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SPECTRAL PROBLEM FOR A GRAPH OF SYMMETRIC STIELTJES STRINGS

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Dedicated to Yury Samoĭlenko on the occasion of his 70th birthday

Abstract. A spectral problem generated by the Stieltjes string recurrence relations with a finite number of point masses on a connected graph is considered with Neumann conditions at pendant vertices and continuity and Kirchhoff conditions at interior vertices. The strings on the edges are supposed to be the same and symmetric with respect to the midpoint of the string. The characteristic function of such a problem is expressed via characteristic functions of two spectral problems on an edge: one with Dirichlet conditions at the both ends and the other one with the Neumann condition at one end and the Dirichlet condition at the other end. This permits to find values of the point masses and the lengths of the subintervals into which the masses divide the string from knowing the spectrum of the problem on the graph and the length of an edge. If the number of vertices is less than five then the spectrum uniquely determines the form of the graph.

1. Introduction

In this paper we describe finite-dimensional analogues of the results in [20].

The notion of a discrete Laplacian (see [4]) is closely related to the notion of adjacency matrix of the classical spectral graph theory [3], namely, for a simple connected graph with no loops,

$$L = I - T^{-1/2} A T^{-1/2},$$

where L is the discrete Laplacian, $T = \mathrm{diag}\{d(v_1), d(v_2), \ldots, d(v_p)\}$, $d(v_i)$ is the degree of the vertex $v_i, i = 1, \ldots, p$, A is the $(p \times p)$ adjacency matrix, p is the number of vertices in the graph.

The quantum graph theory (see [5]) considers spectral problems generated by Sturm-Liouville or Dirac equation on metric graph domains. The mentioned equations on the edges of a graph with boundary and matching conditions (usually continuity and Kirchhoff conditions) at the interior vertices generate an operator which is called continuous Laplacian.

Under the conditions of equal lengths of the edges and equal potentials on the edges and symmetry of the potential with respect to the middle of an edge the problem for a continuous Laplacian can be reduced to the problem for a discrete Laplacian ([1], [6], [2]). This approach has been widely used (see, e.g. [14], [17], [18], [20]).

A finite-dimensional analogue of this theory occurs in description of small vibrations of mechanical systems (nets of Stieltjes strings, of springs connecting point masses [12], [8], [9]).

After M. G. Krein [13], [11] we call *Stieltjes string* a weightless thread bearing point masses $\{m_k\}_{k=0}^{n}$ $(n \leq \infty, m_k > 0)$ at points $\{x_k\}_{k=1}^{n}$ $(0 \leq x_k < x_{k+1})$. We will consider

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only strings with a finite number of masses and denote by $l_k = x_{k+1} - x_k$ the intervals between them. Then $l = \sum_{k=0}^{n} l_k$ is the total length of the string.

We consider small transverse vibrations of a graph consisting of symmetric Stieltjes strings. If point masses are present at each interior vertex then the spectral problem can be reduced to the spectral problem for a discrete Laplacian on the corresponding graph. If our graph is a tree then the problem is related with the so-called tree-patterned matrices [15], [16], [7], [19]. These matrices are generalizations of Jacobi matrices.

It should be mentioned that absence of masses at the vertices of our graph makes the problem a bit more complicated than in [8], [15], [16], [7] because instead of the equation $(\lambda I - L)Y = 0$, where L is the discrete Laplacian, we have the equation $(\lambda M - L)Y = 0$, where $M \geq 0$ is a diagonal matrix.

We use the idea of a connection between the continuous Laplacian and discrete Laplacian to reduce the more complicated discrete operator of the problem of small vibration of a graph of symmetric Stieltjes strings to a simpler finite dimensional operator to solve the related direct and inverse problems. We show how to find the form of the graph and values of the point masses and the subintervals of the Stieltjes string on an edge using the spectrum of the whole graph and the length of an edge.

