Methods of Functional Analysis and Topology
Vol. 20 (2014), no. 1, pp. 59–67

ON THE COMMON POINT SPECTRUM OF PAIRS OF SELF-ADJOINT EXTENSIONS

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Dedicated to Vladimir Koshmanenko on the occasion of his 70th birthday

Abstract. Given two different self-adjoint extensions of the same symmetric operator, we analyse the intersection of their point spectra. Some simple examples are provided.

1. Preliminaries

Given a linear closed operator L, we denote by

$$\mathcal{D}(L)\,,\quad \mathcal{K}(L)\,,\quad \mathcal{R}(L)\,,\quad \mathcal{G}(L)\,,\quad \rho(L)$$

its domain, kernel, range, graph and resolvent set respectively. $\mathcal{H}$ denotes a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$; we also make use of an auxiliary Hilbert space $\mathfrak{h}$ with scalar product $(\cdot,\cdot)$ and corresponding norm $|\cdot|$.

Given a closed, densely defined, symmetric operator

$$S:\mathcal{D}(S)\subseteq\mathcal{H}\to\mathcal{H}$$

with equal deficiency indices, by von Neumann's theory one has (here the direct sums are given w.r.t. the graph inner product of S^*)

$$\mathcal{D}(S^*)=\mathcal{D}(S)\oplus\mathcal{K}_+\oplus\mathcal{K}_-\,,\quad \mathcal{K}_\pm:=\mathcal{K}(-S^*\pm i)\,,$$

$$S^*(\phi_\circ\oplus\phi_+\oplus\phi_-)=S\phi_\circ+i\,\phi_+-i\,\phi_-\,,$$

and any self-adjoint extension of S is of the kind $A_U=S^*|\mathcal{G}(U)$, the restriction of S^* to $\mathcal{G}(U)$, where $U:\mathcal{K}_+\to\mathcal{K}_-$ is unitary. Therefore, fixing a unitary $U_\circ$ and posing $A:=A_{U_\circ}$, one has

$$S=A|\mathcal{K}(\tau_\circ)\,,\quad \tau_\circ:\mathcal{D}(A)\to\mathfrak{h}_\circ\,,$$

where

$$\mathfrak{h}_\circ=\mathcal{K}_+\,,\quad \tau_\circ=P_+\,,$$

and P_+ is the orthogonal (w.r.t. the graph inner product of S^*) projection onto $\mathcal{K}_+$. Since $\mathcal{K}(\tau_\circ)=\mathcal{K}(\tau)$ where $\tau=M\tau_\circ$ and $M:\mathfrak{h}_\circ\to\mathfrak{h}$ is any continuous linear bijection, in the search of the self-adjoint extension of S, we can consider the following equivalent problem: determine all the self-adjoint extensions of $A|\mathcal{K}(\tau)$, where

$$\tau:\mathcal{D}(A)\to\mathfrak{h}$$

is a linear, continuous (with respect to the graph norm on $\mathcal{D}(A)$), surjective map onto an auxiliary Hilbert space $\mathfrak{h}$ with its kernel $\mathcal{K}(\tau)$ dense in $\mathcal{H}$. Typically A is a differential operator, τ is some trace (restriction) operator along a null subset N and $\mathfrak{h}$ is some function space over N.

2000 *Mathematics Subject Classification.* Primary 47B25; Secondary 47B40, 35P99.
Key words and phrases. Self-adjoint extensions, Krein's resolvent formula, point spectrum.

We suppose that the spectrum of A does not coincide with the whole real line and so, by eventually adding a constant to A, we make the following hypothesis:

$$0 \in \rho(A)\,.$$

By the results provided in [9] and [7] (to which we refer for proofs and connections with equivalent formulations, in particular with boundary triplets theory) one has the following

Theorem 1.1. *The set of all self-adjoint extensions of S is parametrized by the set $\mathsf{E}(\mathfrak{h})$ of couples (Π,Θ), where Π is an orthogonal projection in $\mathfrak{h}$ and Θ is a self-adjoint operator in $\mathcal{R}(\Pi)$. If $A^{\Pi,\Theta}$ denotes the self-adjoint extension corresponding to $(\Pi,\Theta)\in\mathsf{E}(\mathfrak{h})$ then*

$$A^{\Pi,\Theta}:\mathcal{D}(A^{\Pi,\Theta})\subseteq\mathcal{H}\to\mathcal{H}\,,\quad A^{\Pi,\Theta}\phi:=A\phi_0\,,$$

$$\mathcal{D}(A^{\Pi,\Theta}):=\{\phi=\phi_0+G_0\xi_\phi\,,\ \phi_0\in\mathcal{D}(A)\,,\,\xi_\phi\in\mathcal{D}(\Theta)\,,\,\Pi\tau\phi_0=\Theta\xi_\phi\}\,,$$

where

$$G_z:\mathfrak{h}\to\mathcal{H}\,,\quad G_z:=\left(\tau(-A+\bar z)^{-1}\right)^*\,,\quad z\in\rho(A)\,.$$

