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# CONTINUOUS DUAL OF $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ AND $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$

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Abstract. A bilateral sequence is a function whose domain is the set $\mathbb{Z}$ of all integers with natural ordering. In this paper we study the continuous dual of the Banach space of $X$-valued bilateral sequence spaces $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

## 1. Introduction

We know that bilateral sequences play a vital role in various branches of mathematical analysis, for instance, in theories connected to the representation of the functions by hypergeometric series, Laurent series and Fourier series which are among many others. Saavedra and Rodriguez, in [13] has worked on the complex bilateral sequence space $\ell^2(\mathbb{Z})$ to obtain various results on hyper-cyclic bilateral weighted shift. In [9], Menet generalized this result to the complex bilateral sequence spaces $\ell^p(\mathbb{Z})$ with $1 \leq p < \infty$ and $c_\circ(\mathbb{Z})$ and afterward, to the complex weighted spaces $\ell^p(v, \mathbb{Z})$ and $c_\circ(v, \mathbb{Z})$. Shkarin, in [12] and [11] used the bilateral sequence spaces $\ell_\infty(\mathbb{Z})$, $\ell_p(\mathbb{Z})$ with $1 \leq p < \infty$ and $c_\circ(\mathbb{Z})$ to obtain various results associated with weighted bilateral shift on these spaces and also used $\{f_j\}_{j\in\mathbb{Z}}$, a sequence of elements of $\mathcal{B}$ where $\mathcal{B}$ is a Banach space. Further utility of bilateral sequences can be found in Simon and Marko [4] which deals with the characterization of various types of operators connected to the Banach space $X-$valued bilateral sequence spaces $c_\circ(\mathbb{Z}^n, X)$, $\ell_p(\mathbb{Z}^n, X)$ and $\ell_\infty(\mathbb{Z}^n, X)$ where the domain of bilateral sequences is $\mathbb{Z}^n$ (see also [1, 8, 10, 14, 15, 16]). In this direction, we have also introduced and developed certain vector valued sequence spaces in [5, 6]. Also we find it interesting that our spaces $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ reduce to Maddox paranorm spaces $c_\circ(p)$ and $c(p)$ if $\mathbb{Z}$, $X$, $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$ and $\bar{p} = (p_k)_{-\infty}^{\infty}$ are replaced by $\mathbb{N}$, $\mathbb{C}$, $\bar{\lambda} = e = (1, 1, 1, \dots)$ and $\bar{p} = (p_k)_0^{\infty}$.

## 2. Preliminaries

A bilateral sequence is usually denoted by $\bar{a} = (a_k)_{-\infty}^{\infty}$. Let $\bar{p} = (p_k)_{-\infty}^{\infty}$ be a bilateral sequence of strictly positive real numbers, $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$ be a bilateral sequence of non-zero complex numbers, $X$ and $Y$ be Banach spaces over the field $\mathbb{C}$ of complex numbers and $B(X, Y)$ be Banach space of all bounded linear operators from $X$ into $Y$. For $T \in B(X, Y)$, the operator norm of $T$ is defined to be

$$||T|| = \sup\{||Tx|| : x \in S\},$$

where $S = \{x \in X : ||x|| \leq 1\}$. The zero element of the Banach spaces $X, Y$ and $B(X, Y)$ will be denoted by $\theta$ and $X^\star$ denotes the continuous dual of $X$, i.e., $B(X, \mathbb{C}) = X^\star$.

By the convergence of the bilateral series $\sum_{-\infty}^{\infty} a_k$ to $s$ written as $\sum_{-\infty}^{\infty} a_k = s$, we mean the convergence of the sequence $(s_n)_{n=1}^{\infty}$ to $s$, where $s_n = \sum_{-n}^{n} a_k$ is the n-th partial sum of the bilateral series $\sum_{-\infty}^{\infty} a_k$.

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The following classes of Banach space $X$-valued bilateral sequences have been introduced by the authors in [5]:

$$c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \{\bar{x} = (x_k)_{-\infty}^{\infty} : x_k \in X, k \in \mathbb{Z}, ||\lambda_k x_k||^{p_k} \to 0, \text{ as } k \to -\infty \text{ as well as } k \to \infty\}$$

and

$$c(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \{\bar{x} = (x_k)_{-\infty}^{\infty} : x_k \in X, k \in \mathbb{Z} \text{ and there exist } l_1, l_2 \in X \text{ such that } ||\lambda_k x_k - l_1||^{p_k} \to 0 \text{ as } k \to -\infty \text{ and } ||\lambda_k x_k - l_2||^{p_k} \to 0 \text{ as } k \to \infty\}.$$

Also we shall frequently use

$$\ell_\infty(\mathbb{Z}, \mathbb{R}) = \{\bar{a} = (a_k)_{-\infty}^{\infty} : a_k \in \mathbb{R},\ k \in \mathbb{Z},\ \sup_k |a_k| < \infty\}.$$

This paper is in continuation of our papers [5] and [7]. In [7] we have introduced the Köthe-Toeplitz duals of a class of vector valued bilateral sequences. Here our aim is to investigate the continuous dual of $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ with the help of Köthe-Toeplitz duals, characterized in [7]. The related results are as follows.

