_\lambda(t)]_1$ and*

$$\varphi(\mu, f) = \int_{\mathcal{I}} (u_1(t,\mu), \dots, u_n(t,\mu))^* \, m[f(t)]\, dt$$

$$= \int_{\mathcal{I}} \sum_{k=0}^{s/2} \left(u_1^{(k)}(t,\mu), \dots, u_n^{(k)}(t,\mu)\right)^* \mathrm{m}_k[f(t)]\, dt$$

*instead of $\varphi(\mu, f)$ (66), where $u_j(t,\lambda)$ see (61), $\mathrm{m}_k[f(t)]$ see (62).*

*Therefore if we represent spectral operator-function $\sigma_{ab}(\mu)$ in matrix form: $\sigma_{a,b}(\mu) = \left\|(\sigma_{ab}(\mu))_{ij}\right\|_{i,j=1}^n$, $(\sigma_{a,b}(\mu))_{ij} \in B(\mathcal{H})$ then, for example[5], the following inversion formula is valid in $L_m^2(0,\infty)$ for any vector-function $f(t) \in \overset{\circ}{H}$ satisfying (67):*

$$f(t) = \overset{\circ}{P} \int_{\mathcal{B}^1} \sum_{i,j=1}^n u_i(t,\mu)\, d(\sigma_{a,b}(\mu))_{ij} \int_{\mathcal{I}} u_j^*(s,\mu)\, m[f(s)]\, ds$$

$$= \overset{\circ}{P} \int_{\mathcal{B}^1} \sum_{i,j=1}^n u_i(t,\mu)\, d(\sigma_{a,b}(\mu))_{ij} \int_{\mathcal{I}} \sum_{k=0}^{s/2} \left(u_j^{(k)}(s,\mu)\right)^* \mathrm{m}_k[f(s)]\, ds.$$

The proof is carried out in the same way as the proofs of Theorem 2.1 and Proposition 2.1 with the help of Proposition 1.1 and Lemma 1.2.

Let us notice that in contrast to operator spectral function $\sigma_{a,b}(\mu)$ from Theorem 2.4 the scalar spectral function in [39] was constructed with the help of different formulae that corresponds to such intervals of real axis where $v(\mu) \neq -1$ or $v(\mu) \neq 1$. But already in matrix case it is impossible to construct the spectral matrix according to [39] since here for some real $\lambda$ (and even for any real $\lambda$) $(v(\lambda) + I_n)^{-1}$ and $(v(\lambda) - I_n)^{-1}$ may simultaneously do not belong to $B(\mathcal{H}^n)$.

[5] Also (65), Parseval equality (69), Bessel inequality (70) can be represented in a similar way.

**Remark 2.2.** *Let contraction* $v(\lambda) \in B(\mathcal{H}^n)$ *is analytic in any point* $\lambda \in \bar{\mathbb{C}}_+ \bigcap \mathcal{B}$ *and is unitary in any point* $\lambda \in \mathcal{B}^1 \neq \varnothing$. *Let* $\dim \mathcal{H} < \infty$. *Then* $v(\lambda)$ *satisfy conditions of Lemma 1.2.*

*If* $\dim \mathcal{H} = \infty$ *in general it is not valid. Namely let domain* $D \subset \mathbb{C}_+$, $\operatorname{dist}\left\{\bar{D}, [-a, a]\right\} > 0 \; \forall a \in \mathbb{R}^1_+$; *set* $\{\lambda_k\}_{k=1}^{\infty} \subset D$ *in dense in* $D$. *Let us consider in* $\mathcal{H}^n = \left(l^2\right)^n$ *the following operator* $v(\lambda) = v_1(\lambda) \oplus I_{n-1}$, *where* $v_1(\lambda) = \operatorname{diag}\left\{\frac{\lambda - \lambda_k}{\lambda - \bar{\lambda}_k}\right\}_{k=1}^{\infty}$. *This operator is analytic in any point* $\lambda \in \bar{\mathbb{C}}_+$, *is a contraction for* $\lambda \in \mathbb{C}_+$ *and is unitary for* $\lambda \in \mathbb{R}^1$. *But for this operator the set* $S = \left\{\lambda \in \mathbb{C}_+ : v^{-1}(\lambda) \notin B(\mathcal{H}^n)\right\} = \bar{D}$.

Finally, we note that, obviously, an analogue of Theorem 2.4 is valid for $\mathcal{I} = (0, b)$, $b < \infty$.