Moreover the resolvent of $A^{\Pi,\Theta}$ is given, for any $z\in\rho(A)\cap\rho(A^{\Pi,\Theta})$, by the Kreĭn's type formula

$$(-A^{\Pi,\Theta}+z)^{-1}=(-A+z)^{-1}+G_z\Pi(\Theta+z\Pi G_0^*G_z\Pi)^{-1}\Pi G_{\bar z}^*\,.$$

Remark 1.2. Notice that the extension corresponding to $\Pi=0$ is A itself. The extension corresponding to $(1,\Theta)$ is denoted by A^Θ and everywhere we omit the index Π in the case $\Pi=1$. By [8], Corollary 3.2, the sub-family $\{A^\Theta:\Theta\text{ self-adjoint}\}$ gives all singular perturbations of A, where we say that $\hat A$ is a singular perturbation of A whenever the set $\{\phi\in\mathcal{D}(A)\cap\mathcal{D}(\hat A):A\phi=\hat A\phi\}$ is dense in $\mathcal{H}$ (see [4]).

Remark 1.3. The operator G_z is injective (by surjectivity of τ) and for any $z\in\rho(A)$ one has (see [7], Remark 2.8)

$$\mathcal{R}(G_z)\cap\mathcal{D}(A)=\{0\}\,, \tag{1.1}$$

so that the decomposition appearing in $\mathcal{D}(A^{\Pi,\Theta})$ is unique. Moreover (see [7], Lemma 2.1)

$$G_w-G_z=(z-w)(-A+w)^{-1}G_z\,. \tag{1.2}$$

2. The common point spectrum

Given a self-adjoint operator A let us denote by

$$\sigma(A)\,,\quad\sigma_p(A)\,,\quad\sigma_d(A)$$

its full, point and discrete spectrum respectively.

Given $\lambda\in\sigma_p(A)$, we denote by P_λ the orthogonal projector onto the corresponding eigenspace $\mathcal{H}_\lambda\subseteq\mathcal{D}(A)$ and pose $P_\lambda^\perp:=1-P_\lambda$.

Given $\lambda\in\sigma_p(A^{\Pi,\Theta})$, we denote by $\mathcal{H}_\lambda^{\Pi,\Theta}\subseteq\mathcal{D}(A^{\Pi,\Theta})$ the corresponding eigenspace.

As regards the eigenvalues of $A^{\Pi,\Theta}$ which are not in the spectrum of A a complete answer is given by the following result which is consequence of Kreĭn's resolvent formula (see [3], Section 2, Propositions 1 and 2, and [8], Theorem 3.4):

Lemma 2.1.

$$\lambda\in\rho(A)\cap\sigma_p(A^{\Pi,\Theta})\iff 0\in\sigma_p(\Theta+\lambda\Pi G_0^*G_\lambda\Pi)\,,$$

$$\mathcal{H}_\lambda^{\Pi,\Theta}=\{G_\lambda\xi\,,\ \xi\in\mathcal{K}(\Theta+\lambda\Pi G_0^*G_\lambda\Pi)\}\,.$$

Here we are interested in the common eigenvalues, i.e. in the points in $\sigma_p(A)\cap\sigma_p(A^{\Pi,\Theta})$. Therefore we take $\lambda\in\sigma_p(A)$ and we look for solutions $\phi\in\mathcal{D}(A^{\Pi,\Theta})$ of the eigenvalue equation

$$A^{\Pi,\Theta}\phi=\lambda\phi\,,$$

i.e., by Theorem 1.1,

$$(A-\lambda)\phi_0=\lambda G_0\xi_\phi\,.$$

By

$$(A-\lambda)P_\lambda\phi_0=0\,,\quad (A-\lambda)P_\lambda^\perp\phi_0\in\mathcal{R}(P_\lambda^\perp)\,,$$

this is equivalent to the couple of equations

$$P_\lambda G_0\xi_\phi=0\,, \tag{2.1}$$

$$(A-\lambda)P_\lambda^\perp\phi_0=\lambda P_\lambda^\perp G_0\xi_\phi \tag{2.2}$$

together with the constraint

$$\xi_\phi\in\mathcal{D}(\Theta)\subseteq\mathcal{R}(\Pi)\,,\quad \Pi\tau\phi_0=\Theta\xi_\phi\,. \tag{2.3}$$

Equation (2.1) gives, for all $\psi\in\mathcal{H}$,

$$0=\langle G_0\xi_\phi,P_\lambda\psi\rangle=-\langle\xi_\phi,\tau A^{-1}P_\lambda\psi\rangle=-\frac{1}{\lambda}\,\langle\xi_\phi,\tau P_\lambda\psi\rangle$$

and so

$$\xi_\phi\in(\mathcal{R}(\tau P_\lambda))^\perp\,.$$

If $\mathcal{R}(\Pi)\cap(\mathcal{R}(\tau P_\lambda))^\perp=\{0\}$ then, since G_0 is injective, one has that in this case ϕ is an eigenvector with eigenvalue λ if and only if $\phi\in\mathcal{H}_\lambda$ and $\Pi\tau\phi=0$.