Throughout the work, we denote $p_k^{-1} = r_k$ and $M = \max(1,\ \sup_k p_k)$. We shall denote by $\mathbb{Z}(m, n)$, the open integer interval defined as

$$\mathbb{Z}(m, n) = \begin{cases} \{m+1, m+2, \ldots, n-2, n-1\}, & \text{if } m+1 \leq n-1, \\ \phi, & \text{otherwise} \end{cases}$$

and its complement by $\mathbb{Z} \backslash \mathbb{Z}(m, n)$.

**Definition 2.1.** Let $X$ and $Y$ be Banach spaces and $(A_k)_{-\infty}^{\infty}$ a bilateral sequence of linear, but not necessarily bounded operators $A_k$ on $X$ into $Y$. Suppose $E(X)$ is a non-empty set of $X$-valued bilateral sequences. Then the $\alpha$-dual of $E(X)$ is defined by

$$E^\alpha(X) = \Big\{\bar{A} = (A_k)_{-\infty}^{\infty} : \sum_{-\infty}^{\infty} ||A_k x_k|| \text{ converges for all } \bar{x} = (x_k)_{-\infty}^{\infty} \in E(X)\Big\}$$

and the $\beta$-dual is defined by

$$E^\beta(X) = \Big\{\bar{A} = (A_k)_{-\infty}^{\infty} : \sum_{-\infty}^{\infty} A_k x_k \text{ converges in the norm of } Y, \text{ for all } \bar{x} \in E(X)\Big\}.$$

Now corresponding to $K$-space and $AK$-spaces for scalar sequences (see Wilansky [3] and Kamthan and Gupta [2]), we define their generalized versions as follows:

**Definition 2.2.** Let $E(X)$ be the linear space of the normed space of $X$-valued bilateral sequences $\bar{x} = (x_k)_{-\infty}^{\infty}$ and $x \in X$. We define

(i) $\delta_n(x) = (\ldots, \theta, x, \theta, \ldots)$, where $x$ is at nth place, $n \in \mathbb{Z}$;

(ii) $E(X)$ equipped with the linear topology $\mathcal{T}$ is said to be a $GK$-space if the map $P_n : E(X) \to X, P_n(\bar{x}) = x_n$, is continuous for each $n \in \mathbb{Z}$.
A $GK$-space is called

(iii) a $GAK$-space if for each $\bar{x} \in E(X)$, $\bar{s}^n(\bar{x}) \to \bar{x}$ as $n \to \infty$ with respect to $\mathcal{T}$, where $\bar{s}^n(\bar{x}) = (\ldots, \theta, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \theta, \ldots)$.

**Theorem 2.3.** $\bar{A} \in c_\circ^\alpha(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ *if and only if*

(i) *there exist* $m, n \in \mathbb{Z}$ *such that* $A_k \in B(X, Y)$ *for all* $k \in \mathbb{Z} \backslash \mathbb{Z}(m, n)$,

(ii) *there exists an integer* $N > 1$ *such that* $\sum_{k \in \mathbb{Z} \backslash \mathbb{Z}(m,n)} |\lambda_k|^{-1} ||A_k|| N^{-r_k} < \infty$.

*Proof.* Suppose (i) and (ii) hold and $\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. We choose integers $q$ and $r$, where $q \leq m < n \leq r$ such that $||\lambda_k x_k||^{p_k} \leq \frac{1}{N}$ for all $k \in \mathbb{Z} \backslash \mathbb{Z}(q, r)$. Now, we easily get that $\sum_{-\infty}^{\infty} ||A_k x_k||$ is convergent and hence $\bar{A} \in c_\circ^\alpha(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

Conversely suppose that $\bar{A} \in c_\circ^\alpha(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. If (i) fails, then there exists a sequence $(k(i))$ such that $k(i) \in \mathbb{Z}\backslash\mathbb{Z}(m,n)$ and $A_{k(i)} \notin B(X,Y)$. For our convenience we assume that for each $i \geq 1$, $k(i) > 0$. Thus for each $i \geq 1$, we can find $z_{k(i)} \in S$ such that $||A_{k(i)}z_{k(i)}|| > |\lambda_{k(i)}|i^{r_{k(i)}}$. Now we define $\bar{x} = (x_k)_{-\infty}^{\infty}$ by

$$x_k = \begin{cases} \lambda_{k(i)}^{-1} i^{-r_{k(i)}} z_{k(i)}, & \text{if } k = k(i),\ i \geq 1, \\ \theta, & \text{otherwise.} \end{cases}$$

We easily see that $\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. But for each $i \geq 1$, $||A_{k(i)}x_{k(i)}|| > 1$ and so $\sum_{-\infty}^{\infty}||A_k x_k|| = \infty$ which contradicts that $\bar{A} \in c_\circ^\alpha(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

Similarly we can prove (ii). This completes the proof. □

If in the above theorem (Theorem 2.3) $A_k \in B(X,Y)$ for all $k \in \mathbb{Z}$ then we have

**Corollary 2.4.** *If $A_k \in B(X,Y)$ for all $k \in \mathbb{Z}$ then*

$$c_\circ^\alpha(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = M_\circ(\mathbb{Z}, B(X,Y), \bar{\lambda}, \bar{p}),$$

*where*

$$M_\circ(\mathbb{Z}, B(X,Y), \bar{\lambda}, \bar{p}) = \bigcup_{N>1} \Big\{ \bar{A} = (A_k)_{-\infty}^{\infty} : A_k \in B(X,Y),\ k \in \mathbb{Z}, \sum_{-\infty}^{\infty} |\lambda_k|^{-1}||A_k||N^{-r_k} < \infty \Big\}.$$

