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DIRECT AND INVERSE SPECTRAL PROBLEMS FOR BLOCK JACOBI TYPE BOUNDED SYMMETRIC MATRICES RELATED TO THE TWO DIMENSIONAL REAL MOMENT PROBLEM

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Abstract. We generalize the connection between the classical power moment problem and the spectral theory of selfadjoint Jacobi matrices. In this article we propose an analog of Jacobi matrices related to some system of orthonormal polynomials with respect to the measure on the real plane. In our case we obtained two matrices that have a block three-diagonal structure and are symmetric operators acting in the space of l_2 type. With this connection we prove the one-to-one correspondence between such measures defined on the real plane and two block three-diagonal Jacobi type symmetric matrices. For the simplicity we investigate in this article only bounded symmetric operators. From the point of view of the two dimensional moment problem this restriction means that the measure in the moment representation (or the measure, connected with orthonormal polynomials) has compact support.

1. Introduction

The investigations follow in accordance with a general plan described in [10] and given in two parts. The first part (Sections 2 and 3) consists of some results about two dimensional real moment problem. The form of the solution is necessary for the understanding of the second part of the article. It is necessary to note that this part is contained in [3] in a general form (the n-dimensional case), but without this part it is complicated to give clear explanations.

The second part of the article (Sections 4, 5 and 6) is a presentation of the direct and inverse spectral problems which is a generalization of similar classical problems for Jacobi matrices and orthogonal polynomials on real axis $\mathbb{R}$ to the case of two block Jacobi type also symmetric matrices and corresponding orthogonal polynomials on the complex plane $\mathbb{R}^2$. This part continues the investigations of the previous article [9] in which the authors considered the unitary matrices and orthogonal polynomials on the unit circle $\mathbb{T} \subset \mathbb{C}$ related to the trigonometric moment problem and [10] in which the authors considered normal matrices and orthogonal polynomials on the complex plane $\mathbb{C}$ related to the power complex moment problem and [8] in which the authors considered commuting unitary and symmetric matrices and orthogonal polynomials on the complex plane $\mathbb{C}$ related to the complex moment problem in the exponential form.

Investigations of the n-dimensional real moment problem can be found in many works, we note here the following [1, 2, 3, 4]. Our proof of the two dimensional moment representation is based on the Berezansky method of the generalized eigenfunction expansion of corresponding couple of commuting selfadjoint operators and will be described in detail in Sections 2 and 3. Let us note that historically this method goes back to the old works of M. G. Krein [23, 24].

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For a better understanding of the second part of our article we give some facts from [10], the direct and inverse spectral problem in the classical case, namely Jacobi matrices and orthogonal polynomials on real axis $\mathbb{R}$ (see, for example, [1, 3, 37]). The classical theory investigated the Hermitian Jacobi matrix

$$(1)\quad J=\begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},\quad b_n\in\mathbb{R},\quad a_n>0,\qquad \begin{array}{r} n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\\ \mathbb{N}=\{1,2,\ldots\},\end{array}$$

defined as an operator on the space l_2 of sequences $f=(f_n)_{n=0}^{\infty}$. This matrix generates the operator J on finite sequences $f\in l_{\mathrm{fin}}\subset l_2$, which is Hermitian with deficiency numbers ether $(1,1)$, or $(0,0)$ and therefore in any case has a selfadjoint extensions in l_2. Under some conditions on J (for example, $\sum_{n=0}^{\infty}a_n^{-1}=\infty$) the closure of J (we denote it also by J) is selfadjoint. For simplicity, we will assume that J is selfadjoint.

