

## PARAMETER-ELLIPTIC PROBLEMS AND INTERPOLATION WITH A FUNCTION PARAMETER

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*Dedicated to Professor Yu. S. Samoilenko on the occasion of his 70th birthday*

ABSTRACT. Parameter-elliptic boundary-value problems are investigated on the extended Sobolev scale. This scale consists of all Hilbert spaces that are interpolation spaces with respect to a Hilbert Sobolev scale. The latter are the Hörmander spaces  $B_{2,k}$  for which the smoothness index  $k$  is an arbitrary radial function RO-varying at  $+\infty$ . We prove that the operator corresponding to this problem sets isomorphisms between appropriate Hörmander spaces provided the absolute value of the parameter is large enough. For solutions to the problem, we establish two-sided estimates, in which the constants are independent of the parameter.

### 1. INTRODUCTION

Interpolation between normed spaces is an efficient and convenient method in the theory of operators. This is due to the fact that boundedness of linear operators, their isomorphism and Fredholm properties remain valid under the interpolation between the spaces on which these operators act. In the theory of differential operators, various methods of the interpolation are used to prove theorems about collections of isomorphisms or Fredholm mappings realized by differential operators on spaces that are intermediate for a given couple of Sobolev spaces. This approach is taken in the monographs by Yu. M. Berezansky [8, Chap. III, Sec. 6], J.-L. Lions and E. Magenes [22, 23], H. Triebel [41], V. A. Mikhailets and A. A. Murach [31, 35].

In the present paper, we give an application of interpolation with a function parameter between Hilbert spaces to an important class of parameter-elliptic boundary-value problems. This class was selected and investigated on a Sobolev scale by S. Agmon, L. Nirenberg [5, 6] and M. S. Agranovich, M. I. Vishik [4]. They proved that the class has the following fundamental property. If the complex parameter is large enough in absolute value, then the operator corresponding to the parameter-elliptic problem sets an isomorphism between appropriate Sobolev spaces. Moreover, the solutions to this problem admit a two-sided estimate with constants that do not depend on the parameter. These properties have important applications to the spectral theory of elliptic operators and parabolic differential equations. Various and more general classes of parameter-elliptic problems are investigated by M. S. Agranovich [1, 2], G. Grubb [16, Ch. 2], A. N. Kozhevnikov [19, 20, 21], R. Denk, R. Mennicken, and L. R. Volevich [12, 13], R. Denk and M. Fairman [14] and others (see also survey [3]).

The purpose of this paper is to prove the above-mentioned fundamental property for the class of all Hilbert spaces that are interpolation spaces with respect to a Hilbert Sobolev scale. It follows from V. I. Ovchinnikov's theorem [38, Sec. 11.4] that these

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spaces are obtained by interpolation with a function parameter between inner product Sobolev spaces. The class of these interpolation spaces is constructively described in [31, 35, Sec. 2.4.2]. It consists of the inner product Hörmander spaces  $B_{2,k}$  [17, Sec. 2.2] for which the smoothness index  $k$  is an arbitrary radial function RO-varying at  $+\infty$ . This class is naturally to call the extended Sobolev scale (by means of interpolation spaces).

Note that various classes of elliptic differential operators given on manifolds without boundary are investigated on this scale in [36, 37, 43] and [31, 35, Sec. 2.4.3].

## 2. STATEMENT OF THE PROBLEM

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ , with  $n \geq 2$ . Suppose that its boundary  $\Gamma := \partial\Omega$  is an infinitely smooth closed manifold of dimension  $n - 1$ . Let  $\nu(x)$  denote the unit vector of the inner normal to  $\partial\Omega$  at a point  $x \in \Gamma$ . As usual,  $\bar{\Omega} := \Omega \cup \Gamma$ .

Let  $q \geq 1$  and  $m_1, \dots, m_q \geq 0$  be arbitrarily chosen integers. We consider the boundary-value problem

$$(1) \quad A(\lambda)u = f \quad \text{in } \Omega, \quad B_j(\lambda)u = g_j \quad \text{on } \Gamma, \quad j = 1, \dots, q,$$

that depends on the parameter  $\lambda \in \mathbb{C}$  as follows:

$$(2) \quad A(\lambda) := \sum_{r=0}^{2q} \lambda^{2q-r} A_r, \quad B_j(\lambda) := \sum_{r=0}^{m_j} \lambda^{m_j-r} B_{j,r}.$$

Here, all  $A_r$  and  $B_{j,r}$  are linear partial differential expressions whose orders do not exceed  $r$ . We write these expressions in the form

$$(3) \quad A_r := A_r(x, D) := \sum_{|\mu| \leq r} a_{r,\mu}(x) D^\mu, \quad x \in \bar{\Omega},$$

$$(4) \quad B_{j,r} := B_{j,r}(x, D) := \sum_{|\mu| \leq r} b_{j,r,\mu}(x) D^\mu, \quad x \in \Gamma.$$

Their coefficients are complex-valued infinitely smooth functions; i.e., all  $a_{r,\mu} \in C^\infty(\bar{\Omega})$  and  $b_{j,r,\mu} \in C^\infty(\Gamma)$ . Put  $B(\lambda) := (B_1(\lambda), \dots, B_q(\lambda))$ .

Note that we use the standard notation in (3), (4), and below; namely, for the multi-index  $\mu = (\mu_1, \dots, \mu_n)$  we let  $|\mu| := \mu_1 + \dots + \mu_n$  and  $D^\mu := D_1^{\mu_1} \dots D_n^{\mu_n}$ , with  $D_k := i \partial / \partial x_k$  for  $k \in \{1, \dots, n\}$  and  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . Moreover, we put  $\xi^\mu := \xi_1^{\mu_1} \dots \xi_n^{\mu_n}$  for the frequency variables  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ , which are dual to the spatial variables  $x = (x_1, \dots, x_n)$  with respect to the Fourier transform

$$\widehat{w}(\xi) := (\mathcal{F}w)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi x} w(x) dx.$$

Following M. S. Agranovich and M. I. Vishik [4, Sec. 4] (see also survey [3, Sec. 3.1]), we recall the notion of parameter-ellipticity in connection with the boundary-value problem (1).

Let us associate certain homogeneous polynomials in  $(\xi, \lambda) \in \mathbb{C}^{n+1}$  with partial differential expressions (2). Namely, we set

$$A^{(0)}(x, \xi, \lambda) := \sum_{r=0}^{2q} \lambda^{2q-r} \sum_{|\mu|=r} a_{r,\mu}(x) \xi^\mu, \quad \text{with } x \in \bar{\Omega},$$

$$B_j^{(0)}(x, \xi, \lambda) := \sum_{r=0}^{m_j} \lambda^{m_j-r} \sum_{|\mu|=r} b_{j,r,\mu}(x) \xi^\mu, \quad \text{with } x \in \Gamma.$$

Let  $K$  be a fixed closed angle in the complex plain with the vertex at the origin; this angle may degenerate into a ray.