Conversely suppose that $\mathcal{R}(\Pi)\cap(\mathcal{R}(\tau P_\lambda))^\perp\neq\{0\}$ and moreover that λ is an isolated eigenvalue. Then $\lambda\in\rho(A|\mathcal{H}_\lambda^\perp)$ and (2.2) gives

$$P_\lambda^\perp\phi_0=-\lambda(-A+\lambda)^{-1}P_\lambda^\perp G_0\xi_\phi\,.$$

By $\Pi\tau\phi_0=\Theta\xi_\phi$ then one gets

$$\Pi\tau P_\lambda\phi_0=\Theta\xi_\phi-\Pi\tau P_\lambda^\perp\phi_0=(\Theta+\lambda\Pi\tau(-A+\lambda)^{-1}P_\lambda^\perp G_0\Pi)\xi\,. \tag{2.4}$$

By defining

$$G_\lambda^\perp:\mathfrak{h}\to\mathcal{H}\,,\quad G_\lambda^\perp:=(\tau(-A+\lambda)^{-1}P_\lambda^\perp)^*\,,$$

and by $(G_\lambda^\perp)^*G_0=G_0^*G_\lambda^\perp$ (this relation is consequence of (1.2)), (2.4) is equivalent to

$$\Pi\tau P_\lambda\phi_0=(\Theta+\lambda\Pi G_0^*G_\lambda^\perp\Pi)\xi\,.$$

Moreover by $(-A+\lambda)^{-1}P_\lambda^\perp G_0=-A^{-1}G_\lambda^\perp$ one has

$$\begin{aligned}P_\lambda\phi_0+P_\lambda^\perp\phi_0+G_0\xi_\phi=&P_\lambda\phi_0+(-\lambda(-A+\lambda)^{-1}P_\lambda^\perp+P_\lambda^\perp)G_0\xi_\phi\\=&P_\lambda\phi_0-A(-A+\lambda)^{-1}P_\lambda^\perp G_0\xi_\phi\\=&P_\lambda\phi_0+G_\lambda^\perp\xi_\phi\,.\end{aligned}$$

In conclusion we have proven the following

Theorem 2.2. *Let $\lambda\in\sigma_p(A)$.*
1) Suppose

$$\mathcal{R}(\Pi)\cap(\mathcal{R}(\tau P_\lambda))^\perp=\{0\}$$

and pose

$$\mathcal{K}_\lambda^\Pi:=\{\psi\in\mathcal{H}_\lambda\,:\,\Pi\tau\psi=0\}\,.$$

Then

$$\lambda\in\sigma_p(A^{\Pi,\Theta})\quad\iff\quad\mathcal{K}_\lambda^\Pi\neq\{0\}$$

and

$$\mathcal{H}_\lambda^{\Pi,\Theta}=\mathcal{K}_\lambda^\Pi\,.$$

2) Suppose

$$\mathcal{R}(\Pi)\cap(\mathcal{R}(\tau P_\lambda))^\perp\neq\{0\}$$

and let λ *be isolated.*

Let $\mathcal{N}_\lambda^{\Pi,\Theta}$ *be the set of couples* $(\psi,\xi)\in\mathcal{H}_\lambda\oplus\mathcal{R}(\Pi)$ *such that*

$$\xi\in D(\Theta)\cap(\mathcal{R}(\tau P_\lambda))^\perp\,, \tag{2.5}$$

$$\Pi\tau\psi=(\Theta+\lambda\Pi G_0^* G_\lambda^\perp\Pi)\,\xi\,. \tag{2.6}$$

Then

$$\lambda\in\sigma_p(A^{\Pi,\Theta})\quad\Longleftrightarrow\quad\mathcal{N}_\lambda^{\Pi,\Theta}\neq\{0\}\,,$$

$$\dim(\mathcal{H}_\lambda^{\Pi,\Theta})=\dim(\mathcal{N}_\lambda^{\Pi,\Theta})$$

and

$$\mathcal{H}_\lambda^{\Pi,\Theta}=\{\phi\in\mathcal{H}\,:\,\phi=\psi+G_\lambda^\perp\xi\,,\ \ (\psi,\xi)\in\mathcal{N}_\lambda^{\Pi,\Theta}\}\,.$$

Remark 2.3. Notice that

$$(\mathcal{R}(\Pi))^\perp\cap\mathcal{R}(\tau P_\lambda)\neq\{0\}\quad\Longrightarrow\quad\mathcal{K}_\lambda^\Pi\neq\{0\}\,.$$

Remark 2.4. Suppose $\lambda\in\sigma_p(A)$ is isolated. Noticing that

$$\mathcal{K}_\lambda^\Pi\oplus(\mathcal{R}(\Pi)\cap(\mathcal{R}(\tau P_\lambda))^\perp\cap\mathcal{K}(\Theta+\lambda\Pi G_0^*G_\lambda^\perp\Pi))\subseteq\mathcal{N}_\lambda^{\Pi,\Theta}\,,$$

one has

$$\mathcal{K}_\lambda^\Pi\neq\{0\}\quad\Longrightarrow\quad\lambda\in\sigma_p(A^{\Pi,\Theta})\,.$$

In particular, in the case λ is simple with eigenvector ψ_λ,

$$\Pi\tau\psi_\lambda=0\quad\Longrightarrow\quad\lambda\in\sigma_p(A^{\Pi,\Theta})\,.$$