In Section 2 we describe well known results on Stieltjes strings vibrations including the method of recovering values of the point masses and the lengths of subintervals using two spectra and the total length due to [11]. In Section 3 we show that in case of a Stieltjes string, which is symmetric with respect to the midpoint, it is possible to find values of the point masses and the lengths of the subintervals using only one spectrum of the Dirichlet problem and the total length of the string. In Section 4 we describe the problem of vibrations of a graph of symmetric Stieltjes strings. In Section 5 we express the characteristic function of the spectral problem on the graph via characteristic functions of the Dirichlet-Dirichlet and Neumann-Dirichlet problem on an edge of the graph (Theorem 5.1) and find its form in the particular case when the graph is bipartite (Theorem 5.2). We show that the spectrum of the problem on the graph and the length of an edge uniquely determine values of the point masses and the subintervals into which the masses divide the string (Theorem 5.3), and show how to find these values (proof of Theorem 5.3). We also discuss the question whether one can judge on the form of the graph knowing the spectrum of the main spectral problem (Corollary after Theorem 5.3).

2. Stieltjes strings

Let us consider a Stieltjes string bearing n point masses $m_1, m_2, \dots, m_n$ ($m_k > 0$), let $l_0, l_1, \dots, l_n$ ($l_k > 0$) be the intervals into which the masses divide the total length l of the string ($\sum_{k=1}^{n} l_k = l$).

Denote by $V_k(t)$ the small transverse displacement of the mass m_k at the time t. Then we have [11]

$$\frac{V_k(t) - V_{k-1}(t)}{l_{k-1}} + \frac{V_k(t) - V_{k+1}(t)}{l_k} + m_k V_k''(t) = 0, \quad k = 1, 2, \dots, n. \tag{1}$$

Substituting $V_k(t) = U_k e^{i\lambda t}$ into (1) we obtain

$$\frac{U_k - U_{k-1}}{l_{k-1}} + \frac{U_k - U_{k+1}}{l_k} = m_k z U_k, \quad k = 1, 2, \dots, n, \tag{2}$$

where U_k is the amplitude of vibrations of the mass m_k, $z = \lambda^2$ is the spectral parameter. If the ends of the string are fixed (Dirichlet-Dirichlet problem) then $V_0(t) = V_{n+1}(t) = 0$ and, consequently,

$$U_0 = 0, \tag{3}$$

$$U_{n+1} = 0. \tag{4}$$

The sequence $\{\nu_j\}_{j=1}^n$ $(0 < \nu_1 < \nu_2 < \cdots < \nu_n)$ of eigenvalues of problem (2)–(4), i.e., the values of z for which there exists an eigenvector $\{U_0, U_1, \ldots, U_n, U_{n+1}\} \neq 0$ is said to be the spectrum of (2)–(4).

Following [11] we look for a solution in the form

$$U_k = R_{2k-2}(z)U_1, \quad k = 1, 2, \ldots, n+1,$$

where $R_{2k-2}(z)$ is a polynomials of degree $k-1$. Then the spectrum of problem (2)–(4) coincides with the set of zeros of the polynomial

$$R_{2n}(z) = \frac{l}{l_0} \prod_{k=1}^{n} \Big(1 - \frac{z}{\nu_k}\Big).$$

Next, we introduce the polynomials of an odd index,

$$R_{2k-1} = \frac{R_{2k}(z) - R_{2k-2}(z)}{l_k}. \tag{5}$$

The polynomials $R_j(z)$ satisfy the following recurrence relations:

$$R_{2k}(z) = l_k R_{2k-1}(z) + R_{2k-2}(z), \tag{6}$$

$$R_{2k-1}(z) = R_{2k-3}(z) - m_k z R_{2k-2}(z) \tag{7}$$

with the initial conditions

$$R_{-1}(z) \equiv \frac{1}{l_0}, \tag{8}$$

$$R_0(z) \equiv 1. \tag{9}$$

The Neumann condition at the right end which corresponds to the situation where the right end of the string is free to move in the direction orthogonal to the equilibrium position of the string is

$$U_{n+1} = U_n. \tag{10}$$

The spectrum $\{\mu_k\}_{k=1}^n$ of problem (2), (3), (10) coincides with the set of zeros of the polynomial

$$R_{2n-1}(z) = \frac{1}{l_0} \prod_{k-1}^{n} \Big(1 - \frac{z}{\mu_k}\Big). \tag{11}$$