The boundary-value problem (1) is called parameter-elliptic in the angle  $K$  if the following two conditions are satisfied:

- i)  $A^{(0)}(x, \xi, \lambda) \neq 0$  for each  $x \in \overline{\Omega}$  and all  $\xi \in \mathbb{R}^n$  and  $\lambda \in K$  with  $|\xi| + |\lambda| \neq 0$ .
- ii) Let  $x \in \Gamma$ ,  $\xi \in \mathbb{R}^n$ , and  $\lambda \in K$  be arbitrarily chosen so that  $\xi$  is tangent to  $\Gamma$  at  $x$  and that  $|\xi| + |\lambda| \neq 0$ . Then the polynomials  $B_j^{(0)}(x, \xi + \tau\nu(x), \lambda)$  in  $\tau$ ,  $j = 1, \dots, q$ , are linearly independent modulo  $\prod_{j=1}^q (\tau - \tau_j^+(x, \xi, \lambda))$ . Here,  $\tau_1^+(x, \xi, \lambda), \dots, \tau_q^+(x, \xi, \lambda)$  are all  $\tau$ -roots of  $A^{(0)}(x, \xi + \tau\nu(x), \lambda)$  with  $\text{Im } \tau > 0$ , each root being taken the number of times equal to its multiplicity.

In connection with Condition ii), it is relevant to mention the following fact [4, Proposition 2.2]. If  $A^{(0)}(x, \xi, \lambda)$  satisfies Condition i), then the polynomial  $A^{(0)}(x, \xi + \tau\nu(x), \lambda)$  in  $\tau$  has  $q$  roots with  $\text{Im } \tau > 0$  and  $q$  roots with  $\text{Im } \tau < 0$  (if we take their multiplicity into account). Thus, the above definition of parameter-ellipticity is reasonably formulated.

Some examples of parameter-elliptic boundary-value problems are given in [3, Sec. 3.1 b]. For instance, the following boundary-value problem:

$$\Delta u + \lambda^2 u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} - \lambda u = g \quad \text{on } \Gamma$$

is parameter-elliptic in the angle  $K$  provided that  $K$  contains neither  $\mathbb{R}_+$  nor  $\mathbb{R}_-$ , where  $\mathbb{R}_\pm := \{\lambda \in \mathbb{R} : \lambda \gtrless 0\}$ . Note also that, if  $A(\lambda)$  satisfies Condition i), then the Dirichlet boundary-value problem for the partial differential equation  $A(\lambda) = f$  is parameter-elliptic in the angle  $K$ .

We investigate properties of the parameter-elliptic boundary-value problem (1) on the extended Sobolev scale.

### 3. THE EXTENDED SOBOLEV SCALE

This scale [34], [35, Sec. 2.4.2] consists of the inner product isotropic Hörmander spaces  $H^\varphi$  for which an arbitrary function parameter  $\varphi \in \text{RO}$  serves as a smoothness index. Here,  $\text{RO}$  is the class of all Borel measurable functions  $\varphi : [1, \infty) \rightarrow (0, \infty)$  such that

$$(5) \quad c^{-1} \leq \frac{\varphi(\lambda t)}{\varphi(t)} \leq c \quad \text{for arbitrary } t \geq 1 \quad \text{and } \lambda \in [1, a]$$

with some numbers  $a = a(\varphi) > 1$  and  $c = c(\varphi) \geq 1$  (certainly,  $a$  and  $c$  are independent of both  $t$  and  $\lambda$ ). Such functions are said to be  $\text{RO}$ -varying in the sense of V. G. Avakumović [7] and are sufficiently investigated [10, 40].

The class  $\text{RO}$  admits the simple description

$$\varphi \in \text{RO} \quad \Leftrightarrow \quad \varphi(t) = \exp\left(\beta(t) + \int_1^t \frac{\gamma(\tau)}{\tau} d\tau\right), \quad t \geq 1,$$

where the real-valued functions  $\beta$  and  $\gamma$  are Borel measurable and bounded on  $[1, \infty)$ . Note also that condition (5) is equivalent to the bilateral inequality

$$(6) \quad c_0 \lambda^{s_0} \leq \frac{\varphi(\lambda t)}{\varphi(t)} \leq c_1 \lambda^{s_1} \quad \text{for arbitrary } t \geq 1 \quad \text{and } \lambda \geq 1,$$

where  $c_0$  and  $c_1$  are certain positive numbers, which do not depend on  $t$  and  $\lambda$ . Hence, for every function  $\varphi \in \text{RO}$ , we can define lower and the upper Matuszewska indices [24] by the formulas

$$\begin{aligned} \sigma_0(\varphi) &:= \sup\{s_0 \in \mathbb{R} : \text{the left-hand inequality in (6) holds}\}, \\ \sigma_1(\varphi) &:= \inf\{s_1 \in \mathbb{R} : \text{the right-hand inequality in (6) holds}\}, \end{aligned}$$

with  $-\infty < \sigma_0(\varphi) \leq \sigma_1(\varphi) < \infty$  (see [10, Theorem 2.2.2]).

Now let  $\varphi \in \text{RO}$  and introduce the function spaces  $H^\varphi$  over  $\mathbb{R}^n$  and then over  $\Omega$  and  $\Gamma$ .

By definition, the linear space  $H^\varphi(\mathbb{R}^n)$  consists of all distributions  $w \in \mathcal{S}'(\mathbb{R}^n)$  such that their Fourier transform  $\widehat{w}$  is locally Lebesgue integrable over  $\mathbb{R}^n$  and satisfies the condition

$$\int_{\mathbb{R}^n} \varphi^2(\langle \xi \rangle) |\widehat{w}(\xi)|^2 d\xi < \infty.$$

Here, as usual,  $\mathcal{S}'(\mathbb{R}^n)$  is the complex linear topological space of tempered distributions given in  $\mathbb{R}^n$ , and  $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$  is the smoothed modulus of  $\xi \in \mathbb{R}^n$ . The inner product in  $H^\varphi(\mathbb{R}^n)$  is defined as follows

$$(w_1, w_2)_{H^\varphi(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \varphi^2(\langle \xi \rangle) \widehat{w}_1(\xi) \overline{\widehat{w}_2(\xi)} d\xi,$$

with  $w_1, w_2 \in H^\varphi(\mathbb{R}^n)$ . It induces the norm  $\|w\|_{H^\varphi(\mathbb{R}^n)} := (w, w)_{H^\varphi(\mathbb{R}^n)}^{1/2}$ .

The space  $H^\varphi(\mathbb{R}^n)$  is a Hilbert and isotropic case of the spaces  $B_{p,k}$  introduced and systematically investigated by L. Hörmander [17, Sec. 2.2] (also see [18, Sec. 10.1]). Namely,  $H^\varphi(\mathbb{R}^n) = B_{p,k}$  provided that  $p = 2$  and  $k(\xi) = \varphi(\langle \xi \rangle)$  for all  $\xi \in \mathbb{R}^n$ . Note that, in the Hilbert case of  $p = 2$ , the Hörmander spaces coincide with the spaces introduced and studied by L. R. Volevich and B. P. Paneah [42, Sec. 2].

If  $\varphi(t) \equiv t^s$ , then  $H^\varphi(\mathbb{R}^n) =: H^{(s)}(\mathbb{R}^n)$  is an inner product Sobolev space of order  $s \in \mathbb{R}$ . In general,

$$(7) \quad s_0 < \sigma_0(\varphi) \leq \sigma_1(\varphi) < s_1 \Rightarrow H^{(s_1)}(\mathbb{R}^n) \hookrightarrow H^\varphi(\mathbb{R}^n) \hookrightarrow H^{(s_0)}(\mathbb{R}^n),$$

both embeddings being continuous and dense.