The direct spectral problem, i.e., the eigenfunction expansion for J is constructed in the following way. We obtain a sequence of polynomials $P(\lambda)=(P_n(\lambda))_{n=0}^{\infty}$, $\forall\lambda\in\mathbb{R}$, as a solution of the equation $JP(\lambda)=\lambda P(\lambda)$, i.e.,

$$(2)\quad a_{n-1}P_{n-1}(\lambda)+b_nP_n(\lambda)+a_nP_{n+1}(\lambda)=\lambda P_n(\lambda),\quad P_0(\lambda)=1,\quad P_{-1}(\lambda)=0,\quad \forall n\in\mathbb{N}.$$

This solution exists and is obtained step by step starting with the initial condition $P_0(\lambda)=1$, (using the convenient form $P_{-1}(\lambda)=0$). This is possible, because all $a_n>0$.

The sequence $P(\lambda)$, $\forall\lambda$ of polynomials belong to $l=\mathbb{C}^{\infty}$ and is a generalized eigenvector of J with the eigenvalue λ. The corresponding Fourier transform (denoted by $\widehat{\ }$) in generalized eigenfunctions of J has the form

$$(3)\qquad l_2\supset l_{\mathrm{fin}}\ni f=(f_n)_{n=0}^{\infty}\longmapsto \hat f(\lambda)=\sum_{n=0}^{\infty}f_nP_n(\lambda)\in L_2(\mathbb{R},d\rho(\lambda)).$$

It is a unitary operator (after taking the closure) between the spaces l_2 and $L_2(\mathbb{R},d\rho(\lambda))$. The image of J is the operator of multiplication by λ on the space $L_2(\mathbb{R},d\rho(\lambda))$. The polynomials $P_n(\lambda)$ are orthonormal w.r.t. $d\rho(\lambda)$.

The inverse problem in this classical case is the following. Let us have a Borel probability measure $d\rho(\lambda)$ on $\mathbb{R}$ for which all the moments exist,

$$(4)\qquad s_n=\int_{\mathbb{R}}\lambda^n\,d\rho(\lambda),\quad n\in\mathbb{N}_0,$$

and the support of $d\rho(\lambda)$ is compact. The problem is to recover the corresponding Jacobi matrix J in such manner that the initial measure $d\rho(\lambda)$ is equal to the spectral measure of the corresponding operator J. For the recovering it is necessary to take the sequence of functions from $L_2(\mathbb{R},d\rho(\lambda))$ according to (4),

$$(5)\qquad 1,\lambda,\lambda^2,\ldots,$$

which are linearly independent and apply to it the classical procedure of Schmidt orthogonalization. As a result, one obtains a sequence of orthonormal polynomials,

$$(6)\qquad P_0(\lambda)=1,P_1(\lambda),P_2(\lambda),\ldots,$$

that is a basis in $L_2(\mathbb{R},d\rho(\lambda))$. And the coefficients of the matrix J are calculated by formulas

$$(7)\qquad a_n=\int_{\mathbb{R}}\lambda P_n(\lambda)P_{n+1}(\lambda)\,d\rho(\lambda),\quad b_n=\int_{\mathbb{R}}\lambda(P_n(\lambda))^2\,d\rho(\lambda),\quad n\in\mathbb{N}_0.$$

The connections described above between Jacobi matrices, classical moment problem, and orthogonal polynomials is very fruitful for investigation of both objects. There are many mathematicians who worked in this direction, but it is necessary to remark the results of M. G. Krein [24, 26], N. I. Achiezer [1], Yu. M. Berezansky [2, 3, 6, 7], M. L. Gorbachuk and V. I. Gorbachuk [20].

The main goal of the second part of this articles (as well as of the whole article) is to generalize the above mentioned classical theory of orthonormal polynomials to the real plane $\mathbb{R}^2$. Roughly speaking it is necessary to pass from selfadjoint operators on l_2 to a couple of commuting symmetric operators, acting on some space, similar to l_2. Instead of the space $l_2 = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$ we take the space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \text{where} \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \tag{8}$$

and instead of one scalar matrix (1) we consider two block Jacobi type matrices J_A and J_B.