**Theorem 2.5.** *Let $\bar{p} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$. Then $\bar{A} \in c_\circ^\beta(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ if and only if*

(i) *there exist $m, n \in \mathbb{Z}$ such that $A_k \in B(X,Y)$ for all $k \in \mathbb{Z}\backslash\mathbb{Z}(m,n)$,*

(ii) *$||R_{m,n}(\lambda, N)|| < \infty$ for some $N > 1$, where*

$$R_{m,n}(\lambda, N) = (\ldots, \lambda_{m-1}^{-1}N^{-r_{m-1}}A_{m-1}, \lambda_m^{-1}N^{-r_m}A_m, \lambda_n^{-1}N^{-r_n}A_n, \lambda_{n+1}^{-1}N^{-r_{n+1}}A_{n+1}, \ldots).$$

*Proof.* Suppose (i) and (ii) hold and $||R_{m,n}(\lambda, N)|| = H < \infty$. Let $\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. For a given $\epsilon > 0$ choose $0 < \eta < 1$ so that $\eta H < \infty$. Then there exist $K_1, K_2 \in \mathbb{Z}\backslash\mathbb{Z}(m,n)$ where $K_1 \leq m$ and $K_2 \geq n$ such that $||\lambda_k x_k||^{p_k} < \eta^M/N$ for $k \in \mathbb{Z}\backslash\mathbb{Z}(K_1, K_2)$. Now for $q, r$ such that $q \leq K_1$, $r \geq K_2$ we have

$$||\sum_{k \in \mathbb{Z}(q-i, r+j)\backslash\mathbb{Z}(q,r)} A_k x_k|| \leq ||R_{q,r}(\lambda, N)|| \max |\lambda_k|N^{r_k}||x_k|| \leq H\eta < \epsilon.$$

Now by the completeness of $Y$, we easily get that $\sum_{-\infty}^{\infty} A_k x_k$ is convergent in $Y$ and hence $\bar{A} \in c_\circ^\beta(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

Conversely, (i) can easily be proved. Now suppose that $\bar{A} \in c_\circ^\beta(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ but $||R_{m,n}(\lambda, N)|| = \infty$, for each $N > 1$. Then $||R_{q,r}(\lambda, N)|| = \infty$ for each $N > 1$ and for every $q, r$ such that $q \leq m$ and $r \geq n$. Thus we can find a sequence $(k(N)), N \geq 1$ in $\mathbb{Z}\backslash\mathbb{Z}(m,n)$ and $k(N) \leq k(N+1)$. Without loss of generality we can assume that $n = k(1) < k(2) < k(3) < \cdots$ such that

$$||\sum_{k \in S(N)} \lambda_k^{-1}N^{-r_k}A_k z_k|| \geq 1 \tag{2.1}$$

for each $N > 1$ where $z_k \in S, k \in \mathbb{Z}$ and $S(N) = \{k(N-1), k(N-1)+1, \ldots, k(N)-1\}$, $N > 1$. Now the sequence $\bar{x} = (x_k)_{-\infty}^{\infty}$ defined by

$$x_k = \begin{cases} \lambda_k^{-1}N^{-r_k}z_k, & \text{if } k \in S(N), N > 1, \\ \theta, & \text{otherwise} \end{cases}$$

belongs to $c_\circ(\mathbb{Z},X,\bar{\lambda},\bar{p})$ as $||\lambda_k x_k||^{p_k} = \frac{1}{N}$, $k \in S(N)$, $N > 1$, but from (2.1) for each $N \geq 2$, $||\sum_{k\in S(N)} A_k x_k|| > 1$ shows that $\sum_n^\infty A_k x_k$ does not converge in $Y$, and consequently $\sum_{-\infty}^\infty A_k x_k$ does not converge. Hence $\bar{A} \notin c_\circ^\beta(\mathbb{Z},X,\bar{\lambda},\bar{p})$, a contradiction. This completes the proof. □

If we take $Y = \mathbb{C}$, i.e., $B(X,\mathbb{C}) = X^\star$, the space of all bounded (continuous) linear functionals on $X$. we define

$$M_\circ(\mathbb{Z},X^\star,\bar{\lambda},\bar{p}) = \bigcup_{N>1} \Big\{ \bar{f} = (f_k)_{-\infty}^\infty : f_k \in X^\star, \sum_{-\infty}^\infty |\lambda_k|^{-1}||f_k||N^{-r_k} < \infty \Big\}.$$

**Theorem 2.6.** *If $f_k \in X^\star$ for all $k \in \mathbb{Z}$ then we have*

$$c_\circ^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p}) = c_\circ^\beta(\mathbb{Z},X,\bar{\lambda},\bar{p}) = M_\circ(\mathbb{Z},X^\star,\bar{\lambda},\bar{p}),$$

*where*

$$M_\circ(\mathbb{Z},X^\star,\bar{\lambda},\bar{p}) = \bigcup_{N>1} \Big\{ \bar{f} = (f_k)_{-\infty}^\infty : f_k \in X^\star, \sum_{-\infty}^\infty |\lambda_k|^{-1}||f_k||N^{-r_k} < \infty \Big\}.$$