Remark 2.5. Suppose $\mathcal{R}(\tau P_\lambda)=\{0\}$. Then $\mathcal{K}_\lambda^\Pi=\mathcal{H}_\lambda$ and so, in case $\lambda\in\sigma_p(A)$ is isolated, $\lambda\in\sigma_p(A^{\Pi,\Theta})$ and

$$\mathcal{H}_\lambda^{\Pi,\Theta}=\{\phi=\psi_\lambda+G_\lambda^\perp\xi\,,\ \psi_\lambda\in\mathcal{H}_\lambda\,,\ \xi\in\mathcal{R}(\Pi)\cap\mathcal{K}(\Theta+\lambda\Pi G_0^*G_\lambda^\perp\Pi)\}\,.$$

Remark 2.6. The papers [1] and [6] contain results related to the ones given by Theorem 2.2 (see Theorem 3.6 in [6] and Theorem 4.7 in [1]). We thank Konstantin Pankrashkin for the communication.

3. Examples

3.1. **Rank-one singular perturbations.** Suppose $\mathfrak{h}=\mathbb{C}$. Then $\Pi=1$, $\Theta=\theta\in\mathbb{R}$ and either $\mathcal{R}(\tau P_\lambda)=\mathbb{C}$ or $\mathcal{R}(\tau P_\lambda)=\{0\}$.

If $\mathcal{R}(\tau P_\lambda)=\mathbb{C}$ then $\lambda\in\sigma_p(A^\theta)$ if and only if $\mathcal{K}_\lambda\neq\{0\}$, where

$$\mathcal{K}_\lambda:=\{\psi\in\mathcal{H}_\lambda\,:\,\tau\psi=0\}\,.$$

Since $\mathcal{R}(\tau P_\lambda)=\{0\}$ if and only if $\mathcal{K}_\lambda=\mathcal{H}_\lambda$, when λ is isolated and $\mathcal{R}(\tau P_\lambda)=\{0\}$ one has

$$\mathcal{N}_\lambda^\theta=\mathcal{K}_\lambda\oplus\{\xi\in\mathbb{C}\,:\,(\theta+\lambda\langle G_0,G_\lambda^\perp\rangle)\,\xi=0\}$$

and so

$$\theta+\lambda\langle G_0,G_\lambda^\perp\rangle=0\quad\Longrightarrow\quad\mathcal{N}_\lambda^\theta=\mathcal{K}_\lambda\oplus\mathbb{C}\equiv\mathcal{H}_\lambda\oplus\mathbb{C}\,,$$

$$\theta+\lambda\langle G_0,G_\lambda^\perp\rangle\neq0\quad\Longrightarrow\quad\mathcal{N}_\lambda^\theta=\mathcal{K}_\lambda\oplus\{0\}\equiv\mathcal{H}_\lambda\,.$$

In conclusion when $\mathfrak{h}=\mathbb{C}$ and $\lambda\in\sigma_p(A)$ is isolated,

$$\lambda\in\sigma_p(A^\theta)\iff\mathcal{K}_\lambda\neq\{0\} \tag{3.1}$$

and

$$\mathcal{H}_\lambda^\theta=\{\psi=\psi_\lambda+G_\lambda^\perp\xi\,,\ \psi_\lambda\in\mathcal{H}_\lambda\,,\ (\theta+\lambda\langle G_0,G_\lambda^\perp\rangle)\xi=0\}\,.$$

In particular if λ is a simple isolated eigenvalue of A with corresponding eigenfunction ψ_λ, then $\lambda \in \sigma_p(A^\theta)$ if and only if $\tau\psi_\lambda = 0$. For example, if $\mathcal{H} = L^2(\Omega)$ and $\tau : \mathcal{D}(A) \to \mathbb{C}$ is the evaluation map at $y \in \Omega$, $\tau\psi := \psi(y)$, then λ is preserved if and only if y belongs to the nodal set (if any) of ψ_λ. Thus if A is (minus) the Dirichlet Laplacian on a bounded open set $\Omega \subset \mathbb{R}^d$, $d \leq 3$, its lowest eigenvalue is never preserved under a point perturbation. Analogous results hold in the case A is the Laplace-Beltrami operator on a compact d-dimensional Riemannian manifold M, $d \leq 3$, thus reproducing the ones given in [2], Theoreme 2, part 1.

3.2. **The Šeba billiard.** Let

$$A = \Delta : \mathcal{D}(A) \subset L^2(R) \to L^2(R)\,,$$

$$\mathcal{D}(A) = \{\phi \in C(\overline{R}) : \Delta\phi \in L^2(R)\,,\ \phi(\mathsf{x}) = 0\,,\ \mathsf{x} \in \partial R\}\,,$$

be the Dirichlet Laplacian on the rectangle $R = (0, a) \times (0, b)$. Then

$$\sigma(A) = \sigma_d(A) = \left\{\lambda_{m,n}\,,\ (m, n) \in \mathbb{N}^2\right\}$$

and

$$\mathcal{H}_{\lambda_{m,n}} = \text{span}\{\psi_{m',n'}\ :\ \lambda_{m',n'} = \lambda_{m,n}\}\,,$$

where

$$\lambda_{m,n} := -\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

and

$$\psi_{m,n}(\mathsf{x}) := \sin\left(\frac{m\pi x_1}{a}\right) \sin\left(\frac{n\pi x_2}{b}\right)\,, \quad \mathsf{x} \equiv (x_1, x_2)\,.$$

Let

$$\tau\psi := \psi(\mathsf{y})\,,$$

so that A^θ describes a "Šeba billiard", i.e. the Dirichlet Laplacian on the rectangle R with a point perturbation placed at the point $\mathsf{y} \equiv (y_1, y_2)$ (see [10]).