Following [34], we call the class of Hilbert function spaces

$$(8) \quad \{H^\varphi(\mathbb{R}^n) : \varphi \in \text{RO}\}$$

an extended Sobolev scale over  $\mathbb{R}^n$ .

Its analogs for the Euclidean domain  $\Omega$  and for the closed manifold  $\Gamma$  are introduced in a standard way on the basis of the class (8) (see [33, Sec. 2] and [31, 35, Sec. 2.4.2]). Let us give necessary definitions; as above,  $\varphi \in \text{RO}$ .

By definition,

$$(9) \quad \begin{aligned} H^\varphi(\Omega) &:= \{w \upharpoonright \Omega : w \in H^\varphi(\mathbb{R}^n)\}, \\ \|u\|_{H^\varphi(\Omega)} &:= \inf \{ \|w\|_{H^\varphi(\mathbb{R}^n)} : w \in H^\varphi(\mathbb{R}^n), w = u \text{ in } \Omega \}, \end{aligned}$$

with  $u \in H^\varphi(\Omega)$ . The linear space  $H^\varphi(\Omega)$  is Hilbert and separable with respect to the norm (9) because  $H^\varphi(\Omega)$  is a factor space of the separable Hilbert space  $H^\varphi(\mathbb{R}^n)$  with respect to its subspace

$$\{w \in H^\varphi(\mathbb{R}^n) : \text{supp } w \subseteq \mathbb{R}^n \setminus \Omega\}.$$

Note that  $C^\infty(\overline{\Omega})$  is dense in  $H^\varphi(\Omega)$ .

By definition, the linear space  $H^\varphi(\Gamma)$  consists of all distributions on  $\Gamma$  that belong to  $H^\varphi(\mathbb{R}^{n-1})$  in local coordinates on  $\Gamma$ . Namely, arbitrarily chose a finite collection of local charts  $\alpha_j : \mathbb{R}^{n-1} \leftrightarrow \Gamma_j$ ,  $j = 1, \dots, \varkappa$ , where the open sets  $\Gamma_j$  form a covering of  $\Gamma$ . Also choose functions  $\chi_j \in C^\infty(\Gamma)$ ,  $j = 1, \dots, \varkappa$ , that satisfy the condition  $\text{supp } \chi_j \subset \Gamma_j$  and that form a partition of unity on  $\Gamma$ . Then

$$H^\varphi(\Gamma) := \{h \in \mathcal{D}'(\Gamma) : (\chi_j h) \circ \alpha_j \in H^\varphi(\mathbb{R}^{n-1}) \text{ for every } j \in \{1, \dots, \varkappa\}\}.$$

Here, as usual,  $\mathcal{D}'(\Gamma)$  is the complex linear topological space of all distributions given on  $\Gamma$ , and  $(\chi_j h) \circ \alpha_j$  is the representation of the distribution  $\chi_j h$  in the local chart  $\alpha_j$ . The inner product in  $H^\varphi(\Gamma)$  is defined by the formula

$$(h_1, h_2)_{H^\varphi(\Gamma)} := \sum_{j=1}^{\varkappa} ((\chi_j h_1) \circ \alpha_j, (\chi_j h_2) \circ \alpha_j)_{H^\varphi(\mathbb{R}^{n-1})},$$

with  $h_1, h_2 \in H^\varphi(\Gamma)$ , and induces the norm  $\|h\|_{H^\varphi(\Gamma)} := (h, h)_{H^\varphi(\Gamma)}^{1/2}$ . The space  $H^\varphi(\Gamma)$  is Hilbert and separable and does not depend (up to equivalence of norms) on our choice of local charts and the partition of unity on  $\Gamma$  [31, 35, Theorem 2.21]. Note also that  $C^\infty(\Gamma)$  is dense in  $H^\varphi(\Gamma)$ .

The above-defined function spaces form the extended Sobolev scales

$$(10) \quad \{H^\varphi(\Omega) : \varphi \in \text{RO}\} \quad \text{and} \quad \{H^\varphi(\Gamma) : \varphi \in \text{RO}\}$$

over  $\Omega$  and  $\Gamma$  respectively. They contain scales of inner product Sobolev spaces; namely, if  $\varphi(t) \equiv t^s$  for some  $s \in \mathbb{R}$ , then  $H^\varphi(\Omega) =: H^{(s)}(\Omega)$  and  $H^\varphi(\Gamma) =: H^{(s)}(\Gamma)$  are the Sobolev spaces of order  $s$ . Property (7) remains true (with the embeddings being continuous and dense) provided that we replace  $\mathbb{R}^n$  with  $\Omega$  or  $\Gamma$  therein.

#### 4. THE MAIN RESULT

Let us state our main result regarding the isomorphism property of the parameter-elliptic problem (1) considered on the extended Sobolev scale.

To avoid mentioning the argument  $t$  in the smoothness indices, we refer to  $\varrho$  as the function  $\varrho(t) := t$  of  $t \geq 1$ . Note that, if  $\varphi \in \text{RO}$  and  $s \in \mathbb{R}$ , then the function  $\varrho^s \varphi$  belongs to  $\text{RO}$ , and its Matuszewska indices satisfy the relation  $\sigma_j(\varrho^s \varphi) = s + \sigma_j(\varphi)$  for each  $j \in \{0, 1\}$ .

The mapping  $u \mapsto (A(\lambda)u, B(\lambda)u)$ , with  $u \in C^\infty(\overline{\Omega})$ , extends uniquely (by continuity) to the bounded linear operator

$$(11) \quad (A(\lambda), B(\lambda)) : H^{\varphi \varrho^{2q}}(\Omega) \rightarrow H^\varphi(\Omega) \oplus \bigoplus_{j=1}^q H^{\varphi \varrho^{2q-m_j-1/2}}(\Gamma) =: \mathcal{H}^\varphi(\Omega, \Gamma)$$

for each  $\lambda \in \mathbb{C}$  and every function parameter  $\varphi \in \text{RO}$  that meets the condition

$$(12) \quad \sigma_0(\varphi) > l := \max\{0, m_1 - 2q + 1/2, \dots, m_q - 2q + 1/2\}.$$

We need to introduce certain norms, which depend on the parameter  $p := |\lambda| \geq 1$ . Let  $\alpha \in \text{RO}$ , with  $\sigma_0(\alpha) > 0$ , and let  $G \in \{\mathbb{R}^n, \Omega, \Gamma\}$ . We define an equivalent Hilbert norm in the space  $H^\alpha(G)$  by the formula

$$(13) \quad \|w\|_{\alpha, p, G} := (\|w\|_{H^\alpha(G)}^2 + \alpha^2(p) \|w\|_{H^{(0)}(G)}^2)^{1/2},$$

with  $w \in H^\alpha(G)$ . The equivalence of norms follows from the continuous embedding  $H^\alpha(G) \hookrightarrow H^{(0)}(G)$ . Note that the space  $H^{(0)}(G) = L_2(G)$  consist of all the functions that are square integrable over  $G$ .

