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SCHATTEN CLASS OPERATORS ON THE BERGMAN SPACE OVER BOUNDED SYMMETRIC DOMAIN

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Abstract. Let Ω be a bounded symmetric domain in $\mathbb{C}^n$ with Bergman kernel $K(z,w)$. Let $dV_\lambda(z) = K(z,z)\frac{dV(z)}{C_\lambda}$, where $C_\lambda = \int_\Omega K(z,z)^\lambda dV(z)$, $\lambda \in \mathbb{R}$, $dV(z)$ is the volume measure of Ω normalized so that $K(z,0) = K(0,w) = 1$. In this paper we have shown that if the Toeplitz operator T_ϕ defined on $L^2_a(\Omega, \frac{dV}{C_0})$ belongs to the Schatten p-class, $1 \leq p < \infty$, then $\widetilde{\phi} \in L^p(\Omega, d\eta)$, where $d\eta(z) = K(z,z)\frac{dV(z)}{C_0}$ and $\widetilde{\phi}$ is the Berezin transform of ϕ. Further if $\phi \in L^p(\Omega, d\eta_\lambda)$, then $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$ and T^λ_ϕ belongs to Schatten p-class. Here $d\eta_\lambda = K(z,z)\frac{dV(z)}{C_\lambda}$, the function $\widetilde{\phi_\lambda}$ is the Berezin transform of ϕ in $L^2_a(\Omega, dV_\lambda)$ and T^λ_ϕ is the Toeplitz operator defined on $L^2_a(\Omega, dV_\lambda)$. We also find conditions on bounded linear operator C defined from $L^2_a(\Omega, dV_\lambda)$ into itself such that C belongs to the Schatten p-class by comparing it with positive Toeplitz operators defined on $L^2_a(\Omega, dV_\lambda)$. Applications of these results are obtained and we also present Schatten class characterization of little Hankel operators defined on $L^2_a(\Omega, dV_\lambda)$.

1. Introduction

Let Ω be a bounded symmetric domain in $\mathbb{C}^n$ with Bergman kernel $K(z,w)$. We assume that Ω is in its standard (Harish-Chandra) representation. Let dV be the volume measure of Ω normalized so that $K(z,0) = K(0,w) = 1$ for all z and w in Ω. By [13] and using the polar co-ordinates representation, there exists a positive number ϵ_Ω such that

$$C_\lambda = \int_\Omega K(z,z)^\lambda dV(z) < +\infty$$

if and only if $\lambda < \epsilon_\Omega$. Let

$$dV_\lambda(z) = C_\lambda^{-1} K(z,z)^\lambda dV(z).$$

Then dV_λ is a probability measure on Ω for all $\lambda < \epsilon_\Omega$. We fix a $\lambda < \epsilon_\Omega$ throughout the paper and consider the weighted Bergman space $L^p_a(\Omega, dV_\lambda), 1 \leq p < +\infty$, consisting of holomorphic functions in $L^p(\Omega, dV_\lambda)$. For $p = 2$, we have an orthogonal projection P_λ from the Hilbert space $L^2(\Omega, dV_\lambda)$ onto the closed subspace $L^2_a(\Omega, dV_\lambda)$. The orthogonal projection P_λ is given by

$$P_\lambda f(z) = \int_\Omega K_\lambda(z,w) f(w)\, dV_\lambda(w),$$

where $K_\lambda(z,w) = K(z,w)^{1-\lambda}$ is the reproducing kernel of $L^2_a(\Omega, dV_\lambda)$. Let $K_\lambda(z,w) = \overline{K_z^{1-\lambda}(w)}$.

Suppose ϕ is a function in $L^\infty(\Omega)$. Then the Toeplitz operator with symbol ϕ is defined by $T^\lambda_\phi(f) = P_\lambda(\phi f), f \in L^2_a(\Omega, dV_\lambda)$ and the Hankel operator H^λ_ϕ with symbol ϕ

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is defined by $H_\phi^\lambda(f) = (I - P_\lambda)(\phi f), f \in L_a^2(\Omega, dV_\lambda)$. Let $\overline{P_\lambda}$ be the orthogonal projection from $L^2(\Omega, dV_\lambda)$ onto $\overline{L_a^2(\Omega, dV_\lambda)} = \left\{\overline{f} : f \in L_a^2(\Omega, dV_\lambda)\right\}$. Then

$$\overline{P_\lambda} f(z) = \int_\Omega \overline{K_\lambda(z,w)} f(w)\, dV_\lambda(w) = \int_\Omega K_\lambda(w,z) f(w)\, dV_\lambda(w).$$

Thus formula also extends $\overline{P_\lambda}$ to $L^1(\Omega, dV_\lambda)$. Given $\phi \in L^\infty(\Omega)$, define the little Hankel operator h_ϕ^λ with domain $L_a^2(\Omega, dV_\lambda)$ as $h_\phi^\lambda(f) = \overline{P_\lambda}(\phi f)$.

For any $a \in \Omega$, let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}}$. The k_a's are called normalized reproducing kernels of $L_a^2(\Omega, dV)$. They are unit vectors in $L_a^2(\Omega, dV)$. It is easy to see that $k_a^{1-\lambda}$ is a unit vector of $L_a^2(\Omega, dV_\lambda)$ for any $a \in \Omega$. Let $\mathcal{L}(L_a^2(\Omega, dV_\lambda))$ be the set of all bounded linear operators from $L_a^2(\Omega, dV_\lambda)$ into itself. For $A \in \mathcal{L}(L_a^2(\Omega, dV_\lambda))$ we define the Berezin transform $\widetilde{A}_\lambda$ of A as

$$\widetilde{A_\lambda}(z) = \left\langle A k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda, \quad z \in \Omega,$$

where $\langle , \rangle_\lambda$ is the inner product in $L_a^2(\Omega, dV_\lambda)$. Since $k_z^{1-\lambda}$ converges to 0 weakly in $L_a^2(\Omega, dV_\lambda)$ as z approaches $\partial\Omega$ (the topological boundary of Ω), it follows that $\widetilde{A_\lambda}$ is bounded on Ω if $A \in \mathcal{L}(L_a^2(\Omega, dV_\lambda))$, and $\widetilde{A_\lambda}(z) \longrightarrow 0$ as $z \longrightarrow \partial\Omega$ if A is compact. For $\phi \in L^\infty(\Omega)$, let $\widetilde{\phi_\lambda}(z) = \left\langle T_\phi^\lambda k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda = \widetilde{T_\phi^\lambda}(z)$, $z \in \Omega$. Hence $\widetilde{\phi_\lambda}$ is the Berezin transform of the Toeplitz operator T_ϕ^λ. We also define for $\phi \in L^\infty(\Omega)$, the operator S_ϕ^λ : $L_a^2(\Omega, dV_\lambda) \longrightarrow L_a^2(\Omega, dV_\lambda)$ as $S_\phi^\lambda(f) = P_\lambda J_\lambda(\phi f)$, where $J_\lambda : L^2(\Omega, dV_\lambda) \longrightarrow L^2(\Omega, dV_\lambda)$ is defined by $J_\lambda f(z) = f(\overline{z})$. The operators $S_\phi^\lambda, h_\phi^\lambda$ are unitarily equivalent. In fact, $J_\lambda S_\phi^\lambda = h_\phi^\lambda$. Hence we shall refer both these operators $S_\phi^\lambda, h_\phi^\lambda$ as little Hankel operators on $L_a^2(\Omega, dV_\lambda)$. Let $d\eta(z) = K(z,z)\frac{dV(z)}{C_\lambda}$.

Let $\mathcal{L}(H)$ be the set of all bounded linear operators from the Hilbert H into itself and $\mathcal{LC}(H)$ be the set of all compact operators in $\mathcal{L}(H)$. For any non-negative integer $k, T \in \mathcal{L}(H)$, let $s_k(T) = \inf\{\|T - R\| : R \in \mathcal{L}(H), \text{rank} R \leq k\}$. The numbers $s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$ are called s-numbers or singular values of T. It is well-known that if $T \in \mathcal{LC}(H)$, then there exist orthonormal vectors $\{u_k\}$ and $\{\sigma_k\}$ in H with $T = \sum_{k=1}^\infty s_k \langle \cdot, u_k \rangle \sigma_k$ for $Tx = \sum_{k=1}^\infty s_k \langle x, u_k \rangle \sigma_k$. For any $1 \leq p < +\infty$, the Schatten ideal $S_p(H) = S_p$ is defined to be the set of all compact operators T on H such that $\displaystyle\sum_{k=1}^{\infty}(s_k(T))^p < +\infty$. The linear space S_p is a Banach space with the norm $\|T\|_p = \|T\|_{S_p} = \left[\sum_{k=1}^\infty (s_k(T))^p\right]^{1/p}$. The space S_p is also a two-sided ideal of the algebra $\mathcal{L}(H)$ and for any $T \in S_p$ and $S, R \in \mathcal{L}(H)$, we have

$$\|STR\|_{S_p} \leq \|S\| \, \|T\|_{S_p} \, \|R\|.$$

The space S_1 is also called the trace class and S_2 is called the Hilbert-Schmidt class. If $T \in S_1$ and $\{u_k\}$ is an orthonormal basis for H, then $\text{tr}(T) = \sum_{k=1}^\infty \langle Tu_k, u_k \rangle$ is convergent and independent of $\{u_k\}$. If $T \in S_1$ and $T \geq 0$, then $\|T\|_{S_1} = \text{tr}(T)$. In general, we have $\|T\|_{S_p} = \left[\text{tr}((T^*T)^{p/2})\right]^{1/p}$. For more information on the Schatten ideals, see [22] for example. Suppose $p \geq 1$ and S_p^λ is the Schatten p-ideal of the Hilbert space $L_a^2(\Omega, dV_\lambda)$. For convenience of notation, we will use S_∞^λ to denote the full algebra of bounded linear operators on the Bergman space $L_a^2(\Omega, dV_\lambda)$. That is, $S_\infty^\lambda = \mathcal{L}(L_a^2(\Omega, dV_\lambda))$. The organization of this paper is as follows. In Section 2, we discuss Schatten p-class Toeplitz operators. We show that if $1 \leq p \leq \infty$ and $\phi \in L^p(\Omega, d\eta_\lambda)$ then $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$. Further if $0 < p < \infty$, $T_\phi^\lambda \in S_p^\lambda$ then $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$ where $d\eta_\lambda(z) = K(z,z)\frac{dV(z)}{C_\lambda}$. In Section 3, we find conditions on $C \in \mathcal{L}(L_a^2(\Omega, dV_\lambda))$ to have membership in the Schatten class with the help of the Schatten class characterization of Toeplitz operators. In Section 4, we concentrate on the Hilbert space $L_a^2(\Omega, \frac{dV}{C_0})$ and prove that if $T_\phi \in S_p$ then

$\widetilde{\phi} \in L^p(\Omega, d\eta)$, $1 \le p < \infty$ where $d\eta(z) = K(z,z)\frac{dV(z)}{C_0}$ and we deduce many important corollaries. Section 5 is devoted to the Schatten class characterization of little Hankel operators.

2. Schatten class Toeplitz operators

In this section we seek to find necessary and sufficient conditions on ϕ which will ensure that the Toeplitz operator belong to S_p^λ. We will concentrate on the special case $\phi \ge 0$.

Proposition 2.1. *Suppose A is a positive operator in $\mathcal{L}(L_a^2(\Omega, dV_\lambda))$ or A is an operator in the trace class of $L_a^2(\Omega, dV_\lambda)$. Then*

$$\operatorname{tr}(A) = \int_\Omega \left\langle Ak_z^{1-\lambda}, k_z^{1-\lambda}\right\rangle_\lambda d\eta_\lambda(z) = \int_\Omega \widetilde{A_\lambda}(z)\, d\eta_\lambda(z),$$

where $\widetilde{A_\lambda}$ is the Berezin symbol of A and $d\eta_\lambda(z) = K(z,z)\frac{dV(z)}{C_\lambda}$.