Since $\sigma(A) = \sigma_d(A)$, by the invariance of the essential spectrum under finite rank perturbations, one has $\sigma(A^\theta) = \sigma_d(A^\theta)$ and, by (3.1), $\lambda_{m,n} \in \sigma(A) \cap \sigma(A^\theta)$ if and only

$$\forall\,(m', n') \text{ s.t. } \lambda_{m',n'} = \lambda_{m,n}\,, \quad \sin\left(\frac{m'\pi y_1}{a}\right) \sin\left(\frac{n'\pi y_2}{b}\right) = 0\,.$$

Equivalently

$$\sigma(A) \cap \sigma(A^\theta) = \emptyset \quad \iff \quad \left(\frac{y_1}{a}, \frac{y_2}{b}\right) \notin \mathbb{Q}^2\,.$$

If there exists relatively prime integers $1 \leq p < q$ such that $\frac{y_1}{a} = \frac{p}{q}$ while $\frac{y_2}{b}$ is irrational, then

$$\sigma(A) \cap \sigma(A^\theta) = \{\lambda_{kq,n}\,,\ (k, n) \in \mathbb{N}^2\}\,.$$

Analogously if $\frac{y_1}{a}$ is irrational and $\frac{y_2}{b} = \frac{p}{q}$ then

$$\sigma(A) \cap \sigma(A_\theta) = \{\lambda_{m,kq}\,,\ (m, k) \in \mathbb{N}^2\}$$

while if $\frac{y_1}{a} = \frac{p}{q}$ and $\frac{y_2}{b} = \frac{r}{s}$, then

$$\sigma(A) \cap \sigma(A^\theta) = \{\lambda_{kq,n}\,,\ (k, n) \in \mathbb{N}^2\} \cup \{\lambda_{m,ks}\,,\ (m, k) \in \mathbb{N}^2\}\,.$$

3.3. **Rank-two singular perturbations.** Let $\mathfrak{h}=\mathbb{C}^2$. Then either $\Pi=1$ or $\Pi=w\otimes w$, $w\in\mathbb{C}^2$, $|w|=1$. Let $\lambda\in\sigma_p(A)$.

1.1) $\mathcal{R}(\tau P_\lambda)=\mathbb{C}^2$, $\Pi=1$. Then $\lambda\in\sigma_p(A^\Theta)$ if and only if there exists $\psi\in\mathcal{H}_\lambda\backslash\{0\}$ such that $\tau\psi=0$.

1.2) $\mathcal{R}(\tau P_\lambda)=\mathbb{C}^2$, $\Pi=w\otimes w$. Then $\lambda\in\sigma_p(A^{\Pi,\Theta})$ if and only if there exists $\psi\in\mathcal{H}_\lambda\backslash\{0\}$ such that $w\cdot\tau\psi=0$.

Now suppose further that $\lambda\in\sigma_p(A)$ is isolated.

2.1) $\mathcal{R}(\tau P_\lambda)=\text{span}(\xi_\lambda)\simeq\mathbb{C}$, $|\xi_\lambda|=1$, $\Pi=1$. Decomposing equation (2.6) w.r.t. the orthonormal base $\{\xi_\lambda,\xi_\lambda^\perp\}$ one gets that $\mathcal{N}_\lambda^\Theta\neq\{0\}$ if and only if there exists $\zeta\equiv(\zeta_1,\zeta_2)\in\mathbb{C}^2\backslash\{0\}$ solving

$$\begin{cases}\zeta_1=(\xi_\lambda\cdot(\Theta+\lambda G_0^*G_\lambda^\perp)\xi_\lambda^\perp)\zeta_2,\\ 0=(\xi_\lambda^\perp\cdot(\Theta+\lambda G_0^*G_\lambda^\perp)\xi_\lambda^\perp)\zeta_2\,.\end{cases}$$

Hence

$$\lambda\in\sigma_p(A^\Theta)\iff(\xi_\lambda^\perp\cdot(\Theta+\lambda G_0^*G_\lambda^\perp)\xi_\lambda^\perp)=0\,.$$

2.2) $\mathcal{R}(\tau P_\lambda)=\text{span}(\xi_\lambda)\simeq\mathbb{C}$, $\Pi=w\otimes w$. Let us use the decomposition $w=w_{||}+w_\perp$ w.r.t. the orthonormal base $\{\xi_\lambda,\xi_\lambda^\perp\}$. If $w_{||}=0$ then $\mathcal{K}_\lambda^\Pi\neq\{0\}$ and so $\lambda\in\sigma_p(A^{\Pi,\Theta})$. If $w_{||}\neq0$ then $\mathcal{K}_\lambda^\Pi=\{0\}$ and $\mathcal{R}(\Pi)\cap(\mathcal{R}(\tau P_\lambda))^\perp=\{0\}$, thus $\lambda\notin\sigma_p(A^{\Pi,\Theta})$. In conclusion

$$\lambda\in\sigma_p(A^{\Pi,\Theta})\qquad\iff\qquad w=\xi_\lambda^\perp\,.$$

3) $\mathcal{R}(\tau P_\lambda)=\{0\}$. In this case $\lambda\in\sigma_p(A^{\Pi,\Theta})$.