The matrix J_A consists of elements a_n, b_n and c_n, that are finite dimensional operators (matrices) and that act between the corresponding spaces $\mathcal{H}_n$ from (8), namely,

$$J_A = \begin{bmatrix} b_0 & c_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & c_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & c_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{array}{lll} a_n & : & \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ b_n & : & \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ c_n & : & \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{9}$$

The matrix (9) generates an operator on finite vectors $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$. The matrices a_n and c_n have the following form and its coefficients satisfy some conditions:

$$\begin{gathered} a_n = \left.\underbrace{\begin{bmatrix} a_{n;0,0} & * & * & \dots & * \\ a_{n;1,0} & * & * & \dots & * \\ 0 & a_{n;2,1} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n;n+1,n} \end{bmatrix}}_{n+1}\right\} n+2, \\ c_n = \left.\underbrace{\begin{bmatrix} c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0 \\ * & * & c_{n;1,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & c_{n;n-1,n} & 0 \\ * & * & * & \dots & * & c_{n;n,n+1} \end{bmatrix}}_{n+2}\right\} n+1, \\ a_{n;0,0}, a_{n;1,1}, \dots, a_{n;n,n} > 0, \quad c_{n;0,0}, c_{n;1,1}, \dots, c_{n;n,n} > 0, \quad n \in \mathbb{N}_0. \end{gathered} \tag{10}$$

In (9) b_n are symmetric $(n+1) \times (n+1)$-matrices $n \in \mathbb{N}_0$.

The matrix J_B consists of elements u_n, w_n and v_n, that are finite dimensional operators (matrices) and that act between the corresponding spaces $\mathcal{H}_n$ from (8), namely:

$$J_B = \begin{bmatrix} w_0 & v_0 & 0 & 0 & 0 & \cdots \\ u_0 & w_1 & v_1 & 0 & 0 & \cdots \\ 0 & u_1 & w_2 & v_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{array}{lll} u_n & : & \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ w_n & : & \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ v_n & : & \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{11}$$

The matrix (11) generates an operator J_B on the finite vectors $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$. The matrices u_n and v_n have the following form and its coefficients satisfy some conditions:

$$u_n = \underbrace{\left.\begin{bmatrix} u_{n;0,0} & * & * & \dots & * \\ 0 & u_{n;1,1} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{n;n,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}\right\}}_{n+1} n+2,$$

(12)
$$v_n = \underbrace{\left.\begin{bmatrix} v_{n;0,0} & 0 & \dots & 0 & 0 & 0 \\ * & v_{n;1,1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & \dots & v_{n;n-1,n-1} & 0 & 0 \\ * & * & \dots & * & v_{n;n,n} & 0 \end{bmatrix}\right\}}_{n+2} n+1,$$

$$u_{n;0,0}, u_{n;1,1}, \dots, u_{n;n,n} > 0, \quad v_{n;0,0}, v_{n;1,1}, \dots, v_{n;n,n} > 0, \quad n \in \mathbb{N}_0.$$

In (11) w_n are symmetric $(n+1)\times(n+1)$-matrices, $n \in \mathbb{N}_0$.

Since we are speaking about symmetric matrices, $u_{n;\alpha,\beta} = v_{n;\beta,\alpha}$, $\beta = 0,1,2,\dots,n$, $\alpha = 0,1,\dots,\beta,\beta+1$, $n \in \mathbb{N}$ and $a_{n;\alpha,\beta} = c_{n;\beta,\alpha}$, $\alpha = 0,1,2,\dots,n$, $\beta = \alpha,\dots,n$, $n \in \mathbb{N}$.

For the sake of simplicity, *we will demand in whole article that norms of all the matrices a_n, b_n, c_n and u_n, w_n, v_n be uniformly bounded and therefore the selfadjoint operators generated by J_A and J_B are bounded on $\mathbf{l}_2$.*

Under some additional conditions on u_n, w_n, v_n, and a_n, b_n, c_n, $n \in \mathbb{N}_n$ (see Section 6) the matrices J_A and J_B are commutative, $J_AJ_B = J_BJ_A$, on finite vectors $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$.

