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ON A CRITERION OF MUTUAL ADJOINTNESS FOR EXTENSIONS OF SOME NONDENSELY DEFINED OPERATORS

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Dedicated to Vladimir Koshmanenko on the occasion of his 70th birthday

Abstract. In the paper the role of initial object is played by a pair of closed linear densely defined operators L_0 and M_0, where $L_0 \subset M_0^* := L$, acting in Hilbert space. A criterion of mutual adjointness for some classes of the extensions of finite-dimensional (non densely defined) restrictions of L_0 and M_0 are established. The main results are based on the theory of linear relations in Hilbert spaces and are formulated in the terms of abstract boundary operators.

1. Introduction

The theory of linear relations (i.e. "multivalued operators") in Hilbert spaces was initiated by R. Arens [1]. Some aspects of the the extension theory of linear relations (in particular, nondensely defined operators, first of all, Hermitian ones) had been developed by many other mathematicians (see, e. g., [2]–[20]).

In this paper, a criterion of mutual adjointness for some classes of extensions of finite-dimensional (non densely defined) restrictions of two given closed linear (densely defined) operators acting in Hilbert space are established.

Through this report we use the following denotations:$D(T), R(T), \ker T$ are, respectively, the domain, range, and kernel of a linear operator ; $\mathcal{B}(X, Y)$ is the set of linear bounded operators $T : X \to Y$ such that $D(T) = X$; $\mathcal{C}(X)$ is a class of closed densely defined linear operators $T : X \to X$; $(\cdot \mid \cdot)_X$ is the inner product in a Hilbert space X; $T \downarrow E$ is the restriction of T onto E; $\mathbb{I}_X$ is the identity in X; $\oplus$ and $\ominus$ are the symbols of orthogonal sum and orthogonal complement, respectively; $AE := \{Ax : x \in E\}$.

If $A : X \to Y_i,\ i = 1, \dots, n$ are linear operators then the notation $A = A_1 \oplus \cdots \oplus A_n$ means that $Ax = (A_1x, \dots, A_nx)$ for every $x \in X$. The role of the initial object is played by a couple (L, L_0) of operators $H \to H$ (H is a fixed complex Hilbert space equipped with the inner product $(\cdot \mid \cdot)$ and the corresponding norm $\| \cdot \|$) such that

$$L, L_0 \in \mathcal{C}(H), \quad L_0 \subset L; \tag{1}$$

$$M_0 := L^*, \quad M := L_0^*. \tag{2}$$

(Here and below T^* means the operator or relation adjoint of the operator or relation T).

By $D[T], T \in \mathcal{C}(H)$, we understand the variety $D(T)$ interpreted as a Hilbert space with the inner product: $\forall y, z \in D(T) (y \mid z)_T = (y \mid z) + (Ty \mid Tz)$ and the corresponding graph-norm $\| \cdot \|_T$. By $\oplus_T$ and $\ominus_T$ we denote the symbols of orthogonal sum and orthogonal complement in $D[T]$.

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Put $H_L = D[L] \ominus_L D[L_0], H_M = D[M] \ominus_M D[M_0]$ and denote by P_L, P_M , respectively, the orthoprojections

$$D[L] \to H_L, D[M] \to H_M.$$

Furthermore, for each $W \in \mathcal{B}(D, G)$, where D coincides with $D[L]$ or $D[M]$ and G is an (auxiliary) Hilbert space, the adjoint operator will be denoted by W', consequently

$$\forall y \in D(L), \quad \forall g \in G (Wy \mid g)_G = (y \mid W'g)_L.$$

Remark 1. It is known (see [1]) that for each couple (L, L_0) satisfying (1) there exists a so-called boundary pair (G, U). It means that G is an auxiliary Hilbert space,

$$U \in \mathcal{B}(D[L], G), \quad R(U) = G, \quad \ker U = D(L_0). \tag{3}$$

Moreover, if M and M_0 are defined by (2), and $(G_L, U), (G_M, V)$ are boundary pairs for $(L, L_0), (M, M_0)$, respectively, then there exists a unique operator E satisfying the following conditions:

i)

$$E \in \mathcal{B}(G_L, G_M), \quad E^{-1} \in \mathcal{B}(G_M, G_L); \tag{4}$$

ii)

$$\forall y \in D(L), \quad \forall z \in D(M)$$

$$(Ly \mid z) - (y \mid Mz) = (EUy \mid Vz)_G = (Uy \mid E^* Vz)_G. \tag{5}$$

Further, let H_0^L, H_0^M be finite-dimensional subspaces of H. Put

$$\begin{aligned} S_0 &= L_0 \downarrow H \ominus H_0^{(L)}, \quad \hat{M}_0 = M_0 \downarrow H \ominus H_0^{(M)}, \\ S &= \left\{ (y, Ly + \phi^{(M)}) : y \in D(L), \phi^{(M)} \in H_0^{(M)} \right\}, \\ T &= \left\{ (z, Mz + \phi^{(L)}) : z \in D(M), \phi^{(L)} \in H_0^{(L)} \right\} \end{aligned}$$

and denote by $P_0^{(L)}, P_0^{(M)}$, respectively, the orthoprojections

$$H \to H_0^{(L)} (H \to H_0^{(M)}).$$

Note that $S_0^* = \hat{M}, T_0^* = \hat{L}$ (see [2]).

2. One abstract analogue of the Green's formula for the pairs of nondensely defined operators

Definition 1. ([18]) Let G_S be an auxiliary Hilbert space and $U_S \in \mathcal{B}(S, G_S)$. A pair $\mathcal{B}(G_S, U_S)$ is called a boundary pair for (S, S_0), if $R(U_S) = G_S, \ker U_S = S_0$. In this case G_S is said to be a boundary space for (S, S_0).

Remark 2. It should be noted that

- in the case where L_0 is a symmetric operator with equal defect numbers and $L = L_0^*$, the latter definition may be interpreted as some generalization of the notion of boundary triplet, exposed in [2], [7], [9], [16] (Yu. M. Arlinskii, V. A. Derkach, M. M. Malamud, S. Hassi, H. S. V. de Snoo and others);
- in the situation if, in addition, $L_0 = S_0$, the concept of boundary triplet (=boundary value space) had been developed by V. I. Gorbachuk and M. L. Gorbachuk [6], F. S. Rofe-Beketov [19], A. N. Kochubei [10], V. M. Bruk [4], V. A. Mikhailets [17] and other mathematicians (see [15] and reference therein).

Lemma 1. *Defined on S the norms $\|\cdot\|_{\hat{L}}$, $\|\cdot\|_{\mathcal{L}}$, where*

$$\forall (y, Ly+\phi^{(M)}) \in S \quad \|(y, Ly+\phi^{(M)})\|^2_{H^2} = \|y\|^2 + \|Ly+\phi^{(M)}\|^2,$$

$$\forall (y, Ly+\phi^{(M)}) \in S \quad \|(y, Ly+\phi^{(M)})\|^2_{S} = \|y\|^2 + \|Ly\|^2 + \|\phi^{(M)}\|^2,$$

are equivalent.

Proof. We have

$$\begin{aligned}\|(y, Ly+\phi^{(M)})\|^2_{H^2} &= \|y\|^2 + \|Ly+\phi^{(M)}\|^2 \\ &\le \|y\|^2 + \|Ly\|^2 + 2\|Ly\|\cdot\|\phi^{(M)}\| + \|\phi^{(M)}\|^2 \\ &\le \|y\|^2 + \|Ly\|^2 + \|Ly\|^2 + \|\phi^{(M)}\|^2 + \|\phi^{(M)}\|^2 \\ &\le 2\left(\|y\|^2 + \|Ly\|^2 + \|\phi^{(M)}\|^2\right) = 2\|(y, Ly+\phi^{(M)})\|^2_S.\end{aligned}$$