## References


1. S. Albeverio, M. Malamud, V. Mogilevskii, *On Titchmarsh-Weyl functions and eigenfunction expansions of first-order symmetric systems*, Integr. Equ. Oper. Theory **77** (2013), no. 3, 303–354.
2. F. Atkinson, *Discrete and Continuous Boundary Problems*, Acad. Press, New York—London, 1964. (Russian transl. Mir, Moscow, 1988)
3. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
4. Yu. M. Berezanskii, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, Amer. Math. Soc., Providence, RI, 1986. (Russian edition: Naukova Dumka, Kiev, 1978)
5. F. Brauer, *Singular self-adjoint boundary value problems for the differential equation* $Lx = \lambda Mx$, Trans. Amer. Math. Soc. **88** (1958), no. 12, 331–345.
6. F. Brauer, *Spectral theory for the differential equation* $Lu = \lambda Mu$, Canad. J. Math. **10** (1958), no. 13, 431–446.
7. F. E. Browder, *Eigenfunction expansions for formally self-adjoint partial differential operators*, Proc. Nat. Acad. Sci. USA **42** (1956), no. 10, 769–771; **42** (1956), no. 11, 870–872.
8. V. M. Bruk, *The generalized resolvents and spectral functions of even order differential operators in a space of vector-valued functions*, Mat. Zametki **15** (1974), no. 6, 945–954. (Russian); English transl. Math. Notes, **15** (1974), no. 6, 563–568.
9. E. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill Book Company, New York—Toronto—London, 1955.
10. V. A. Derkach, M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), no. 1, 1–95.
11. V. Derkach, S. Hassi, M. Malamud, H. de Snoo, *Boundary relations and generalized resolvents of symmetric operators*, Russ. J. Math. Phys. **16** (2009), no. 1, 17–60.
12. V. A. Derkach, S. Hassi, M. M. Malamud, H. S. V. de Snoo, *Boundary triplets and Weyl functions. Recent developments*, Operator Methods for Boundary Value Problems, Hendrik S. V. de Snoo, Seppo Hassi, and Franciszek Hugon Szafraniec, eds., Cambridge University Press, Cambridge, 2012, pp. 161–220.
13. A. Dijksma, H. de Snoo, *Self-adjoint extensions of symmetric subspaces*, Pacific J. Math. **54** (1974), 71–100.
14. N. Dunford, J. T. Schwartz, *Linear Operators.* Part II*: Spectral Theory. Self-Adjoint Operators in Hilbert Space*, Interscience Publishers John Wiley & Sons, New York—London, 1963. (Russian transl. Mir, Moscow, 1966)
15. I. M. Gelfand, A. G. Kostyuchenko, *Expansion in eigenfunctions of differential and other operators*, Doklady Akad. Nauk SSSR **103** (1955), no. 3, 349–352.
16. M. L. Gorbachuk, *On spectral functions of a second order differential equation with operator coefficients*, Ukrain. Mat. Zh. **18** (1966), no. 2, 3–21. (Russian); English transl. Amer. Math. Soc. Transl. Ser 2 **72** (1968), 177–202.
17. V. I. Gorbachuk and M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984)
18. P. R. Halmos, *A Hilbert Space Problem Book*, Princeton, New Jersey, 1967. (Russian edition: Moscow, Mir, 1970)
19. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin—Heidelberg—New York, 1966. (Russian transl. Mir, Moscow, 1972)

