ON THE COMMON POINT SPECTRUM OF PAIRS OF SELF-ADJOINT EXTENSIONS

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Dedicated to Vladimir Koshmanenko on the occasion of his 70th birthday

ABSTRACT. Given two different self-adjoint extensions of the same symmetric operator, we analyse the intersection of their point spectra. Some simple examples are provided.

1. Preliminaries

Given a linear closed operator L, we denote by

$$\mathcal{D}(L), \quad \mathcal{K}(L), \quad \mathcal{R}(L), \quad \mathcal{G}(L), \quad \rho(L)$$

its domain, kernel, range, graph and resolvent set respectively. \mathcal{H} denotes a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$; we also make use of an auxiliary Hilbert space \mathfrak{h} with scalar product (\cdot, \cdot) and corresponding norm $|\cdot|$.

Given a closed, densely defined, symmetric operator

$$S:\mathcal{D}(S)\subseteq\mathcal{H}\to\mathcal{H}$$

with equal deficiency indices, by von Neumann's theory one has (here the direct sums are given w.r.t. the graph inner product of S^*)

$$\mathcal{D}(S^*) = \mathcal{D}(S) \oplus \mathcal{K}_+ \oplus \mathcal{K}_-, \quad \mathcal{K}_\pm := \mathcal{K}(-S^* \pm i),$$

$$S^*(\phi_\circ \oplus \phi_+ \oplus \phi_-) = S\phi_\circ + i\phi_+ - i\phi_-,$$

and any self-adjoint extension of S is of the kind $A_U = S^*|\mathcal{G}(U)$, the restriction of S^* to $\mathcal{G}(U)$, where $U : \mathcal{K}_+ \to \mathcal{K}_-$ is unitary. Therefore, fixing a unitary U_\circ and posing $A := A_{U_\circ}$, one has

$$S = A | \mathcal{K}(\tau_{\circ}), \quad \tau_{\circ} : \mathcal{D}(A) \to \mathfrak{h}_{\circ},$$

where

$$\mathfrak{h}_{\circ} = \mathcal{K}_{+}, \quad \tau_{\circ} = P_{+},$$

and P_+ is the orthogonal (w.r.t. the graph inner product of S^*) projection onto \mathcal{K}_+ . Since $\mathcal{K}(\tau_\circ) = \mathcal{K}(\tau)$ where $\tau = M\tau_\circ$ and $M : \mathfrak{h}_\circ \to \mathfrak{h}$ is any continuous linear bijection, in the search of the self-adjoint extension of S, we can consider the following equivalent problem: determine all the self-adjoint extensions of $A|\mathcal{K}(\tau)$, where

$$\tau:\mathcal{D}(A)\to\mathfrak{h}$$

is a linear, continuous (with respect to the graph norm on $\mathcal{D}(A)$), surjective map onto an auxiliary Hilbert space \mathfrak{h} with its kernel $\mathcal{K}(\tau)$ dense in \mathcal{H} . Typically A is a differential operator, τ is some trace (restriction) operator along a null subset N and \mathfrak{h} is some function space over N.

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We suppose that the spectrum of A does not coincide with the whole real line and so, by eventually adding a constant to A, we make the following hypothesis:

 $0 \in \rho(A)$.

By the results provided in [9] and [7] (to which we refer for proofs and connections with equivalent formulations, in particular with boundary triplets theory) one has the following

Theorem 1.1. The set of all self-adjoint extensions of S is parametrized by the set $\mathsf{E}(\mathfrak{h})$ of couples (Π, Θ) , where Π is an orthogonal projection in \mathfrak{h} and Θ is a self-adjoint operator in $\mathcal{R}(\Pi)$. If $A^{\Pi,\Theta}$ denotes the self-adjoint extension corresponding to $(\Pi,\Theta) \in \mathsf{E}(\mathfrak{h})$ then

$$A^{\Pi,\Theta}: \mathcal{D}(A^{\Pi,\Theta}) \subseteq \mathcal{H} \to \mathcal{H}, \quad A^{\Pi,\Theta}\phi := A\phi_0,$$

$$\mathcal{D}(A^{\Pi,\Theta}) := \{ \phi = \phi_0 + G_0 \xi_{\phi} , \phi_0 \in \mathcal{D}(A) , \xi_{\phi} \in \mathcal{D}(\Theta) , \Pi \tau \phi_0 = \Theta \xi_{\phi} \} ,$$

where

$$G_z : \mathfrak{h} \to \mathcal{H}, \quad G_z := \left(\tau (-A + \bar{z})^{-1}\right)^*, \quad z \in \rho(A)$$

Moreover the resolvent of $A^{\Pi,\Theta}$ is given, for any $z \in \rho(A) \cap \rho(A^{\Pi,\Theta})$, by the Krein's type formula

$$(-A^{\Pi,\Theta} + z)^{-1} = (-A + z)^{-1} + G_z \Pi(\Theta + z \Pi G_0^* G_z \Pi)^{-1} \Pi G_{\bar{z}}^*$$

Remark 1.2. Notice that the extension corresponding to $\Pi = 0$ is A itself. The extension corresponding to $(1,\Theta)$ is denoted by A^{Θ} and everywhere we omit the index Π in the case $\Pi = 1$. By [8], Corollary 3.2, the sub-family $\{A^{\Theta} : \Theta \text{ self-adjoint}\}$ gives all singular perturbations of A, where we say that \hat{A} is a singular perturbation of A whenever the set $\{\phi \in \mathcal{D}(A) \cap \mathcal{D}(\hat{A}) : A\phi = \hat{A}\phi\}$ is dense in \mathcal{H} (see [4]).