Thus $\|\cdot\|_S$ is stronger than $\|\cdot\|_{H^2}$. On the other hand, S is a closed relation, therefore the space $(S, \|\cdot\|_{H^2})$ is complete. Moreover, L is a closed operator and $H_0^{(M)}$ is a finite-dimensional subspaces of H, consequently $(S, \|\cdot\|_{H^2})$ is a complete space too. To complete the proof it is sufficiently to apply Banach inverse operator theorem. □

Remark 3. In the sequel we assume that by setting

$$(y, Ly+\phi^{(M)}) \leftrightarrow (y, \phi^{(M)}), \quad (z, Mz+\phi^{(L)}) \leftrightarrow (z, \phi^{(L)})$$

the identifications

$$S \leftrightarrow D[L] \oplus H_0^{(M)}, \quad T \leftrightarrow D[M] \oplus H_0^{(L)}$$

(therefore the identifications $L \leftrightarrow D[L], L_0 \leftrightarrow D[L_0], S_0 \leftrightarrow D[S_0]$ and $M \leftrightarrow D[M], M_0 \leftrightarrow D[M_0], T_0 \leftrightarrow D[T_0]$) are provided. The latter lemma shows that the mappings

$$S \supset (y, Ly+\phi^{(M)}) \mapsto (y, \phi^{(M)}) \in D[L] \oplus H_0^{(M)},$$

$$T \supset (z, Mz+\phi^{(L)}) \mapsto (z, \phi^{(L)}) \in D[M] \oplus H_0^{(L)}$$

are homeomorphic ones.

Theorem 1. *Suppose that (G, U) is a boundary pair for (L, L_0) and*

$$\forall (y, Ly+\phi^{(M)}) \in S \quad U_S(y, Ly+\phi^{(M)}) = (Uy, P_0^{(L)}y, \phi^{(M)}).$$

Then (G_S, U_S), where $G_S = G \oplus H_0^{(L)} \oplus H_0^{(M)}$, is a boundary pair for (S, S_0).

Proof. **i)** $U_S \in \mathcal{B}(S, G_S)$.

For each $(y, Ly+\phi^{(M)}) \in S$ we have

$$\begin{aligned}\|U_S(y, Ly+\phi^{(M)})\|^2_{G_S} &= \|(Uy, P_0^{(L)}y, \phi^{(M)})\|^2_{G_S} \\ &= \|Uy\|^2_G + \|P_0^{(L)}y\|^2 + \|\phi^{(M)}\|^2 \\ &\le c^2\left(\|y\|^2 + \|Ly\|^2\right) + \|y\|^2 + \|\phi^{(M)}\|^2 \\ &\le c_1^2\left(\|y\|^2 + \|Ly\|^2 + \|\phi^{(M)}\|^2\right) = c_1^2\left\|(y, Ly+\phi^{(M)})\right\|^2_S\end{aligned}$$

(for some $c > 0$, $c_1 > 0$), consequently $U_S \in \mathcal{B}(S, G_S)$. Now the proof follows from Lemma 1.

ii) $R(U_S) = G_S$.

Suppose that $(g, h_L, h_M) \in G \oplus H_0^{(L)} \oplus H_0^{(M)} = \hat{G}$. Since $R(U) = G$ and $P_0^{(L)}$ is a bounded finite-dimensional operator, there exists $y \in D(L)$ such that $Uy = g, P_0^{(L)}y = h_L$ (see, e. g, [1, p. 195]). Put $\phi^{(M)} = h_M$. It is clear that $U_S(y, Ly+\phi^{(M)}) = (g, h_L, h_M)$.

iii) $\ker U_S = \hat{L}_0$.

Indeed,

$$U_S(y, Ly+\phi^{(M)}) = 0 \Leftrightarrow (Uy=0, P_0^{(L)}=0) \wedge (\phi^{(M)}=0)$$
$$\Leftrightarrow (y \in D(L_0) \wedge (\phi^{(M)}=0)) \Leftrightarrow (y, Ly+\phi^{(M)}) = (y, S_0 y) \in S_0.$$

(Let us recall that in the theory of linear relations the operator and its graph are identified). □