*Proof.* By Corollary 2.4, we immediately get $c_\circ^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p}) = M_\circ(\mathbb{Z},X^\star,\bar{\lambda},\bar{p})$. Further since $\mathbb{C}$ is complete therefore $c_\circ^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p}) \subset c_\circ^\beta(\mathbb{Z},X,\bar{\lambda},\bar{p})$. We now prove that $c_\circ^\beta(\mathbb{Z},X,\bar{\lambda},\bar{p}) = M_\circ(\mathbb{Z},X^\star,\bar{\lambda},\bar{p})$. Suppose on the contrary $\bar{f} \in c_\circ^\beta(\mathbb{Z},X,\bar{\lambda},\bar{p})$ but $\bar{f} \notin M_\circ(\mathbb{Z},X^\star,\bar{\lambda},\bar{p})$ and so $\sum_{-\infty}^\infty |\lambda_k|^{-1}||f_k||N^{-r_k} = \infty$ for each $N > 1$. Then we can choose $n = k(1) < k(2) < k(3) < \cdots$ such that $\sum_{k\in S(N)} |\lambda_k|^{-1}||f_k||N^{-r_k} > 2$ where $S(N) = \{k(N-1), k(N-1)+1, \ldots, k(N)-1\}$, $N > 1$.

Moreover for each $k \in \mathbb{Z}$, there exists $z_k \in S$ such that $||f_k|| < 2|f_k(z_k)|$. Thus the bilateral sequence $\bar{x} = (x_k)_{-\infty}^\infty$ defined by

$$x_k = \begin{cases} \operatorname{sgn}(f_k(z_k))|\lambda_k|^{-1}N^{-r_k}z_k, & \text{if } k \in S(N), N > 1,\\ \theta, & \text{otherwise}\end{cases}$$

is in $c_\circ(\mathbb{Z},X,\bar{\lambda},\bar{p})$ but $\sum_{k=-\infty}^\infty f_k(x_k) > \sum_{N=2}^\infty 1$. This shows that $\bar{f} \notin c_\circ^\beta(\mathbb{Z},X,\bar{\lambda},\bar{p})$. Hence $c_\circ^\beta(\mathbb{Z},X,\bar{\lambda},\bar{p}) \subset M_\circ(\mathbb{Z},X^\star,\bar{\lambda},\bar{p})$ and it completes the proof. □

**Theorem 2.7.** $\bar{A} \in c^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p})$ if and only if

(i) there exist $m,n \in \mathbb{Z}$ such that $A_k \in B(X,Y)$ for all $k \in \mathbb{Z}\backslash\mathbb{Z}(m,n)$,
(ii) $\sum_{k\in\mathbb{Z}\backslash\mathbb{Z}(m,n)} |\lambda_k|^{-1}||A_k||N^{-r_k} < \infty$ for each $N > 1$,
(iii) $\sum_{k\in\mathbb{Z}\backslash\mathbb{Z}(m,n)} |\lambda_k|^{-1}||A_k(x)|| < \infty$ for every $x \in X$.

*Proof.* Since $c^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p}) \subset c_\circ^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p})$ therefore the necessity of (i) and (ii) follows from Theorem 2.3 and necessity of (iii) follows from the fact that for every $x \in X$, $\bar{x} = (\lambda_k^{-1}x)_{-\infty}^\infty \in c(\mathbb{Z},X,\bar{\lambda},\bar{p})$. Now suppose (i), (ii) and (iii) hold. Then from (i) and (ii) we get that $\bar{A} \in c_\circ^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p})$ (see Theorem 2.3). Let $\bar{x} \in c(\mathbb{Z},X,\bar{\lambda},\bar{p})$. Suppose for $l_1, l_2 \in X$ we have $||\lambda_k x_k - l_1||^{p_k} \to 0$ as $k \to -\infty$ and $||\lambda_k x_k - l_2||^{p_k} \to 0$ as $k \to \infty$. Thus for given $N > 1$ there exists $K_1 \leq m$ and $K_2 \geq n$ such that

$$||\lambda_k x_k - l_1||^{p_k} < \frac{1}{N}, \quad \text{for all} \quad k \leq K_1 \quad \text{and} \quad ||x_k - \lambda_k^{-1}l_1||^{p_k} < \frac{1}{N|\lambda_k|^{p_k}},$$

$$||\lambda_k x_k - l_2||^{p_k} < \frac{1}{N}, \quad \text{for all} \quad k \geq K_2 \quad \text{and} \quad ||x_k - \lambda_k^{-1}l_2||^{p_k} < \frac{1}{N|\lambda_k|^{p_k}}.$$

Hence

$$\begin{aligned}\textstyle\sum_{-\infty}^\infty ||A_k x_k|| &= \textstyle\sum_{-\infty}^{K_1} ||A_k x_k|| + \sum_{K_1}^{K_2} ||A_k x_k|| + \sum_{K_2}^\infty ||A_k x_k||\\ &\leq \textstyle\sum_{-\infty}^{K_1} ||A_k(x_k - \lambda_k^{-1}l_1)|| + \sum_{-\infty}^{K_1} ||A_k\lambda_k^{-1}l_1|| + \sum_{K_1}^{K_2} ||A_k x_k||\\ &\quad \textstyle+ \sum_{K_2}^\infty ||A_k(x_k - \lambda_k^{-1}l_2)|| + \sum_{K_2}^\infty ||A_k\lambda_k^{-1}l_2|| < \infty.\end{aligned}$$