Proof. Let $\left\{e_n^\lambda\right\}_{n=0}^\infty$ be an orthonormal basis for $L_a^2(\Omega, dV_\lambda)$. Hence

$$\begin{aligned}
\operatorname{tr}(A) &= \sum_{n=1}^\infty \left\langle Ae_n^\lambda, e_n^\lambda\right\rangle_\lambda = \sum_{n=1}^\infty \int_\Omega (Ae_n^\lambda)(z)\overline{e_n^\lambda(z)}\, dV_\lambda(z) \\
&= \sum_{n=1}^\infty \int_\Omega \left\langle Ae_n^\lambda, K_z^{1-\lambda}\right\rangle_\lambda \overline{e_n^\lambda(z)}\, dV_\lambda(z) \\
&= \int_\Omega \left\langle A\Big(\sum_{n=1}^\infty e_n^\lambda(z)\overline{e_n^\lambda(z)}\Big), K_z^{1-\lambda}\right\rangle_\lambda dV_\lambda(z) = \int_\Omega \left\langle AK_z^{1-\lambda}, K_z^{1-\lambda}\right\rangle_\lambda dV_\lambda(z) \\
&= \int_\Omega \left\langle Ak_z^{1-\lambda}, k_z^{1-\lambda}\right\rangle_\lambda K(z,z)\frac{dV(z)}{C_\lambda} = \int_\Omega \widetilde{A_\lambda}(z)\, d\eta_\lambda(z).
\end{aligned}$$

□

Corollary 2.2. *If ϕ is a non-negative function on Ω then*

$$\operatorname{tr}(T_\phi^\lambda) = \int_\Omega \phi(w)\, d\eta_\lambda(w).$$

Proof. By Proposition 2.1 and Fubini's theorem [19], we have

$$\begin{aligned}
\operatorname{tr}(T_\phi^\lambda) &= \int_\Omega \widetilde{\phi_\lambda}(z)K(z,z)\frac{dV(z)}{C_\lambda} \\
&= \int_\Omega K(z,z)\frac{dV(z)}{C_\lambda}\int_\Omega \left|k_z^{1-\lambda}(w)\right|^2 \phi(w)\, dV_\lambda(w) \\
&= \int_\Omega \frac{dV(z)}{C_\lambda}\int_\Omega \frac{\left|K^{1-\lambda}(z,w)\right|^2}{K^{1-\lambda}(z,z)}K(z,z)\phi(w)\frac{1}{C_\lambda}K^\lambda(w,w)\, dV(w) \\
&= \int_\Omega \phi(w)\, dV(w)\int_\Omega \left|K^{1-\lambda}(z,w)\right|^2 K^\lambda(z,z)\frac{1}{C_\lambda}K^\lambda(w,w)\, dV(z) \\
&= \int_\Omega \phi(w)K^\lambda(w,w)\frac{dV(w)}{C_\lambda}\int_\Omega \left|K^{1-\lambda}(z,w)\right|^2 K^\lambda(z,z)\frac{1}{C_\lambda}dV(z) \\
&= \int_\Omega \phi(w)K^\lambda(w,w)K^{1-\lambda}(w,w)\frac{dV(w)}{C_\lambda} \\
&= \int_\Omega \phi(w)K(w,w)\frac{dV(w)}{C_\lambda} = \int_\Omega \phi(w)\, d\eta_\lambda(w).
\end{aligned}$$

□

The above results are very useful in the study of Schatten class operators on the Bergman space $L_a^2(\Omega, dV_\lambda)$, especially when combined with the inequalities given in (2.2) and (2.3).

Lemma 2.3. *If $p \geq 1$ and $\phi \in L^p(\Omega, d\eta_\lambda)$, then T_ϕ^λ is in the Schatten class S_p^λ.*

Proof. By interpolation, we only need to prove the result for the case $p = 1$. The case $p = +\infty$ is trivial. Suppose $\phi \in L^1(\Omega, d\eta_\lambda)$ and $\{e_m^\lambda\}_{m=0}^\infty$ is an orthonormal basis in $L_a^2(\Omega, dV_\lambda)$. For any $m \geq 1$, $\left\langle T_\phi^\lambda e_m^\lambda, e_m^\lambda \right\rangle_\lambda = \int_\Omega \left| e_m^\lambda(z) \right|^2 \phi(z)\, dV_\lambda(z)$. It follows that

$$\begin{aligned}
\sum_{m=0}^\infty \left| \langle T_\phi^\lambda e_m^\lambda, e_m^\lambda \rangle_\lambda \right| &\leq \int_\Omega \sum_{m=0}^\infty \left| e_m^\lambda(z) \right|^2 |\phi(z)|\, dV_\lambda(z) \\
&\leq \int_\Omega K^{1-\lambda}(z,z)\, |\phi(z)|\, \frac{1}{C_\lambda} K^\lambda(z,z)\, dV(z) \\
&= \int_\Omega |\phi(z)|\, \frac{1}{C_\lambda} K(z,z)\, dV(z) = \int_\Omega |\phi(z)|\, d\eta_\lambda(z).
\end{aligned}$$

By [22], the operator $T_\phi^\lambda \in S_1^\lambda$ and $\left\| T_\phi^\lambda \right\|_{S_1{}^\lambda} \leq \int_\Omega |\phi(z)|\, d\eta_\lambda(z)$. □

Let $h > 1$. The generalized Kantorvich constant $K(p)$ is defined by

$$K(p) = \frac{h^p - h}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h} \right)^p \tag{2.1}$$

for any real number p and it is known that $K(p) \in (0, 1]$ for $p \in [0, 1]$. We state below the known results on the generalized Kantorvich constant $K(p)$. Let A be a strictly positive operator satisfying $MI \geq A \geq mI > 0$, where $M > m > 0$. Put $h = \frac{M}{m} > 1$. Then the following [10] inequalities (2.2) and (2.3) hold for every unit vector x and are equivalent:

$$K(p) \langle Ax, x \rangle^p \geq \langle A^p x, x \rangle \geq \langle Ax, x \rangle^p \quad \text{for any } p > 1 \quad \text{or} \quad \text{any } p < 0; \tag{2.2}$$

$$\langle Ax, x \rangle^p \geq \langle A^p x, x \rangle \geq K(p) \langle Ax, x \rangle^p \quad \text{for any } p \in (0, 1]. \tag{2.3}$$

The Kantorvich constant $K(p)$ is symmetric with respect to $p = \frac{1}{2}$ and $K(p)$ is an increasing function of p for $p \geq \frac{1}{2}$, $K(p)$ is a decreasing function of p for $p \leq \frac{1}{2}$, and $K(0) = K(1) = 1$. Further, $K(p) \geq 1$ for $p \geq 1$ or $p \leq 0$, and $1 \geq K(p) \geq \frac{2h^{\frac{1}{4}}}{(h^{\frac{1}{2}}+1)}$ for $p \in [0, 1]$.

Corollary 2.4. *Suppose ϕ is a non-negative function on $\Omega, 1 \leq p \leq +\infty$ and $T_\phi^\lambda \in S_p^\lambda$. Then $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$.*

Proof. The case $p = \infty$ is not difficult to verify. So suppose $1 \leq p < \infty$ and $T_\phi^\lambda \in S_p^\lambda$. Then $(T_\phi^\lambda)^p \in S_1^\lambda$ since T_ϕ^λ is positive. By Proposition 2.1,

$$\operatorname{tr}((T_\phi^\lambda)^p) = \int_\Omega \left\langle (T_\phi^\lambda)^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) < +\infty.$$

By (2.2),

$$\begin{aligned}
\int_\Omega \left[\widetilde{\phi_\lambda}(z) \right]^p d\eta_\lambda(z) &= \int_\Omega \left\langle T_\phi^\lambda k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda^p d\eta_\lambda(z) \\
&\leq \int_\Omega \left\langle (T_\phi^\lambda)^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) < +\infty.
\end{aligned}$$

□

Proposition 2.5. *Let T_ϕ^λ be strictly positive satisfying $MI \geq T_\phi^\lambda \geq mI > 0$, where $M > m > 0$. The following hold:*

(i) *If $0 < p < \infty$ and $T_\phi^\lambda \in S_p^\lambda$ then $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$.*
(ii) *If $0 < p \leq 1, \widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$ then $T_\phi^\lambda \in S_p^\lambda$.*
(iii) *Let $p \in [1, \infty)$ be such that $K(p) < \infty$. If $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$ then $T_\phi^\lambda \in S_p^\lambda$.*

Proof. To prove (i), suppose $0 < p \leq 1$ and $T_\phi^\lambda \in S_p^\lambda$. Then

$$\int_\Omega \left\langle (T_\phi^\lambda)^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) = \int_\Omega \left\langle \left|T_\phi^\lambda\right|^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) < +\infty.$$

Hence from (2.3), it follows that $K(p) \int_\Omega \left\langle T_\phi^\lambda k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda^p d\eta_\lambda(z) < +\infty$. Since $K(p) \in (0,1]$ for $p \in [0,1]$, hence $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$. Suppose $p > 1$ and $T_\phi^\lambda \in S_p^\lambda$. Then

$$\int_\Omega \left\langle (T_\phi^\lambda)^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) = \int_\Omega \left\langle \left|T_\phi^\lambda\right|^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) < +\infty.$$

Hence by (2.2), $\int_\Omega \left\langle T_\phi^\lambda k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda^p d\eta_\lambda(z) < +\infty$. That is, $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$. To prove (ii), assume $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$. Then if $0 < p \leq 1$ then by (2.3), we have $\int_\Omega \left\langle \left|T_\phi^\lambda\right|^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) < +\infty$ and hence $T_\phi^\lambda \in S_p^\lambda$. To prove (iii), suppose $1 \leq p < +\infty, K(p) < +\infty$ and $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$. Then by (2.2) and (2.3), we have

$$\int_\Omega \left\langle \left|T_\phi^\lambda\right|^p k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda d\eta_\lambda(z) < +\infty \quad \text{and} \quad T_\phi^\lambda \in S_p^\lambda.$$

□

The Berezin transform of a bounded linear operator on the Bergman space $L_a^2(\Omega, dV_\lambda)$ contains a lot of information about the operator. It is one of the most useful tools in the study of Toeplitz operators. Another useful tool is Carleson measures on Bergman spaces. The characterization of boundedness and compactness of a positive Toeplitz operator on the Bergman spaces in terms of Carleson measures appears first in [16] and in terms of the Berezin transform appears first in [23]. For more details about Carleson measures, see [15] and [1].

We will denote by $\beta(z, w)$ the Bergman distance function on Ω. For any z in Ω and $r > 0$, let

$$E(z, r) = \{w \in \Omega : \beta(z, w) < r\}.$$

We denote by $|E(z,r)|$ the normalized volume of $E(z,r)$, that is, $|E(z,r)| = \int_{E(z,r)} dV(w)$. It is not difficult to see that $|E(z,r)|^{1-\lambda}$ is equivalent [23] to $V_\lambda(E(z,r))$ for any fixed $r > 0$.

Let $\mu \geq 0$ be a finite Borel measure on Ω. We say that μ is a Carleson measure on $L_a^p(\Omega, dV_\lambda)$ if there exists a constant $M > 0$ such that

$$\int_\Omega |f(z)|^p \, d\mu(z) \leq M \int_\Omega |f(z)|^p \, dV_\lambda(z)$$

for all f in $L_a^p(\Omega, dV_\lambda)$. The following theorem gives a geometric description of Carleson measures on $L_a^p(\Omega, dV_\lambda)$. In particular, it implies that Carleson measures on $L_a^p(\Omega, dV_\lambda)$ only depends on λ, not on p.

Theorem 2.6. *Suppose $\mu \geq 0$ is a finite Borel measure on $\Omega, p \geq 1$, then μ is a Carleson measure on $L_a^p(\Omega, dV_\lambda)$ if and only if $\dfrac{\mu(E(z,r))}{|E(z,r)|^{1-\lambda}}$ is bounded on Ω (as a function of z) for all (or some) $r > 0$. Moreover, the following quantities are equivalent for any fixed $r > 0$ and $p \geq 1$:*

(i) $\sup \left\{ \dfrac{\mu(E(z,r))}{|E(z,r)|^{1-\lambda}} : z \in \Omega \right\}$;

(ii) $\sup \left\{ \dfrac{\int_\Omega |f(z)|^p \, d\mu(z)}{\int_\Omega |f(z)|^p \, dV_\lambda(z)} : f \in L_a^p(\Omega, dV_\lambda) \right\}$.

Proof. For proof see [23]. □

Let $BT_\Omega^\lambda = \left\{ f \in L^1(\Omega, dV_\lambda) : \|f\|_{BT_\Omega^\lambda} = \sup_{z\in\Omega} \widetilde{|f|}_\lambda(z) < \infty \right\}$. The space $L^\infty(\Omega)$ is properly contained in BT_Ω^λ since if $\phi \in L^\infty(\Omega)$ then for all $z \in \Omega$,

$$\widetilde{|\phi|}_\lambda(z) = \left| \left\langle T_{|\phi|}^\lambda k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda \right| \le \left\| T_{|\phi|}^\lambda \right\| \le \| |\phi| \|_\infty = \|\phi\|_\infty < \infty.$$

It also follows that if $f \in L^1(\Omega, dV_\lambda)$ then $f \in BT_\Omega^\lambda$ if and only if $|f| dV_\lambda$ is a Carleson measure on Ω. In the following proposition we verify that if $\phi \in BT_\Omega^\lambda$ then T_ϕ^λ is bounded on $L_a^2(\Omega, dV_\lambda)$ and there is a constant C such that $\|T_\phi^\lambda\| \le C\|\phi\|_{BT_\Omega^\lambda}$.