Theorem 2. *Let $(G_L, U), (G_M, U), E$ be as above, in particular (5) is fulfilled;*

$$G_S := G_L \oplus H_0^{(L)} \oplus H_0^{(M)},$$
$$\forall (y, Ly+\phi^{(M)}) \in S \quad U_S(y, Ly+\phi^{(M)}) = (Uy, P_0^{(L)}y, \phi^{(M)});$$
$$G_T := G_M \oplus H_0^{(M)} \oplus H_0^{(L)},$$
$$\forall (z, Mz+\phi^{(L)}) \in T \quad V_T(z, Mz+\phi^{(L)}) = (Vz, P_0^{(M)}z, \phi^{(L)}).$$

Then

i) (G_S, U_S) *is a boundary pair for* (S, S_0);
ii) (G_T, V_T) *is a boundary pair for* (T, T_0);
iii) $\forall (y, Ly+\phi^{(M)}) \in S$, $\forall (z, Mz+\phi^{(L)}) \in T$

$$\begin{aligned}(6)\qquad &(Ly+\phi^{(M)} \mid z) - (y \mid Mz+\phi^{(L)})\\ &= (E_S U_S(y, Ly+\phi^{(M)}) \mid V_T(z, Mz+\phi^{(L)}))_{G_T}\\ &= (U_S(y, Ly+\phi^{(M)}) \mid E_S^* V_T(z, Mz+\phi^{(L)}))_{G_S}\end{aligned}$$

where

$$(7)\qquad E_S = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & \mathbb{I}_{H_0^{(M)}} \\ 0 & -\mathbb{I}_{H_0^{(L)}} & 0 \end{pmatrix},$$

$E_S \in \mathcal{B}(G_S, G_T)$.

Proof. The statement i) was shown before (see Theorem 1). The proof of the second statement is analogous. Further, in view of (5) for each $y \in D(L), z \in D(M), \phi^{(L)} \in H_0^{(L)}, \phi^{(M)} \in H_0^{(M)}$ we have

$$\begin{aligned}&(Ly+\phi^{(M)} \mid z) - (y \mid Mz+\phi^{(L)}) = (Ly \mid z) - (y \mid Mz) - (y\phi^{(L)})\\ &= (EUy \mid Vz)_{G_M} + (\phi^{(M)} \mid P_0^{(M)}z)_{H_0^{(M)}} - (P_0^{(L)}y \mid \phi^{(L)})_{H_0^{(L)}}\\ &= (Uy \mid E^*Vz)_{G_L} + (\phi^{(M)} \mid P_0^{(M)}z)_{H_0^{(M)}} - (P_0^{(L)}y \mid \phi^{(L)})_{H_0^{(L)}}.\end{aligned}$$

But

$$\begin{aligned}&(EUy \mid Vz)_{G_M} + (\phi^{(M)} \mid P_0^{(M)}z)_{H_0^{(M)}} - (P_0^{(L)}y \mid \phi^{(L)})_{H_0^{(L)}}\\ &= \left(\begin{pmatrix} EUy \\ \phi^{(M)} \\ -P_0^{(L)}y \end{pmatrix} \mid \begin{pmatrix} Vz \\ P_0^{(M)}z \\ \phi^{(L)} \end{pmatrix} \right)_{G_M \oplus H_0^{(M)} \oplus H_0^{(M)}}\\ &= \left(\begin{pmatrix} E & 0 & 0 \\ 0 & 0 & 1_M \\ 0 & -1_L & 0 \end{pmatrix} \begin{pmatrix} Uy \\ P_0^{(L)}y \\ \phi^{(M)} \end{pmatrix} \mid \begin{pmatrix} Vz \\ P_0^{(M)}z \\ \phi^{(L)} \end{pmatrix} \right)_{G_T}\\ &= (E_S U_S(y, Ly+\phi^{(M)}) \mid V_T(z, Mz+\phi^{(L)}))_{G_T}\\ &= (U_S(y, Ly+\phi^{(M)}) \mid E_S^* V_S(z, Mz+\phi^{(L)}))_{G_S}\end{aligned}$$

(here and below $1_L := \mathbb{I}_{H_0^{(L)}}, 1_M := \mathbb{I}_{H_0^{(M)}}$). The Theorem is proved. □