3.4. **The Laplacian on a bounded interval.** Let

$$A:\mathcal{D}(A)\subseteq L^2(0,a)\to L^2(0,a)\,,\quad A\phi=\phi''\,,$$

$$\mathcal{D}(A)=\{\phi\in C^1[0,a]\,:\,\phi''\in L^2(0,a)\,,\ \phi(0)=\phi(a)=0\},$$

be the Dirichlet Laplacian on the bounded interval $(0,a)$ and pose

$$\tau:\mathcal{D}(A)\to\mathbb{C}^2\,,\quad\tau\phi\equiv\gamma_1\phi:=(\phi'(0),-\phi'(a))\ .$$

Therefore $S=A|\mathcal{K}(\tau)$ is the minimal Laplacian with domain

$$\mathcal{D}(S)=\{\phi\in C^1[0,a]\,:\,\phi''\in L^2(0,a)\,,\ \phi(0)=\phi'(0)=\phi(a)=\phi'(a)=0\}$$

and the self-adjoint extensions of S are rank-two perturbations of the Dirichlet Laplacian A. One has

$$\sigma(A)=\sigma_d(A)=\{\lambda_n\}_1^\infty\,,\quad\lambda_n=-\left(\frac{n\pi}{a}\right)^2$$

and the normalized eigenvector corresponding to λ_n is

$$\psi_n(x)=\sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi x}{a}\right)\,.$$

By Theorem 1.1 and by the change of extension parameter (here P_0 represents the Dirichlet-to-Neumann operator)

$$(\Pi,\Theta)\mapsto(\Pi,B)\,,\quad B:=\Theta-\Pi P_0\Pi\,,\quad P_0\equiv\frac{1}{a}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

any self-adjoint extension of the minimal Laplacian S is of the kind $A^{\Pi,B}$, $(\Pi,B)\in\mathsf{E}(\mathbb{C}^2)$, where

$$A^{\Pi,B}:\mathcal{D}(A^{\Pi,B})\subset L^2(0,a)\to L^2(0,a)\,,\quad A^{\Pi,B}\phi=\phi''\,,$$

$$\mathcal{D}(A^{\Pi,B})=\{\phi\in C^1[0,a]:\phi''\in L^2(0,a)\,,\ \gamma_0\phi\in\mathcal{R}(\Pi)\,,\ \Pi\gamma_1\phi=B\gamma_0\phi\}$$

(see e.g. [9], Example 5.1). Here $\gamma_0\phi:=(\phi(0),\phi(a))$.

The case $\Pi = 0$ reproduces A itself, the case $\Pi = 1$, $B = \begin{pmatrix} b_{11} & b_{12} \\ \bar b_{12} & b_{22} \end{pmatrix}$, $b_{11}, b_{22} \in \mathbb{R}$, $b_{12} \in \mathbb{C}$, gives the boundary conditions

$$\begin{cases} b_{11}\,\phi(0) - \phi'(0) + b_{12}\,\phi(a) = 0\,, \\ \bar b_{12}\,\phi(0) + b_{22}\,\phi(a) + \phi'(a) = 0\,, \end{cases}$$

and the case $\Pi = w \otimes w$, $w \equiv (w_1, w_2) \in \mathbb{C}^2$, $|w_1|^2 + |w_2|^2 = 1$, $B \equiv b \in \mathbb{R}$, gives the boundary conditions

$$\begin{cases} w_2\,\phi(0) - w_1\,\phi(a) = 0\,, \\ \bar w_1\,(b\,\phi(0) - \phi'(0)) + \bar w_2\,(b\,\phi(a) + \phi'(a)) = 0\,. \end{cases}$$

By the invariance of the essential spectrum under finite rank perturbations, $\sigma(A^{\Pi,B}) = \sigma_d(A^{\Pi,B})$. Now we use the results given in subsection 3.3. One has

$$\mathcal{R}(\tau P_{\lambda_n}) = \operatorname{span}(\hat\xi_n)\,, \quad \hat\xi_n \equiv \frac{1}{\sqrt{2}}\left(1, (-1)^{n-1}\right)\,.$$