Proposition 2.7. *Suppose $1 < p < \infty$ and $\phi \in BT_\Omega^\lambda$. Then T_ϕ^λ is bounded on $L_a^p(\Omega, dV_\lambda)$ and there is a constant C (depending only on p and λ) such that $\|T_\phi^\lambda\|_p \le C\|\phi\|_{BT_\Omega^\lambda}$.*

Proof. It is well-known that the dual of L_a^p is L_a^q (see [1]) where $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L_a^p$ and $g \in L_a^q$, by Holder's inequality

$$\begin{aligned} \left| \left\langle T_\phi^\lambda f, g \right\rangle_\lambda \right| &= |\langle \phi f, g\rangle_\lambda| = \left| \int_\Omega f(z)\overline{g(z)}\phi(z)\, dV_\lambda(z) \right| \\ &\le \int_\Omega |\phi(z)|\, |f(z)|\, |g(z)|\, dV_\lambda(z) \\ &\le \Big(\int_\Omega |f(z)|^p\, |\phi(z)|\, dV_\lambda(z) \Big)^{1/p} \Big(\int_\Omega |g(z)|^q\, |\phi(z)|\, dV_\lambda(z) \Big)^{1/q}. \end{aligned}$$

Thus it follows that $\left| \left\langle T_\phi^\lambda f, g \right\rangle_\lambda \right| \le C \|\phi\|_{BT_\Omega^\lambda} \|f\|_p \|g\|_q$ where C is a constant depending only on p and λ. This shows that T_ϕ^λ is bounded on $L_a^p(\Omega, dV_\lambda)$ and $\left\| T_\phi^\lambda \right\|_p \le C \|\phi\|_{BT_\Omega^\lambda}$. □

Proposition 2.8. *For $1 \le p \le \infty$, if $\phi \in L^p(\Omega, d\eta_\lambda)$ then $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$.*

Proof. Suppose $\phi \in L^1(\Omega, d\eta_\lambda)$. Then

$$\begin{aligned} \int_\Omega \left| \widetilde{\phi_\lambda}(w) \right| d\eta_\lambda(w) &= \int_\Omega \left| \widetilde{\phi_\lambda}(w) \right| K(w,w) \frac{dV(w)}{C_\lambda} \\ &\le \int_\Omega \Big(\int_\Omega |\phi(z)| \frac{\left| K^{1-\lambda}(z,w) \right|^2}{K^{1-\lambda}(w,w)} \frac{1}{C_\lambda} K^\lambda(z,z)\, dV(z) \Big) K(w,w) \frac{dV(w)}{C_\lambda} \\ &= \int_\Omega |\phi(z)| \int_\Omega \left| K^{1-\lambda}(z,w) \right|^2 K^\lambda(w,w) \frac{dV(w)}{C_\lambda} K^\lambda(z,z) \frac{dV(z)}{C_\lambda} \\ &= \int_\Omega |\phi(z)|\, K^\lambda(z,z) \Big(\int_\Omega \left| K^{1-\lambda}(z,w) \right|^2 K^\lambda(w,w) \frac{dV(w)}{C_\lambda} \Big) \frac{dV(z)}{C_\lambda} \\ &= \int_\Omega |\phi(z)|\, K^\lambda(z,z) K^{1-\lambda}(z,z) \frac{dV(z)}{C_\lambda} \\ &= \int_\Omega |\phi(z)|\, K(z,z) \frac{dV(z)}{C_\lambda} = \int_\Omega |\phi(z)|\, d\eta_\lambda(z). \end{aligned}$$

The change of order of integration is justified by the positivity of the integrand. Hence $\widetilde{\phi_\lambda} \in L^1(\Omega, d\eta_\lambda)$. Similarly if $\phi \in L^\infty(\Omega)$ then $\widetilde{\phi_\lambda} \in L^\infty(\Omega)$ as

$$\begin{aligned} \left| \widetilde{\phi_\lambda}(w) \right| &= \left| \langle \phi k_w^{1-\lambda}, k_w^{1-\lambda} \rangle_\lambda \right| \\ &\le \left\| \phi k_w^{1-\lambda} \right\|_{L_a^2(\Omega, dV_\lambda)} \left\| k_w^{1-\lambda} \right\|_{L_a^2(\Omega, dV_\lambda)} \\ &\le \|\phi\|_\infty \left\| k_w^{1-\lambda} \right\|^2_{L_a^2(\Omega, dV_\lambda)} = \|\phi\|_\infty . \end{aligned}$$

By Marcinkiewicz interpolation theorem [22] it follows that if $\phi \in L^p(\Omega, d\eta_\lambda)$ then $\widetilde{\phi_\lambda} \in L^p(\Omega, d\eta_\lambda)$ for $1 \le p \le \infty$. □

3. Schatten class operators in $\mathcal{L}(L^2_a(\Omega, dV_\lambda))$

In this section we find conditions on bounded linear operator $C \in \mathcal{L}(L^2_a(\Omega, dV_\lambda))$ such that $C \in S^\lambda_p$, $1 \le p < \infty$ by comparing it with positive Toeplitz operators defined on $L^2_a(\Omega, dV_\lambda)$ and applications of the result are also obtained.

Theorem 3.1. *Let $\phi \in L^p(\Omega, d\eta_\lambda), \psi \in L^q(\Omega, d\eta_\lambda)$, where $1 \le p, q < \infty$. Let $C \in \mathcal{L}(L^2_a(\Omega, dV_\lambda))$ is such that*

$$\left|\left\langle CK_z^{1-\lambda}, K_w^{1-\lambda}\right\rangle_\lambda\right|^2 \le \left\langle T^\lambda_{|\phi|}K_z^{1-\lambda}, K_z^{1-\lambda}\right\rangle_\lambda \left\langle T^\lambda_{|\psi|}K_w^{1-\lambda}, K_w^{1-\lambda}\right\rangle_\lambda \tag{3.1}$$

for all $z, w \in \Omega$. Then $C \in S^\lambda_{2r}$ and $\|C\|^2_{2r} \le \left\|T^\lambda_{|\phi|}\right\|_p \left\|T^\lambda_{|\psi|}\right\|_q$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Proof. First we show that (3.1) implies

$$|\langle Cf, g\rangle_\lambda|^2 \le \left\langle T^\lambda_{|\phi|}f, f\right\rangle_\lambda \left\langle T^\lambda_{|\psi|}g, g\right\rangle_\lambda$$

for all $f, g \in L^2_a(\Omega, dV_\lambda)$. Let $f = \sum_{j=1}^n c_j K_{z_j}^{1-\lambda}$ where c_j are constants, $z_j \in \Omega$ for $j = 1, 2, \ldots, n$ and $g = \sum_{i=1}^m d_i K_{w_i}^{1-\lambda}$ where d_i are constants, $w_i \in \Omega$ for $i = 1, 2, \ldots, m$. Then

$$\begin{aligned}
|\langle Cf, g\rangle_\lambda| &= \left|\left\langle C\Big(\sum_{j=1}^n c_j K_{z_j}^{1-\lambda}\Big), \sum_{i=1}^m d_i K_{w_i}^{1-\lambda}\right\rangle_\lambda\right| = \Big|\sum_{i=1,j=1}^{m,n} c_j \overline{d_i} \left\langle CK_{z_j}^{1-\lambda}, K_{w_i}^{1-\lambda}\right\rangle_\lambda \Big| \\
&\le \sum_{i=1,j=1}^{m,n} |c_j||d_i| \left|\left\langle CK_{z_j}^{1-\lambda}, K_{w_i}^{1-\lambda}\right\rangle_\lambda\right| \\
&\le \sum_{i=1,j=1}^{m,n} |c_j||d_i| \left\langle T^\lambda_{|\phi|}K_{z_j}^{1-\lambda}, K_{z_j}^{1-\lambda}\right\rangle_\lambda^{1/2} \left\langle T^\lambda_{|\psi|}K_{w_i}^{1-\lambda}, K_{w_i}^{1-\lambda}\right\rangle_\lambda^{1/2} \\
&= \left\langle T^\lambda_{|\phi|}\Big(\sum_{j=1}^n c_j K_{z_j}^{1-\lambda}\Big), \sum_{j=1}^n c_j K_{z_j}^{1-\lambda}\right\rangle_\lambda^{1/2} \left\langle T^\lambda_{|\psi|}\Big(\sum_{i=1}^m d_i K_{w_i}^{1-\lambda}\Big), \sum_{i=1}^m d_i K_{w_i}^{1-\lambda}\right\rangle_\lambda^{1/2} \\
&= \left\langle T^\lambda_{|\phi|}f, f\right\rangle_\lambda^{1/2} \left\langle T^\lambda_{|\psi|}g, g\right\rangle_\lambda^{1/2}.
\end{aligned}$$

Since the set of vectors $\Big\{\sum c_j K_{w_j}^{1-\lambda}, w_j \in \Omega, j = 1, \ldots, n\Big\}$ is dense in $L^2_a(\Omega, dV_\lambda)$, hence $|\langle Cf, g\rangle_\lambda|^2 \le \langle T^\lambda_{|\phi|}f, f\rangle_\lambda \langle T^\lambda_{|\psi|}g, g\rangle_\lambda$ for all $f, g \in L^2_a(\Omega, dV_\lambda)$. If $\phi \in L^p(\Omega, d\eta_\lambda)$, then $T^\lambda_{|\phi|} \in S^\lambda_p$ and

$$\left\|T^\lambda_{|\phi|}\right\|_p = \Big(\text{trace } (T^\lambda_{|\phi|})^p\Big)^{\frac{1}{p}} < \infty.$$

Similarly, since $\psi \in L^q(\Omega, d\eta_\lambda)$ then

$$\left\|T^\lambda_{|\psi|}\right\|_q = \Big(\text{trace } (T^\lambda_{|\psi|})^q\Big)^{\frac{1}{q}} < \infty.$$

Let $\{u_n^\lambda\}_{n=0}^\infty$ and $\{\sigma_n^\lambda\}_{n=0}^\infty$ be two orthonormal sequences in $L_a^2(\Omega, dV_\lambda)$. Then using Holder's inequality, we obtain that

$$\begin{aligned}\sum_{n=0}^\infty \left|\langle Cu_n^\lambda, \sigma_n^\lambda\rangle_\lambda\right|^{2r} &\leq \sum_{n=0}^\infty \left\langle T_{|\phi|}^\lambda u_n^\lambda, u_n^\lambda\right\rangle_\lambda^r \left\langle T_{|\psi|}^\lambda \sigma_n^\lambda, \sigma_n^\lambda\right\rangle_\lambda^r \\ &\leq \Big(\sum_{n=0}^\infty \left\langle T_{|\phi|}^\lambda u_n^\lambda, u_n^\lambda\right\rangle_\lambda^p\Big)^{\frac{r}{p}} \Big(\sum_{n=0}^\infty \left\langle T_{|\psi|}^\lambda \sigma_n^\lambda, \sigma_n^\lambda\right\rangle_\lambda^q\Big)^{\frac{r}{q}} \\ &\leq \Big(\sum_{n=0}^\infty \left\langle (T_{|\phi|}^\lambda)^p u_n^\lambda, u_n^\lambda\right\rangle_\lambda\Big)^{\frac{r}{p}} \Big(\sum_{n=0}^\infty \left\langle (T_{|\psi|}^\lambda)^q \sigma_n^\lambda, \sigma_n^\lambda\right\rangle_\lambda\Big)^{\frac{r}{q}} \\ &\leq \Big(\text{trace } (T_{|\phi|}^\lambda)^p\Big)^{\frac{r}{p}} \Big(\text{trace } (T_{|\psi|}^\lambda)^q\Big)^{\frac{r}{q}} = \|T_{|\phi|}^\lambda\|_p^r \|T_{|\psi|}^\lambda\|_q^r\end{aligned}$$

if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Thus $\|C\|_{2r} \leq \|T_{|\phi|}^\lambda\|_p^{\frac{1}{2}} \|T_{|\psi|}^\lambda\|_q^{\frac{1}{2}}$. □

Corollary 3.2. *If $\phi, \psi \in L^p(\Omega, d\eta_\lambda)$ and $C \in \mathcal{L}(L_a^2(\Omega, dV_\lambda))$ is such that*

$$\left|\langle CK_z^{1-\lambda}, K_w^{1-\lambda}\rangle_\lambda\right|^2 \leq \left\langle T_{|\phi|}^\lambda K_z^{1-\lambda}, K_z^{1-\lambda}\right\rangle_\lambda \left\langle T_{|\psi|}^\lambda K_w^{1-\lambda}, K_w^{1-\lambda}\right\rangle_\lambda$$

for all $z, w \in \Omega$ then $\|C\|_p^2 \leq \|T_{|\phi|}^\lambda\|_p \|T_{|\psi|}^\lambda\|_p$.

Proof. The proof follows from Theorem 3.1 if we assume $p = q$. □

Corollary 3.3. *If A, B are two positive operators in $\mathcal{L}(L_a^2(\Omega, dV_\lambda)$ and $A \in S_p^\lambda, B \in S_q^\lambda, 1 \leq p, q < \infty$ and $C \in \mathcal{L}(L_a^2(\Omega, dV_\lambda)$ is such that*

$$\left|\langle CK_z^{1-\lambda}, K_w^{1-\lambda}\rangle_\lambda\right|^2 \leq \langle AK_z^{1-\lambda}, K_z^{1-\lambda}\rangle_\lambda \langle BK_w^{1-\lambda}, K_w^{1-\lambda}\rangle_\lambda$$

for all $z, w \in \Omega$ then $\|C\|_{2r}^2 \leq \|A\|_p \|B\|_q$ if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $p = q$, then $\|C\|_p^2 \leq \|A\|_p \|B\|_p$.

Proof. Proceeding similarly as in Theorem 3.1 and Corollary 3.2 by replacing $T_{|\phi|}^\lambda$ by A and $T_{|\psi|}^\lambda$ by B, the corollary follows. □

Corollary 3.4. *If $A, B \in \mathcal{L}(L_a^2(\Omega, dV_\lambda)), 0 \leq A, A \in S_p^\lambda, 1 \leq p < \infty$ and (3.1) holds for $z, w \in \Omega$, then $\|C\|_{2p}^2 \leq \|A\|_p \|B\|$.*

Proof. Let $\{u_n^\lambda\}_{n=0}^\infty$ and $\{\sigma_n^\lambda\}_{n=0}^\infty$ be two orthonormal bases for $L_a^2(\Omega, dV_\lambda)$, then

$$\left|\langle Cu_n^\lambda, \sigma_n^\lambda\rangle_\lambda\right|^2 \leq \langle Au_n^\lambda, u_n^\lambda\rangle_\lambda \langle B\sigma_n^\lambda, \sigma_n^\lambda\rangle_\lambda \leq \langle Au_n^\lambda, u_n^\lambda\rangle_\lambda \|B\|.$$

Then $\left|\langle Cu_n^\lambda, \sigma_n^\lambda\rangle_\lambda\right|^{2p} \leq \|B\|^p \langle Au_n^\lambda, u_n^\lambda\rangle_\lambda^p$. Hence

$$\sum_{n=0}^\infty \left|\langle Cu_n^\lambda, \sigma_n^\lambda\rangle_\lambda\right|^{2p} \leq \|B\|^p \sum_{n=0}^\infty \langle Au_n^\lambda, u_n^\lambda\rangle_\lambda^p$$

and $\|C\|_{2p}^2 \leq \|B\| \|A\|_p$. □

If $\phi \in L^p(\Omega, d\eta_\lambda)$ then $T_\phi^\lambda \in S_p^\lambda$. Hence $|T_\phi^\lambda| \in S_p^\lambda$. Thus if $B \in \mathcal{L}(L_a^2(\Omega, dV_\lambda)), C \in \mathcal{L}(L_a^2(\Omega, dV_\lambda))$ are such that

$$\left|\langle CK_z^{1-\lambda}, K_w^{1-\lambda}\rangle_\lambda\right|^2 \leq \langle |T_\phi^\lambda| K_z^{1-\lambda}, K_z^{1-\lambda}\rangle_\lambda \langle BK_w^{1-\lambda}, K_w^{1-\lambda}\rangle_\lambda$$

for all $z, w \in \Omega$ then $C \in S_{2p}^\lambda$ and $\|C\|_{2p}^2 \leq \|B\| \||T_\phi^\lambda|\|_p$.