**Isomorphism Theorem.** *Let the boundary-value problem (1) be parameter-elliptic in the angle  $K$ . Then there exists a number  $\lambda_1 \geq 1$  such that, for each  $\lambda \in K$  with  $|\lambda| \geq \lambda_1$  and for every  $\varphi \in \text{RO}$  subject to (12), we have the isomorphism*

$$(14) \quad (A(\lambda), B(\lambda)) : H^{\varphi \varrho^{2q}}(\Omega) \leftrightarrow \mathcal{H}^\varphi(\Omega, \Gamma).$$

Moreover, for each fixed  $\varphi \in \text{RO}$  satisfying (12), there exists a number  $c = c(\varphi) \geq 1$  such that

$$(15) \quad \begin{aligned} c^{-1} \|u\|_{\varphi \varrho^{2q}, |\lambda|, \Omega} &\leq \|A(\lambda)u\|_{\varphi, |\lambda|, \Omega} + \sum_{j=1}^q \|B_j(\lambda)u\|_{\varphi \varrho^{2q-m_j-1/2}, |\lambda|, \Gamma} \\ &\leq c \|u\|_{\varphi \varrho^{2q}, |\lambda|, \Omega} \end{aligned}$$

for every  $\lambda \in K$  with  $|\lambda| \geq \lambda_1$  and for arbitrary  $u \in H^{\varphi \varrho^{2q}}(\Omega)$ . Here, the number  $c$  does not depend on  $\lambda$  and  $u$ .

In the Sobolev case of  $\varphi(t) \equiv t^s$ , this theorem was proved by M. S. Agranovich and M. I. Vishik [4, § 4 and 5] with the additional assumption that  $s$  is integer. The

assumption is not obligatory [3, Sec. 3.2]. V. A. Mikhailets and the second author of the present paper proved this theorem in the case where the function parameter  $\varphi$  varies regularly in the sense of J. Karamata at infinity and where each  $m_j \leq 2q - 1$  (see [28, Sec. 7] and [31, 35, Sec. 4.1.4]).

In Section 6, we will deduce Isomorphism Theorem from the Sobolev case with the help of the interpolation with a function parameter between Sobolev spaces.

Remark that the right-hand inequality in (15) remains true for every  $\lambda \in \mathbb{C}$  without the assumption that (1) is parameter-elliptic on  $K$  (cf. [4, Proposition 4.1] or [3, Sec. 3.2 a]).

Isomorphism Theorem implies the following important fact (cf. [4, Sec. 6.4]).

**Fredholm Property.** *Suppose that the boundary-value problem (1) is parameter-elliptic on a certain ray  $K := \{\lambda \in \mathbb{C} : \arg \lambda = \text{const}\}$ . Then the operator (11) is Fredholm with zero index for each  $\lambda \in \mathbb{C}$  and for every  $\varphi \in \text{RO}$  subject to (12).*

We will prove these remark and property in Section 6 as well.

Note that the theory of general elliptic boundary-value problems is built for a narrower class of Hörmander spaces (called the refined Sobolev scale) by V. A. Mikhailets and A. A. Murach in a series of papers, among them we mention the articles [26–29], survey [32], and monographs [31, 35]. This class consists of the spaces  $H^\varphi(\Omega)$  such that  $\varphi$  varies regularly at infinity. Specifically, parameter-elliptic problems is investigated in [28, Sec. 7] under additional assumption that all  $m_j \leq 2q - 1$ .

## 5. METHOD OF INTERPOLATION WITH A FUNCTION PARAMETER

This method will play a key role in our proof of Isomorphism Theorem. Namely, we will use the fact that every space  $H^\alpha(G)$ , with  $\alpha \in \text{RO}$  and  $G \in \{\mathbb{R}^n, \Omega, \Gamma\}$ , can be obtained by means of the interpolation (with an appropriate function parameter) between the Sobolev spaces  $H^{(s_0)}(G)$  and  $H^{(s_1)}(G)$  with  $s_0 < \sigma_0(\alpha)$  and  $s_1 > \sigma_1(\alpha)$ .

The method of interpolation with a function parameter between normed spaces was introduced by C. Foiaş and J.-L. Lions in [15] and was then developed and investigated by some researchers; see monographs [11, 38] and the literature given therein.

For our purposes it is sufficient to use this interpolation for separable Hilbert spaces. We recall its definition and some of its properties; see monographs [31, 35, Sec. 1.1] or paper [30, Sec. 2], in which this matter is systematically set forth.

Let  $X := [X_0, X_1]$  be a given ordered couple of separable complex Hilbert spaces  $X_0$  and  $X_1$  that satisfy the continuous and dense embedding  $X_1 \hookrightarrow X_0$ . We say that this couple is admissible. For  $X$  there exists an isometric isomorphism  $J : X_1 \leftrightarrow X_0$  such that  $J$  is a self-adjoint positive operator on  $X_0$  with the domain  $X_1$ . This operator is uniquely determined by the couple  $X$  and is called a generating operator for  $X$ .

By  $\mathcal{B}$  we denote the set of all Borel measurable functions  $\psi : (0, \infty) \rightarrow (0, \infty)$  such that  $\psi$  is bounded on each compact interval  $[a, b]$ , with  $0 < a < b < \infty$ , and that  $1/\psi$  is bounded on every set  $[r, \infty)$ , with  $r > 0$ .

Let  $\psi \in \mathcal{B}$ , and consider the operator  $\psi(J)$ , which is defined (and positive) in  $X_0$  as the Borel function  $\psi$  of  $J$ . Denote by  $[X_0, X_1]_\psi$  or, simply, by  $X_\psi$  the domain of the operator  $\psi(J)$  endowed with the inner product  $(u_1, u_2)_{X_\psi} := (\psi(J)u_1, \psi(J)u_2)_{X_0}$  and the corresponding norm  $\|u\|_{X_\psi} = \|\psi(J)u\|_{X_0}$ . The space  $X_\psi$  is Hilbert and separable.

A function  $\psi \in \mathcal{B}$  is called an interpolation parameter if the following condition is fulfilled for all admissible couples  $X = [X_0, X_1]$  and  $Y = [Y_0, Y_1]$  of Hilbert spaces and for an arbitrary linear mapping  $T$  given on  $X_0$ : if the restriction of  $T$  to  $X_j$  is a bounded operator  $T : X_j \rightarrow Y_j$  for each  $j \in \{0, 1\}$ , then the restriction of  $T$  to  $X_\psi$  is also a bounded operator  $T : X_\psi \rightarrow Y_\psi$ . In this case we say that  $X_\psi$  is obtained by the interpolation with the function parameter  $\psi$  of the couple  $X$  (or, otherwise speaking, between the spaces  $X_0$  and  $X_1$ ).

The function  $\psi \in \mathcal{B}$  is an interpolation parameter if and only if  $\psi$  is pseudoconcave on a neighborhood of  $+\infty$ , i.e.  $\psi(t) \asymp \psi_1(t)$  with  $t \gg 1$  for a certain positive concave function  $\psi_1(t)$ . (As usual, the designation  $\psi \asymp \psi_1$  means that both the functions  $\psi/\psi_1$  and  $\psi_1/\psi$  are bounded on the indicated set). This fundamental fact follows from J. Peetre's [39] description of all interpolation functions of positive order (see also the monograph [9, Sec. 5.4]).

Now we can formulate the above-mentioned interpolation property of the extended Sobolev as follows.