Remark 1.3. The operator G_z is injective (by surjectivity of τ) and for any $z \in \rho(A)$ one has (see [7], Remark 2.8)

(1.1)
$$\mathcal{R}(G_z) \cap \mathcal{D}(A) = \{0\}$$

so that the decomposition appearing in $\mathcal{D}(A^{\Pi,\Theta})$ is unique. Moreover (see [7], Lemma 2.1)

(1.2)
$$G_w - G_z = (z - w)(-A + w)^{-1}G_z.$$

2. The common point spectrum

Given a self-adjoint operator A let us denote by

$$\sigma(A), \quad \sigma_p(A), \quad \sigma_d(A)$$

its full, point and discrete spectrum respectively.

Given $\lambda \in \sigma_p(A)$, we denote by P_{λ} the orthogonal projector onto the corresponding eigenspace $\mathcal{H}_{\lambda} \subseteq \mathcal{D}(A)$ and pose $P_{\lambda}^{\perp} := 1 - P_{\lambda}$. Given $\lambda \in \sigma_p(A^{\Pi,\Theta})$, we denote by $\mathcal{H}_{\lambda}^{\Pi,\Theta} \subseteq \mathcal{D}(A^{\Pi,\Theta})$ the corresponding eigenspace. As regards the eigenvalues of $A^{\Pi,\Theta}$ which are not in the spectrum of A a complete

answer is given by the following result which is consequence of Kreĭn's resolvent formula (see [3], Section 2, Propositions 1 and 2, and [8], Theorem 3.4):

Lemma 2.1.

$$\lambda \in \rho(A) \cap \sigma_p(A^{\Pi,\Theta}) \quad \Longleftrightarrow \quad 0 \in \sigma_p(\Theta + \lambda \Pi G_0^* G_\lambda \Pi) \,,$$
$$\mathcal{H}_{\lambda}^{\Pi,\Theta} = \{ G_\lambda \xi \,, \, \xi \in \mathcal{K}(\Theta + \lambda \Pi G_0^* G_\lambda \Pi) \} \,.$$

Here we are interested in the common eigenvalues, i.e. in the points in $\sigma_p(A) \cap \sigma_p(A^{\Pi,\Theta})$. Therefore we take $\lambda \in \sigma_p(A)$ and we look for solutions $\phi \in \mathcal{D}(A^{\Pi,\Theta})$ of the eigenvalue equation

$$A^{\Pi,\Theta}\phi = \lambda\phi\,,$$

i.e., by Theorem 1.1,

$$(A - \lambda)\phi_0 = \lambda G_0 \xi_\phi \,.$$

By

$$(A - \lambda)P_{\lambda}\phi_0 = \mathbf{0}, \quad (A - \lambda)P_{\lambda}^{\perp}\phi_0 \in \mathcal{R}(P_{\lambda}^{\perp}),$$

this is equivalent to the couple of equations

$$(2.1) P_{\lambda}G_0\xi_{\phi} = 0\,,$$

(2.2)
$$(A - \lambda)P_{\lambda}^{\perp}\phi_0 = \lambda P_{\lambda}^{\perp}G_0\xi_{\phi}$$

together with the constraint

(2.3)
$$\xi_{\phi} \in \mathcal{D}(\Theta) \subseteq \mathcal{R}(\Pi), \quad \Pi \tau \phi_0 = \Theta \xi_{\phi}.$$

Equation (2.1) gives, for all $\psi \in \mathcal{H}$,

$$0 = \langle G_0 \xi_{\phi}, P_{\lambda} \psi \rangle = -\langle \xi_{\phi}, \tau A^{-1} P_{\lambda} \psi \rangle = -\frac{1}{\lambda} \langle \xi_{\phi}, \tau P_{\lambda} \psi \rangle$$

and so

$$\xi_{\phi} \in (\mathcal{R}(\tau P_{\lambda}))^{\perp}$$

If $\mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda}))^{\perp} = \{0\}$ then, since G_0 is injective, one has that in this case ϕ is an eigenvector with eigenvalue λ if and only if $\phi \in \mathcal{H}_{\lambda}$ and $\Pi \tau \phi = 0$.