4. Schatten p-class operators in $\mathcal{L}(L^2_a(\Omega, \frac{dV}{C_0}))$

In this section we assume that Ω is in its standard realization so that $0 \in \Omega$ and Ω is circular. The domain Ω is also starlike; i.e., $z \in \Omega$ implies that $tz \in \Omega$ for all $t \in [0,1]$. Let $\mathrm{Aut}(\Omega)$ be the Lie group of all automorphisms (biholomorphic mappings) of Ω, and G_0, the isotropy subgroup at 0; i.e., $G_0 = \{\psi \in \mathrm{Aut}(\Omega) : \psi(0) = 0\}$. It is well known [14] that G_0 is compact and that G_0 is a subgroup of the unitary group $\mathcal{U}_n$ of $\mathbb{C}^n$. Since Ω is bounded symmetric, we can canonically define [4] for each a in Ω an automorphism ϕ_a in $\mathrm{Aut}(\Omega)$ such that

(i) $\phi_a \circ \phi_a(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in Ω.

Actually, the above three conditions completely characterize the ϕ_a's as the set of all (holomorphic) geodesic symmetrics of Ω.

For any $a \in \Omega$, let γ_a be the unique geodesic such that $\gamma_a(0) = 0$, $\gamma_a(1) = a$. Since Ω is Hermitian symmetric, there exists a unique $\phi_a \in \mathrm{Aut}(\Omega)$ such that $\phi_a o \phi_a(z) \equiv z$ and $\gamma_a(\frac{1}{2})$ is an isolated fixed point of ϕ_a and ϕ_a is the geodesic symmetry at $\gamma_a(\frac{1}{2})$. In particular, $\phi_a(0) = a$ and $\phi_a(a) = 0$. If $a = 0$, then we have $\phi_a(z) = -z$ for all z in Ω. A good reference for this is [12]. We denote by m_a the geodesic midpoint $\gamma_a(\frac{1}{2})$ of 0 and a. Given $\psi \in \mathrm{Aut}(\Omega)$, let $a = \psi^{-1}(0)$, then we have $\psi \circ \phi_a(0) = \psi(a) = 0$, thus $\psi \circ \phi_a \in G_0$ and so there exists a unitary matrix U such that $\psi = U\phi_a, U \in G_0$. If $\psi \in \mathrm{Aut}(\Omega)$ has an isolated fixed point in Ω, then ψ has a unique fixed point and each ϕ_a has m_a as a unique fixed point. Further, for any a and b in Ω, there exists a unitary $U \in G_0$ such that $\phi_b \circ \phi_a = U\phi_{\phi_a(b)}$ and $\phi_{m_a} \circ \phi_a = -\phi_{m_a}$ for any $a \in \Omega$. If $a \in \Omega$ and $U \in G_0$, then $U\phi_a = \phi_{Ua}U$.

For any $\psi \in \mathrm{Aut}(\Omega)$, we denote by $J_\psi(z)$ the complex Jacobian determinant of the mapping $\psi : \Omega \longrightarrow \Omega$. If $a \in \Omega$, then by a result of [4], there exists a unimodulus constant $\theta(a)$ such that

$$J_{\phi_a}(z) = \theta(a)k_a(z)$$

for all $z \in \Omega$. In the simplest case $\Omega = \mathbb{D}$, we have $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ and $J_{\phi_a}(z) = \phi_a'(z) = -k_a(z)$, thus $\theta(a) = -1$ is independent of a. This is also true for any bounded symmetric domain Ω. In fact $\theta(a) = (-1)^n$ for any $a \in \Omega$, where n is the (complex) dimension of Ω. Suppose $\psi \in \mathrm{Aut}(\Omega)$, there exists a unitary $U \in G_0$ such that $\psi = U\phi_a$ with $a = \psi^{-1}(0)$. Taking complex Jacobian determinant of this equality, we get

$$J_\psi(z) = \det(U)J_{\phi_a}(z) = (-1)^n \det(U)k_a(z).$$

In this section we shall assume $\lambda = 0$. Then $dV_0(z) = \frac{dV(z)}{C_0}$ is the normalized Lebesgue measure on Ω. Let $P_0 = P$ we define the Toeplitz and Hankel operators in the usual way. We write $T^0_\phi, H^0_\phi, h^0_\phi, S^0_\phi$ as T_ϕ, H_ϕ, h_ϕ and S_ϕ respectively for notational simplicity. For $A \in \mathcal{L}(L^2_a(\Omega, \frac{dV}{C_0}))$, let $\widetilde{A}(z) = \langle Ak_z, k_z\rangle_{L^2_a(\Omega, \frac{dV}{C_0})}$ for $z \in \Omega$, the Berezin symbol of A. That is, $\widetilde{A_0}(z) = \widetilde{A}(z)$ for all $z \in \Omega$. Here $k_z(w) = \frac{K(w,z)}{\sqrt{K(z,z)}}$, where $K(z,w) = \overline{K_z(w)}$ is the reproducing kernel of $L^2_a(\Omega, \frac{dV(z)}{C_0})$. Let $\widetilde{\phi}(z) = \langle T_\phi k_z, k_z\rangle_{L^2_a(\Omega, \frac{dV}{C_0})}$ where T_ϕ is the Toeplitz operator with symbol $\phi \in L^\infty(\Omega)$ on $L^2_a(\Omega, \frac{dV}{C_0})$. Let $S^0_p = S_p$, the Schatten p-class in $\mathcal{L}(L^2_a(\Omega, \frac{dV}{C_0}))$. In this section we show that if $1 \le p < \infty$, $d\eta(z) = K(z,z)\frac{dV(z)}{C_0}$ and $T_\phi \in S_p$ then $\widetilde{\phi} \in L^p(\Omega, d\eta)$.

Given $z \in \Omega$ and f any measurable function on Ω, we define a function $U_z f$ on Ω by $U_z f(w) = k_z(w)f(\phi_z(w))$. Since $|k_z|^2$ is real Jacobian determinant of the mapping ϕ_z (see [4]), U_z is easily seen to be a unitary operator on $L^2(\Omega, \frac{dV}{C_0})$ and $L^2_a(\Omega, \frac{dV}{C_0})$. It is to check that $U_z^* = U_z$, thus U_z is a self-adjoint unitary operator. This implies that spectrum $\sigma(U_z) = \{-1, 1\}$. We can check easily that $U_z \neq \pm I$. If $\phi \in L^\infty(\Omega, \frac{dV}{C_0})$ and $z \in \Omega$ then $U_z T_\phi = T_{\phi\circ\phi_z} U_z$. This is so as $PU_z = U_z P$ and for $f \in L^2_a(\Omega, \frac{dV}{C_0})$,

$$\begin{aligned} T_{\phi\circ\phi_z} U_z f &= T_{\phi\circ\phi_z}((f \circ \phi_z)k_z) \\ &= P((\phi \circ \phi_z)(f \circ \phi_z)k_z) = P(U_z(\phi f)) \\ &= U_z P(\phi f) = U_z T_\phi f. \end{aligned}$$

Theorem 4.1. *Suppose* $1 \le p < \infty$ *and* $d\eta(z) = K(z, z)\frac{dV(z)}{C_0}$. *If* $T_\phi \in S_p$ *then* $\widetilde{\phi} \in L^p(\Omega, d\eta)$.

Proof. Suppose $T_\phi \in S_p$. Then $\int_\Omega \langle |T_\phi|^p k_w, k_w\rangle_{L^2_a(\Omega, \frac{dV}{C_0})} d\eta(w) < \infty$. (Henceforth in the proof the inner product and norm is evaluated in the space $L^2(\Omega, \frac{dV}{C_0})$.) That is, $\int_\Omega \left\langle \left(T^*_\phi T_\phi\right)^{p/2} k_w, k_w \right\rangle d\eta(w) < \infty$. If $2 \le p < \infty$, then

$$\int_\Omega \langle T^*_\phi T_\phi k_w, k_w\rangle^{p/2} d\eta(w) \le \int_\Omega \left\langle \left(T^*_\phi T_\phi\right)^{p/2} k_w, k_w \right\rangle d\eta(w) < \infty.$$

This implies

$$\begin{aligned} \int_\Omega \|P(\phi \circ \phi_w)\|^p d\eta(w) &= \int_\Omega \|P(U_w(\phi k_w))\|^p d\eta(w) \\ &= \int_\Omega \|U_w T_\phi k_w\|^p d\eta(w) = \int_\Omega \|T_\phi k_w\|^p d\eta(w) \\ &= \int_\Omega \langle T^*_\phi T_\phi k_w, k_w\rangle^{p/2} d\eta(w) < \infty. \end{aligned}$$

Now

$$\begin{aligned} \|P(\phi \circ \phi_w)(0)\| &= |\langle P(\phi \circ \phi_w), 1\rangle| = |\langle U_w(T_\phi k_w), 1\rangle| \\ &= |\langle T_\phi k_w, U_w 1\rangle| = |\langle T_\phi k_w, k_w\rangle| \\ &\le \|T_\phi k_w\| = \|P(\phi \circ \phi_w)\|. \end{aligned}$$

Thus $\int_\Omega |P(\phi \circ \phi_w(0))|^p d\eta(w) < \infty$. That is, $\int_\Omega \left|\widetilde{\phi}(w)\right|^p d\eta(w) < \infty$ and $\widetilde{\phi} \in L^p(\Omega, d\eta)$. Suppose $1 \le p < 2$. Then by Heinz inequality [11], [9] it follows that

$$\begin{aligned} \infty > \int_\Omega \langle |T_\phi|^p k_w, k_w\rangle d\eta(w) &= \int_\Omega \left\langle |T_\phi|^{2(\frac{p}{2})} k_w, k_w\right\rangle d\eta(w) \\ \ge \int_\Omega \frac{|\langle T_\phi k_w, k_w\rangle|^2}{\left\langle \left|T^*_\phi\right|^{2(1-p/2)} k_w, k_w\right\rangle} d\eta(w) &= \int_\Omega \frac{\left|\widetilde{\phi}(w)\right|^2}{\left\|P(\overline{\phi} \circ \phi_w)\right\|^{2-p}} d\eta(w) \\ = \int_\Omega \left|\widetilde{\phi}(w)\right|^2 \left\|P\left(\overline{\phi} \circ \phi_w\right)\right\|^{p-2} d\eta(w) &\ge \int_\Omega \frac{\left|\widetilde{\phi}(w)\right|^2}{\left\|P\left(\overline{\phi} \circ \phi_w\right)\right\|^2} \left\|P\left(\overline{\phi} \circ \phi_w\right)\right\|^p d\eta(w) \\ \ge \int_\Omega \frac{\left|\widetilde{\phi}(w)\right|^2}{C^2 \|\phi\|^2_{BT}} |P(\phi \circ \phi_w)(0)|^p d\eta(w) &= \int_\Omega \frac{\left|\widetilde{\phi}(w)\right|^2}{C^2 \|\phi\|^2_{BT}} \left|\widetilde{\phi}(w)\right|^p d\eta(w) \end{aligned}$$

since

$$\begin{aligned} \left\langle \left|T^*_\phi\right|^{2-p} k_w, k_w\right\rangle &= \left\langle \left|T^*_\phi\right|^{2(\frac{2-p}{2})} k_w, k_w\right\rangle \le \left\langle \left|T^*_\phi\right|^2 k_w, k_w\right\rangle^{\frac{2-p}{2}} \\ &= \left\langle T_\phi T^*_\phi k_w, k_w\right\rangle^{\frac{2-p}{2}} = \left\|T^*_\phi k_w\right\|^{2-p} = \left\|P\left(\overline{\phi} \circ \phi_w\right)\right\|^{2-p}. \end{aligned}$$

Hence $\int_\Omega \left|\widetilde{\phi}(w)\right|^{p+2} d\eta(w) < \infty$ and therefore $\int_\Omega \left|\widetilde{\phi}(w)\right|^p d\eta(w) < \infty$. Thus $\widetilde{\phi} \in L^p(\Omega, d\eta)$.

□

Let $\pi : \mathcal{L}(L^2_a(\Omega)) \longrightarrow \mathcal{L}(L^2_a(\Omega))/\mathcal{LC}(L^2_a(\Omega))$ be the natural surjection onto the Calkin algebra $\mathcal{L}(L^2_a(\Omega))/\mathcal{LC}(L^2_a(\Omega))$.