**Proposition 1.** *Let a function  $\alpha \in \text{RO}$  and real numbers  $s_0 < \sigma_0(\alpha)$  and  $s_1 > \sigma_1(\alpha)$  be given. Put*

$$(16) \quad \psi(t) := \begin{cases} t^{-s_0/(s_1-s_0)} \alpha(t^{1/(s_1-s_0)}) & \text{for } t \geq 1, \\ \alpha(1) & \text{for } 0 < t < 1. \end{cases}$$

Then  $\psi \in \mathcal{B}$  is an interpolation parameter, and

$$(17) \quad [H^{(s_0)}(G), H^{(s_1)}(G)]_\psi = H^\alpha(G)$$

with equivalence of norms provided that  $G \in \{\mathbb{R}^n, \Omega, \Gamma\}$ . Moreover, if  $G = \mathbb{R}^n$ , then (17) holds with equality of norms.

This proposition is proved in [31, 35, Theorems 2.18 and 2.22] for  $G \in \{\mathbb{R}^n, \Gamma\}$  and in [33, Theorem 5.1] for  $G = \Omega$ .

Besides, the extended Sobolev scale  $\{H^\alpha(G) : \alpha \in \text{RO}\}$  possesses the following important interpolation properties (see [31, 35, Sec. 2.4.2] and [33, Sec. 2 and 5]). This scale is closed with respect to the above method of interpolation with a function parameter and coincides (up to equivalence of norms) with the class of all Hilbert spaces that are interpolation spaces for the couple of the Sobolev spaces  $H^{(s_0)}(G)$  and  $H^{(s_1)}(G)$  with  $s_0, s_1 \in \mathbb{R}$  and  $s_0 < s_1$ . The latter property follows from V. I. Ovchinnikov's theorem [38, Sec. 11.4], which gives a description of all Hilbert spaces that are interpolation spaces for an arbitrarily chosen couple of Hilbert spaces. Recall that the property of a (Hilbert) space  $H$  to be an interpolation space for an admissible couple  $X = [X_0, X_1]$  means the following: a) the continuous and dense embeddings  $X_1 \hookrightarrow H \hookrightarrow X_0$  hold; b) every linear operator bounded on each of the spaces  $X_0$  and  $X_1$  should be bounded on  $H$  as well.

Thus, the extended Sobolev scale is the completed extension of the Sobolev scale  $\{H^{(s)}(G) : s \in \mathbb{R}\}$  by the interpolation within the category of Hilbert spaces.

We will also use two properties of interpolation between abstract Hilbert spaces [31, 35, Sec 1.1.8 and 1.1.5].

The first of them is an important estimate of the operator norm in interpolation spaces. Recall that an admissible couple of Hilbert spaces  $X = [X_0, X_1]$  is said to be normal if  $\|u\|_{X_0} \leq \|u\|_{X_1}$  for each  $u \in X_1$ .

**Proposition 2.** *For every interpolation parameter  $\psi \in \mathcal{B}$  there exists a number  $\tilde{c} = \tilde{c}(\psi) > 0$  such that*

$$\|T\|_{X_\psi \rightarrow Y_\psi} \leq \tilde{c} \max \{ \|T\|_{X_j \rightarrow Y_j} : j = 0, 1 \}.$$

Here,  $X = [X_0, X_1]$  and  $Y = [Y_0, Y_1]$  are arbitrary normal admissible couples of Hilbert spaces, and  $T$  is an arbitrary linear mapping which is given on  $X_0$  and defines the bounded operators  $T : X_j \rightarrow Y_j$ , with  $j = 0, 1$ . The number  $c_\psi > 0$  does not depend on  $X, Y$ , and  $T$ .

The second property is useful when we interpolate between direct sums of Hilbert spaces.

**Proposition 3.** *Let  $[X_0^{(j)}, X_1^{(j)}]$ , with  $j = 1, \dots, r$ , be a finite collection of admissible couples of Hilbert spaces. Then, for every function  $\psi \in \mathcal{B}$ , we have*

$$\left[ \bigoplus_{j=1}^r X_0^{(j)}, \bigoplus_{j=1}^r X_1^{(j)} \right]_{\psi} = \bigoplus_{j=1}^r [X_0^{(j)}, X_1^{(j)}]_{\psi} \quad \text{with equality of norms.}$$

## 6. PROOFS

Previously, we will prove an interpolation property of the norm (13) used in Isomorphism Theorem.

Let  $\alpha \in \text{RO}$ , with  $\sigma_0(\alpha) > 0$ , and let  $p \geq 1$  and  $G \in \{\mathbb{R}^n, \Omega, \Gamma\}$ . By  $H^\alpha(G, p)$  we denote the space  $H^\alpha(G)$  endowed with the equivalent Hilbert norm (13). In the Sobolev case where  $\alpha(t) \equiv t^s$  for some  $s > 0$ , we also use the notation  $H^{(s)}(G, p)$  and  $\|\cdot\|_{(s), p, G}$  for the space  $H^\alpha(G, p)$  and norm in it.

**Interpolation Lemma.** *Let positive numbers  $s_0 < \sigma_0(\alpha)$  and  $s_1 > \sigma_1(\alpha)$  be given. Define the interpolation parameter  $\psi$  by formula (16) and denote, for the sake of brevity,  $E(G, p) := [H^{(s_0)}(G, p), H^{(s_1)}(G, p)]_{\psi}$ . Then there exists a number  $c_0 \geq 1$  such that*

$$(18) \quad c_0^{-1} \|w\|_{E(G, p)} \leq \|w\|_{\alpha, p, G} \leq c_0 \|w\|_{E(G, p)}$$

for every  $p \geq 1$  and all  $w \in H^\alpha(G)$ . The number  $c_0$  is independent of  $p$  and  $w$ .

This lemma reinforces Proposition 1 in the sense that the constants in the estimates of the equivalent norms in  $H^\alpha(G, p)$  and  $E(G, p)$  can be chosen to be independent of the parameter  $p$ .

*Proof of Interpolation Lemma.* First, we prove this lemma for  $G = \mathbb{R}^n$ . Then we deduce it in the  $G \in \{\Omega, \Gamma\}$  cases, which will be used in our proof of Isomorphism Theorem.

Suppose that  $G = \mathbb{R}^n$ , with  $n \geq 1$ . Let  $p \geq 1$  and  $w \in H^\alpha(\mathbb{R}^n)$ . Note that

$$\|w\|_{\alpha, p, \mathbb{R}^n} = \left( \int_{\mathbb{R}^n} (\alpha^2(\langle \xi \rangle) + \alpha^2(p)) |\widehat{w}(\xi)|^2 d\xi \right)^{1/2}.$$

Along with this norm, we consider another Hilbert norm

$$(19) \quad \|w\|'_{\alpha, p, \mathbb{R}^n} := \left( \int_{\mathbb{R}^n} \alpha^2(\langle \xi \rangle + p) |\widehat{w}(\xi)|^2 d\xi \right)^{1/2}.$$

They are equivalent to each other; moreover,

$$(20) \quad c_1^{-1} \|w\|'_{\alpha, p, \mathbb{R}^n} \leq \|w\|_{\alpha, p, \mathbb{R}^n} \leq c_1 \|w\|'_{\alpha, p, \mathbb{R}^n}$$

with some number  $c_1 = c_1(\alpha) \geq 1$  being independent of  $p$  and  $w$ .