Let $\Pi = 1$ and $\hat\xi_n^\perp \equiv \frac{1}{\sqrt{2}}\,(1, (-1)^n)$. By point 2.1 in subsection 3.3 we known that $\lambda_n \in \sigma(A^B)$ if and only if $\hat\xi_n^\perp \cdot (B + P_0 + \lambda_n G_0^* G_{\lambda_n}^\perp)\hat\xi_n^\perp = 0$. Since the resolvent of A is explicitly known, $\hat\xi_n^\perp \cdot (B + P_0 + \lambda_n G_0^* G_{\lambda_n}^\perp)\hat\xi_n^\perp$ can be calculated. However we use here a short cut which avoids any calculation: the Neumann Laplacian corresponds to $B = 0$ and we know that its spectrum is $\{0\} \cup \sigma(A)$, thus

$$\hat\xi_n^\perp \cdot (P_0 + \lambda_n G_0^* G_{\lambda_n}^\perp)\hat\xi_n^\perp = 0. \tag{3.2}$$

Therefore we obtain

$$\lambda_n \in \sigma(A^B) \quad \Longleftrightarrow \quad b_{11} + b_{22} + 2\,(-1)^n \operatorname{Re}(b_{12}) = 0\,.$$

If $\Pi = w \otimes w$ by point 2.2 in subsection 3.3 one has

$$\lambda_n \in \sigma(A^{\Pi,B}) \quad \Longleftrightarrow \quad w = \hat\xi_n^\perp\,.$$

In both cases

$$\lambda_n \in \sigma(A^{\Pi,B}) \quad \Longleftrightarrow \lambda_{n+2} \in \sigma(A^{\Pi,B})\,.$$

Moreover

$$\sigma(A) \subseteq \sigma(A^{\Pi,B}) \quad \Longleftrightarrow \quad \Pi = 1 \text{ and } b_{11} + b_{22} = 0,\ \operatorname{Re}(b_{12}) = 0\,.$$

3.5. **Equilateral quantum graphs.** Let $\mathcal{H} = \oplus_{k=1}^N L^2(0,a)$ and $A_N = \oplus_{k=1}^N A$, where A is defined as in subsection 3.4 (to which we refer for notations). Then $\sigma(A_N) = \sigma_d(A_N) = \sigma(A)$ and the eigenfunctions corresponding to the N-fold degenerate eigenvalue λ_n are

$$\Psi_{k,n} = \oplus_{i=1}^N \psi_{i,k,n}\,, \quad k = 1, \dots, N\,, \quad \psi_{i,k,n} = \begin{cases} 0\,, & i \neq k, \\ \psi_n\,, & i = k\,. \end{cases}$$

By taking

$$\tau : \mathcal{D}(A_N) \equiv \oplus_{k=1}^N \mathcal{D}(A) \to \oplus_{k=1}^N \mathbb{C}^2 \equiv \mathbb{C}^{2N}\,, \quad \tau = \oplus_{k=1}^N \gamma_1\,,$$

one gets, by Theorem 1.1, self-adjoint extensions describing quantum graphs (see e.g. [5]) with N edges of the same length a . By Theorem 1.1 and by the change of extension parameter

$$(\Pi, \Theta) \mapsto (\Pi, B)\,, \quad B := \Theta - \Pi(\oplus_{k=1}^N P_0)\Pi\,,$$

such extensions are of the kind $A^{\Pi,B}$, $(\Pi, B) \in \mathsf{E}(\mathbb{C}^{2N})$, where (see [9], Example 5.2).

$$A^{\Pi,B} : \mathcal{D}(A^{\Pi,B}) \subset \oplus_{k=1}^N L^2(0,a) \to \oplus_{k=1}^N L^2(0,a)\,,$$

$$A^{\Pi,B}(\oplus_{k=1}^N \phi_k) = \oplus_{k=1}^N \phi_k''\,,$$

$$\mathcal{D}(A^{\Pi,B}) = \Big\{ \oplus_{k=1}^{N} \phi_k : \phi_k \in C^1[0,a]\,,\ \phi_k'' \in L^2(0,a)\,,$$
$$(\oplus_{k=1}^{N}\gamma_0\phi_k) \in \mathcal{R}(\Pi)\,,\ \Pi(\oplus_{k=1}^{N}\gamma_1\phi_k) = B(\oplus_{k=1}^{N}\gamma_0\phi_k)\Big\}\,.$$

The couple (Π, B) represents the connectivity of the quantum graph.

1) $\Pi = 1$. Given $\lambda_n \in \mathcal{D}(A)$, we pose

$$\mathbb{C}^{2N}_{||} := \oplus_{k=1}^{N}\text{span}(\hat{\xi}_n) \simeq \mathbb{C}^N\,, \quad \mathbb{C}^{2N}_{\perp} := \oplus_{k=1}^{N}\text{span}(\hat{\xi}_n^{\perp}) \simeq \mathbb{C}^N\,,$$

so that $\mathcal{R}(\tau P_{\lambda_n}) = \mathbb{C}^{2N}_{||}$, $(\mathcal{R}(\tau P_{\lambda_n}))^{\perp} = \mathbb{C}^{2N}_{||}$, $\mathbb{C}^{2N} = \mathbb{C}^{2N}_{||} \oplus \mathbb{C}^{2N}_{\perp}$ and for any linear operator $L : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ we can consider the block decomposition $L = \begin{pmatrix} L_{||} & L_{||\perp} \\ (L_{||\perp})^* & L_{\perp}\end{pmatrix}$. By using such decompositions in equation (2.6) one gets that $\mathcal{N}^{\Theta}_{\lambda_n} \neq \{0\}$, $\Theta = B + \oplus_{k=1}^{N} P_0$, if and only if there exists $\zeta \neq \{0\}$, $\zeta = \zeta_{||} \oplus \zeta_{\perp} \in \mathbb{C}^{2N}_{||} \oplus \mathbb{C}^{2N}_{\perp}$ solving

$$\begin{cases} \zeta_{||} = (B + \oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G^{\perp}_{\lambda_n})_{||\perp}\,\zeta_{\perp}\,, \\ 0 = (B + \oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G^{\perp}_{\lambda_n})_{\perp}\,\zeta_{\perp}\,. \end{cases}$$