Conversely suppose that $\mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda}))^{\perp} \neq \{0\}$ and moreover that λ is an isolated eigenvalue. Then $\lambda \in \rho(A|\mathcal{H}_{\lambda}^{\perp})$ and (2.2) gives

$$P_{\lambda}^{\perp}\phi_0 = -\lambda(-A+\lambda)^{-1}P_{\lambda}^{\perp}G_0\xi_{\phi}$$

By $\Pi \tau \phi_0 = \Theta \xi_\phi$ then one gets

(2.4)
$$\Pi \tau P_{\lambda} \phi_0 = \Theta \xi_{\phi} - \Pi \tau P_{\lambda}^{\perp} \phi_0 = (\Theta + \lambda \Pi \tau (-A + \lambda)^{-1} P_{\lambda}^{\perp} G_0 \Pi) \xi$$

By defining

$$G_{\lambda}^{\perp}:\mathfrak{h}\to\mathcal{H},\quad G_{\lambda}^{\perp}:=(\tau(-A+\lambda)^{-1}P_{\lambda}^{\perp})^{*},$$

and by $(G_{\lambda}^{\perp})^* G_0 = G_0^* G_{\lambda}^{\perp}$ (this relation is consequence of (1.2)), (2.4) is equivalent to

$$\mathrm{I}\tau P_{\lambda}\phi_0 = (\Theta + \lambda \Pi G_0^* G_{\lambda}^{\perp} \Pi) \xi \,.$$

Moreover by $(-A + \lambda)^{-1} P_{\lambda}^{\perp} G_0 = -A^{-1} G_{\lambda}^{\perp}$ one has

$$P_{\lambda}\phi_{0} + P_{\lambda}^{\perp}\phi_{0} + G_{0}\xi_{\phi} = P_{\lambda}\phi_{0} + (-\lambda(-A+\lambda)^{-1}P_{\lambda}^{\perp} + P_{\lambda}^{\perp})G_{0}\xi_{\phi}$$
$$= P_{\lambda}\phi_{0} - A(-A+\lambda)^{-1}P_{\lambda}^{\perp}G_{0}\xi_{\phi}$$
$$= P_{\lambda}\phi_{0} + G_{\lambda}^{\perp}\xi_{\phi} .$$

In conclusion we have proven the following

Theorem 2.2. Let $\lambda \in \sigma_p(A)$. 1) Suppose

$$\mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda}))^{\perp} = \{0\}$$

and pose

$$\mathcal{K}^{\Pi}_{\lambda} := \left\{ \psi \in \mathcal{H}_{\lambda} : \Pi \tau \psi = 0 \right\}.$$

and

Then

$$\lambda \in \sigma_p(A^{\Pi,\Theta}) \quad \iff \quad \mathcal{K}^{\Pi}_{\lambda} \neq \{0\}$$

 $\mathcal{H}^{\Pi,\Theta}_{\lambda} = \mathcal{K}^{\Pi}_{\lambda}$.

2) Suppose

$$\mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda}))^{\perp} \neq \{0\}$$

and let λ be isolated. Let $\mathcal{N}_{\lambda}^{\Pi,\Theta}$ be the set of couples $(\psi,\xi) \in \mathcal{H}_{\lambda} \oplus \mathcal{R}(\Pi)$ such that

(2.5)
$$\xi \in D(\Theta) \cap (\mathcal{R}(\tau P_{\lambda}))^{\perp},$$

(2.6)
$$\Pi \tau \psi = (\Theta + \lambda \Pi G_0^* G_\lambda^{\perp} \Pi) \xi$$

Then

$$\lambda \in \sigma_p(A^{\Pi,\Theta}) \iff \mathcal{N}_{\lambda}^{\Pi,\Theta} \neq \{0\}, \\ \dim(\mathcal{H}_{\lambda}^{\Pi,\Theta}) = \dim(\mathcal{N}_{\lambda}^{\Pi,\Theta})$$

and

$$\mathcal{H}_{\lambda}^{\Pi,\Theta} = \{ \phi \in \mathcal{H} \, : \, \phi = \psi + G_{\lambda}^{\perp} \xi \, , \ (\psi,\xi) \in \mathcal{N}_{\lambda}^{\Pi,\Theta} \} \, .$$

Remark 2.3. Notice that

$$(\mathcal{R}(\Pi))^{\perp} \cap \mathcal{R}(\tau P_{\lambda}) \neq \{0\} \implies \mathcal{K}_{\lambda}^{\Pi} \neq \{0\}.$$

Remark 2.4. Suppose $\lambda \in \sigma_p(A)$ is isolated. Noticing that

$$\mathcal{K}^{\Pi}_{\lambda} \oplus (\mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda}))^{\perp} \cap \mathcal{K}(\Theta + \lambda \Pi G_{0}^{*} G_{\lambda}^{\perp} \Pi)) \subseteq \mathcal{N}^{\Pi,\Theta}_{\lambda}$$

one has

$$\mathcal{K}^{\Pi}_{\lambda} \neq \{0\} \implies \lambda \in \sigma_p(A^{\Pi,\Theta}).$$

In particular, in the case λ is simple with eigenvector ψ_{λ} ,

$$\Pi \tau \psi_{\lambda} = 0 \quad \Longrightarrow \quad \lambda \in \sigma_p(A^{\Pi, \Theta}) \,.$$