It is convenient a vector $x \in \mathcal{H}_n = \mathbb{C}^{n+1}$ denote by $x = (x_0, x_1, \dots, x_n)$.

Let $x \in \mathbb{R}$ belong to the spectrum of J_A, $y \in \mathbb{R}$ belong to the spectrum of J_B and $P(x,y) = (P_n(x,y))_{n=0}^{\infty}$ be a corresponding generalized eigenvector of J_A and J_B. Here $P_n(x,y) \in \mathcal{H}_n$ is a vector-valued polynomial with respect to x, y, i.e., its coordinates are some linear combinations of x^jy^k, $0 \le j+k \le n$. According to the generalized eigenvectors expansion theorem it is some solution of the system of two equations of type (2) (but with matrix coefficients),

$$J_AP(x,y) = xP(x,y), \quad J_BP(x,y) = yP(x,y). \tag{13}$$

The corresponding Fourier transform (denoted also by $\widehat{\ }$) like (3) has now the form

$$\mathbf{l}_2 \supset \mathbf{l}_{\text{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(x,y) = \sum_{n=0}^{\infty} (f_n, P_n(x,y))_{\mathcal{H}_n} \in L_2(\mathbb{R}^2, d\rho(x,y)) =: L_2, \tag{14}$$

where $d\rho(x,y)$ is a spectral measure of the operators J_A and J_B with compact support. Transformation (14) is a unitary operator (after taking the closure) between $\mathbf{l}_2$ and L_2. The polynomials $P_n(x,y)$ are orthonormal with respect to $d\rho(x,y)$ and form a basis in the space L_2. Note that these results are formulated in Theorem 7, but it is convenient for a moment to denote here these polynomials by

$$\begin{aligned}(\overline{Q_{n;0}(x,y)}, \overline{Q_{n;1}(x,y)}, \dots, \overline{Q_{n;n}(x,y)}) \\ = (P_{n;0}(x,y), P_{n;1}(x,y), \dots, P_{n;n}(x,y)) = P_n(x,y).\end{aligned}$$

So, the results described above make a direct spectral problem for J_A an J_B of type (9) with conditions (10) and (11) with conditions (12).

For two dimensional real moment problem the inverse spectral problem is now formulated in the following way. Let us have a probability Borel measure $d\rho(x,y)$ with compact support on $\mathbb{R}^2$; assume that all the moments

$$s_{m,n} = \int_{\mathbb{R}^2} x^m y^n \, d\rho(x,y), \quad m,n \in \mathbb{N}_0, \tag{15}$$

exist and the support of $d\rho(x,y)$ is such that all the functions $x^j y^k$, $j,k \in \mathbb{N}_0$ (which belonging to $L_2(\mathbb{R}^2, d\rho(x,y))$, see (15)), are linearly independent in this space. The problem is to construct the Jacobi type block matrix (9) and (11) with properties (10) and (12) correspondingly in such a way that for the selfadjoint (in our case bounded) operators J_A and J_B the spectral measure is equal to the initial measure.

As in the classical case we apply to the sequence of the functions from L_2,

$$(x^j y^k)_{j,k=0}^{\infty}, \tag{16}$$

(instead of (5)) the standard Schmidt orthogonalization procedure in the space L_2. Since the sequence (16) is two-indexes, there is a need to choose a convenient global linear order for (16). The order is as in [35],

$$x^0y^0 = 1; \quad x^0y^1, x^1y^0; \quad x^0y^2, x^1y^1, x^2y^0; \quad \dots; \quad x^0y^n, x^1y^{n-1}, \dots, x^ny^0; \quad \dots \tag{17}$$

(see Figure 1 and (51)).

After such an orthogonalization we get a sequence of polynomials,

$$P_n(x,y) = (P_{n;0}(x,y), P_{n;1}(x,y), \dots, P_{n;n}(x,y)), \quad n \in \mathbb{N}_0,,$$

and matrices (9) with conditions (10) and (11) with conditions (12) are reconstructed by using the formulas of the type (7).