Hence $\bar{A} \in c^\alpha(\mathbb{Z},X,\bar{\lambda},\bar{p})$. This completes the proof. □

**Theorem 2.8.** $\bar{A} \in c^{\beta}(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, if and only if

(i) there exist $m, n \in \mathbb{Z}$ such that $A_k \in B(X, Y)$ for all $k \in \mathbb{Z} \backslash \mathbb{Z}(m, n)$,
(ii) $\sum_{m}^{-\infty} \lambda_k^{-1} A_k(x)$ and $\sum_{n}^{\infty} \lambda_k^{-1} A_k(x)$ are convergent in $Y$ for every $x \in X$.

*Proof.* To show (ii) is necessary, let $x \in X$. Then considering the bilateral sequences $\bar{x} = (x_k)_{-\infty}^{\infty}$ and $\bar{y} = (y_k)_{-\infty}^{\infty}$ defined by

$$x_k = \begin{cases} \lambda_k^{-1} x, & \text{if } k \le m, \\ \theta, & \text{otherwise,} \end{cases}$$

and

$$y_k = \begin{cases} \lambda_k^{-1} x, & \text{if } k \ge n, \\ \theta, & \text{otherwise.} \end{cases}$$

We see that $\bar{x} = (x_k)_{-\infty}^{\infty}$ and $\bar{y} = (y_k)_{-\infty}^{\infty}$ are in $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, and we immediately get the necessity of (ii).

For the converse part let $\bar{x} \in c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Then there exist $l_1, l_2 \in X$ such that $||\lambda_k x_k - l_1||^{p_k} \to 0$ as $k \to -\infty$ and $||\lambda_k x_k - l_2||^{p_k} \to 0$ as $k \to \infty$. Consider the sequence $\bar{u} = (u_k)_{-\infty}^{\infty}$, defined by

$$u_k = \begin{cases} x_k - \lambda_k^{-1} l_1, & \text{if } k \le m, \\ \theta, & m < k < n, \\ x_k - \lambda_k^{-1} l_2, & \text{if } k \ge n. \end{cases}$$

Clearly $\bar{u} \in c_{\circ}(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and therefore by Corollary 2.4 $\sum_{-\infty}^{\infty} A_k u_k$ is convergent in $Y$.

Further by (ii) we have $\sum_{m}^{-\infty} \lambda_k^{-1} A_k(l_1)$ and $\sum_{n}^{\infty} \lambda_k^{-1} A_k(l_2)$ are convergent in $Y$. We now easily get that $\sum_{-\infty}^{\infty} A_k x_k$ is convergent in $Y$ because

$$\sum_{-\infty}^{\infty} A_k x_k = \sum_{-\infty}^{\infty} A_k u_k + \sum_{m+1}^{n-1} A_k x_k + \sum_{m}^{-\infty} \lambda_k^{-1} A_k(l_1) + \sum_{n}^{\infty} \lambda_k^{-1} A_k(l_2)$$

and all the four series on the right hand side are convergent. This completes the proof. □

**Corollary 2.9.** Let $Y = \mathbb{C}$ and $f_k \in X^{\star}$ for all $k \in \mathbb{Z}$. If

(a) $\bar{f} \in M_{\circ}(\mathbb{Z}, X^{\star}, \bar{\lambda}, \bar{p})$,
(b) $\sum_{-\infty}^{\infty} |\lambda_k|^{-1} |f_k(x)| < \infty$ for every $x \in X$,
(c) $\sum_{-\infty}^{\infty} \lambda_k^{-1} f_k(x) < \infty$ for every $x \in X$.

Then

(i) $\bar{f} \in c^{\alpha}(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ if and only if (a) and (b) hold and
(ii) $\bar{f} \in c^{\beta}(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ if and only if (a) and (c) hold.

*Proof.* It can easily be proved with the help of the Theorems 2.7 and 2.8. □

The topological linear space structures of $c_{\circ}(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ have been studied in [5], when the topology is induced by the natural paranorm

$$P_{\bar{\lambda}, \bar{p}}(\bar{x}) = \sup_k ||\lambda_k x_k||^{p_k / M}.$$

**Lemma 2.10.** $c_{\circ}(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a linear space with co-ordinate-wise vector operations i.e., $\bar{x} + \bar{y} == (x_k + y_k)_{-\infty}^{\infty}$ and $\alpha \bar{x} = (\alpha x_k)_{-\infty}^{\infty}$ if and only if $\bar{p} \in \ell_{\infty}(\mathbb{Z}, \mathbb{R})$.