It is known [11] that

$$0 < \mu_1 < \nu_1 < \mu_2 < \nu_2 < \cdots < \mu_n < \nu_n. \tag{12}$$

The Neumann condition at the left end is

$$U_0 = U_1. \tag{13}$$

We look for a solution of (2) which satisfies (13) in the form

$$U_k = Q_{2k-2}(z)U_1, \quad k = 1, 2, \ldots, n,$$

where $Q_{2k-2}(z)$ are polynomials of degree $k-1$ $(k = 1, 2, \ldots, n)$ which together with the polynomials of an odd degree,

$$Q_{2k-1} = \frac{Q_{2k}(z) - Q_{2k-2}(z)}{l_k}, \tag{14}$$

satisfy the recurrence relations

$$Q_{2k}(z) = l_k Q_{2k-1}(z) + Q_{2k-2}(z), \tag{15}$$

$$Q_{2k-1}(z) = Q_{2k-3}(z) - m_k \lambda Q_{2k-2}(z) \tag{16}$$

with the initial conditions

$$Q_0(z) \equiv 1, \tag{17}$$

$$Q_{-1}(z) \equiv 0. \tag{18}$$

The zeros $\{\eta_k\}_{k=1}^n$ of

$$Q_{2n}(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{\eta_k}\right) \tag{19}$$

are eigenvalues of Neumann-Dirichlet problem (2), (4), (13). Therefore,

$$0 < \eta_1 < \nu_1 < \eta_2 < \nu_2 < \cdots < \eta_n < \nu_n.$$

Let us denote by $\{\xi_k\}_{k=1}^n$ the set of zeros of

$$Q_{2n-1}(z) = z \prod_{k=2}^{n} \left(1 - \frac{z}{\xi_k}\right) \sum_{k=1}^{n} m_k,$$

which are eigenvalues of the Neumann-Neumann problem (2), (10), (13). It is known (see [11]) that

$$0 = \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_n < \eta_n,$$
$$\xi_1 < \mu_1 < \xi_2 < \mu_2 < \cdots < \xi_n < \mu_n.$$

The Lagrange identity is

$$R_{2k-1}(z)Q_{2k}(z) - R_{2k}(z)Q_{2k-1}(z) = \frac{1}{l_0}. \tag{20}$$

The following theorem gives an algorithm for recovering the sets $\{m_k\}_{k=1}^n$ and $\{l_k\}_{k=0}^n$ using the spectra $\{\mu_k\}_{k=1}^n$ and $\{\nu_k\}_{k=1}^n$ and the total length l of the string.

Theorem 2.1. ([11]). *For two sequences of positive numbers $\{\mu_k\}_{k=1}^n$ and $\{\nu_k\}_{k=1}^n$ to be spectra of problems (2), (3), (10) and (2), (3), (4), respectively, it is necessary and sufficient that they are interlaced in sense of (12). Under this condition the collection of the corresponding sets $\{m_k\}_{k=1}^n$ and $\{l_k\}_{k=0}^n$ is unique for a given value of the total length of the string $l > 0$. The masses and the subintervals can be found using the continued fraction*

$$\frac{R_{2n}(z)}{R_{2n-1}(z)} = l_n + \cfrac{1}{-m_n z + \cfrac{1}{l_{n-1} + \cfrac{1}{-m_{n-1}z + \ldots + \cfrac{1}{l_1 + \cfrac{1}{-m_1 z + \frac{1}{l_0}}}}}}.$$

3. Symmetric Stieltjes string

In the sequel we consider Stieltjes strings symmetric with respect to their midpoints. This means that

1) if n is even then:

$$m_k = m_{n-k+1}, \quad k = 1, \ldots, \frac{n}{2},$$
$$l_k = l_{n-k}, \quad k = 0, \ldots, \frac{n}{2} - 1.$$

2) if n is odd then:

$$m_k = m_{n-k+1}, \quad k = 1, \ldots, \left[\frac{n}{2}\right],$$
$$l_k = l_{n-k}, \quad k = 0, \ldots, \left[\frac{n}{2}\right],$$

where $[a]$ denotes the integer part of a.