Remark 2.5. Suppose $\mathcal{R}(\tau P_{\lambda}) = \{0\}$. Then $\mathcal{K}^{\Pi}_{\lambda} = \mathcal{H}_{\lambda}$ and so, in case $\lambda \in \sigma_p(A)$ is isolated, $\lambda \in \sigma_p(A^{\Pi,\Theta})$ and

$$\mathcal{H}_{\lambda}^{\Pi,\Theta} = \left\{ \phi = \psi_{\lambda} + G_{\lambda}^{\perp} \xi \,, \, \psi_{\lambda} \in \mathcal{H}_{\lambda} \,, \, \xi \in \mathcal{R}(\Pi) \cap \mathcal{K}(\Theta + \lambda \Pi G_{0}^{*} G_{\lambda}^{\perp} \Pi) \right\}.$$

Remark 2.6. The papers [1] and [6] contain results related to the ones given by Theorem 2.2 (see Theorem 3.6 in [6] and Theorem 4.7 in [1]). We thank Konstantin Pankrashkin for the communication.

3. Examples

3.1. Rank-one singular perturbations. Suppose $\mathfrak{h} = \mathbb{C}$. Then $\Pi = 1, \Theta = \theta \in \mathbb{R}$ and either $\mathcal{R}(\tau P_{\lambda}) = \mathbb{C}$ or $\mathcal{R}(\tau P_{\lambda}) = \{0\}.$

If $\mathcal{R}(\tau P_{\lambda}) = \mathbb{C}$ then $\lambda \in \sigma_p(A^{\theta})$ if and only if $\mathcal{K}_{\lambda} \neq \{0\}$, where

$$\mathcal{K}_{\lambda} := \{ \psi \in \mathcal{H}_{\lambda} : \tau \psi = 0 \}.$$

Since $\mathcal{R}(\tau P_{\lambda}) = \{0\}$ if and only if $\mathcal{K}_{\lambda} = \mathcal{H}_{\lambda}$, when λ is isolated and $\mathcal{R}(\tau P_{\lambda}) = \{0\}$ one has

$$\mathcal{N}_{\lambda}^{\theta} = \mathcal{K}_{\lambda} \oplus \{\xi \in \mathbb{C} : (\theta + \lambda \langle G_0, G_{\lambda}^{\perp} \rangle) \xi = 0\}$$

and so

$$\theta + \lambda \langle G_0, G_\lambda^\perp \rangle = 0 \quad \Longrightarrow \quad \mathcal{N}_\lambda^\theta = \mathcal{K}_\lambda \oplus \mathbb{C} \equiv \mathcal{H}_\lambda \oplus \mathbb{C} \,,$$

$$\theta + \lambda \langle G_0, G_\lambda^\perp \rangle \neq 0 \implies \mathcal{N}_\lambda^\theta = \mathcal{K}_\lambda \oplus \{0\} \equiv \mathcal{H}_\lambda.$$

In conclusion when $\mathfrak{h} = \mathbb{C}$ and $\lambda \in \sigma_p(A)$ is isolated,

(3.1)
$$\lambda \in \sigma_p(A^{\theta}) \iff \mathcal{K}_{\lambda} \neq \{0\}$$

and

$$\mathcal{H}_{\lambda}^{\theta} = \{ \psi = \psi_{\lambda} + G_{\lambda}^{\perp} \xi, \ \psi_{\lambda} \in \mathcal{H}_{\lambda}, \ (\theta + \lambda \langle G_0, G_{\lambda}^{\perp} \rangle) \xi = 0 \}.$$

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In particular if λ is a simple isolated eigenvalue of A with corresponding eigenfunction ψ_{λ} , then $\lambda \in \sigma_p(A^{\theta})$ if and only if $\tau \psi_{\lambda} = 0$. For example, if $\mathcal{H} = L^2(\Omega)$ and $\tau : \mathcal{D}(A) \to \mathbb{C}$ is the evaluation map at $y \in \Omega, \tau \psi := \psi(y)$, then λ is preserved if and only if y belongs to the nodal set (if any) of ψ_{λ} . Thus if A is (minus) the Dirichlet Laplacian on a bounded open set $\Omega \subset \mathbb{R}^d, d \leq 3$, its lowest eigenvalue is never preserved under a point perturbation. Analogous results hold in the case A is the Laplace-Beltrami operator on a compact d-dimensional Riemannian manifold $M, d \leq 3$, thus reproducing the ones given in [2], Theoreme 2, part 1.

3.2. The Šeba billiard. Let

$$A = \Delta : \mathcal{D}(A) \subset L^2(R) \to L^2(R)$$

$$\mathcal{D}(A) = \{ \phi \in C(\overline{R}) : \Delta \phi \in L^2(R) \,, \ \phi(\mathsf{x}) = 0 \,, \ \mathsf{x} \in \partial R \} \,,$$

be the Dirichlet Laplacian on the rectangle $R = (0, a) \times (0, b)$. Then

$$\sigma(A) = \sigma_d(A) = \left\{ \lambda_{m,n} \,, \ (m,n) \in \mathbb{N}^2 \right\}$$

and

$$\mathcal{H}_{\lambda_{m,n}} = \operatorname{span}\{\psi_{m',n'} : \lambda_{m',n'} = \lambda_{m,n}\},\$$

where

$$\lambda_{m,n} := -\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

and

$$\psi_{m,n}(\mathsf{x}) := \sin\left(\frac{m\pi x_1}{a}\right) \sin\left(\frac{n\pi x_2}{b}\right), \quad \mathsf{x} \equiv (x_1, x_2).$$

Let

$$\tau\psi:=\psi(\mathsf{y})\,,$$

so that A^{θ} describes a "Šeba billiard", i.e. the Dirichlet Laplacian on the rectangle R with a point perturbation placed at the point $y \equiv (y_1, y_2)$ (see [10]).