Denotations. Let us introduce the following denotations.

$$L_u := L \downarrow H_L, \quad M_v := M \downarrow H_M, \quad 1_L := \mathbb{I}_{H_0^{(L)}}, \quad 1_M := \mathbb{I}_{H_0^{(M)}},$$

$$S_u := \begin{pmatrix} L_u & 0 & 0 \\ 0 & 0 & 1_M \\ 0 & -1_L & 0 \end{pmatrix}, \quad T_v := \begin{pmatrix} M_v & 0 & 0 \\ 0 & 0 & 1_L \\ 0 & -1_M & 0 \end{pmatrix}. \tag{8}$$

Corollary 1. *Assume that $(G_S L, U_S)$ and (G_T, V_T) are boundary pairs for (S, S_0) and (T, T_0), respectively. The following assertions are equivalent (up to the identifications $S \leftrightarrow D[L] \oplus H_0^{(M)}$, $T \leftrightarrow D[M] \oplus H_0^{(L)}$):*

(9) i) *the relation* (6) *holds*;

(10) ii) $U_S T_v V_T^* = -E_S^{-1}$;

(11) iii) $V_T^* E_S U_S \downarrow H_S = S_u$;

(12) iv) $V_T S_u U_S^* = (E_S^*)^{-1}$;

(13) v) $U_S^* E_S^* V_T \downarrow H_T = -T_v$,

where $H_S = H_L \oplus H_0^{(L)} \oplus H_0^{(M)}$, $H_T = H_M \oplus H_0^{(M)} \oplus H_0^{(L)}$.

Proof. At first let us remind that here and below S and T are treated as Hilbert spaces equipped with the inner products generating the norms

$$\forall (y, Ly + \phi^{(M)}) \in S \quad \|(y, Ly + \phi^{(M)})\|_S^2 = \|y\|^2 + \|Ly\|^2 + \|\phi^{(M)}\|^2$$

and

$$\forall (z, Mz + \phi^{(L)}) \in T \quad \|(z, Mz + \phi^{(L)})\|_T^2 = \|z\|^2 + \|Mz\|^2 + \|\phi^{(L)}\|^2$$

respectively.

The Theorem 2 shows that the relations (5) and (6) are equivalent. On the other hand, (5) is equivalent to each of following equalities:

vi) $UM_vV' = -E^{-1}$;

vii) $V'EU \downarrow H_L = L_u$;

viii) $VL_uU' = (E^*)^{-1}$;

ix) $U'E^*V \downarrow H_M = -M_v$

(see [1]). Further, $L_u^* = -M_v, L_u M_v = -\mathbb{I}_{H_M}, M_v L_u = -\mathbb{I}_{H_L}$ (it is proved in [7]; see also [1, p. 158]), consequently one can readily check by calculations that

$$S_u^* = -T_v, S_u(-T_v) = \begin{pmatrix} \mathbb{I}_{H_M} & 0 & 0 \\ 0 & 1_M & 0 \\ 0 & 0 & 1_L \end{pmatrix}, \quad (-T_v)S_u = \begin{pmatrix} \mathbb{I}_{H_M} & 0 & 0 \\ 0 & 1_L & 0 \\ 0 & 0 & 1_M \end{pmatrix}. \tag{14}$$

Furthermore, it is clear that

$$U_S \downarrow H_S = \begin{pmatrix} U_{H_L} & 0 & 0 \\ 0 & 1_L & 0 \\ 0 & 0 & 1_M \end{pmatrix}, \quad V_T \downarrow H_T = \begin{pmatrix} V_{H_M} & 0 & 0 \\ 0 & 1_M & 0 \\ 0 & 0 & 1_L \end{pmatrix}. \tag{15}$$

In addition, (7) implies

$$E_S^* = \begin{pmatrix} E^* & 0 & 0 \\ 0 & 0 & -1_L \\ 0 & 1_M & 0 \end{pmatrix}, \quad \hat{E}^{-1} = \begin{pmatrix} E^{-1} & 0 & 0 \\ 0 & 0 & -1_L \\ 0 & 1_M & 0 \end{pmatrix},$$