Corollary 4.2. *Suppose $1 \leq p \leq \infty, I - T^*_\phi T_\phi \in S_p$ and $\sigma(T_\phi)$ does not fill $\mathbb{D}$. Then $\phi \notin L^p(\Omega, d\eta)$ and $T_\phi = W + R$ where W is unitary, $R \in S_p$. In addition if $T^{-1}_\phi \in \mathcal{L}(L^2_a(\Omega))$ and $\lambda \in \sigma(T_\phi)$ with $|\lambda| \neq 1$ then λ is an isolated eigenvalue of T_ϕ.*

Proof. Suppose $I - T^*_\phi T_\phi \in S_p$ and $\phi \in L^p(\Omega, d\eta)$. Then by Lemma 2.3, $T_\phi \in S_p$ and therefore $I \in S_p$. But this is not true. Thus $\phi \notin L^p(\Omega, d\eta)$. Now since $I - T^*_\phi T_\phi \in S_p$, hence $\pi(T_\phi)$ is an isometry and further since $\sigma(T_\phi)$ does not fill $\mathbb{D}$, hence $\pi(T_\phi)$ is unitary. By [6], $T_\phi = U + K$ where $K \in \mathcal{LC}(L^2_a(\Omega))$ and U is unitary or a shift or the adjoint of a shift. As $\sigma(T_\phi)$ does not fill $\mathbb{D}$, hence the operator U is unitary. Thus the Fredholm index of $T_\phi = \operatorname{ind}(T_\phi) = \dim\ker T_\phi - \dim\ker T^*_\phi = 0$ and $T_\phi = VS$ where V is unitary and $S^2 = T^*_\phi T_\phi$. From the hypothesis $I - T^*_\phi T_\phi \in S_p$ it follows that $I - S \in S_p$. Hence $T_\phi = VS = V - V(I - S) = V + R$ where V is unitary and $R = -V(I - S) \in S_p$. Now suppose $\lambda \in \sigma(T_\phi)$ but $|\lambda| \neq 1, T^{-1}_\phi \in \mathcal{L}(L^2_a(\Omega))$ and $I - T^*_\phi T_\phi \in S_p$. As $T_\phi = V + R$, we have $I = T^{-1}_\phi T_\phi = T^{-1}_\phi V + T^{-1}_\phi R$. Therefore, $V^* = T^{-1}_\phi V V^* + T^{-1}_\phi R V^*$ where $R \in S_p$. That is, $T^{-1}_\phi = V^* - T^{-1}_\phi R V^*$ where $R \in S_p$. By [18], each $\lambda \in \sigma(T_\phi)$ with $|\lambda| > 1$ is an isolated eigenvalue and $\sigma(T_\phi) \bigcap \mathbb{D}$ is either $\mathbb{D}$ or a countable set of isolated eigenvalues of T_ϕ. Hence each $\lambda \in \sigma(T_\phi) \bigcap \mathbb{D}$ is also an isolated eigenvalue of T_ϕ. □

Corollary 4.3. *Suppose $\phi \geq 0$ and there exists $z \in \Omega$ such that $T_\phi - U_z \in S_p$, $1 \leq p < \infty$. If $\lambda \in \sigma(T_\phi)$ and $\lambda \neq \pm 1$ then λ is an isolated eigenvalue of T_ϕ with finite multiplicity.*

Proof. The operator U_z is unitary and $\sigma(U_z) = \{-1, 1\}$. For proof see [21]. Since $\phi \geq 0$, hence T_ϕ is positive and therefore a normal operator. Notice that T_ϕ is a compact perturbation of U_z. According to Weyl's theorem for normal operators, T_ϕ and U_z have same Weyl spectrum [2]. For the normal operator T_ϕ the Weyl spectrum coincides with the points of $\sigma(T_\phi)$ which are not isolated eigenvalues with finite multiplicity [2]. The operators for which the above set coincides with the Weyl spectrum are characterized in [20]. Since the Weyl spectrum of U_z and, hence the Weyl spectrum of T_ϕ is contained in $\sigma(U_z) = \{-1, 1\}$, the conclusion of the corollary follows. □

Lemma 4.4. *If $\{A_n\}, \{B_n\}$ are sequences in S^λ_p and $A_n \xrightarrow{w} A$ and $B_n \xrightarrow{s} B$ then $A_n B_n \xrightarrow{w} AB$.*

Proof. Fix $f, g \in L^2_a(\Omega, dV_\lambda)$. Then

$$\langle A_n B_n f, g\rangle_\lambda = \langle A_n (B_n - B) f, g\rangle_\lambda + \langle A_n B f, g\rangle_\lambda .$$

Since $\langle A_n B f, g\rangle_\lambda \longrightarrow \langle A B f, g\rangle_\lambda$ and $|\langle A_n (B_n - B) f, g\rangle_\lambda| \leq M \|(B_n - B) f\| \|g\|$, where $M = \sup_n \{\|A_n\|\} < \infty$, by the uniform boundedness principle, we obtain that $\langle A_n B_n f, g\rangle_\lambda \longrightarrow \langle A B f, g\rangle_\lambda$. □

Lemma 4.5. *Let $\mathcal{L}$ denote either the space of all operators on $L^2_a(\Omega, dV_\lambda)$, with the weak operator topology, or any of the Banach spaces S^λ_p $(1 < p < \infty)$ with its weak topology. If $\{A_n\}, \{B_n\} \subset \mathcal{L}$, with $A_n \longrightarrow A$ and $B_n \longrightarrow B$ weakly, and if each B_n has the upper triangular form, then $A_n B_n \longrightarrow AB$ weakly.*

Proof. We denote the matrices of the operators $A_n, B_n, A_n B_n$ and AB as $(\widehat{A_n}(i, j))$, $(\widehat{B_n}(i, j))$, $(d_n(i, j))$ and $(d(i, j))$ respectively. One verifies that if $\{A_n\} \subset \mathcal{L}$ then $A_n \xrightarrow{w} A$ if and only if $\{\|A_n\|_{\mathcal{L}}\}$ is a bounded sequence, and $\widehat{A_n}(i, j) \longrightarrow \widehat{A}(i, j)$ for all i, j. Thus to complete the proof we have to show that $\|A_n B_n\|_{\mathcal{L}}$ are bounded and $d_n(i, j) \longrightarrow d(i, j)$ for all i, j. We recall that in S_p we have

$$\|A_n B_n\|_p \leq \|A_n B_n\|_{p/2} \leq \|A_n\|_p \|B_n\|_p .$$

Thus $\{\|A_nB_n\|_{\mathcal{L}}\}$ is a bounded sequence. Further since each B_n is upper triangular,

$$d_n(i,j) = \sum_{k=1}^{j} \widehat{A_n}(i,k)\widehat{B_n}(k,j)$$

and

$$d(i,j) = \sum_{k=1}^{j} \widehat{A}(i,k)\widehat{B}(k,j),$$

where $(\widehat{A}(i,j))$ and $(\widehat{B}(i,j))$ denote the matrices of A and B respectively. Thus, for each fixed choice of i, j, $d_n(i,j) \longrightarrow d(i,j)$. $\square$

Lemma 4.6. *Let $p \geq 1$, $A \in \mathcal{L}(L^2_a(\Omega, dV_\lambda))$ and $A_n \in S^\lambda_p$ for all $n \in \mathbb{N}$. If $A_n \longrightarrow A$ in weak operator topology and $\|A_n\|_p \leq C < \infty$ for all $n \in \mathbb{N}$ and for some constant $C > 0$ then $A \in S^\lambda_p$ and $\|A\|_p \leq C$.*

Proof. For each $n \in \mathbb{N}$, define

$$\xi_n(K) = \operatorname{tr}(A_nK).$$

Then $\xi_n \in S^*_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\xi_n\| = \|A_n\|_p \leq C < \infty$. By Banach-Alaoglu's theorem [8], there exists a subsequence $\{\xi_{n_k}\}$ such that $\xi_{n_k} \longrightarrow \xi$ in w^*-topology and $\xi \in (S^\lambda_q)^*$. Therefore $\operatorname{tr}(A_{n_k}K) = \xi_{n_k}(K) \longrightarrow \xi(K)$, for all $K \in S^\lambda_q$ and $|\xi(K)| \leq M\|K\|_q$, for some constant $M > 0$. On the other hand, since $A_n \longrightarrow A$ in weak operator topology, $\operatorname{tr}(A_nK) \longrightarrow \operatorname{tr}(AK)$ for all operators K of finite rank. The lemma follows since

$$\|A\|_p = \sup\{|\operatorname{tr}(AK)| : \operatorname{rank}(K) < \infty \text{ and } \|K\|_q \leq 1\} < \infty.$$

$\square$

5. Schatten class little Hankel operators

In this section we find conditions on $\phi \in L^2(\Omega, \frac{dV}{C_0})$ such that the little Hankel operator $S_{\overline{\phi}}$ defined on $L^2_a(\Omega, \frac{dV}{C_0})$ belong to the class $S_p, 1 \leq p < \infty$. We then extend the result to obtain Schatten class characterization of little Hankel operators defined on $L^2_a(\Omega, dV_\lambda)$. We also present many applications of these characterizations. Recall that for $\phi \in L^\infty(dV)$, we define the little Hankel operator S_ϕ from $L^2_a(\Omega, \frac{dV}{C_0})$ into $L^2_a(\Omega, \frac{dV}{C_0})$ as $S_\phi f = P(J(\phi f))$ where $J : L^2(\Omega, \frac{dV}{C_0}) \longrightarrow L^2(\Omega, \frac{dV}{C_0})$ is defined as $Jf(z) = f(\overline{z})$ and P is the orthogonal projection from $L^2(\Omega, \frac{dV}{C_0})$ onto $L^2_a(\Omega, \frac{dV}{C_0})$ and

$$Pf(z) = \int_\Omega K(z,w)f(w)\, dV(w).$$

The above integral formula extends P to $L^1(\Omega, \frac{dV}{C_0})$. The little Hankel operator S_ϕ can also be defined for $\phi \in L^2(\Omega, \frac{dV}{C_0})$ as $S_\phi f = P(J(\phi f))$ for $f \in L^2_a(\Omega, \frac{dV}{C_0})$. Notice that if $\phi \in L^2(\Omega, \frac{dV}{C_0})$, then $S_{\overline{\phi}} = S_{\overline{P\phi}}$ in the sense that $S_{\overline{\phi}}g = S_{\overline{P\phi}}g$ for all $g \in H^\infty(\Omega)$ which is dense in $L^2_a(\Omega, \frac{dV}{C_0})$. Let $\overline{P}$ be the orthogonal projection from $L^2(\Omega, \frac{dV}{C_0})$ onto $\overline{L^2_a(\Omega, \frac{dV}{C_0})} = \left\{\overline{f} : f \in L^2_a(\Omega, \frac{dV}{C_0})\right\}$. Then

$$\overline{P}f(z) = \int_\Omega \overline{K(z,w)}f(w)\frac{dV(w)}{C_0} = \int_\Omega K(w,z)f(w)\frac{dV(w)}{C_0}.$$

This formula also extends $\overline{P}$ to $L^1(\Omega, \frac{dV}{C_0})$. Given $\phi \in L^2(\Omega, \frac{dV}{C_0})$, define the operators H_ϕ and h_ϕ with domain $L^2_a(\Omega, \frac{dV}{C_0})$ as follows : $H_\phi f = (I-P)(\phi f); h_\phi f = \overline{P}(\phi f)$, where I is the identity operator. The operator H_ϕ is called the Hankel operator with symbol ϕ and h_ϕ is called the reduced (or little) Hankel operator with symbol ϕ. The word "reduced" (or little) is justified by the inequality $\overline{P} - P_0 \le I - P$, where P_0 is the orthogonal projection of rank one from $L^2(\Omega, \frac{dV}{C_0})$ onto the constants, that is,

$$P_0 f(z) = \int_\Omega f(z) \frac{dV(z)}{C_0}.$$

We refer both the operators S_ϕ and h_ϕ as "little" Hankel operators since $JS_\phi = h_\phi$ and J is a unitary operator.

The main purpose of this section is to demonstrate that there exists an integral transform on Ω which carries a lot of information on the little Hankel operators. We give a unified treatment on the size estimates of S_ϕ using the integral transform W defined as follows. Given $f \in L^1(\Omega, \frac{dV}{C_0}), Wf$ is the function on Ω defined by

$$Wf(z) = \lambda_\Omega \int_\Omega f(w) \overline{k_z^2(w)} \frac{dV(w)}{C_0}, \quad z \in \Omega,$$

where $\lambda_\Omega^{-1} = \int_\Omega K(z,z)^{-1} \frac{dV(z)}{C_0}$. Notice that for $\phi \in L^2(\Omega, \frac{dV}{C_0})$, we always have $S_{\overline{\phi}} = S_{\overline{P\phi}}$, where P is the Bergman projection. Thus in considering little Hankel operators, we can content ourselves with antiholomorphic symbols. We collect here some of the basic properties of the integral transform W as follows. If $f \in L^2(\Omega, \frac{dV}{C_0})$, then

(i) $PWf = Pf$;
(ii) $WPf = Wf$;
(iii) $W^2 f = Wf$;
(iv) W is a bounded operator on $L^p\big(\Omega, K(z,z)\frac{dV(z)}{C_0}\big)$ for all $1 \le p \le +\infty$ and W is an orthogonal projection on the Hilbert space $L^2\big(\Omega, K(z,z)\frac{dV(z)}{C_0}\big)$.