The above presented results are explained in Sections 4 and 5. The orthogonalization procedure is described in the Section 4. The references connected with the projection spectral theorem are given in Sections 2 and 5. Note that we give now references to some results concerning the related topic, [14, 33, 12]. Note also that the theory of block Jacobi matrices that are either Hermitian or selfadjoint operators acting on the spaces $l_2(\mathcal{H}) = \mathcal{H} \oplus \mathcal{H} \oplus \cdots$, where $\mathcal{H}$ is any Hilbert space, was investigated at first in [25] in the case $\dim\mathcal{H} < \infty$ and in [2, 3] in the case $\dim\mathcal{H} \leq \infty$. For results on families of commuting selfadjoint operators acting on the symmetric Fock space, see [6]. The Fock space has the form (8) with $\mathcal{H}_n$ that are, for $n > 0$, n-particle infinite-dimensional Hilbert space.

Remark 1. It is interesting to develop the spectral theory of block Jacobi type Hermitian bounded matrices J_A of the form (9) and J_B of the form (10) in the case of unbounded operators. What are the conditions on elements of the matrices J_A and J_B that would guarantee that the operators J_A and J_B are essentially selfadjoint and commute ? In what terms is it possible to describe all commuting selfadjoint extensions of J_A and J_B on $\mathbf{l}_2$ similarly to classical Jacobi matrices in the case where J_A and J_B are not essentially selfadjoint ?

Remark 2. It is necessary to stress that in this article we consider the Jacobi type symmetry bounded matrices J_A and J_B in "a general situation", where the functions (17) are linearly independent in the space L_2 constructed from the spectral measure $d\rho(x,y)$ of the bounded selfadjoint operators J_A and J_B. This condition in terms of these operators means that if for some coefficients $c_{j,k} \in \mathbb{R}^2$,

$$\sum_{j,k=0}^{n} s_{j,k} A^j B^k = 0, \quad n \in \mathbb{N}, \tag{18}$$

then $c_{j,k} = 0$, $\forall j,k \in \{0,1,\dots,n\}$. The last condition is equivalent to linearly independence of (17). So, let $dE(x,y)$ be a resolution of identity for J_A and J_B, then it is possible to rewrite (18) as follows:

$$\int_{\mathbb{R}^2} \left| \sum_{j,k=0}^{n} c_{j,k} x^j y^k \right|^2 d(E(x,y)f,f)_{\mathbf{l}_2} = 0, \quad \forall f \in \mathbf{l}_2.$$

Using boundedness of the support of $E(\alpha)$ we conclude that the last equality means that $\sum_{j,k=0}^{n} s_{j,k} x^j y^k \in L_2$ and equals to 0. By our assumption all $s_{j,k} = 0$, i.e., functions (17) are linearly independent in L_2. The inverse conclusion is also true.

A similar situation is encountered when classical Jacobi matrices are considered, if the corresponding selfadjoint operator S on the space l_2 has the property $\sum_{j=0}^{n} s_j S^j = 0$ with some $s_j \in \mathbb{R}$, $(s_0, \dots, s_n) \neq 0$, then its matrix generates an operator on a finite-dimensional subspace of l_2.

Let us now to compare our results with the closely related results from [10]. In [10] the authors considered the above mentioned theory in the case where the matrix of the form (9) is a normal operator. In such a case, the spectrum of this operator lies in the complex plane $\mathbb{C}$. Instead of the space $l_2 = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$ in [10], the space under consideration is

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \text{where} \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \tag{19}$$

but the scalar matrix (1) was replaced in [10] with the following Jacobi type block matrix with elements p_n, r_n and q_n, which are finite dimensional operators (matrices) and act between the spaces $\mathcal{H}_n$ from (19), namely,

$$J_N = \begin{bmatrix} r_0 & q_0 & 0 & 0 & 0 & \cdots \\ p_0 & r_1 & q_1 & 0 & 0 & \cdots \\ 0 & p_1 & r_2 & q_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad \begin{array}{lcl} p_n & : & \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \\ r_n & : & \mathcal{H}_n \longrightarrow \mathcal{H}_n, \\ q_n & : & \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, \quad n \in \mathbb{N}_0. \end{array} \tag{20}$$