For a symmetric Stieltjes string $\mu_k = \eta_k$ for all k, therefore, with account of (11) and (19) we obtain

Lemma 3.1. *If the Stieltjes string is symmetric then*

$$R_{2n-1}(z) = \frac{1}{l_0} Q_{2n}(z). \tag{21}$$

Using (21) we see that the Lagrange identity (20) for a symmetric Stieltjes string attains the form

$$\frac{1}{l_0} Q_{2n}^2(z) - \frac{1}{l_0} = R_{2n}(z) Q_{2n-1}(z). \tag{22}$$

Proposition 3.2. *Any sequence of distinct positive numbers $\{\nu_k\}_{k=1}^n$ is the spectrum of a Dirichlet-Dirichlet problem for a symmetric Stieltjes string of n masses and of a prescribed total length. The spectrum $\{\nu_k\}_{k=-n,k\neq 0}^n$ of the Dirichlet-Dirichlet problem (2)–(4) and the total length $l > 0$ uniquely determine the point masses $\{m_k\}_{k=1}^n$ and subintervals $\{l_k\}_{k=0}^n$ of a symmetric Stieltjes string.*

Proof. 1. In the case of even n we easily obtain

$$R_{2n}(z) = 2R_n(z) R_{n-1}(z). \tag{23}$$

Since the zeros of $R_{n-1}(z)$ and of $R_n(z)$ interlace we have

$$R_{n-1}(z) = \frac{1}{l_0} \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{z}{\nu_{2k-1}}\right),$$

$$R_n(z) = \frac{l}{2l_0} \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{z}{\nu_{2k}}\right)$$

and, consequently,

$$\frac{R_n(z)}{R_{n-1}(z)} = l_{\frac{n}{2}} + \frac{1}{m_{\frac{n}{2}} z + \frac{1}{l_{\frac{n}{2}-1}.+\cdots+\frac{1}{-m_1 z + \frac{1}{l_0}}}}.$$

2. In the case of odd n, we obtain, instead of (23), that

$$R_{2n}(z) = R_{2[\frac{n}{2}]}(z)(2R_{2[\frac{n}{2}]-1}(z) - R_{2[\frac{n}{2}]}(z) m_{[\frac{n}{2}]+1} z).$$

Since zeros of $R_{2[\frac{n}{2}]}(z)$ and $(2R_{2[\frac{n}{2}]-1}(z) - R_{2[\frac{n}{2}]}(z) m_{[\frac{n}{2}]+1} z)$ interlace,

$$R_{2[n/2]}(z) = \frac{l}{2l_0} \prod_{k=1}^{[\frac{n}{2}]} \left(1 - \frac{z}{\nu_{2k}}\right),$$

$$2R_{2[\frac{n}{2}]-1}(z) - R_{2[\frac{n}{2}]}(z) m_{[\frac{n}{2}]+1} z = \frac{1}{l_0} \prod_{k=1}^{[\frac{n}{2}]+1} \left(1 - \frac{z}{\nu_{2k-1}}\right)$$

and, consequently,

$$\begin{aligned} \frac{R_{2[\frac{n}{2}]}(z)}{2R_{2[\frac{n}{2}]-1}(z) - R_{2[\frac{n}{2}]}(z) m_{[\frac{n}{2}]+1} z} &= \frac{l}{2} \frac{\prod_{k=1}^{[\frac{n}{2}]} \left(1 - \frac{z}{\nu_{2k}}\right)}{\prod_{k=1}^{[\frac{n}{2}]+1} \left(1 - \frac{z}{\nu_{2k-1}}\right)} \\ &= \frac{1}{-m_{[\frac{n}{2}]+1} z + \frac{1}{2^{-1} l_{[\frac{n}{2}]} + \frac{1}{-2m_{[\frac{n}{2}]} z + \frac{1}{\cdots + \frac{1}{2^{-1} l_0}}}}}. \end{aligned}$$

□

4. Graph of symmetric Stieltjes strings

Now we consider an oriented connected plane graph G each edge of which is the same symmetric Stieltjes string bearing n point masses. The graph is stretched and vibrate in the direction orthogonal to the equilibrium plane of the graph. The orientation of the edges of the graph is arbitrary.

Denote the vertices of G by $v_i, i = 1, 2, \ldots, p$, where p is the number of vertices in G, by e_j the edges of G ($j = 1, 2, \ldots, g$ where g is the number of edges).

For each i denote by $d(v_i)$ the degree of the vertex v_i, by $d^+(v_i)$ the indegree, i.e., the number of edges incoming into v_i, by $d^-(v_i)$ the outdegree, i.e., the number of edges outgoing from v_i.

Let J be the set of pendant vertices, K the set of interior vertices of G, W_i^+ the set of numbers of edges incoming into v_i and W_i^- be the set of numbers of edges outgoing from v_i $(i = \overline{1, \ldots, p})$.