Indeed, since  $\sigma_0(\alpha) > 0$ , then there is a number  $c_2 = c_2(\alpha) \geq 1$  such that

$$(21) \quad c_2^{-1} \alpha(t_1 + t_2) \leq \alpha(t_1) + \alpha(t_2) \leq c_2 \alpha(t_1 + t_2) \quad \text{for all } t_1, t_2 \geq 1.$$

This immediately implies (20) with  $c_1 := \sqrt{2} c_2$ . The demonstration of property (21) is simple and will be given at once after the proof of this lemma.

Let  $H^\alpha(\mathbb{R}^n, p, 1)$  denote the space  $H^\alpha(\mathbb{R}^n)$  endowed with the equivalent Hilbert norm (19). In the Sobolev case where  $\alpha(t) \equiv t^s$  for some  $s > 0$ , we also use the notation  $H^{(s)}(\mathbb{R}^n, p, 1)$  and  $\|\cdot\|'_{(s), p, \mathbb{R}^n}$  for the space  $H^\alpha(\mathbb{R}^n, p, 1)$  and norm in it. Put, for the sake of brevity,

$$E(\mathbb{R}^n, p, 1) := [H^{(s_0)}(\mathbb{R}^n, p, 1), H^{(s_1)}(\mathbb{R}^n, p, 1)]_{\psi}.$$

Let us prove the equality

$$(22) \quad \|w\|_{E(\mathbb{R}^n, p, 1)} = \|w\|'_{\alpha, p, \mathbb{R}^n}.$$



To this end, calculate the norm in the interpolation space  $E(\mathbb{R}^n, p, 1)$ . It is directly verified that the pseudodifferential operator

$$J : u \mapsto \mathcal{F}^{-1} [(\langle \xi \rangle + p)^{s_1 - s_0} \widehat{u}(\xi)], \quad \text{with } u \in H^{(s_1)}(\mathbb{R}^n),$$

is the generating operator for the admissible couple of the Hilbert spaces  $H^{(s_0)}(\mathbb{R}^n, p, 1)$  and  $H^{(s_1)}(\mathbb{R}^n, p, 1)$ . Here,  $\mathcal{F}^{-1}$  is the inverse Fourier transform in  $\xi \in \mathbb{R}^n$ . Applying the isometric isomorphism

$$\mathcal{F} : H^{(s_0)}(\mathbb{R}^n, p, 1) \leftrightarrow L_2(\mathbb{R}^n, (\langle \xi \rangle + p)^{2s_0} d\xi),$$

we reduce the operator  $\psi(J)$  to the form of multiplication by the function

$$\psi((\langle \xi \rangle + p)^{s_1 - s_0}) = (\langle \xi \rangle + p)^{-s_0} \alpha(\langle \xi \rangle + p) \quad \text{of } \xi,$$

in view of (16). Therefore,

$$\begin{aligned} \|w\|_{E(\mathbb{R}^n, p, 1)} &= \|\psi(J)w\|'_{(s_0), p, \mathbb{R}^n} \\ &= \left( \int_{\mathbb{R}^n} \psi^2((\langle \xi \rangle + p)^{s_1 - s_0}) |\widehat{w}(\xi)|^2 (\langle \xi \rangle + p)^{2s_0} d\xi \right)^{1/2} \\ &= \left( \int_{\mathbb{R}^n} \alpha^2(\langle \xi \rangle + p) |\widehat{w}(\xi)|^2 d\xi \right)^{1/2} = \|w\|'_{\alpha, p, \mathbb{R}^n}. \end{aligned}$$

Equality (22) is proved.

We note that

$$(23) \quad c_3^{-1} \|w\|_{E(\mathbb{R}^n, p)} \leq \|w\|_{E(\mathbb{R}^n, p, 1)} \leq c_3 \|w\|_{E(\mathbb{R}^n, p)}$$

with some number  $c_3 \geq 1$ , this number being independent of  $p$  and  $w$ . Indeed, the identity operator  $I$  sets the isomorphism

$$I : H^{(s)}(\mathbb{R}^n, p) \leftrightarrow H^{(s)}(\mathbb{R}^n, p, 1) \quad \text{for each } s > 0.$$

The norms of this operator and its inverse do not exceed  $2^s$ . Consider this isomorphism for each  $s \in \{s_0, s_1\}$  and apply the interpolation with the function parameter  $\psi$ . We obtain the isomorphism

$$I : E(\mathbb{R}^n, p) \leftrightarrow E(\mathbb{R}^n, p, 1)$$

and conclude by Proposition 2 that the norms of this operator and its inverse do not exceed  $2^{s_1} \tilde{c}$ , where the number  $\tilde{c} = \tilde{c}(\psi)$  depends only on  $\psi$ . This means (23) with  $c_3 := 2^{s_1} \tilde{c}$ .

Formulas (20), (22), and (23) directly imply the required estimate (18) for  $G = \mathbb{R}^n$  and  $c_0 := c_1 c_3$ , with  $c_0$  being independent of  $p$  and  $w$ . Thus,

$$(24) \quad (c_1 c_3)^{-1} \|w\|_{E(\mathbb{R}^n, p)} \leq \|w\|_{\alpha, p, \mathbb{R}^n} \leq c_1 c_3 \|w\|_{E(\mathbb{R}^n, p)}$$

for all  $w \in H^\alpha(\mathbb{R}^n)$  and  $p \geq 1$ .

Now, let us deduce Interpolation Lemma in the case of  $G = \Omega$ . As above,  $p \geq 1$ . Let  $R$  denote the operator that restricts distributions  $w \in \mathcal{S}'(\mathbb{R}^n)$  to  $\Omega$ . We have the bounded linear operators

$$(25) \quad R : H^\alpha(\mathbb{R}^n, p) \rightarrow H^\alpha(\Omega, p),$$

$$(26) \quad R : H^{(s)}(\mathbb{R}^n, p) \rightarrow H^{(s)}(\Omega, p), \quad \text{with } s > 0.$$

Certainly, their norms do not exceed 1.

Consider the operator (26) for each  $s \in \{s_0, s_1\}$  and apply the interpolation with the parameter  $\psi$ . We obtain the bounded operator

$$R : E(\mathbb{R}^n, p) \rightarrow E(\Omega, p), \quad \text{whose norm } \leq \tilde{c}$$

by Proposition 2. Hence, in view of (24), we have the bounded operator

$$(27) \quad R : H^\alpha(\mathbb{R}^n, p) \rightarrow E(\Omega, p), \quad \text{whose norm} \leq c_4 := c_1 c_3 \tilde{c}.$$

Further, we need to use a bounded linear operator which is a right-inverse of both the operators (25) and (27). As is known, for each integer  $k \geq 1$  there exists a linear mapping  $T_k$  that extends every function  $u \in H^{(0)}(\Omega)$  onto  $\mathbb{R}^n$  and that sets the bounded operator

$$(28) \quad T_k : H^{(s)}(\Omega) \rightarrow H^{(s)}(\mathbb{R}^n) \quad \text{for every } s \in [0, k].$$

Such a mapping is constructed in [41, Sec 4.2.2], for example.