Since $\sigma(A) = \sigma_d(A)$, by the invariance of the essential spectrum under finite rank perturbations, one has $\sigma(A^{\theta}) = \sigma_d(A^{\theta})$ and, by (3.1), $\lambda_{m,n} \in \sigma(A) \cap \sigma(A^{\theta})$ if and only

$$\forall (m',n') \text{ s.t. } \lambda_{m',n'} = \lambda_{m,n}, \quad \sin\left(\frac{m'\pi y_1}{a}\right) \sin\left(\frac{n'\pi y_2}{b}\right) = 0.$$

Equivalently

$$\sigma(A) \cap \sigma(A^{\theta}) = \emptyset \quad \iff \quad \left(\frac{y_1}{a}, \frac{y_2}{b}\right) \notin \mathbb{Q}^2 \,.$$

If there exists relatively prime integers $1 \le p < q$ such that $\frac{y_1}{a} = \frac{p}{q}$ while $\frac{y_2}{b}$ is irrational, then

$$\sigma(A) \cap \sigma(A^{\theta}) = \{\lambda_{kq,n}, \ (k,n) \in \mathbb{N}^2\}.$$

Analogously if $\frac{y_1}{a}$ is irrational and $\frac{y_2}{b} = \frac{p}{q}$ then

$$\sigma(A) \cap \sigma(A_{\theta}) = \{\lambda_{m,kq}, (m,k) \in \mathbb{N}^2\}$$

while if $\frac{y_1}{a} = \frac{p}{q}$ and $\frac{y_2}{b} = \frac{r}{s}$, then

$$\sigma(A) \cap \sigma(A^{\theta}) = \{\lambda_{kq,n}, (k,n) \in \mathbb{N}^2\} \cup \{\lambda_{m,ks}, (m,k) \in \mathbb{N}^2\}.$$

3.3. Rank-two singular perturbations. Let $\mathfrak{h} = \mathbb{C}^2$. Then either $\Pi = 1$ or $\Pi = w \otimes w$, $w \in \mathbb{C}^2$, |w| = 1. Let $\lambda \in \sigma_p(A)$.

1.1) $\mathcal{R}(\tau P_{\lambda}) = \mathbb{C}^2$, $\Pi = 1$. Then $\lambda \in \sigma_p(A^{\Theta})$ if and only if there exists $\psi \in \mathcal{H}_{\lambda} \setminus \{0\}$ such that $\tau \psi = 0$.

1.2) $\mathcal{R}(\tau P_{\lambda}) = \mathbb{C}^2$, $\Pi = w \otimes w$. Then $\lambda \in \sigma_p(A^{\Pi,\Theta})$ if and only if there exists $\psi \in \mathcal{H}_{\lambda} \setminus \{0\}$ such that $w \cdot \tau \psi = 0$.

Now suppose further that $\lambda \in \sigma_p(A)$ is isolated.

2.1) $\mathcal{R}(\tau P_{\lambda}) = \operatorname{span}(\xi_{\lambda}) \simeq \mathbb{C}, |\xi_{\lambda}| = 1, \Pi = 1$. Decomposing equation (2.6) w.r.t. the orthonormal base $\{\xi_{\lambda}, \xi_{\lambda}^{\perp}\}$ one gets that $\mathcal{N}_{\lambda}^{\Theta} \neq \{0\}$ if and only if there exists $\zeta \equiv (\zeta_1, \zeta_2) \in \mathbb{C}^2 \setminus \{0\}$ solving

$$\begin{cases} \zeta_1 = (\xi_\lambda \cdot (\Theta + \lambda G_0^* G_\lambda^\perp) \xi_\lambda^\perp) \zeta_2, \\ 0 = (\xi_\lambda^\perp \cdot (\Theta + \lambda G_0^* G_\lambda^\perp) \xi_\lambda^\perp) \zeta_2. \end{cases}$$

Hence

$$\lambda \in \sigma_p(A^{\Theta}) \iff (\xi_{\lambda}^{\perp} \cdot (\Theta + \lambda G_0^* G_{\lambda}^{\perp}) \xi_{\lambda}^{\perp}) = 0.$$

2.2) $\mathcal{R}(\tau P_{\lambda}) = \operatorname{span}(\xi_{\lambda}) \simeq \mathbb{C}, \Pi = w \otimes w$. Let us use the decomposition $w = w_{||} + w_{\perp}$ w.r.t. the orthonormal base $\{\xi_{\lambda}, \xi_{\lambda}^{\perp}\}$. If $w_{||} = 0$ then $\mathcal{K}_{\lambda}^{\Pi} \neq \{0\}$ and so $\lambda \in \sigma_p(A^{\Pi,\Theta})$. If $w_{||} \neq 0$ then $\mathcal{K}_{\lambda}^{\Pi} = \{0\}$ and $\mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda}))^{\perp} = \{0\}$, thus $\lambda \notin \sigma_p(A^{\Pi,\Theta})$. In conclusion

$$\lambda \in \sigma_p(A^{\Pi,\Theta}) \quad \iff \quad w = \xi_\lambda^\perp \,.$$

3) $\mathcal{R}(\tau P_{\lambda}) = \{0\}$. In this case $\lambda \in \sigma_p(A^{\Pi,\Theta})$.