*Proof.* Let $\bar{p} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$ and $\bar{x} = (x_k)_{-\infty}^{\infty}, \bar{y} = (y_k)_{-\infty}^{\infty} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Then $||\lambda_k x_k||^{p_k} \to 0$ as $k \to -\infty$ as well as $k \to \infty$. Now considering

$$||\lambda_k(x_k + y_k)||^{p_k/M} \leq ||\lambda_k x_k||^{p_k/M} + ||\lambda_k y_k||^{p_k/M}$$

we see that $||\lambda_k(x_k + y_k)||^{p_k/M} \to 0$ as $k \to -\infty$ as well as $k \to \infty$ and hence $\bar{x} + \bar{y} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Also, it is clear that, for any scalar $\alpha$, $\alpha\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, since

$$||\alpha\lambda_k x_k||^{p_k/M} = |\alpha|^{p_k/M}||\lambda_k x_k||^{p_k/M} \leq A(\alpha)||\lambda_k x_k||^{p_k/M}$$

for each $k \in \mathbb{Z}$. Conversely if $\bar{p} \notin \ell_\infty(\mathbb{Z}, \mathbb{R})$, then without loss of generality we can find a sequence $(k(n))$, $k(n+1) \geq k(n)$ such that for each $n \geq 1$, $p_{k(n)} > n$. Now taking $z \in X$, we define a sequence $\bar{x} = (x_k)_{-\infty}^{\infty}$ by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-r_{k(n)}} z, & \text{if } k = k(n),\ n \geq 1 \text{ and} \\ \theta, & \text{otherwise.} \end{cases}$$

Then we see that $\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ but for the scalar $\alpha = 2$

$$||\lambda_{k(n)}(\alpha x_{k(n)})||^{p_{k(n)}} = |2|^{p_{k(n)}} \frac{1}{n} > \frac{2^n}{n} > 1, \quad \text{for each} \quad n \geq 1.$$

This shows that $\alpha\bar{x} \notin c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. This completes the proof. □

Similarly we can prove that $\bar{p} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$ is a necessary and sufficient condition for the linearity of $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ also. Therefore throughout the next section we shall take $\bar{p} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$.

## 3. Continuous dual of $c_0(\mathbb{Z}, X, \overline{\lambda}, \overline{p})$ and $c(\mathbb{Z}, X, \overline{\lambda}, \overline{p})$

In this section we shall investigate continuous dual of the spaces $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

**Theorem 3.1.** Let $\bar{p} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$. Then $c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, the continuous dual of $(c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), P_{\bar{\lambda},\bar{p}})$, is isomorphic to $M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$.

*Proof.* Let $F \in c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Since $(c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), P_{\bar{\lambda},\bar{p}})$ is a $GAK$-space therefore $\bar{s}^{(n)} \to \bar{x}$ as $n \to \infty$ where $\bar{s}^{(n)} = (\dots, \theta, x_{-n}, x_{-n+1}, \dots, x_{-1}, x_0, x_1, \dots, x_n, \theta, \dots)$. Hence for $\delta_k(x) = (\dots, \theta, x, \theta, \dots)$, $x \in X$ is at $k^{th}$-place, $\bar{s}^{(n)} = \sum_{k=-n}^{n} \delta_k(x_k)$ and

$$F(\bar{x}) = \lim_{n\to\infty} F(\bar{s}^{(n)}) = \lim_{n\to\infty} \sum_{-n}^{n} F(\delta_k(x_k)) = \sum_{-\infty}^{\infty} f_k(x_k), \tag{3.1}$$

where we write $F(\delta_k(x)) = f_k(x)$, $k \in \mathbb{Z}$. Clearly for each $k \in \mathbb{Z}$, $f_k$ is a linear functional on $X$.

Further if $x_i \to \theta$ in $X$ as $i \to \infty$ then $\delta_k(x_i) \to \delta_k(\theta) = \bar{\theta}$ in $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ with respect to $P_{\bar{\lambda},\bar{p}}$ and so $F(\delta_k(x_i)) \to F(\bar{\theta}) = 0$ i.e., $f_k(x_i) \to 0$ as $i \to \infty$ whence $f_k \in X^\star$, for all $k \in \mathbb{Z}$. Thus $\bar{f} = (f_k)_{-\infty}^{\infty}$ is a bilateral sequence in $X^\star$ and by (3.1) $\sum_{-\infty}^{\infty} f_k(x_k)$ is convergent for every $\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ i.e., $\bar{f} \in c_\circ^\beta(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, so by Theorem 2.6, we have $\bar{f} \in M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$. Hence each $F \in c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ determines an $\bar{f} \in M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$.

On the other hand if $\bar{f} \in M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$, i.e., there exists $N > 1$ such that $\sum_{-\infty}^{\infty} |\lambda_k|^{-1}\ ||f_k||\ N^{-r_k} < \infty$, then by Theorem 2.6, $\sum_{-\infty}^{\infty} f_k(x_k)$ is convergent for every $\bar{x} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Now define $F$ on $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ by $F(\bar{x}) = \sum_{-\infty}^{\infty} f_k(x_k)$. Clearly $F$ is linear. For the continuity of $F$, let $(\bar{x}^{(n)})$ be a sequence in $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ converging to $\bar{\theta}$ with respect to $P_{\bar{\lambda},\bar{p}}$. Now for $\epsilon > 0$ and $N > 1$ choose $0 < \eta < 1$ such that

$\eta \sum_{-\infty}^{\infty} |\lambda_k|^{-1} ||f_k|| \ N^{-r_k} < \epsilon$. Thus for $\eta \ N^{-1/M} > 0$, $(M = \max(1, \sup_k p_k))$ there exists $n_\circ$ such that for all $n \geq n_\circ$

$$P_{\bar{\lambda},\bar{p}}(\bar{x}^{(n)}) = \sup_k ||\lambda_k x_k^{(n)}||^{p_k/M} < \eta N^{-1/M}.$$