We enumerate the point masses m_k $(k = 1, 2, \ldots, n)$ and the subintervals l_k $(k = 0, 1, \ldots, n)$ on an edge successively in the direction of the edge.

We assume absence of point masses in the vertices. For a point mass indexed by k which lies on the edge j we have

$$\frac{V_k^j(t) - V_{k-1}^j(t)}{l_{k-1}} + \frac{V_k^j(t) - V_{k+1}^j(t)}{l_k} = m_k {V_k^j}''(t), \tag{1}$$

where $k = 1, 2, \ldots, n, j = 1, 2, \ldots, g$, $V_k^j(t)$ is the transverse displacement of this mass.

Let the pendant vertices (if any) be free to move in the direction orthogonal to the equilibrium position of the graph. Then we impose the Neumann condition at a pendant vertex with an incoming edge,

$$V_n^{j^+}(t) = V_{n+1}^{j^+}(t) \tag{2}$$

and at a pendant vertex with an outgoing edge,

$$V_0^{j^-}(t) = V_1^{j^-}(t). \tag{3}$$

At an interior vertex v_i we impose the continuity conditions

$$V_0^{j_1^-}(t) = V_0^{j_2^-}(t) = \cdots = V_0^{j^-_{d^-(v_i)}}(t) = V_{n+1}^{j_1^+}(t) = V_{n+1}^{j_2^+}(t) = \cdots = V_{n+1}^{j^+_{d^+(v_i)}}(t), \tag{4}$$

where $\{j_1^-, \ldots, j^-_{d^-(v_i)}\} \in W_i^-; \{j_1^+, \ldots, j^+_{d^+(v_i)}\} \in W_i^+$ and the balance of forces condition

$$\sum_{m=1}^{d^+(v_i)} \frac{V_{n+1}^{(j_m^+)}(t) - V_n^{(j_m^+)}(t)}{l_n} - \sum_{m=1}^{d^-(v_i)} \frac{V_1^{(j_m^-)}(t) - V_0^{(j_m^-)}(t)}{l_0} = 0. \tag{5}$$

Substituting $V_k^j(t) = U_k^j e^{ipt}$ into (1) – (5) we obtain the following spectral problem:

$$\frac{U_k^j - U_{k-1}^j}{l_{k-1}} + \frac{U_k^j - U_{k+1}^j}{l_k} = -m_k z U_k^j, \tag{6}$$

$$U_n^{j_m^+} = U_{n+1}^{j_m^+}, \tag{7}$$

$$U_0^{j_m^-} = U_1^{j_m^-}, \tag{8}$$

$$U_0^{j_1^-} = U_0^{j_2^-} = \cdots = U_0^{j^-_{d^-(v_i)}} = U_{n+1}^{j_1^+} = U_{n+1}^{j_2^+} = \cdots = U_{n+1}^{j^+_{d^+(v_i)}}, \tag{9}$$

$$\sum_{m=1}^{d^+(v_i)} \frac{U_{n+1}^{(j_m^+)} - U_n^{(j_m^+)}(t)}{l_n} - \sum_{m=1}^{d^-(v_i)} \frac{U_1^{(j_m^-)} - U_0^{(j_m^-)}}{l_0} = 0, \tag{10}$$

where $k = 1, 2, \ldots, n; i = 1, 2, \ldots, p; j_m^- \in W_i^-, m = j_1^-, \ldots, j_{d^-(v_i)}^-; j_m^+ \in W_i^+, m = j_1^+, \ldots, j_{d^+(v_i)}^+$ and U_k^j is the amplitude of vibrations of the mass m_k located on the edge e_j, $z = \rho^2$ is the spectral parameter.

5. Main results

It is convenient to introduce the following solutions of (6):

$$U_k^j(z) = \frac{B^j - A^j Q_{2n}(z)}{R_{2n}(z)} R_{2k-2}(z) + A^j Q_{2k-2}(z), \tag{11}$$

where A^j, B^j are constants independent of k and z. These solutions exist for all z which are not zeros of $R_{2n}(z)$.