20. V. I. Khabustovskii, *The spectral matrix of a periodic symmetric system with a degenerate weight on the axis*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **35** (1981), 111–119. (Russian)
21. V. I. Khabustovskii, *Eigenfunction expansions of periodic systems with weight*, Dokl. Akad. Nauk Ukrain. SSR Ser. A (1984), no. 5, 26–29. (Russian)
22. V. I. Khabustovskii, *Spectral analysis of periodic systems with degenerate weight on the axis and half-axis*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **44** (1985), no. 4, 122–133. (Russian); English transl. J. Soviet Math. **48** (1990), no. 3, 345–355.
23. V. I. Khabustovskii, *On the characteristic matrix of Weyl-Titchmarsh type for differential-operator equations with the spectral parameter which contains the spectral parameter in linear or Nevanlinna's manner*, Mat. Fiz. Anal. Geom. **10** (2003), no. 2, 205–227. (Russian)
24. V. I. Khrabustovsky, *On the Characteristic Operators and Projections and on the Solutions of Weyl Type of Dissipative and Accumulative Operator Systems.* I. *General Case*, J. Math. Phys. Anal. Geom. **2** (2006), no. 2, 149–175; II. *Abstract Theory*, ibid, no. 3, 299–317; III. *Separated Boundary Conditions*, ibid, no. 4, 449–473.
25. V. I. Khrabustovskyi, *On the limit of regular dissipative and self-adjoint boundary value problems with nonseparated boundary conditions when an iterval streches to the semiaxis*, J. Math. Phys. Anal. Geom. **5** (2009), no. 1, 54–81.
26. V. I. Khrabustovskyi, *Expansion in eigenfunctions of relations generated by pair of operator differential expressions*, Methods Funct. Anal. Topology **15** (2009), no. 2, 137–151.
27. V. I. Khrabustovskyi, *Analogs of generalized resolvents and eigenfunction expansions of relations generated by pair of differential operator expressions one of which depends on spectral parameter in nonlinear manner*; arXiv:1301.2926.
28. V. I. Khrabustovskyi, *Analogs of generalized resolvents for relations generated by pair of differential operator expressions one of which depends on spectral parameter in nonlinear manner*, J. Math. Phys. Anal. Geom. **9** (2013), no. 4, 496–535.
29. V. E. Lyantse, O. G. Storozh, *Methods of the Theory of Unbounded Operators*, Naukova Dumka, Kiev, 1983. (Russian)
30. V. A. Marchenko, *Sturm-Liouville Operators and Their Applications*, Oper. Theory Adv. Appl., vol. 22, Birkhäuser Verlag, Basel, 1986. (Russian edition: Naukova Dumka, Kiev, 1977)
31. V. Mogilevskii, *Boundary pairs and boundary conditions for general (not necessarily definite) first-order symmetric systems with arbitrary deficiency indices*, Math. Nachr. **285** (2012), no. 14–15, 1895–1931.
32. M. A. Naimark, *Linear Differential Operators*, 2nd rev. ed., Nauka, Moscow, 1969. (Russian); English transl. Ungar, New York, 1967.
33. S. A. Orlov, *Description of the Green functions of canonical differential systems.* I. II., Teor. Funktsii Funktsional. Anal. i Prilozhen. **51** (1989), 78–88. (Russian); **52** (1989), 33–39. (Russian); English transl. J. Sov. Math. **52** (1990), no. 6, 3500–3508; **52** (1990), no. 5, 3372–3377.
34. F. S. Rofe-Beketov, *Expansion in eigenfunctions of infinite systems of differential equations in the non-self-adjoint and self-adjoint cases*, Mat. Sb. **51** (1960), no. 3, 293–342. (Russian)
35. F. S. Rofe-Beketov, *Self-adjoin extensions of differential operators in a space of vector-valued functions*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **8** (1969), 3–24. (Russian)
36. F. S. Rofe-Beketov, *Square-integrable solutions, self-adjoint extensions and spectrum of differential systems*, Proc. International Conference on Differential Equations, Uppsala, 1977, pp. 169–178.
37. F. Rofe-Beketov and A. Kholkin, *Spectral Analysis of Differential Operators. Interplay Between Spectral and Oscillatory Properties*, World Scientific Monograph Series in Mathematics, vol. 7, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
38. L. A. Sakhnovich, *Spectral Theory of Canonical Differential Systems. Method of Operator Idetities*, Oper. Theory Adv. Appl., vol. 107, Birkhäuser Verlag, Basel, 1999.
39. A. V. Shtraus, *On eigenfunction expansion of a second order boundary problem of a semi-axis*, Izv. Akad. Nauk SSSR Ser. Mat. **20** (1956), 783–792.
40. A. V. Shtraus, *On the generalized resolvents and spectral functions of even-order differential operators*, Izv. Akad. Nauk SSSR Ser. Mat., **21** (1957), 785–808. (Russian); English transl. Amer. Math. Soc. Transl. Ser. 2. **16** (1960), 462–464.
41. B. Sz.-Nagy, C. Foias, *Analyse harmonique des operateurs de lespace de Hilbert*, Masson et Cie, Paris, 1967. (Russian transl. Mir, Moscow, 1977)


Ukrainian State Academy of Railway Transport, Kharkiv, Ukraine
*E-mail address*: `v_khrabustovskyi@ukr.net`

Received 09/07/2013; Revised 07/09/2013