The boundedness of W on $L^p(\Omega, K(z,z)\frac{dV(z)}{C_0}), 1 \le p \le +\infty$ implies that

$$\int_\Omega (Wf)(z)\overline{g(z)} K(z,z) \frac{dV(z)}{C_0} = \int_\Omega f(z) \overline{Wg(z)} K(z,z) \frac{dV(z)}{C_0}$$

for all $f \in L^p(\Omega, K(z,z)\frac{dV(z)}{C_0}); g \in L^q(\Omega, K(z,z)\frac{dV(z)}{C_0})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Under the usual integral pairing $\langle , \rangle$ (with respect to $\frac{dV}{C_0}$), we have

$$W^* f(z) = \lambda_\Omega \int_\Omega \frac{K(z,w)^2}{K(w,w)} f(w) \frac{dV(w)}{C_0} = Qf(z),$$

where Q is a bounded projection from $L^1(\Omega, \frac{dV}{C_0})$ onto $L^1_a(\Omega, \frac{dV}{C_0})$. It is also not difficult to check that (i)$S_{\overline{\phi}} = S_{\overline{P\phi}}$ (ii)$S_{\overline{\phi}} = S_{\overline{W\phi}}$ and (iii)$\overline{W\phi(z)} = \lambda_\Omega \langle S_{\overline{\phi}} k_z, \overline{k_z}\rangle$. We verify now that if $\phi \in L^2(\Omega, \frac{dV}{C_0})$ then $S_{\overline{\phi}}$ is bounded if and only if $W\phi(z)$ is bounded in Ω. Since each k_z is a unit vector in $L^2(\Omega, \frac{dV}{C_0})$, we have for all $z \in \Omega$,

$$|W\phi(z)| = \lambda_\Omega \big|\big\langle S_{\overline{\phi}} k_z, \overline{k_z}\big\rangle\big| \le \lambda_\Omega \big\|S_{\overline{\phi}} k_z\big\| \le \lambda_\Omega \big\|S_{\overline{\phi}}\big\|.$$

Hence $\|W\phi\|_\infty \le \lambda_\Omega \|S_{\overline{\phi}}\|$. On the other hand, $S_{\overline{\phi}} = S_{\overline{P\phi}} = S_{\overline{PW\phi}} = S_{\overline{W\phi}}$. Thus $\|S_{\overline{\phi}}\| = \|S_{\overline{W\phi}}\|$. It is easy to see that $\|S_{\overline{\psi}}\| \le \|\psi\|_\infty$ for all $\psi \in L^\infty(\Omega)$. Hence we also have $\|S_{\overline{\phi}}\| \le \|W\phi\|_\infty$.

Theorem 5.1. *Suppose* $1 \le p \le +\infty$. *Then* $S_{\overline{\phi}} \in S_p$ *if and only if* $W\phi \in L^p\big(\Omega, K(z,z)\frac{dV(z)}{C_0}\big)$.

Proof. We shall first show that if $W\phi \in L^p(\Omega, K(z,z)\frac{dV(z)}{C_0})$ then $S_{\overline{\phi}} \in S_p$. We have already proved the case $p = \infty$. We need only to show for $1 \le p < \infty$. Since $S_{\overline{\phi}} = S_{\overline{W\phi}}$, it suffices to show that $S_{\overline{\phi}}$ is in S_p whenever $\phi \in L^p(\Omega, K(z,z)\frac{dV(z)}{C_0})$. From Heinz inequality [11], [9], it follows that

$$\begin{aligned}
\left|\left\langle S_{\overline{\phi}}k_z, k_w\right\rangle\right|^2 &\leq \left\langle |S_{\overline{\phi}}|k_z, k_z\right\rangle \left\langle |S^*_{\overline{\phi}}|k_w, k_w\right\rangle = \left\langle \left(S^*_{\overline{\phi}}S_{\overline{\phi}}\right)^{1/2} k_z, k_z\right\rangle \left\langle \left(S_{\overline{\phi}}S^*_{\overline{\phi}}\right)^{1/2} k_w, k_w\right\rangle \\
&\leq \left\langle \left(S^*_{\overline{\phi}}S_{\overline{\phi}}\right) k_z, k_z\right\rangle^{1/2} \left\langle \left(S_{\overline{\phi}}S^*_{\overline{\phi}}\right) k_w, k_w\right\rangle^{1/2} \\
&= \left\|S_{\overline{\phi}}k_z\right\|_{L^2(\Omega, \frac{dV}{C_0})} \left\|S_{\overline{\phi}^+}k_w\right\|_{L^2(\Omega, \frac{dV}{C_0})} \\
&= \left\|PJ(\overline{\phi}k_z)\right\|_{L^2(\Omega, \frac{dV}{C_0})} \left\|PJ(\overline{\phi}^+k_w)\right\|_{L^2(\Omega, \frac{dV}{C_0})} \\
&\leq \left\|\overline{\phi}k_z\right\|_{L^2(\Omega, \frac{dV}{C_0})} \left\|\overline{\phi}^+k_w\right\|_{L^2(\Omega, \frac{dV}{C_0})} \\
&= \Big(\int_\Omega |\phi(u)|^2|k_z(u)|^2\frac{dV(u)}{C_0}\Big)^{1/2} \Big(\int_\Omega |\overline{\phi}^+(v)|^2|k_w(v)|^2\frac{dV(v)}{C_0}\Big)^{1/2} \\
&= \left\langle T_{|\phi|^2}k_z, k_z\right\rangle^{1/2} \left\langle T_{|\phi^+|^2}k_w, k_w\right\rangle^{1/2} \\
&= \left\langle M_{|\phi|^2}k_z, k_z\right\rangle^{1/2} \left\langle M_{|\phi^+|^2}k_w, k_w\right\rangle^{1/2} \\
&= \left\langle M^2_{|\phi|}k_z, k_z\right\rangle^{1/2} \left\langle M^2_{|\phi^+|}k_w, k_w\right\rangle^{1/2} \\
&\leq d\left\langle M_{|\phi|}k_z, k_z\right\rangle \left\langle M_{|\phi^+|}k_w, k_w\right\rangle = d\left\langle T_{|\phi|}k_z, k_z\right\rangle \left\langle T_{|\phi^+|}k_w, k_w\right\rangle
\end{aligned}$$

for some constant $d \geq 0$. The last inequality follows from the Kantorvich inequality $\langle Ax, x\rangle^p \geq \langle A^p x, x\rangle \geq K(p)\langle Ax, x\rangle^p, p \in (0,1], \|x\| = 1$. Taking $p = \frac{1}{2}$, we have $\langle Ax, x\rangle^{\frac{1}{2}} \leq \frac{1}{K(\frac{1}{2})}\left\langle A^{\frac{1}{2}}x, x\right\rangle$ and $K(\frac{1}{2}) \in (0,1]$. Thus

$$\left|\left\langle S_{\overline{\phi}}K_z, K_w\right\rangle\right|^2 \leq d\left\langle T_{|\phi|}K_z, K_z\right\rangle \left\langle T_{|\phi^+|}K_w, K_w\right\rangle.$$

Now $\phi \in L^p(\Omega, K(z,z)dV(z))$ implies $|\phi|, |\phi^+| \in L^p(\Omega, K(z,z)dV(z))$. Hence $T_{|\phi|}, T_{|\phi^+|} \in S_p$. Hence by Theorem 3.1, $S_{\overline{\phi}} \in S_p$. Now we shall prove that if $1 \leq p \leq +\infty$, then $S_{\overline{\phi}} \in S_p$ implies $W\phi \in L^p(\Omega, K(z,z)dV(z))$. We have already settled the case $p = +\infty$. Now we assume $2 \leq p < \infty$ and $S_{\overline{\phi}} \in S_p$. Then

$$\begin{aligned}
&\int_\Omega |(W\phi)(z)|^p K(z,z)\,dV(z) = \int_\Omega \lambda^p_\Omega \left|\left\langle S_{\overline{\phi}}k_z, \overline{k_z}\right\rangle\right|^p K(z,z)\,dV(z) \\
&\leq \lambda^p_\Omega \int_\Omega \left\|S_{\overline{\phi}}k_z\right\|^p K(z,z)\,dV(z) = \lambda^p_\Omega \int_\Omega \left\langle S_{\overline{\phi}}k_z, S_{\overline{\phi}}k_z\right\rangle^{p/2} K(z,z)\,dV(z) \\
&= \lambda^p_\Omega \int_\Omega \left\langle S^*_{\overline{\phi}}S_{\overline{\phi}}k_z, k_z\right\rangle^{p/2} K(z,z)\,dV(z) \leq \lambda^p_\Omega \int_\Omega \left\langle (S^*_{\overline{\phi}}S_{\overline{\phi}})^{p/2}k_z, k_z\right\rangle K(z,z)\,dV(z) \\
&= \lambda^p_\Omega \int_\Omega \left\langle |S_{\overline{\phi}}|^p k_z, k_z\right\rangle K(z,z)\,dV(z).
\end{aligned}$$

Thus $\|W\phi\|_{L^p(\Omega, K(z,z)dV(z))} \leq \lambda_\Omega \left(\int_\Omega \left\langle |S_{\overline{\phi}}|^p k_z, k_z\right\rangle K(z,z)\,dV(z)\right)^{1/p} < \infty$ as $S_{\overline{\phi}} \in S_p$. Hence $W\phi \in L^p(\Omega, K(z,z)dV(z))$.

The proof for $1 \leq p < 2$ is very tricky. Fix a sequence of points $\{a_n\}$ in Ω such that (1) $\Omega = \bigcup_{n=1}^\infty E(a_n, r)$, where $E(a_n, r)$ is the Bergman metric ball with center at a_n and radius r, a fixed positive number;
(2) There exists a constant $C > 0$ such that every function $f \in L^2_a(\Omega, dV(z))$ can be written as $f(z) = \sum_{n=1}^\infty c_n k_{a_n}(z)$ with $\|f\|_2 \leq C \inf\left\{\sqrt{\sum_{n=1}^\infty |c_n|^2} : f = \sum_{n=1}^\infty c_n k_{a_n}\right\}$.

One can refer [7] for the construction of such a sequence $\{a_n\}$. Define an operator A on $L^2_a(\Omega, dV(z))$ by letting $Ae_n = k_{a_n}, n = 1, 2, \ldots$, where $\{e_n\}_{n=1}^\infty$ is a fixed orthonormal basis of $L^2_a(\Omega, dV(z))$. If $f \in L^2_a(\Omega, dV(z))$ with $f = \sum_{n=1}^\infty f_n e_n$, then $Af = \sum_{n=1}^\infty f_n k_{a_n}$ and by (2) above,

$$\|Af\| \le C \inf \left\{ \sqrt{\sum_{n=1}^\infty |c_n|^2} : Af = \sum_{n=1}^\infty c_n k_{a_n} \right\} \le C \sqrt{\sum_{n=1}^\infty |f_n|^2} = C\|f\|.$$

Thus A is a bounded linear operator. Let $\overline{A}$ be the operator on $\overline{L^2_a(\Omega, dV(z))}$ defined by $\overline{A}\overline{e_n} = \overline{k_{a_n}}$; then $\overline{A}$ is also bounded. Suppose $S_{\overline{\phi}} \in S_p$ with $1 \le p < 2$. Then we also have $\overline{A}^* S_{\overline{\phi}} A \in S_p$. This implies

$$\sum_{n=1}^\infty \left| \left\langle \overline{A}^* S_{\overline{\phi}} A e_n, \overline{e_n} \right\rangle \right|^p < +\infty \quad \text{or} \quad \sum_{n=1}^\infty \left| \left\langle S_{\overline{\phi}} k_{a_n}, \overline{k_{a_n}} \right\rangle \right|^p < +\infty.$$

That is, $\sum_{n=1}^\infty |W\phi(a_n)|^p < +\infty$. It is not difficult to show that [23], $W\phi(z)$ behaves like $W\phi(a_n)$ for $z \in E(a_n, r)$. Also [23], the Bergman kernel $K(z,z)$ behaves like $K(a_n, a_n) \cong \frac{1}{E(a_n,r)}$ for $z \in E(a_n, r)$. It thus follows that

$$\begin{aligned}
\int_\Omega |W\phi(z)|^p K(z,z)\, dV(z) &\le \sum_{n=1}^\infty \int_{E(a_n,r)} |W\phi(z)|^p K(z,z)\, dV(z) \\
&\le C_1 \sum_{n=1}^\infty \frac{1}{|E(a_n,r)|} \int_{E(a_n,r)} |W\phi(z)|^p dV(z) \\
&\le C_2 \sum_{n=1}^\infty \frac{1}{|E(a_n,r)|} \int_{E(a_n,r)} |W\phi(a_n)|^p dV(z) = C_2 \sum_{n=1}^\infty |W\phi(a_n)|^p < \infty
\end{aligned}$$

and $W\phi \in L^p\big(\Omega, K(z,z) dV(z)\big)$. This completes the proof. $\square$

Corollary 5.2. *If $1 \le p \le \infty$ and $\phi \in L^p(\Omega, d\eta_\lambda)$ then $S^\lambda_{\overline{\phi}} \in S^\lambda_p$.*