In view of (8) and (9), equation (6) for $k = 0$ implies $R_{-2}(z) \equiv 0$ while in view of (17) and (18) equation (15) implies $Q_{-2}(z) \equiv 1$. Substituting these into (11) we have

$$U_0^j(z) = \frac{B^j - A^j Q_{2n}(z)}{R_{2n}(z)} R_{-2}(z) + A^j Q_{-2}(z) = A^j. \tag{12}$$

In the same way, for $k = n+1$,

$$U_{n+1}^j(z) = \frac{B^j - A^j Q_{2n}(z)}{R_{2n}(z)} R_{2n}(z) + A^j Q_{2n}(z) = B^j. \tag{13}$$

Accounting for (9), (17), (11) and (12), the Neumann condition at a pendant vertex with an outgoing edge attains the form

$$U_0^j(z) - U_1^j(z) = \frac{A^j Q_{2n}(z) - B^j}{R_{2n}(z)} = 0.$$

Using (5), (14), (20), (11) and that $l_n = l_0$, the Neumann condition at a pendant vertex with an incoming edge become

$$U_n^j(z) - U_{n+1}^j(z) = \frac{A^j - l_n B^j R_{2n-1}(z)}{R_{2n}(z)} = 0. \tag{14}$$

Continuity conditions (9) look now as

$$A^{j_1^-} = A^{j_2^-} = \cdots = A^{j_{d^-(v_i)}^-} = B^{j_1^+} = B^{j_2^+} = \cdots = B^{j_{d^+(v_i)}^+} =: \Phi(v_i). \tag{15}$$

Balance of forces equation (10) with account of (12), (13), (15) attains the form

$$\sum_{m=1}^{d^+(v_i)} (l_n B^j R_{2n-1}(z) - A^j) - \sum_{m=1}^{d^-(v_i)} (B^j - A^j Q_{2n}(z)) = 0. \tag{16}$$

Using definition (15) of $\Phi(v_j)$ and (21) we rewrite (16) as follows:

$$\begin{aligned} &\sum_{m=1}^{d^+(v_i)} (l_n \Phi(v_i) R_{2n-1}(z) - A^j) - \sum_{m=1}^{d^-(v_i)} (B^j - \Phi(v_i) Q_{2n}(z)) \\ &= \sum_{m=1}^{d^+(v_i)} (\Phi(v_i) Q_{2n}(z) - A^j) - \sum_{m=1}^{d^-(v_i)} (B^j - \Phi(v_i) Q_{2n}(z)) = 0 \end{aligned}$$

or

$$Q_{2n}(z) d(v_i) \Phi(v_i) - \sum_{v_j \sim v_i} \Phi(v_i) = 0.$$

Finally, we obtain using the notation $\zeta = Q_{2n}(z)$, $F = \{\Phi(v_1), \dots, \Phi(v_p)\}^T$, $T = \operatorname{diag}\{d(v_1), \dots, d(v_p)\}$ and denoting by A the adjacency matrix of our graph that

$$\zeta TF - AF = 0. \tag{17}$$

Let z_0 be not a zero of $R_{2n}(z)$, then it is an eigenvalue of problem (6)–(10) if and only if $\zeta_0 =: Q_{2n}(z_0)$ is an eigenvalue of the matrix equation (17). This means that the spectrum of problem (6)–(10) consists of zeros of $R_{2n}(z)$ and of zeros of the polynomials $Q_{2n}(z) - \alpha_s$, where α_s are $(s = 1, 2, \dots, p)$ the eigenvalues of (17).

Since there are no isolated vertices in our graph, i.e., $d(v_i) \geq 1$ for each vertex, the matrix T is invertible and there exists $T^{-1/2}$. Therefore, equation (17) can be rewritten as

$$(\zeta I - \widetilde{A})\widetilde{F} = 0,$$

where

$$\widetilde{A} = T^{-1/2} A T^{-1/2}, \quad \widetilde{F} = T^{\frac{1}{2}} F.$$

The spectrum of the matrix $\widetilde{A}$ consists of p eigenvalues, counting the multiplicities. Thus, if $R_{2n}(z_k) \neq 0$, then z_k is an eigenvalue of problem (6) – (10) if and only if $Q_{2n}(z_k) = \alpha_s$ for some s, where α_s $(\alpha_s \leq \alpha_{s+1},\ s = 1, 2, \dots, p)$ are zeros of the polynomial $P_p(z) = \det(zI - \widetilde{A})$ of degree p.