3.4. The Laplacian on a bounded interval. Let

$$A: \mathcal{D}(A) \subseteq L^2(0,a) \to L^2(0,a), \quad A\phi = \phi'',$$

$$\mathcal{D}(A) = \{ \phi \in C^1[0, a] : \phi'' \in L^2(0, a), \ \phi(0) = \phi(a) = 0 \},\$$

be the Dirichlet Laplacian on the bounded interval (0, a) and pose

$$: \mathcal{D}(A) \to \mathbb{C}^2, \quad \tau \phi \equiv \gamma_1 \phi := (\phi'(0), -\phi'(a))$$

Therefore $S = A | \mathcal{K}(\tau)$ is the minimal Laplacian with domain

$$\mathcal{D}(S) = \{ \phi \in C^1[0, a] : \phi'' \in L^2(0, a), \ \phi(0) = \phi'(0) = \phi(a) = \phi'(a) = 0 \}$$

and the self-adjoint extensions of S are rank-two perturbations of the Dirichlet Laplacian A. One has

$$\sigma(A) = \sigma_d(A) = \{\lambda_n\}_1^\infty, \quad \lambda_n = -\left(\frac{n\pi}{a}\right)^2$$

and the normalized eigenvector corresponding to λ_n is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \,.$$

By Theorem 1.1 and by the change of extension parameter (here P_0 represents the Dirichlet-to-Neumann operator)

$$(\Pi, \Theta) \mapsto (\Pi, B), \quad B := \Theta - \Pi P_0 \Pi, \quad P_0 \equiv \frac{1}{a} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

any self-adjoint extension of the minimal Laplacian S is of the kind $A^{\Pi,B}$, $(\Pi, B) \in \mathsf{E}(\mathbb{C}^2)$, where

$$A^{\Pi,B} : \mathcal{D}(A^{\Pi,B}) \subset L^2(0,a) \to L^2(0,a) , \quad A^{\Pi,B}\phi = \phi'' ,$$
$$\mathcal{D}(A^{\Pi,B}) = \{\phi \in C^1[0,a] : \phi'' \in L^2(0,a) , \ \gamma_0\phi \in \mathcal{R}(\Pi) , \ \Pi\gamma_1\phi = B\gamma_0\phi\}$$

(see e.g. [9], Example 5.1). Here $\gamma_0 \phi := (\phi(0), \phi(a))$.

The case $\Pi = 0$ reproduces A itself, the case $\Pi = 1$, $B = \begin{pmatrix} b_{11} & b_{12} \\ \overline{b}_{12} & b_{22} \end{pmatrix}$, $b_{11}, b_{22} \in \mathbb{R}$, $b_{12} \in \mathbb{C}$, gives the boundary conditions

$$\begin{cases} b_{11} \phi(0) - \phi'(0) + b_{12} \phi(a) = 0, \\ \bar{b}_{12} \phi(0) + b_{22} \phi(a) + \phi'(a) = 0, \end{cases}$$

and the case $\Pi = w \otimes w$, $w \equiv (w_1, w_2) \in \mathbb{C}^2$, $|w_1|^2 + |w_2|^2 = 1$, $B \equiv b \in \mathbb{R}$, gives the boundary conditions

$$\begin{cases} w_2 \phi(0) - w_1 \phi(a) = 0, \\ \bar{w}_1 (b \phi(0) - \phi'(0)) + \bar{w}_2 (b \phi(a) + \phi'(a)) = 0. \end{cases}$$

By the invariance of the essential spectrum under finite rank perturbations, $\sigma(A^{\Pi,B}) = \sigma_d(A^{\Pi,B})$. Now we use the results given in subsection 3.3. One has

$$\mathcal{R}(\tau P_{\lambda_n}) = \operatorname{span}(\hat{\xi}_n), \quad \hat{\xi}_n \equiv \frac{1}{\sqrt{2}} \left(1, (-1)^{n-1} \right).$$

Let $\Pi = 1$ and $\hat{\xi}_n^{\perp} \equiv \frac{1}{\sqrt{2}} (1, (-1)^n)$. By point 2.1 in subsection 3.3 we known that $\lambda_n \in \sigma(A^B)$ if and only if $\hat{\xi}_n^{\perp} \cdot (B + P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp}) \hat{\xi}_n^{\perp} = 0$. Since the resolvent of A is explicitly known, $\hat{\xi}_n^{\perp} \cdot (B + P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp}) \hat{\xi}_n^{\perp}$ can be calculated. However we use here a short cut which avoids any calculation: the Neumann Laplacian corresponds to B = 0 and we know that its spectrum is $\{0\} \cup \sigma(A)$, thus