By (3.2) one obtains $(\oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G^{\perp}_{\lambda_n})_{\perp} = 0$. Therefore one gets

$$\lambda_n \in \sigma(A^B) \iff \det(B_{\perp}) = 0\,.$$

2) $\Pi \neq 1$. Given $\lambda_n \in \mathcal{D}(A)$ we pose

$$\mathcal{R}(\Pi)_{||} := \mathcal{R}(\Pi) \cap (\oplus_{k=1}^{N}\text{span}(\hat{\xi}_n)), \quad \mathcal{R}(\Pi)_{\perp} := \mathcal{R}(\Pi) \cap (\oplus_{k=1}^{N}\text{span}(\hat{\xi}_n^{\perp}))\,,$$

so that $\mathcal{R}(\Pi) \cap \mathcal{R}(\tau P_{\lambda_n}) = \mathcal{R}(\Pi)_{||}$, $\mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda_n}))^{\perp} = \mathcal{R}(\Pi)_{\perp}$, $\mathcal{R}(\Pi) = \mathcal{R}(\Pi)_{||} \oplus \mathcal{R}(\Pi)_{\perp}$ and for any linear operator $L : \mathcal{R}(\Pi) \to \mathcal{R}(\Pi)$ we can consider the block decomposition $L = \begin{pmatrix} L_{||} & L_{||\perp} \\ (L_{||\perp})^* & L_{\perp}\end{pmatrix}$.

Define $\hat{\xi}_{k,n} = \oplus_{i=1}^{n}\hat{\xi}_{i,k,n} \in \mathbb{C}^{2N}$ and $\hat{\xi}^{\perp}_{k,n} = \oplus_{i=1}^{n}\hat{\xi}^{\perp}_{i,k,n} \in \mathbb{C}^{2N}$, $k = 1, \dots, N$, by

$$\hat{\xi}_{i,k,n} := \begin{cases} 0\,, & i \neq k, \\ \hat{\xi}_n\,, & i = k\,, \end{cases} \quad \hat{\xi}^{\perp}_{i,k,n} := \begin{cases} 0\,, & i \neq k, \\ \hat{\xi}^{\perp}_n\,, & i = k\,. \end{cases}$$

If $\Pi\hat{\xi}^{\perp}_{k,n} = 0$ for all k then $\mathcal{R}(\Pi)_{\perp} = \{0\}$ and in this case

$$\lambda \in \sigma_p(A^{\Pi,B}) \iff \exists k \text{ s.t } \Pi\hat{\xi}_{k,n} = 0\,.$$

If there exists k' such that $\Pi\hat{\xi}^{\perp}_{k',n} \neq 0$ then $\mathcal{R}(\Pi)_{\perp} \neq \{0\}$. By Remark 2.3

$$\exists k \text{ s.t } \Pi\hat{\xi}_{k,n} = 0 \implies \lambda \in \sigma_p(A^{\Pi,B})\,.$$

Suppose now $\Pi\hat{\xi}_{k,n} \neq 0$ for all k, i.e. $\mathcal{K}^{\Pi}_{\lambda_n} = \{0\}$. Then, using the above decompositions in equation (2.6) one gets that $\mathcal{N}^{\Pi,\Theta}_{\lambda_n} \neq \{0\}$, $\Theta = B + \Pi(\oplus_{k=1}^{N} P_0)\Pi$, if and only if there exists $\zeta \neq 0$, $\zeta = \zeta_{||} \oplus \zeta_{\perp} \in \mathcal{R}(\Pi)_{||} \oplus \mathcal{R}(\Pi)_{\perp}$ solving

$$\begin{cases} \zeta_{||} = (B + \Pi(\oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G^{\perp}_{\lambda_n})\Pi)_{||\perp}\,\zeta_{\perp}\,, \\ 0 = (B + \Pi(\oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G^{\perp}_{\lambda_n})\Pi)_{\perp}\,\zeta_{\perp}\,. \end{cases}$$

By (3.2) one obtains $(\Pi(\oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G^{\perp}_{\lambda_n})\Pi)_{\perp} = 0$. Therefore one gets, in case there exists k' such that $\Pi\hat{\xi}^{\perp}_{k',n} \neq 0$ and $\Pi\hat{\xi}_{k,n} \neq 0$ for all k

$$\lambda_n \in \sigma(A^{\Pi,B}) \iff \det(B_{\perp}) = 0\,.$$

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Received 15/09/2013; Revised 17/12/2013