Let  $k := [s_1] + 1$ , and consider the operator (28) for each  $s \in \{s_0, s_1\}$ . Applying the interpolation with the parameter  $\psi$  and Proposition 1, we obtain the bounded operator

$$(29) \quad T_k : H^\alpha(\Omega) \rightarrow H^\alpha(\mathbb{R}^n).$$

Let  $c_5$  stand for the maximum of the norms of operators (29) and (28) with  $s \in \{0, s_0, s_1\}$ . We have the bounded operators

$$(30) \quad T_k : H^\alpha(\Omega, p) \rightarrow H^\alpha(\mathbb{R}^n, p), \quad \text{whose norm} \leq c_5,$$

and

$$(31) \quad T_k : H^{(s)}(\Omega, p) \rightarrow H^{(s)}(\mathbb{R}^n, p), \quad s \in \{s_0, s_1\}, \quad \text{whose norms} \leq c_5.$$

Applying the interpolation with the parameter  $\psi$  and Proposition 2 to (31), we obtain the bounded operator

$$T_k : E(\Omega, p) \rightarrow E(\mathbb{R}^n, p), \quad \text{whose norm} \leq c_5 \tilde{c}.$$

Hence, in view of (24), we have the bounded operator

$$(32) \quad T_k : E(\Omega, p) \rightarrow H^\alpha(\mathbb{R}^n, p), \quad \text{whose norm} \leq c_6 := c_1 c_3 c_5 \tilde{c}.$$

Now, considering the product of the operators (32) and (25), and then, the product of the operators (30) and (27), we arrive at the identity operators

$$I = RT_k : E(\Omega, p) \rightarrow H^\alpha(\Omega, p), \quad \text{whose norm} \leq c_6,$$

and

$$I = RT_k : H^\alpha(\Omega, p) \rightarrow E(\Omega, p), \quad \text{whose norm} \leq c_4 c_5.$$

This gives the required estimate (18) for  $G = \Omega$  and  $c_0 := \max\{c_4 c_5, c_6\}$ , with  $c_0$  being independent of  $p$  and  $w$ .

Finally, let us deduce Interpolation Lemma in the case where  $G = \Gamma$ . We fix a finite collection of local charts  $\{\alpha_j\}$  on  $\Gamma$  and corresponding partition of unity  $\{\chi_j\}$  used in the definition of the extended Sobolev scale on  $\Gamma$ ; here,  $j$  runs over the values  $1, \dots, \varkappa$ .

Consider the linear mapping

$$T : h \mapsto ((\chi_1 h) \circ \alpha_1, \dots, (\chi_\varkappa h) \circ \alpha_\varkappa),$$

with  $h \in \mathcal{D}'(\Gamma)$ . We can directly verify that this mapping defines the isometric operators

$$(33) \quad T : H^\alpha(\Gamma, p) \rightarrow (H^\alpha(\mathbb{R}^{n-1}, p))^\varkappa,$$

$$(34) \quad T : H^{(s)}(\Gamma, p) \rightarrow (H^{(s)}(\mathbb{R}^{n-1}, p))^\varkappa, \quad \text{with } s > 0.$$

Consider the operator (34) for each  $s \in \{s_0, s_1\}$  and apply the interpolation with the parameter  $\psi$ . According to Propositions 2 and 3, we obtain the bounded operator

$$T : E(\Gamma, p) \rightarrow (E(\mathbb{R}^{n-1}, p))^\varkappa, \quad \text{whose norm} \leq \tilde{c}.$$

Hence, in view of (24), we have the bounded operator

$$(35) \quad T : E(\Gamma, p) \rightarrow (H^\alpha(\mathbb{R}^{n-1}, p))^\varkappa, \quad \text{whose norm} \leq c_4 = c_1 c_3 \tilde{c}.$$

Along with  $T$ , consider the linear mapping

$$Q : (w_1, \dots, w_\varkappa) \mapsto \sum_{j=1}^{\varkappa} \Theta_j((\eta_j w_j) \circ \alpha_j^{-1}),$$

with  $w_1, \dots, w_\varkappa \in \mathcal{S}'(\mathbb{R}^n)$ . Here, the function  $\eta_j \in C^\infty(\mathbb{R}^n)$  is equal to 1 on the set  $\alpha_j^{-1}(\text{supp } \chi_j)$  and is compactly supported, whereas  $\Theta_j$  denotes the operator of extension by zero from  $\Gamma_j$  onto  $\Gamma$ . The mapping  $Q$  is a left-inverse of  $T$ . Indeed,

$$QTh = \sum_{j=1}^{\varkappa} \Theta_j((\eta_j((\chi_j h) \circ \alpha_j)) \circ \alpha_j^{-1}) = \sum_{j=1}^{\varkappa} \Theta_j((\chi_j h) \circ \alpha_j \circ \alpha_j^{-1}) = \sum_{j=1}^{\varkappa} \chi_j h = h$$

for every  $h \in \mathcal{D}'(\Gamma)$ .

We have the bounded operators

$$(36) \quad Q : (H^\alpha(\mathbb{R}^{n-1}))^\varkappa \rightarrow H^\alpha(\Gamma),$$

$$(37) \quad Q : (H^{(s)}(\mathbb{R}^{n-1}))^\varkappa \rightarrow H^{(s)}(\Gamma), \quad \text{with } s \in \mathbb{R}.$$

The boundedness of (37) is a known property of Sobolev spaces (see, e.g., [17, Sec. 2.6]). The boundedness of (36) follows from this property with the help of interpolation by virtue of Propositions 1 and 3.

Let  $c_7$  denote the maximum of the norms of operators (36) and (37) with  $s \in \{0, s_0, s_1\}$ . We have the bounded operators

$$(38) \quad Q : (H^\alpha(\mathbb{R}^{n-1}, p))^\varkappa \rightarrow H^\alpha(\Gamma, p), \quad \text{whose norm} \leq c_7,$$

and

$$(39) \quad Q : (H^{(s)}(\mathbb{R}^{n-1}, p))^\varkappa \rightarrow H^{(s)}(\Gamma, p), \quad s \in \{s_0, s_1\}, \quad \text{whose norms} \leq c_7.$$

Applying the interpolation with the parameter  $\psi$  to (39), we obtain the bounded operator

$$Q : (E(\mathbb{R}^{n-1}, p))^\varkappa \rightarrow E(\Gamma, p), \quad \text{whose norm} \leq c_7 \tilde{c}$$

by virtue of Propositions 2 and 3. Hence, in view of (24), we have the bounded operator

$$(40) \quad Q : (H^\alpha(\mathbb{R}^{n-1}, p))^\varkappa \rightarrow E(\Gamma, p), \quad \text{whose norm} \leq c_8 := c_1 c_3 c_7 \tilde{c}.$$

Thus, multiplying (40) by the isometric operator (33), we obtain the bounded identity operator

$$I = QT : H^\alpha(\Gamma, p) \rightarrow E(\Gamma, p), \quad \text{whose norm} \leq c_8.$$

Besides, taking the product of the operators (38) and (35), we get the bounded identity operator

$$I = QT : E(\Gamma, p) \rightarrow H^\alpha(\Gamma, p), \quad \text{whose norm} \leq c_4 c_7.$$

This yields the required estimate (18) for  $G = \Gamma$  and  $c_0 := \max\{c_8, c_4 c_7\}$ , with  $c_0$  being independent of  $p$  and  $w$ .  $\square$

As we have promised, let us prove that every function  $\alpha \in \text{RO}$  with  $\sigma_0(\alpha) > 0$  satisfies property (21).