(3.2)
$$\hat{\xi}_n^{\perp} \cdot (P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp}) \hat{\xi}_n^{\perp} = 0.$$

$$\lambda_n \in \sigma(A^B) \quad \iff \quad b_{11} + b_{22} + 2 (-1)^n \operatorname{Re}(b_{12}) = 0.$$

If $\Pi = w \otimes w$ by point 2.2 in subsection 3.3 one has

$$\lambda_n \in \sigma(A^{\Pi,B}) \quad \iff \quad w = \hat{\xi}_n^\perp$$

In both cases

$$\lambda_n \in \sigma(A^{\Pi,B}) \quad \iff \lambda_{n+2} \in \sigma(A^{\Pi,B}).$$

Moreover

$$\sigma(A) \subseteq \sigma(A^{\Pi,B}) \quad \iff \quad \Pi = 1 \text{ and } b_{11} + b_{22} = 0, \operatorname{Re}(b_{12}) = 0.$$

3.5. Equilateral quantum graphs. Let $\mathcal{H} = \bigoplus_{k=1}^{N} L^2(0, a)$ and $A_N = \bigoplus_{k=1}^{N} A$, where A is defined as in subsection 3.4 (to which we refer for notations). Then $\sigma(A_N) = \sigma_d(A_N) = \sigma(A)$ and the eigenfunctions corresponding to the N-fold degenerate eigenvalue λ_n are

$$\Psi_{k,n} = \bigoplus_{i=1}^{N} \psi_{i,k,n}, \quad k = 1, \dots, N, \quad \psi_{i,k,n} = \begin{cases} 0, & i \neq k, \\ \psi_n, & i = k. \end{cases}$$

By taking

$$\tau: \mathcal{D}(A_N) \equiv \bigoplus_{k=1}^N \mathcal{D}(A) \to \bigoplus_{k=1}^N \mathbb{C}^2 \equiv \mathbb{C}^{2N}, \quad \tau = \bigoplus_{k=1}^N \gamma_1,$$

one gets, by Theorem 1.1, self-adjoint extensions describing quantum graphs (see e.g. [5]) with N edges of the same length a. By Theorem 1.1 and by the change of extension parameter

$$(\Pi, \Theta) \mapsto (\Pi, B), \quad B := \Theta - \Pi(\bigoplus_{k=1}^{N} P_0) \Pi,$$

such extensions are of the kind $A^{\Pi,B}$, $(\Pi,B) \in \mathsf{E}(\mathbb{C}^{2N})$, where (see [9], Example 5.2).

$$A^{\Pi,B}: \mathcal{D}(A^{\Pi,B}) \subset \bigoplus_{k=1}^{N} L^2(0,a) \to \bigoplus_{k=1}^{N} L^2(0,a) ,$$
$$A^{\Pi,B}(\bigoplus_{k=1}^{N} \phi_k) = \bigoplus_{k=1}^{N} \phi_k'' ,$$

$$\mathcal{D}(A^{\Pi,B}) = \left\{ \bigoplus_{k=1}^{N} \phi_k : \phi_k \in C^1[0,a], \ \phi_k'' \in L^2(0,a), \\ (\bigoplus_{k=1}^{N} \gamma_0 \phi_k) \in \mathcal{R}(\Pi), \ \Pi(\bigoplus_{k=1}^{N} \gamma_1 \phi_k) = B(\bigoplus_{k=1}^{N} \gamma_0 \phi_k) \right\}.$$

The couple (Π, B) represents the connectivity of the quantum graph.

1) $\Pi = 1$. Given $\lambda_n \in \mathcal{D}(A)$, we pose

$$\mathbb{C}_{||}^{2N} := \bigoplus_{k=1}^{N} \operatorname{span}(\hat{\xi}_{n}) \simeq \mathbb{C}^{N}, \quad \mathbb{C}_{\perp}^{2N} := \bigoplus_{k=1}^{N} \operatorname{span}(\hat{\xi}_{n}^{\perp}) \simeq \mathbb{C}^{N},$$

so that $\mathcal{R}(\tau P_{\lambda_n}) = \mathbb{C}^{2N}_{||}, \ (\mathcal{R}(\tau P_{\lambda_n}))^{\perp} = \mathbb{C}^{2N}_{||}, \ \mathbb{C}^{2N} = \mathbb{C}^{2N}_{||} \oplus \mathbb{C}^{2N}_{\perp}$ and for any linear operator $L: \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ we can consider the block decomposition $L = \begin{pmatrix} L_{||} & L_{||\perp} \\ (L_{||\perp})^* & L_{\perp} \end{pmatrix}$. By using such decompositions in equation (2.6) one gets that $\mathcal{N}_{\lambda_n}^{\Theta} \neq \{0\}, \Theta = B + \bigoplus_{k=1}^{N} P_0$, if and only if there exists $\zeta \neq \{0\}, \zeta = \zeta_{||} \oplus \zeta_{\perp} \in \mathbb{C}_{||}^{2N} \oplus \mathbb{C}_{\perp}^{2N}$ solving

$$\begin{cases} \zeta_{||} = (B + \bigoplus_{k=1}^{N} P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp})_{||\perp} \zeta_{\perp}, \\ 0 = (B + \bigoplus_{k=1}^{N} P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp})_{\perp} \zeta_{\perp}. \end{cases}$$