This implies that for all $n \geq n_\circ$

$$\begin{aligned} |F(\bar{x}^{(n)})| &\leq \sum_{-\infty}^{\infty} |f_k(x_k^{(n)})| \leq \sum_{-\infty}^{\infty} ||f_k|| \ ||x_k^{(n)}|| \\ &\leq \sum_{-\infty}^{\infty} |\lambda_k|^{-1} ||f_k|| \ N^{-r_k} \eta^{M/p_k} \leq \eta \sum_{-\infty}^{\infty} |\lambda_k|^{-1} ||f_k|| N^{-r_k} < \epsilon, \end{aligned}$$

i.e., $|F(\bar{x}^{(n)})| < \epsilon$, for all $n \geq n_\circ$. Thus $F(\bar{x}^{(n)}) \to 0$ as $n \to \infty$ and hence $F \in c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. This shows that each $\bar{f} \in M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$ corresponds to an $F \in c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Now, $\phi : c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \to M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$ defined by $\phi(F) = \bar{f}$ clearly defines an isomorphism of $c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ onto $M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$. This completes the proof. □

**Theorem 3.2.** *Let $\bar{p} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$ with $d = \inf_k p_k > 0$. Then $F \in c^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, the continuous dual of $(c(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), P_{\bar{\lambda},\bar{p}})$, if and only if there exists $\bar{f} \in M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$ and $g$ and $h \in X^\star$ such that*

$$F(\bar{x}) = g(l_1) + h(l_2) + \sum_{-\infty}^{\infty} f_k(x_k)$$

*for every $\bar{x} \in c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, where $l_1, l_2 \in X$ satisfy $||\lambda_k x_k - l_1||^{p_k} \to 0$ as $k \to -\infty$ and $||\lambda_k x_k - l_2||^{p_k} \to 0$ as $k \to \infty$.*

*Proof.* Let $F \in c^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, $\bar{x} \in c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $l_1, l_2 \in X$ satisfying $||\lambda_k x_k - l_1||^{p_k} \to 0$ as $k \to -\infty$ and $||\lambda_k x_k - l_2||^{p_k} \to 0$ as $k \to \infty$. We know that $c^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and therefore $F \in c_\circ^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Consider $\bar{z} = (z_k)_{-\infty}^{\infty}$ such that

$$z_k = \begin{cases} x_k - \lambda_k^{-1} l_1, & k \leq 0, \\ x_k - \lambda_k^{-1} l_2, & k > 0, \end{cases}$$

clearly $\bar{z} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Now the existence of $\bar{f} \in M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$ such that $F(\bar{y}) = \sum_{-\infty}^{\infty} f_k y_k$ is convergent for all $\bar{y} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ follows from Theorem 3.1 and so in particular for $\bar{z} = (z_k)_{-\infty}^{\infty} \in c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, $F(\bar{z}) = \sum_{-\infty}^{\infty} f_k z_k$ is convergent.

Further since $\bar{f} \in M_\circ(\mathbb{Z}, X^\star, \bar{\lambda}, \bar{p})$ therefore $\sum_{-\infty}^{\infty} |\lambda_k|^{-1} \ ||f_k|| \ N^{-r_k} < \infty$, for some $N > 1$ and hence using $\inf_k p_k = d > 0$, we have $\sum_{-\infty}^{\infty} |\lambda_k^{-1} f_k(x)| < \infty$, i.e., $\sum_{-\infty}^{\infty} f_k(\lambda_k^{-1} x)$ is convergent for each $x \in X$. Now consider $\bar{u} = (u_k)_{-\infty}^{\infty}$ and $\bar{v} = (v_k)_{-\infty}^{\infty}$ defined by

$$u_k = \begin{cases} \lambda_k^{-1} l_1, & k \leq 0, \\ \theta, & k > 0, \end{cases} \qquad v_k = \begin{cases} \theta, & k \leq 0, \\ \lambda_k^{-1} l_2, & k > 0. \end{cases}$$

We easily see that $\bar{x} = \bar{z} + \bar{u} + \bar{v}$ where $\bar{z}, \bar{u}$ and $\bar{v} \in c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Further since $F$ is linear we get

$$\begin{aligned} F(\bar{x}) &= F(\bar{z}) + F(\bar{u}) + F(\bar{v}) \\ &= F(\bar{u}) + F(\bar{v}) + \sum_{-\infty}^{\infty} f_k x_k - \sum_{0}^{-\infty} f_k(\lambda_k^{-1} l_1) - \sum_{1}^{\infty} f_k(\lambda_k^{-1} l_2) \end{aligned}$$

$$= \Big[F(\bar{u}) - \sum_{0}^{-\infty} f_k(\lambda_k^{-1} l_1)\Big] + \Big[F(\bar{v}) - \sum_{1}^{\infty} f_k(\lambda_k^{-1} l_2)\Big] + \sum_{-\infty}^{\infty} f_k x_k$$
$$= g(l_1) + h(l_2) + \sum_{-\infty}^{\infty} f_k x_k,$$

where we write $g(l_1) = F(\bar{u}) - \sum_0^{-\infty} f_k(\lambda_k^{-1} l_1)$ and $h(l_2) = F(\bar{v}) - \sum_1^{\infty} f_k(\lambda_k^{-1} l_2)$.