(16)

$$(E_S^*)^{-1} = \begin{pmatrix} (E^*)^{-1} & 0 & 0 \\ 0 & 0 & 1_M \\ 0 & -1_L & 0 \end{pmatrix}$$

and the equalities (15) imply

$$U_S^* = (U_S \downarrow H_S)^* = \begin{pmatrix} U' & 0 & 0 \\ 0 & 1_L & 0 \\ 0 & 0 & 1_M \end{pmatrix}, \quad V_T^* = (V_T \downarrow H_T)^* = \begin{pmatrix} V' & 0 & 0 \\ 0 & 1_M & 0 \\ 0 & 0 & 1_L \end{pmatrix}. \tag{17}$$

Taking into account (15)–(17), we obtain

$$V_T S_u U_S^* = \begin{pmatrix} V L_u U' & 0 & 0 \\ 0 & 0 & 1_M \\ 0 & -1_L & 0 \end{pmatrix}, \quad U_S T_v V_T^* = \begin{pmatrix} U M_v V' & 0 & 0 \\ 0 & 0 & 1_L \\ 0 & -1_M & 0 \end{pmatrix},$$

$$V_T^* E_S U_S \downarrow H_S = \begin{pmatrix} V' E U \downarrow H_L & 0 & 0 \\ 0 & 0 & 1_M \\ 0 & -1_L & 0 \end{pmatrix},$$

$$U_S E_S^* V_T \downarrow H_T = \begin{pmatrix} U' E^* V \downarrow H_M & 0 & 0 \\ 0 & 0 & -1_L \\ 0 & 1_M & 0 \end{pmatrix}.$$

Now the proof follows from the equalities vi)–ix). □

Remark 4. Taking into account (14), it is easy to conclude that (up to the mentioned identifications)

$$S_0^* = S^* \oplus S_u(S \ominus S_0), \quad T_0^* = T^* \oplus T_v(T \ominus T_0).$$

Corollary 2. *Suppose that the boundary pair (G_S, U_S) for (S, S_0) is as above and there exist the orthogonal decomposition $G_S = G_1 \oplus G_2$ and the operators $U_i \in \mathcal{B}(S, G_i)$ $(i = 1, 2)$ such that $U_s = U_1 \oplus U_2$. Then*

a) *there exist unique $\tilde{U}_1 \in \mathcal{B}(T, G_2)$,$\tilde{U}_2 \in \mathcal{B}(T, G_1)$ such that $(\tilde{G}_S, \tilde{U}_S)$ where $\tilde{G}_S = G_2 \oplus G_1$,$\tilde{U}_S = U_2 \oplus U_2$. is a boundary pair for (T, T_0) and*

$$\begin{aligned} &\forall (y, Ly + \phi^{(M)}) \in S, \quad \forall (z, Mz + \phi^{(L)}) \in T \\ &(Ly + \phi^{(M)} \mid z) - (y \mid Mz + \phi^{(L)}) \\ &\qquad = (iJ_S U_S(y, Ly + \phi^{(M)}) \mid \tilde{U}_S(z, Mz + \phi^{(L)}))_{\tilde{G}_S} \\ &\qquad = (U_S(y, Ly + \phi^{(M)}) \mid -iJ_S^* \tilde{U}_S(z, Mz + \phi^{L}))_{G_S} \\ &\qquad = (U_1(y, Ly + \phi^{(M)}) \mid \tilde{U}_2(z, Mz + \phi^{(L)}))_{G_1} \\ &\qquad - (U_2(y, Ly + \phi^{(M)}) \mid \tilde{U}_1(z, Mz + \phi^{(L)}))_{G_2} \end{aligned} \tag{18}$$

where

$$(\forall g_1 \in G_1)\ (\forall g_2 \in G_2) \quad J_S(g_1, g_2) = (ig_2, -ig_1). \tag{19}$$

b) *Let* $(\tilde{G}_S, \tilde{U}_S)$ *where* $\tilde{G}_S = G_2 \oplus G_1$, $\tilde{U}_S = \tilde{U}_1 \oplus \tilde{U}_2$ *is a boundary pair for* (T, T_0). *The following statements are equivalent*:

i) *the relation* (18) *holds*;

ii) $U_S T_v \tilde{U}_S^* = iJ_S^*$;

iii) $\tilde{U}_S^* J_S U_S \downarrow H_S = -iS_u$;

iv) $\tilde{U}_S S_u U_S^* = iJ_S$;

v) $U_S^* J_S^* \tilde{U}_S \downarrow H_T = -iT_v$.