Since  $\sigma_0(\alpha) > 0$ , then, by (6), there exists a number  $c' > 0$  such that

$$(41) \quad c' \leq \frac{\alpha(\lambda t)}{\alpha(t)} \quad \text{for all } t \geq 1 \quad \text{and} \quad \lambda \geq 1.$$

Hence, for arbitrary  $t_1, t_2 \geq 1$ , we can write  $c' \alpha(t_j) \leq \alpha(t_1 + t_2)$  with  $j \in \{1, 2\}$ , which gives the right-hand inequality in (21).

Besides, applying properties (41) and (5) of  $\alpha \in \text{RO}$ , we deduce the left-hand inequality in (21), namely,

$$c' \alpha(t_1 + t_2) \leq \alpha(2t) \leq c'' \alpha(t) \leq c'' (\alpha(t_1) + \alpha(t_2)) \quad \text{for all } t_1, t_2 \geq 1.$$

Here,  $t := \max\{t_1, t_2\}$ , and  $c''$  is a certain positive number, which does not depend on  $t_1$  and  $t_2$ . Thus, property (21) is proved.

Now, applying Interpolation Lemma, we will give

*Proof of Isomorphism Theorem.* In the Sobolev case of  $\varphi(t) \equiv t^s$ , this theorem was proved by M. S. Agranovich and M. I. Vishik (see [4, § 4 and 5] and [3, Sec. 3.2]). Using spaces  $H^{(s)}(G, |\lambda|)$ , we can reformulate their result in the following way. Let the boundary-value problem (1) be parameter-elliptic in the angle  $K$ . Then there exists a number  $\lambda_1 \geq 1$  that the isomorphism

$$(42) \quad \begin{aligned} (A(\lambda), B(\lambda)) : H^{(s+2q)}(\Omega, |\lambda|) &\leftrightarrow H^{(s)}(\Omega, |\lambda|) \oplus \bigoplus_{j=1}^q H^{(s+2q-m_j-1/2)}(\Gamma, |\lambda|) \\ &=: \mathcal{H}^{(s)}(\Omega, \Gamma, |\lambda|) \end{aligned}$$

holds for every  $s > l$  and each  $\lambda \in K$  with  $|\lambda| \geq \lambda_1$ . Moreover, the norms of the operator (42) and its inverse are uniformly bounded with respect to  $\lambda$ .

Deduce Isomorphism Theorem for the extended Sobolev scale. Let  $\varphi \in \text{RO}$  meet condition (12), and let  $\lambda \in K$  satisfy  $|\lambda| \geq \lambda_1$ . We choose  $l_0, l_1 \in \mathbb{R}$  so that  $l < l_0 < \sigma_0(\varphi)$  and  $l_1 > \sigma_1(\varphi)$ . Define the interpolation parameter  $\psi \in \mathcal{B}$  by formula (16), in which  $\alpha := \varphi$ ,  $s_0 := l_0$ , and  $s_1 := l_1$ .

Consider the isomorphism (42) for each  $s \in \{l_0, l_1\}$  and apply the interpolation with the function parameter  $\psi$ . According to Propositions 2 and 3, we obtain the isomorphism

$$(43) \quad \begin{aligned} (A(\lambda), B(\lambda)) : [H^{(l_0+2q)}(\Omega, |\lambda|), H^{(l_1+2q)}(\Omega, |\lambda|)]_{\psi} &\leftrightarrow [H^{(l_0)}(\Omega, |\lambda|), H^{(l_1)}(\Omega, |\lambda|)]_{\psi} \\ &\oplus \bigoplus_{j=1}^q [H^{(l_0+2q-m_j-1/2)}(\Gamma, |\lambda|), H^{(l_1+2q-m_j-1/2)}(\Gamma, |\lambda|)]_{\psi}. \end{aligned}$$

The norms of the operator (43) and its inverse are uniformly bounded with respect to  $\lambda$ .

Hence, we deduce by Interpolation Lemma that the isomorphism (43) acts between the following spaces:

$$(44) \quad \begin{aligned} (A(\lambda), B(\lambda)) : H^{\varphi \varrho^{2q}}(\Omega, |\lambda|) &\leftrightarrow H^{\varphi}(\Omega, |\lambda|) \oplus \bigoplus_{j=1}^q H^{\varphi \varrho^{2q-m_j-1/2}}(\Gamma, |\lambda|) \\ &:= \mathcal{H}^{\varphi}(\Omega, \Gamma, |\lambda|). \end{aligned}$$

Moreover, the norms of the operator (44) and its inverse are uniformly bounded with respect to  $\lambda$ . Note that we have also applied this lemma in the case where

$$\alpha = \varphi \varrho^{2q}, \quad s_1 = l_1 + 2q, \quad \text{and} \quad s_0 = l_0 + 2q,$$

and in the case where

$$\alpha = \varphi \varrho^{2q-m_j-1/2}, \quad s_0 = l_0 + 2q - m_j - 1/2, \quad \text{and} \quad s_1 = l_1 + 2q - m_j - 1/2.$$

In these cases the function  $\psi$  still satisfies (16).

Owing to the properties of the operator (44), we can state that Isomorphism Theorem is proved.  $\square$

Analysing this proof, we note the following. According to [3, Sec. 3.2 a], the mapping  $u \mapsto (A(\lambda)u, B(\lambda)u)$ , with  $u \in C^\infty(\overline{\Omega})$ , extends uniquely (by continuity) to the bounded linear operator

$$(45) \quad (A(\lambda), B(\lambda)) : H^{(s+2q)}(\Omega, |\lambda|) \rightarrow \mathcal{H}^{(s)}(\Omega, \Gamma, |\lambda|)$$

for every  $s > l$  and each  $\lambda \in \mathbb{C}$ . Moreover, the norm of this operator is uniformly bounded with respect to  $\lambda$ . Here, we do not suppose that the boundary-value problem (1) is

parameter-elliptic in  $K$ . Applying the interpolation and reasoning as in the latter proof, we obtain the bounded operator

$$(46) \quad (A(\lambda), B(\lambda)) : H^{\varphi e^{2q}}(\Omega, |\lambda|) \rightarrow \mathcal{H}^{\varphi}(\Omega, \Gamma, |\lambda|)$$

for each  $\lambda \in \mathbb{C}$  and every  $\varphi \in \text{RO}$  with  $\sigma_0(\varphi) > l$ . Its norm is uniformly bounded with respect to  $\lambda$ . This means that the right-hand inequality in (15) remains true for indicated  $\lambda$  and  $\varphi$  without the assumption about parameter-ellipticity.

It remains to give

*Proof of Fredholm Property.* Let the boundary-value problem (1) be parameter-elliptic on a certain closed ray  $K := \{\lambda \in \mathbb{C} : \arg \lambda = \text{const}\}$ . According to [4, Sec. 6.4], the bounded operator (45) is Fredholm with zero index for every  $s > l$  and each  $\lambda \in \mathbb{C}$ . Moreover, the kernel of this operator lies in  $C^\infty(\overline{\Omega})$  and, hence, does not depend on  $s$ . Both the Fredholm property and index are preserved under interpolation of normed spaces and bounded operators provided that these operators have the same index and the same kernel (see, e.g., [31, 35, Sec. 1.1.7]). Therefore, using the interpolation with a function parameter and applying Proposition 1, we conclude that the bounded operator (46) is Fredholm with zero index for each  $\lambda \in \mathbb{C}$  and every  $\varphi \in \text{RO}$  with  $\sigma_0(\varphi) > l$ .  $\square$

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