Clearly $g$ and $h$ are linear on $X$. Now we prove continuity of $h$. Suppose $w_n \to \theta$ in $X$ as $n \to \infty$. Consider $\bar{w}^{(n)}$ where

$$w_k^{(n)} = \begin{cases} \lambda_k^{-1} w_n, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then we easily see that $\bar{w}^{(n)} \in c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ for each $n \geq 1$ and

$$P_{\bar{\lambda},\bar{p}}(\bar{w}^{(n)}) = \sup_k ||\lambda_k \lambda_k^{-1} w_n||^{p_k/M} \leq ||w_n||^{p_k/M} \leq ||w_n||^{d/M}$$

shows that $\bar{w}^{(n)}$ converges to $\bar{\theta}$ in $(c(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), P_{\bar{\lambda},\bar{p}})$ and hence $F(\bar{w}^{(n)})$ will converge to 0 as $n \to \infty$. Moreover

$$\Big|\sum_{1}^{\infty} \lambda_k^{-1} f_k(w_n)\Big| \leq ||w_n|| \Big(N^{1/d} \sum_{1}^{\infty} |\lambda_k|^{-1} ||f_k|| N^{-r_k}\Big)$$

implies that $\sum_1^{\infty} \lambda_k^{-1} f_k(w_n)$ converges to 0 as $n \to \infty$. Thus $h(w_n) \to 0$ as $n \to \infty$ and hence $h$ is continuous on $X$. Similarly we can prove that $g$ is continuous on $X$.

For converse suppose that $g$ and $h \in X^{\star}$ and $\bar{f} \in M_{\circ}(\mathbb{Z}, X^{\star}, \bar{\lambda}, \bar{p})$. Then there exists an integer $N > 1$ such that $\sum_{-\infty}^{\infty} |\lambda_k|^{-1} ||f_k|| N^{-r_k} < \infty$. Now define $F$ on $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ by

$$F(\bar{x}) = g(l_1) + h(l_2) + \sum_{-\infty}^{\infty} f_k(x_k),$$

where for $\bar{x} \in c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, $||\lambda_k x_k - l_1||^{p_k} \to 0$ as $k \to -\infty$ and $||\lambda_k x_k - l_2||^{p_k} \to 0$ as $k \to \infty$. By Corollary 2.9, $F$ is well defined and $F$ is linear. Let $0 < \epsilon < 1$ and $\bar{x} \in c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Suppose that $P_{\bar{\lambda},\bar{p}}(\bar{x}) = \sup_k ||\lambda_k x_k||^{p_k/M} < \frac{\epsilon}{2}$, $||\lambda_k x_k - l_1||^{p_k/M} < \frac{\epsilon}{2}$, for all $k \leq K_0$ and $||\lambda_k x_k - l_2||^{p_k/M} < \frac{\epsilon}{2}$ for all $k \geq G_0$. Then for $k \leq K_0$

$$||l_1||^{p_k/M} \leq ||\lambda_k x_k - l_1||^{p_k/M} + ||\lambda_k x_k||^{p_k/M} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

implies that $||l_1|| < \epsilon$. Similarly we can show that $||l_2|| < \epsilon$. Therefore

$$\begin{aligned}
|F(\bar{x})| &\leq |g(l_1)| + |h(l_2)| + \Big|\sum_{-\infty}^{\infty} f_k(x_k)\Big| \\
&\leq ||g||\ ||l_1|| + ||h||\ ||l_2|| + \sum_{-\infty}^{\infty} ||f_k||\ ||x_k||\ |\lambda_k|\ |\lambda_k|^{-1} N^{-r_k} N^{r_k} \\
&\leq ||g||\ ||l_1|| + ||h||\ ||l_2|| + \Big(\sum_{-\infty}^{\infty} |\lambda_k|^{-1} ||f_k|| N^{-r_k}\Big) \sup_k (||\lambda_k x_k||\ N^{r_k}) \\
&\leq \epsilon ||g|| + \epsilon ||h|| + N^{1/d} \epsilon \Big(\sum_{-\infty}^{\infty} |\lambda_k|^{-1} ||f_k|| N^{-r_k}\Big) \\
&\leq \epsilon \Big[||g|| + ||h|| + N^{1/d} \Big(\sum_{-\infty}^{\infty} |\lambda_k|^{-1} ||f_k|| N^{-r_k}\Big)\Big].
\end{aligned}$$

Thus taking $L = \Big[||g|| + ||h|| + N^{1/d}\Big(\sum_{-\infty}^{\infty} |\lambda_k|^{-1}||f_k||N^{-r_k}\Big)\Big]$, we see $|F(\bar{x})| < \epsilon L$, where $L$ is independent of $\bar{x}$. This shows that $F$ is continuous on $(c(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), P_{\bar{\lambda},\bar{p}})$ whence $F \in c^\star(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. This completes the proof. □

## 4. Conclusion

In this paper we have generalized the conventional Maddox type sequence spaces in which sequences are defined on the set of natural number to the bilateral sequence spaces in which the sequences are defined on the set of integers. We have investigated their linear space structures and have also characterized their $\alpha$- and $\beta$- duals in our other papers. With the help of these properties, in this paper we have investigated continuous dual of the above defined spaces.

Further we may obtain matrix transformation for these bilateral sequence spaces. We may study these spaces in the theory of weighted bilateral shifts and also in the operator theory in future.

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