Proof. The proof of Corollary 2 can be obtained from Theorem 2 and Corollary 1 by substituting $(G_T, V_T) = (\tilde{G}_S, \tilde{U}_S)$, $E_S = iJ_S$ into the corresponding formulas. □

3. The general form of mentioned above relation

Proposition 1. *Let* $G_i, U_i, \tilde{U}_i$ $(i = 1, 2)$ *be as in Corollary 2. Put* $S_1 = \ker U_1$. *Then* $S_1^* = \ker \tilde{U}_1$.

Proof. The inclusion $S_0 \subset S_1$ implies $S_1^* \subset T$. Further, (18) yields $\ker \tilde{U}_1 \subset S_1^*$. Conversely, assume that $(z, Mz + \phi^{(L)})$. Taking into account (18), we conclude that

$$\forall (y, Ly + \phi^{(M)}) \in S_1 = \ker U_1 \quad \Big(U_2(y, Ly + \phi^{(M)}) \mid \tilde{U}_1(z, Mz + \phi^{(L)}) \Big)_{G_2} = 0.$$

But the equalities $R(U_1 \oplus U_2) = G_1 \oplus G_2 = R(U_1) \oplus R(U_2)$ show that $R(U_2 \downarrow \ker U_1) = R(U_2) = G_2$, therefore $\tilde{U}_1(z, Mz + \phi^{(L)}) = 0$. In other words, $(z, Mz + \phi^{(L)}) \in \ker \tilde{U}_1$; thus $S_1^* \subset \ker \tilde{U}_1$. □

Theorem 3. *Assume that* $S_0 \subset S_1 = \overline{S_1} \subset S$ *and* G_S *is a boundary space for* (S, S_0). *Then*

i) *there exist the orthogonal decomposition* $G_S = G_1 \oplus G_2$ *and the operators*

$$U_1 \in \mathcal{B}(S, G_1), \quad V_1 \in \mathcal{B}(T, G_2) \tag{20}$$

such that

$$S_1 = \ker U_1, \quad S_1^* = \ker V_1, \tag{21}$$

sequently

$$\ker U_1 \supset S_0, \quad \ker V_1 \supset T_0; \tag{22}$$

ii) ii) *with the loss of generality, we may assume that*

$$R(U_1) = G_1, \quad R(V_1) = G_2. \tag{23}$$

Proof. Let (G_S, U_S) be a boundary pair for (S, S_0) . Put $G_2 = \big\{U_S(y, Ly + \phi^{(M)}) : (y, Ly + \phi^{(M)}) \in S_1\big\} = \big\{(U_S \downarrow H_S)(y, Ly + \phi^{(M)}) : (y, Ly + \phi^{(M)}) \in S_1 \ominus S_0\big\}$.

Since $U_S \downarrow H_S$ is a homeomorphism $H_S \to G_2 (\subset G_S)$, G_2 is a closed linear space of G_S . Put $G_1 = G_S \ominus G_2$, $U_i = P_i U_S$ where P_i $(i = 1, 2)$ are the orthoprojections $G_S \to G_i$, and denote by $\tilde{U}_1 \in \mathcal{B}(T, G_2)$, $\tilde{U}_2 \in \mathcal{B}(T, G_1)$ the operators uniquely determined by $U_i \in \mathcal{B}(S, G_i)$ $(i = 1, 2)$ from (18).