By (3.2) one obtains $(\bigoplus_{k=1}^{N} P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp})_{\perp} = 0$. Therefore one gets

$$\lambda_n \in \sigma(A^B) \quad \iff \quad \det(B_\perp) = 0$$

2) $\Pi \neq 1$. Given $\lambda_n \in \mathcal{D}(A)$ we pose

$$\mathcal{R}(\Pi)_{||} := \mathcal{R}(\Pi) \cap (\oplus_{k=1}^{N} \operatorname{span}(\hat{\xi}_{n})), \quad \mathcal{R}(\Pi)_{\perp} := \mathcal{R}(\Pi) \cap (\oplus_{k=1}^{N} \operatorname{span}(\hat{\xi}_{n}^{\perp})),$$

so that $\mathcal{R}(\Pi) \cap \mathcal{R}(\tau P_{\lambda_n}) = \mathcal{R}(\Pi)_{||}, \mathcal{R}(\Pi) \cap (\mathcal{R}(\tau P_{\lambda_n}))^{\perp} = \mathcal{R}(\Pi)_{\perp}, \mathcal{R}(\Pi) = \mathcal{R}(\Pi)_{||} \oplus \mathcal{R}(\Pi)_{\perp}$ and for any linear operator $L : \mathcal{R}(\Pi) \to \mathcal{R}(\Pi)$ we can consider the block decomposition $L = \begin{pmatrix} L_{||} & L_{||\perp} \\ (L_{||\perp})^* & L_{\perp} \end{pmatrix}$. Define $\hat{\xi}_{k,n} = \bigoplus_{i=1}^n \hat{\xi}_{i,k,n} \in \mathbb{C}^{2N}$ and $\hat{\xi}_{k,n}^{\perp} = \bigoplus_{i=1}^n \hat{\xi}_{i,k,n}^{\perp} \in \mathbb{C}^{2N}$, $k = 1, \dots, N$, by

$$\hat{\xi}_{i,k,n} := \begin{cases} 0, & i \neq k, \\ \hat{\xi}_n, & i = k, \end{cases} \quad \hat{\xi}_{i,k,n}^{\perp} := \begin{cases} 0, & i \neq k, \\ \hat{\xi}_n^{\perp}, & i = k \end{cases}$$

If $\Pi \hat{\xi}_{k,n}^{\perp} = 0$ for all k then $\mathcal{R}(\Pi)_{\perp} = \{0\}$ and in this case

$$\lambda \in \sigma_p(A^{\Pi,B}) \quad \iff \quad \exists k \text{ s.t } \Pi \hat{\xi}_{k,n} = 0.$$

If there exists k' such that $\Pi \hat{\xi}_{k',n}^{\perp} \neq 0$ then $\mathcal{R}(\Pi)_{\perp} \neq \{0\}$. By Remark 2.3

$$\exists k \text{ s.t } \Pi \hat{\xi}_{k,n} = 0 \quad \Longrightarrow \quad \lambda \in \sigma_p(A^{\Pi,B}) \,.$$

Suppose now $\Pi \hat{\xi}_{k,n} \neq 0$ for all k, i.e. $\mathcal{K}_{\lambda_n}^{\Pi} = \{0\}$. Then, using the above decompositions in equation (2.6) one gets that $\mathcal{N}_{\lambda_n}^{\Pi,\Theta} \neq \{0\}, \Theta = B + \Pi(\bigoplus_{k=1}^N P_0)\Pi$, if and only if there exists $\zeta \neq 0, \zeta = \zeta_{||} \oplus \zeta_{\perp} \in \mathcal{R}(\Pi)_{||} \oplus \mathcal{R}(\Pi)_{\perp}$ solving

$$\begin{cases} \zeta_{||} = (B + \Pi(\oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp}) \Pi)_{||\perp} \zeta_{\perp}, \\ 0 = (B + \Pi(\oplus_{k=1}^{N} P_0 + \lambda_n G_0^* G_{\lambda_n}^{\perp}) \Pi)_{\perp} \zeta_{\perp}. \end{cases}$$

By (3.2) one obtains $(\Pi(\oplus_{k=1}^{N}P_0 + \lambda_n G_0^*G_{\lambda_n}^{\perp})\Pi)_{\perp} = 0$. Therefore one gets, in case there exists k' such that $\Pi \hat{\xi}_{k',n}^{\perp} \neq 0$ and $\Pi \hat{\xi}_{k,n} \neq 0$ for all k

$$\lambda_n \in \sigma(A^{\Pi,B}) \quad \iff \quad \det(B_\perp) = 0.$$

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