This means that the characteristic polynomial of problem (6)–(10) is

$$\phi(z) = R_{2n}^{g-p}(z) P_p(Q_{2n}(z)).$$

Theorem 5.1. *The characteristic polynomial of problem (6)–(10) is of the form*

$$\phi(z) = R_{2n}^{g-p}(z)(Q_{2n}(z) - 1)\widetilde{P}_{p-1}(Q_{2n}(z)),$$

where $\widetilde{P}_{p-1}(\zeta)$ is a polynomial of degree $p-1$ with $\tilde{P}_{p-1}(1) \neq 0$.

Proof. It was shown in [4] that for the eigenvalues $\{\chi_k\}_{k=0}^{p-1}$ of the discrete Laplacian $\mathcal{L} = I - \widetilde{A}$ the following inequalities are true:

$$0 = \chi_0 \leq \chi_1 \leq \dots \leq \chi_{p-1} \leq 2.$$

It is clear that

$$\chi_{k-1} = 1 - z_k,$$

where z_k $(z_1 \geq z_2 \geq \dots \geq z_p)$ are eigenvalues of $\tilde{A}$.

Since our graph is connected, its adjacency matrix is irreducible with all elements being nonnegative. Therefore, by the Frobenius theorem ([10], p. 335), we obtain that $z_1 = 1$ is simple. Theorem is proved. □

Theorem 5.2. *The representation*

$$\phi(z) = R_{2n}^{g-p}(z)(Q_{2n}^2(z) - 1)Q_{2n}^m(z)\hat{P}_{\frac{p-m}{2}-1}(Q_{2n}^2(z)),$$

where $m \in N \cup \{0\}, m + p$ is an odd number, $\hat{P}_{\frac{p-m}{2}-1}$ is a polynomial of degree $\frac{p-m}{2} - 1$ with $\hat{P}_{\frac{p-m}{2}-1}(1) \neq 0$, is true if and only if the graph is bipartite.

Proof. Let the graph be bipartite. With an account of the previous theorem we need to prove that $z_p = -1$.

Since the graph is bipartite, we can attribute the value $+1$ to the vertices of one part and -1 to the vertices of the second part. We direct the edges from the vertices with $+1$ to the vertices with -1.

Let us consider a Neumann-Neumann problem, i.e., the problem generated by (2) with conditions (10), (13) on one edge.

It is known ([10]) that according to the Sturm oscillatory theorem the eigenvector corresponding to the eigenvalue z_k changes the sign of its elements $k-1$ times. In particular, it is true for the second eigenvector which we denote by $U_2 = \{u_0, u_1, \ldots, u_{n+1}\}$. Due to the symmetry of our string we may assume that $u_0 = 1$ and $u_{n+1} = -1$.

Now placing such eigenvectors on edges of our graph such that u_0 appears at each vertex assigned $+1$ and u_{n+1} at each vertex assigned -1 we obtain an $(n+2)q$ -dimensional vector

$$U = (u_0, u_1, \ldots, u_{n+1}, u_0, u_1, \ldots, u_{n+1}, \ldots, u_0, u_1, \ldots, u_{n+1}).$$

This vector is an eigenvector of problem (6) - (10) because, by the construction, it satisfies (6) on each edge, it satisfies the continuity conditions at the interior vertices because at these vertices we have $v_0 = 1$ or $v_{n+1} = -1$, respectively, on each incident edge. Moreover, since $u_0 = u_1$ and $n_{n+1} = u_n$ on all edges, both sums in (10) are equal to zero, thus, (10) is true. Since $u_{n+1} = Q_{2n}(z_k)u_1$ and $u_{n+1} = -1$ and $u_0 = 1$ we conclude that for this z_k we have $Q_{2n}(z_k) = -1$.

Now let z_k be an eigenvalue of problem (6)–(10) and $Q_{2n}(z_k) = -1$. Then by (21), $R_{2n-1}(z_k) = -\frac{1}{l_0}$ and (22) implies that $R_{2n}(z_k) = 0$. Then any component of the eigenvector of problem (6)–(10) corresponding to z_k, being of the form $(C_{j,1}Q_{-2}(z_k) + C_{j,2}R_{-2}(z_k), C_{j,1}Q_2(z_k)+C_{j,2}R_2(z_k), \ldots, C_{j,1}Q_{2n}(z_k)+C_{j,2}R_{2n}(z_k))^T$, attains the values $C_{j,1}Q_{-2}(z_k) = C_{j,1}$ and $C_{j,1}Q_{2n}(z_k) = -C_{j,1}$ of the opposite signs. This means that the graph is bipartite. Theorem is proved. □