To complete the proof it is sufficient to substitute $V_1 = \tilde{U}_1$ into (20)–(23) and to apply Proposition 1. □

4. On mutual adjointness of considered relations

In this item the following problem is considered: under what conditions two closed relations satisfying the inclusions

$$S_0 \subset S_1 \subset S, \quad T_0 \subset T_1 \subset T$$

are mutually adjoint. Taking into account Theorem 3, we see that this problem may be

$$S_1 = \ker U_1, \quad T_1 = \ker V_1 \tag{24}$$

formulated in such way: assume that are as above (see (20), (22)), (24) (cf. (21)); establish the criterion of mutual adjointness of $\hat{L}_1$ and $\hat{M}_1$. Before to solve this problem let us introduce the following notations:

$$\begin{cases} X_1 = S_1 \ominus S_0, \quad X_2 = S \ominus S_1, \\ Y_1 = T_1 \ominus T_0, \quad Y_2 = T \ominus T_1. \end{cases} \tag{25}$$

It is clear that

$$H_S = X_1 \oplus X_2, \quad H_T = Y_1 \oplus Y_2, \tag{26}$$

$$S_0 \oplus X_1 = S_1 = \ker U_1, \quad T_0 \oplus Y_1 = T_1 = \ker V_1. \tag{27}$$

Moreover, by virtue of (14), $H_S = T_v H_T$. Whence using (26) and the unitarity of T_v we obtain

$$H_S = T_v H_T = T_v(Y_1 \oplus Y_2) = T_v Y_1 \oplus T_v Y_2. \tag{28}$$

Lemma 2.

$$T_1^* = S_0 \oplus T_v Y_2 = S_0 \oplus T_v \overline{R(V_1^*)}. \tag{29}$$

Proof. Applying the assertion from Remark 4 to the pair (T, T_1) (instead) (T, T_0) , we obtain $T_1^* = S_0 \oplus T_v(T \oplus T_1)$. Taking into account (24), (25), we have

$$Y_2 = T \ominus T_1 = T \ominus \ker V_1 = \overline{R(V_1^*)}.$$

This completes the proof of the lemma. □

Lemma 3. *The following statements are equivalent*:

i) $S_1 \supset T_1^*$;

ii) $U_1 T_v V_1^* = 0$;

iii) $\ker U_1 \supset S_0 \oplus T_v \overline{R(V_1^*)}$;

iv) $X_1 \supset T_v Y_2$.

In this case

$$S \ominus T_1^* = \ker U_1 \ominus \left(S_0 \oplus T_v \overline{R(V_1^*)}\right) = X \ominus T_v Y_2. \tag{30}$$

Proof. Taking into account (26)–(29) and the inclusion $\ker U_1 \supset S_0$ we obtain

$$\begin{aligned} U_1 T_v V_1^* = 0 &\Leftrightarrow \ker U_1 \supset T_v R(V_1^*) \Leftrightarrow \ker U_1 \supset T_v \overline{R(V_1^*)} \\ &\Leftrightarrow \ker U_1 \supset S_0 \oplus T_v \overline{R(V_1^*)} \Leftrightarrow S_1 \supset T_1^* \\ &\Leftrightarrow S_1 \ominus S_0 \supset T_1^* \ominus S_0 \Leftrightarrow X_1 \supset T_v Y_2. \end{aligned}$$

Therefore, the conditions i)–iv) are equivalent. Suppose these conditions take place. From (27) and (29) the equalities (30) are derived. □

Now we are able to formulate the main result of present paper.

Theorem 4.

$$S_1 = T_1^* \Leftrightarrow \ker U_1 = S_0 \oplus T_v \overline{R(V_1^*)} \Leftrightarrow X_1 = T_v Y_2. \tag{31}$$

Proof. Proof follows immediately from (30). □

Corollary 3. *Under the conditions of Theorem 4 suppose that* $\dim H_L, \infty$ *and equalities (23) hold. In this case* $S_1 = T_1^* \Leftrightarrow U_1 T_v V_1^* = 0$. *Proof can be obtained from Theorem 3 in the same way as in [15] the proof of Corollary 4.6.5 was obtained from Corollary 4.6.3.*

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