Proof. Suppose $\phi \in L^p(\Omega, d\eta_\lambda)$ and $1 \le p < \infty$. From Heinz inequality [9], [11], it follows that

$$\begin{aligned}
\left| \left\langle S^\lambda_{\overline{\phi}} k_z^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda \right|^2 &\le \left\langle |S^\lambda_{\overline{\phi}}| k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda \left\langle |(S^\lambda_{\overline{\phi}})^*| k_w^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda \\
&= \left\langle \left(\left(S^\lambda_{\overline{\phi}} \right)^* S^\lambda_{\overline{\phi}} \right)^{1/2} k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda^{1/2} \left\langle \left(S^\lambda_{\overline{\phi}} \left(S^\lambda_{\overline{\phi}} \right)^* \right) k_w^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda^{1/2} \\
&= \left\| S^\lambda_{\overline{\phi}} k_z^{1-\lambda} \right\|_{L^2(\Omega, dV_\lambda)} \left\| S^\lambda_{\overline{\phi}^+} k_w^{1-\lambda} \right\|_{L^2(\Omega, dV_\lambda)} \\
&= \left\| P_\lambda J_\lambda (\overline{\phi} k_z^{1-\lambda}) \right\|_{L^2(\Omega, dV_\lambda)} \left\| P_\lambda J_\lambda (\overline{\phi}^+ k_w^{1-\lambda}) \right\|_{L^2(\Omega, dV_\lambda)} \\
&\le \left\| \overline{\phi} k_z^{1-\lambda} \right\|_{L^2(\Omega, dV_\lambda)} \left\| \overline{\phi}^+ k_w^{1-\lambda} \right\|_{L^2(\Omega, dV_\lambda)} \\
&= \left(\int_\Omega |\phi(u)|^2 |k_z^{1-\lambda}(u)|^2 dV_\lambda \right)^{1/2} \left(\int_\Omega |\overline{\phi}^+(v)|^2 |k_w^{1-\lambda}(v)|^2 dV_\lambda \right)^{1/2} \\
&= \left\langle T^\lambda_{|\phi|^2} k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda^{1/2} \left\langle T^\lambda_{|\phi^+|^2} k_w^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda^{1/2} \\
&= \left\langle M^\lambda_{|\phi|^2} k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda^{1/2} \left\langle M^\lambda_{|\phi^+|^2} k_w^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda^{1/2} \\
&= \left\langle \left(M^\lambda_{|\phi|} \right)^2 k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda^{1/2} \left\langle \left(M^\lambda_{|\phi^+|} \right)^2 k_w^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda^{1/2}
\end{aligned}$$

$$\leq d_\lambda \left\langle M^\lambda_{|\phi|} k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda \left\langle M^\lambda_{|\phi^+|} k_w^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda$$
$$= d_\lambda \left\langle T^\lambda_{|\phi|} k_z^{1-\lambda}, k_z^{1-\lambda} \right\rangle_\lambda \left\langle T^\lambda_{|\phi^+|} k_w^{1-\lambda}, k_w^{1-\lambda} \right\rangle_\lambda$$

for some constant $d_\lambda \geq 0$. Here $\phi^+(z) = \overline{\phi(\overline{z})}$ and M^λ_ϕ denote the multiplication operator defined on $L^2_a(\Omega, dV_\lambda)$ with symbol $\phi \in L^\infty(\Omega)$. The last inequality follows from Kantorvich's inequality. Thus

$$\left| \left\langle S^\lambda_{\overline{\phi}} K_z^{1-\lambda}, K_w^{1-\lambda} \right\rangle_\lambda \right|^2 \leq d_\lambda \left\langle T^\lambda_{|\phi|} K_z^{1-\lambda}, K_z^{1-\lambda} \right\rangle_\lambda \left\langle T^\lambda_{|\phi^+|} K_w^{1-\lambda}, K_w^{1-\lambda} \right\rangle_\lambda .$$

Now since $\phi \in L^p(\Omega, d\eta_\lambda)$ we have $|\phi|, |\phi^+| \in L^p(\Omega, d\eta_\lambda)$. Hence from Lemma 2.3, it follows that $T^\lambda_{|\phi|}, T^\lambda_{|\phi^+|} \in S^\lambda_p$. From Theorem 3.1, $S\overline{\phi}^\lambda \in S^\lambda_p$. Now if $f \in L^2(\Omega, dV_\lambda)$,

$$\left\| S^\lambda_{\overline{\phi}} f \right\|_{L^2(\Omega, dV_\lambda)} = \left\| P_\lambda J_\lambda(\overline{\phi} f) \right\|_{L^2(\Omega, dV_\lambda)} \leq \|P_\lambda\| \, \|J_\lambda\| \, \left\|\overline{\phi}\right\|_{L^\infty(\Omega)} \|f\|_{L^2(\Omega, dV_\lambda)} .$$

Hence $\left\| S^\lambda_{\overline{\phi}} \right\| \leq \|\phi\|_{L^\infty(\Omega)}$. The corollary follows. □

Corollary 5.3. *Let $\phi \in L^p(\Omega, d\eta_\lambda), 1 < p < \infty$ and $\phi = \phi^+$ where $\phi^+(z) = \overline{\phi(\overline{z})}$. Then there exists an operator $S \in \mathcal{L}(L^2_a(\Omega, dV_\lambda))$ such that $T^\lambda_{|\phi|} S = S T^\lambda_{|\phi|}$ and $\|T^\lambda_{|\phi|} S\|_p \leq r(S) \|T^\lambda_{|\phi|}\|_p$ where $r(S)$ is the spectral radius of S .*

Proof. Since $\phi \in L^p(\Omega, d\eta_\lambda)$ and $\phi^+ = \phi$, hence from Lemma 2.3, Corollary 4.2 it follows that $T^\lambda_{|\phi|}$ and S^λ_ϕ are self-adjoint operators, $T^\lambda_{|\phi|} \in S^\lambda_p$ and $S^\lambda_\phi \in S^\lambda_p$. Let $\mathfrak{N}$ be the group of unitary operators on $L^2_a(\Omega, dV_\lambda)$. Let $\mathfrak{N}_A = \{UAU^* : U \in \mathfrak{N}\}$, the unitary orbit of an operator $A \in \mathcal{L}(L^2_a(\Omega, dV_\lambda))$. Define $f(X) = \left\| T^\lambda_{|\phi|} - X \right\|_p$ for all $X \in S^\lambda_p$. Then f attains its minimum at some $S \in S^\lambda_p$ on $\mathfrak{N}_{S_\phi} = \left\{ U S^\lambda_\phi U^* : U \in \mathfrak{N} \right\}$ and $T^\lambda_{|\phi|} S = S T^\lambda_{|\phi|}$. This follows from [5]. The operator S is self-adjoint. To prove the corollary we have to show that for any two orthonormal sequences $\left\{u^\lambda_n\right\}_{n=0}^\infty$ and $\left\{\sigma^\lambda_n\right\}_{n=0}^\infty$ in $L^2_a(\Omega, dV_\lambda)$,

$$\sum_{n=0}^\infty \left| \left\langle T^\lambda_{|\phi|} S u^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda \right|^p \leq r(S)^p \left\| T^\lambda_{|\phi|} \right\|_p^p .$$

Notice that since $T^\lambda_{|\phi|} S = S T^\lambda_{|\phi|}$ and $S = S^*$ we obtain

$$\begin{aligned}
\left| \left\langle T^\lambda_{|\phi|} S u^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda \right|^2 &= \left| \left\langle T^\lambda_{|\phi|} (S u^\lambda_n), \sigma^\lambda_n \right\rangle_\lambda \right|^2 \leq \left\langle T^\lambda_{|\phi|} (S u^\lambda_n), S u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} \sigma^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda \\
&= \left\langle S^* T^\lambda_{|\phi|} S u^\lambda_n, u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} \sigma^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda \\
&= \left\langle T^\lambda_{|\phi|} S^2 u^\lambda_n, u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} \sigma^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda .
\end{aligned}$$

Repeating this process we obtain

$$\begin{aligned}
&\left| \left\langle T^\lambda_{|\phi|} S u^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda \right|^{2^{m+1}} = \left(\left| \left\langle T^\lambda_{|\phi|} S u^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda \right|^{2^m} \right)^2 \\
&\leq \left[\left\langle T^\lambda_{|\phi|} S^{2^m} u^\lambda_n, u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} u^\lambda_n, u^\lambda_n \right\rangle_\lambda^{(2^{m-1})-1} \left\langle T^\lambda_{|\phi|} \sigma^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda^{2^{m-1}} \right]^2 \\
&\leq \left\langle T^\lambda_{|\phi|} S^{2^m} u^\lambda_n, S^{2^m} u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} u^\lambda_n, u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} u^\lambda_n, u^\lambda_n \right\rangle_\lambda^{2^m - 2} \left\langle T^\lambda_{|\phi|} \sigma^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda^{2^m} \\
&= \left\langle {S^*}^{2^m} T^\lambda_{|\phi|} S^{2^m} u^\lambda_n, u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} u^\lambda_n, u^\lambda_n \right\rangle_\lambda^{2^m - 1} \left\langle T^\lambda_{|\phi|} \sigma^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda^{2^m} \\
&= \left\langle T^\lambda_{|\phi|} S^{2^{m+1}} u^\lambda_n, u^\lambda_n \right\rangle_\lambda \left\langle T^\lambda_{|\phi|} u^\lambda_n, u^\lambda_n \right\rangle_\lambda^{2^m - 1} \left\langle T^\lambda_{|\phi|} \sigma^\lambda_n, \sigma^\lambda_n \right\rangle_\lambda^{2^m} .
\end{aligned}$$

Thus

$$\left|\left\langle T^\lambda_{|\phi|}Su^\lambda_n,\sigma^\lambda_n\right\rangle_\lambda\right|^{2^m}\le\|T^\lambda_{|\phi|}\|\|S^{2^m}\|\|u^\lambda_n\|^2\left\langle T^\lambda_{|\phi|}u^\lambda_n,u^\lambda_n\right\rangle_\lambda^{(2^{m-1})-1}\left\langle T^\lambda_{|\phi|}\sigma^\lambda_n,\sigma^\lambda_n\right\rangle_\lambda^{2^{m-1}}$$

and

$$\left|\left\langle T^\lambda_{|\phi|}Su^\lambda_n,\sigma^\lambda_n\right\rangle_\lambda\right|\le\|T^\lambda_{|\phi|}\|^{\frac{1}{2^m}}\|S^{2^m}\|^{\frac{1}{2^m}}\|u^\lambda_n\|^{\frac{2}{2^m}}\left\langle T^\lambda_{|\phi|}u^\lambda_n,u^\lambda_n\right\rangle_\lambda^{\frac12-\frac{1}{2^m}}\left\langle T^\lambda_{|\phi|}\sigma^\lambda_n,\sigma^\lambda_n\right\rangle_\lambda^{\frac12}.$$

Letting $m\longrightarrow\infty$, we obtain

$$\left|\left\langle T^\lambda_{|\phi|}Su^\lambda_n,\sigma^\lambda_n\right\rangle_\lambda\right|^2\le[r(S)]^2\left\langle T^\lambda_{|\phi|}u^\lambda_n,u^\lambda_n\right\rangle_\lambda\left\langle T^\lambda_{|\phi|}\sigma^\lambda_n,\sigma^\lambda_n\right\rangle_\lambda.$$

Hence proceeding as in Theorem 3.1 and Corollary 3.4, one can show that

$$\left\|T^\lambda_{|\phi|}S\right\|_p\le r(S)\left\|T^\lambda_{|\phi|}\right\|_p.$$

□

Let $\mathcal{B}$ denote the unit ball in n-dimensional complex space $\mathbb{C}^n$ and dz be the normalized Lebesgue volume measure on $\mathcal{B}$. The Bergman space $L^2_a(\mathcal{B},dz)$ is the space of analytic functions h on $\mathcal{B}$ which are square-integrable with respect to Lebesgue volume measure. For $z=(z_1,\cdots,z_n)\in\mathbb{C}^n$, let $\langle z,w\rangle=\sum_{i=1}^n z_i\overline{w_i}$ and $\|z\|^2=\langle z,z\rangle$. For $z\in\mathcal{B}$, let P_z be the orthogonal projection of $\mathbb{C}^n$ onto the subspace $[z]$ generated by z and let $Q_z=I-P_z$. Then

$$\phi_z(w)=\frac{z-P_z(w)-(1-\|z\|^2)^{\frac12}Q_z(w)}{1-\langle w,z\rangle}$$

is the automorphism of $\mathcal{B}$ that interchanges 0 and z. The reproducing kernel in $L^2_a(\mathcal{B},dz)$ is given by

$$K^{\mathcal{B}}_z(w)=\frac{1}{(1-\langle w,z\rangle)^{n+1}}$$

for $z,w\in\mathcal{B}$ and the normalized reproducing kernel $k^{\mathcal{B}}_z$ is $\frac{K^{\mathcal{B}}_z(w)}{\|K^{\mathcal{B}}_z(\cdot)\|_2}$.