Theorem 5.3. *Let G be a cyclically connected not bipartite graph with the same edges, which is not a simple cycle. Let the same even number of masses be placed on each edge symmetrically with respect to the midpoint of each edge. Then the spectrum of problem (6)–(10) and the length of an edge uniquely determine the masses $\{m_k\}_{k=1}^n$ and the subintervals $\{l_k\}_{k=0}^n$ between them.*

Proof. We need to identify the zeros of $R_{2n}(z)$ among all the zeros of $\phi(z)$. Each zero of $\phi(z)$ is a zero of at least one of the polynomials $R_{2n}^{g-p}(z)$, $(Q_{2n}(z)-1)$, $(Q_{2n}(z)-\alpha_k)$ $(k = 1, 2, \ldots, p-1)$.

Let $\{\nu_k\}_{k=1}^n$ be the zeros of $R_{2n}(z)$. Then (22) implies

$$Q_{2n}(\nu_k) = \pm 1.$$

Location of the zeros $\{z_k\}_{k=1}^n$ of $\phi(z)$ is given in Fig. 1.

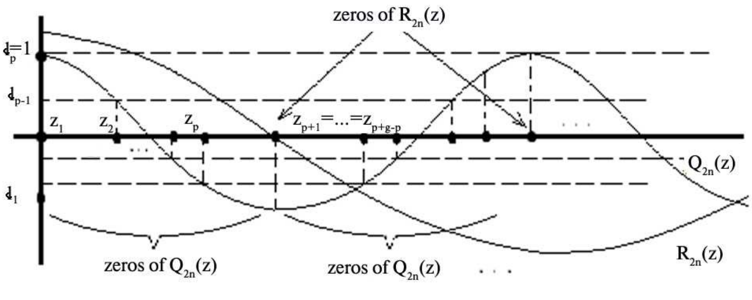

Figure 1

It is clear that

$$z_1 < z_2 \leq \cdots < z_{p+1} = \cdots = z_g < \cdots \leq z_{g+p-1} \leq z_{g+p} = \cdots = z_{2g+1} < \cdots .$$

Here the zero z_1 is the lowest zero of the polynomial $Q_{2n}(z) - \alpha_p = Q_{2n}(z) - 1$, the zeros $z_2, z_3, \ldots, z_p$ are the lowest zeros of the polynomials $Q_{2n}(z) - \alpha_{p-1}$, $Q_{2n}(z) - \alpha_{p-2}$, ...,

$Q_{2n}(z) - \alpha_1$, respectively. To the right of them, we have a zero of $R_{2n}(z)$ of multiplicity $p - q$. Also it is clear that $\{z_{kg}\}_{k=1}^n$ belongs to the set of zeros of $R_{2n}(z)$. Then using the procedure described in the proof of Proposition 3.2 and the known value of the edge length we can construct the sequences $\{m_k\}_{k=1}^n$ and $\{l_k\}_{k=0}^n$. Theorem is proved. □

Corollary. *If the conditions of Theorem 5.3 are satisfied and in addition to the spectrum of problem (6)–(10) and the length of an edge, the number p of vertices is given then we can find the number of edges g in the graph. If $p < 5$, then the form of the graph is determined by these data.*

Proof. Using Theorem 5.3 we can find the multiplicity of z_g, because

$$\nu_1 = z_g, \quad \nu_2 = z_{2g}, \quad \dots, \quad \nu_n = z_{ng}.$$

According to Theorem 5.1 this multiplicity equals $g - p$. Therefore, knowing the number of vertices, p, and the multiplicity of z_g $(= \nu_1)$ we can find the number of the edges, g.

Knowing the sequences $\{m_k\}_{k=1}^n$ and $\{l_k\}_{k=0}^n$ we can solve the direct problem for an edge and find the characteristic polynomial $Q_{2n}(z)$. Then we can find all α_k $(k = 1, 2, \dots, p - 1)$:

$$\alpha_k = Q_{2n}(z_{p-k+1}), \quad k = 1, 2, \dots, p - 1.$$

In case of $p < 5$ using α_k $(k = 1, 2, \dots, p - 1)$ and $\alpha_p = 1$ we can find the form of the graph according to [3], Sec. 6.1. □

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