Given $\phi\in L^\infty(\mathcal{B})$, the Toeplitz operator T_ϕ is defined on $L^2_a(\mathcal{B},dz)$ by $T_\phi f=P_{\mathcal{B}}(\phi f)$ where $P_{\mathcal{B}}$ denotes the orthogonal projection of $L^2(\mathcal{B},dz)$ onto $L^2_a(\mathcal{B},dz)$ and the little Hankel operator S_ϕ from $L^2_a(\mathcal{B},dz)$ into $L^2_a(\mathcal{B},dz)$ is defined as $S_\phi f=P_{\mathcal{B}}(J_{\mathcal{B}}(\phi f))$ where $J_{\mathcal{B}}:L^2(\mathcal{B},dz)\longrightarrow L^2(\mathcal{B},dz)$ is defined as $J_{\mathcal{B}}f(z_1,\ldots,z_n)=f(\overline{z_1},\ldots,\overline{z_n})$. We have used the same notation T_ϕ,S_ϕ to denote Toeplitz operators and little Hankel operators defined on $L^2_a(\Omega,\frac{dV}{C_0})$ and $L^2_a(\mathcal{B},dz)$. The context will make it clear on which space we considering these operators. For $z\in\mathcal{B}$ and a non-negative integer m, let

$$K^{\mathcal{B},m}_z(u)=\frac{1}{(1-\langle u,z\rangle)^{m+n+1}},\quad u\in\mathcal{B}$$

and define the m-Berezin transform of an operator $S\in\mathcal{L}(L^2_a(\mathcal{B},dz))$ by

$$B_mS(z)=\binom{m+n}{n}\left(1-\|z\|^2\right)^{m+n+1}\sum_{|k|=0}^m C_{m,k}\left\langle S(u^kK^{\mathcal{B},m}_z),u^kK^{\mathcal{B},m}_z\right\rangle,$$

where

$$C_{m,k}=\binom{m}{|k|}(-1)^{|k|}\frac{|k|!}{k_1!\cdots k_n!},\quad u\in\mathcal{B},$$

$k=(k_1,\ldots,k_n)\in\mathbb{Z}^n_+$, where $\mathbb{Z}_+$ is the set of non-negative integers, $|k|=\sum_{i=0}^n k_i$, $u^k=u_1^{k_1}\cdots u_n^{k_n}$, $k!=k_1!\cdots k_n!$. Clearly, $B_m:\mathcal{L}(L^2_a(\mathcal{B},dz))\longrightarrow L^\infty(\mathcal{B})$ is a bounded linear operator and for $\phi\in L^\infty(\mathcal{B})$, define $B_m(\phi)(z)=B_m(T_\phi)(z)$. In fact, from [17] it follows that for $\phi\in L^\infty(\mathcal{B})$,

$$B_m(\phi)(z)=\int_{\mathcal{B}}(\phi\circ\phi_z)(u)\,dA_m(u),$$

$z \in \mathcal{B}$, where $dA_m(u) = \binom{m+n}{n}(1-\|u\|^2)^m du$. Berezin first introduced the Berezin transform $\mathcal{B}_0(S)$ of bounded operators S and the m-Berezin transform of functions in [3]. Clearly, for $S \in \mathcal{L}(L^2_a(\mathcal{B}, dz))$, $\|B_m S\|_\infty \le C(m,n)\|S\|$ where $C(m,n)$ is a constant depending only on m and n. Thus $B_m : \mathcal{L}(L^2_a(\mathcal{B}, dz)) \longrightarrow L^\infty(\mathcal{B})$ is a bounded linear operator and for $m \ge 0$

$$\|B_m\| = \binom{m+n}{n} \sum_{|k|=0}^{m} |C_{m,k}| \frac{n!\ k!}{(n+|k|)!}\ .$$

Let

$$d\eta_{\mathcal{B}}(z) = \frac{1}{(1-\|z\|^2)^{n+1}} dz, \quad z \in \mathcal{B}.$$

Corollary 5.4. *Suppose* $2 \le p < \infty$, $\phi, \psi \in L^\infty(\mathcal{B})$, $B_m S_\psi \in L^p(\mathcal{B}, d\eta_{\mathcal{B}})$ *and* $B_m\phi \in L^p(\mathcal{B}, d\eta_{\mathcal{B}})$ *for all* $m \ge 0$. *Suppose*

$$\max\left\{\|T_{B_m S_\psi}\|_p, \|T_{B_m\phi}\|_p\right\} < M \tag{5.1}$$

for some constant $M > 0$ *independent of* m. *The following hold.*

(i) $S_\psi \in S_p$.

(ii) $T_{B_m\phi} T_{B_m S_\psi} \xrightarrow{w} T_\phi S_\psi$ *and* $T_\phi S_\psi \in S_p$.

(iii) *If* $C_m \in \mathcal{L}(L^2_a(\mathcal{B}, dz)), m \ge 0$, $C_m \xrightarrow{w} C$ *and if* C_m *is a sequence of upper triangular matrices then* $T_{B_m S_\psi} C_m \xrightarrow{w} S_\psi C$ *and* $S_\psi C \in S_p$.

(iv) *If* $B_m S_\psi \ge 0, B_m\phi \ge 0$ *for all* $m \ge 0$ *and* $\|T_{B_m S_\psi} - S_\psi\|_p \longrightarrow 0$, $\|T_{B_m\phi} - T_\phi\|_p \longrightarrow 0$ *as* $m \longrightarrow \infty$ *and* $\mathrm{Range} T_{B_m S_\psi} \subset \ker T_{B_m\phi}$, $\mathrm{Range} T_{B_m\phi} \subset \ker T_{B_m S_\psi}$ *then* $\mathrm{Range} S_\psi \subset \ker T_\phi$ *and* $\mathrm{Range} T_\phi \subset \ker S_\psi$.

(v) *If* $B_m S_\psi \ge 0$ *for all* $m \ge 0$ *and* $\{C_m\}$ *is a sequence of positive operators in* S_p *such that* $C_m \xrightarrow{w} C$ *and* $\mathrm{Range} T_{B_m S_\psi} \subset \ker C_m$ *and* $\mathrm{Range} C_m \subset \ker T_{B_m S_\psi}$ *for all* $m \ge 0$ *then* $\mathrm{Range} S_\psi \subset \ker C$ *and* $\mathrm{Range} C \subset \ker S_\psi$.

Proof. Since $B_m S_\psi \in L^p(\mathcal{B}, d\eta_{\mathcal{B}})$, hence by Lemma 2.3, $T_{B_m S_\psi} \in S_p$. Further, since $\|T_{B_m S_\psi}\|_p < M$ for all $m \ge 0$, we have

$$\|T_{B_m S_\psi}\|_p^p = \int_{\mathcal{B}} \left\langle |T_{B_m S_\psi}|^p k_z^{\mathcal{B}}, k_z^{\mathcal{B}} \right\rangle d\eta_{\mathcal{B}}(z) < M^p.$$

Since $2 \le p < \infty$, we obtain

$$\begin{aligned}
&\int_{\mathcal{B}} \|P_{\mathcal{B}}(B_m S_\psi \circ \phi_z)\|^p \, d\eta_{\mathcal{B}}(z) = \int_{\mathcal{B}} \left\|P_{\mathcal{B}}(U_z(B_m S_\psi) k_z^{\mathcal{B}})\right\|^p d\eta_{\mathcal{B}}(z)\\
&= \int_{\mathcal{B}} \left\|U_z T_{B_m S_\psi} k_z^{\mathcal{B}}\right\|^p d\eta_{\mathcal{B}}(z) = \int_{\mathcal{B}} \left\|T_{B_m S_\psi} k_z^{\mathcal{B}}\right\|^p d\eta_{\mathcal{B}}(z)\\
&= \int_{\mathcal{B}} \left\langle T^*_{B_m S_\psi} T_{B_m S_\psi} k_z^{\mathcal{B}}, k_z^{\mathcal{B}}\right\rangle^{\frac{p}{2}} d\eta_{\mathcal{B}}(z) \le \int_{\mathcal{B}} \left\langle (T^*_{B_m S_\psi} T_{B_m S_\psi})^{\frac{p}{2}} k_z^{\mathcal{B}}, k_z^{\mathcal{B}}\right\rangle d\eta_{\mathcal{B}}(z)\\
&= \int_{\mathcal{B}} \left\langle |T_{B_m S_\psi}|^p k_z^{\mathcal{B}}, k_z^{\mathcal{B}}\right\rangle d\eta_{\mathcal{B}}(z) < M^p.
\end{aligned}$$

This implies

$$\sup_{z\in\mathcal{B}} \left\|T_{B_m S_\psi \circ \phi_z} 1\right\|_p = \sup_{z\in\mathcal{B}} \|P_{\mathcal{B}}(B_m S_\psi \circ \phi_z)\|_p < M.$$

Since $\left\|T^*_{B_m S_\psi}\right\|_p = \left\|T_{B_m S_\psi}\right\|_p$, hence $\sup_{z\in\mathcal{B}} \left\|T^*_{B_m S_\psi \circ \phi_z} 1\right\|_p < M$. From [17], it thus follows that $T_{B_m S_\psi} \longrightarrow S_\psi$ as $m \longrightarrow \infty$ in $\mathcal{L}(L^2_a(\mathcal{B}, dz))$-norm and from Lemma 4.6, it follows that $S_\psi \in S_p$. This proves (i). To prove (ii) observe that since (5.1) holds, we have $\|T_{B_m\phi\circ\phi_z} 1\|_p < M$ and $\left\|T^*_{B_m\phi\circ\phi_z} 1\right\|_p < M$ for all $m \ge 0$. Hence from [17], it follows that $T_{B_m\phi} \longrightarrow T_\phi$ as $m \longrightarrow \infty$ in $\mathcal{L}(L^2_a(\mathcal{B}, dz))$-norm. Thus using Lemma 4.4, we obtain $T_{B_m\phi}T_{B_mS_\psi} \xrightarrow{w} T_\phi S_\psi$. Since $S_\psi \in S_p$, we have $T_\phi S_\psi \in S_p$. Now we shall prove (iii). From Lemma 4.5 it follows that $T_{B_mS_\psi}C_m \xrightarrow{w} S_\psi C$ and since $S_\psi \in S_p$, we obtain $S_\psi C \in S_p$. To prove (iv), we first notice that $T_{B_mS_\psi}$ and $T_{B_m\phi}$ are positive operators for all $m \geq 0$. This is so since $B_mS_\psi \geq 0$ and $B_m\phi \geq 0$ for all $m \geq 0$. Given that $T_{B_mS_\psi} \longrightarrow S_\psi$ and $T_{B_m\phi} \longrightarrow T_\phi$ in S_p as $m \longrightarrow \infty$. As $\|T\|_{\mathcal{L}(L^2_a(\mathcal{B},dz))} \leq \|T\|_p$ for all $T \in S_p$, hence $T_{B_mS_\psi} \longrightarrow S_\psi$ and $T_{B_m\phi} \longrightarrow T_\phi$ in norm, $S_\psi \geq 0$ and $T_\phi \geq 0$. It thus follows that $T_{B_m\phi}T_{B_mS_\psi} \longrightarrow T_\phi S_\psi$ and $T_{B_mS_\psi}T_{B_m\phi} \longrightarrow S_\psi T_\phi$ in norm as $m \longrightarrow \infty$. The reason for this is as follows:

$$\begin{aligned}\left\|T_{B_m\phi}T_{B_mS_\psi} - T_\phi S_\psi\right\| &= \left\|T_{B_m\phi}T_{B_mS_\psi} - T_\phi T_{B_mS_\psi} + T_\phi T_{B_mS_\psi} - T_\phi S_\psi\right\| \\ &\leq \left\|\left(T_{B_m\phi} - T_\phi\right)T_{B_mS_\psi}\right\| + \left\|T_\phi\left(T_{B_mS_\psi} - S_\psi\right)\right\| \\ &\leq \left\|T_{B_m\phi} - T_\phi\right\|\left\|T_{B_mS_\psi}\right\| + \left\|T_\phi\right\|\left\|T_{B_mS_\psi} - S_\psi\right\| \longrightarrow 0\end{aligned}$$

as $m \longrightarrow \infty$, since $\sup_m \left\|T_{B_mS_\psi}\right\| \leq L$ for some $L > 0$ by uniform boundedness principle. Further

$$\left\|T_{B_mS_\psi}T_{B_m\phi} - S_\psi T_\phi\right\| = \left\|\left(T_{B_m\phi}T_{B_mS_\psi}\right)^* - \left(T_\phi S_\psi\right)^*\right\| = \left\|T_{B_m\phi}T_{B_mS_\psi} - T_\phi S_\psi\right\| \longrightarrow 0$$

as $m \longrightarrow \infty$. Now since Range$T_{B_mS_\psi} \subset \ker T_{B_m\phi}$ and Range$T_{B_m\phi} \subset \ker T_{B_mS_\psi}$ we obtain $T_{B_m\phi}T_{B_mS_\psi} = T_{B_mS_\psi}T_{B_m\phi} = 0$. Taking limit $m \longrightarrow \infty$, we obtain $T_\phi S_\psi = S_\psi T_\phi = 0$ and (iv) follows. To prove (v), assume that $B_mS_\psi \geq 0$ for all $m \geq 0$ and $\{C_m\}$ is a sequence in S_p such that $C_m \xrightarrow{w} C$. If Range$T_{B_mS_\psi} \subset \ker C_m$ and Range$C_m \subset \ker T_{B_mS_\psi}$ for all $m \geq 0$, then $C_mT_{B_mS_\psi} = T_{B_mS_\psi}C_m = 0$ for all $m \geq 0$. From Lemma 4.4 it follows that $C_mT_{B_mS_\psi} \longrightarrow CS_\psi$. That is, for all $f, g \in L^2_a(\mathcal{B}, dz)$,

$$\left\langle C_mT_{B_mS_\psi}f, g\right\rangle \longrightarrow \left\langle CS_\psi f, g\right\rangle. \tag{5.2}$$

Thus since $C_mT_{B_mS_\psi} = 0$ for all $m \geq 0$, hence $CS_\psi = 0$. Further, from (5.2), it follows that for all $f, g \in L^2_a(\mathcal{B}, dz)$,

$$\left\langle f, T^*_{B_mS_\psi}C^*_mg\right\rangle \longrightarrow \left\langle f, S^*_\psi C^*g\right\rangle.$$

That is,

$$\left\langle f, T_{B_mS_\psi}C_mg\right\rangle \longrightarrow \left\langle f, S_\psi Cg\right\rangle$$

for all $f, g \in L^2_a(\mathcal{B}, dz)$. Thus $S_\psi C = 0$ and the result (v) follows. □

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