This matrix (20) generates a normal operator J_N on finite vectors $\mathbf{l}_{\text{fin}} \subset \mathbf{l}_2$ (19). The matrices p_n and q_n have the following form and its coefficients satisfy some conditions:

$$p_n = \underbrace{\left.\begin{bmatrix} p_{n;0,0} & * & * & \dots & * \\ 0 & p_{n;1,1} & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p_{n;n,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}\right\}}_{n+1} n+2,$$

$$q_n = \underbrace{\left.\begin{bmatrix} q_{n;0,0} & q_{n;0,1} & 0 & \dots & 0 & 0 \\ * & * & q_{n;1,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \dots & q_{n;n-1,n} & 0 \\ * & * & * & \dots & * & q_{n;n,n+1} \end{bmatrix}\right\}}_{n+2} n+1, \tag{21}$$

$$p_{n;0,0}, p_{n;1,1}, \dots, p_{n;n,n} > 0, \quad q_{n;0,1}, q_{n;1,2}, \dots, q_{n;n,n+1} > 0, \quad n \in \mathbb{N}_0.$$

In (20) r_n are $(n+1) \times (n+1)$-matrices $n \in \mathbb{N}_0$ such that J_N commutes with J_N^* on finite vectors $\mathbf{l}_{\text{fin}}$.

Remark 3. Comparing matrices (20) of J_N with (9) of J_A and (11) of J_B we conclude that

1) the form of the matrices J_A and $\mathrm{Re}(J_N)$ (the real part of J_N) are the same, but the forms of the matrices J_B and $\mathrm{Im}(J_N)$ (the imaginary part of J_N) are different;

2) what is important is that the matrices J_A, J_B and J_N are defined in different basis, i.e., $x^n y^m$ and $z^n \bar{z}^m$ correspondingly;

3) and hence the matrices J_A, J_B are not a simple implication of J_N and vice versa on the spaces $L_2(\mathbb{R}^2, d\rho(x,y))$ and $L_2(\mathbb{C}, d\varrho(z,\bar{z}))$.

This article connected with a vast number of works is devoted to polynomials orthogonal w.r.t. some measure on the real plane. We will not consider here these relations and indicate only the well known works that contain main results about such polynomials [19, 35, 36].

It is important to stress that many proofs in this article are similar to [9, 10, 11]. But we wanted to make reading of this article as much independent of other papers as possible.

2. Auxiliary results

Let $\mathcal{H}$ be a separable Hilbert space and let A and B be a commuting selfadjoint operators defined on $\mathrm{Dom}(A)$ and $\mathrm{Dom}(B)$ in $\mathcal{H}$. Consider a rigging of $\mathcal{H}$

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset \mathcal{D}, \tag{22}$$

such that $\mathcal{H}_+$ is a Hilbert space topologically and quasinuclearly embedded into $\mathcal{H}$ (topologically means densely and continuously; quasinuclearly means that the inclusion operator is of Hilbert-Schmidt type); $\mathcal{H}_-$ is the dual of $\mathcal{H}_+$ with respect to space $\mathcal{H}$; $\mathcal{D}$ is a linear, topological space, topologically embedded into $\mathcal{H}_+$.

The operators A and B are called standardly connected with the chain (22) if $\mathcal{D} \subset \mathrm{Dom}(A)$, $\mathcal{D} \subset \mathrm{Dom}(B)$ and restrictions $A \upharpoonright \mathcal{D}$, $B \upharpoonright \mathcal{D}$ act from $\mathcal{D}$ into $\mathcal{H}_+$ continuously.

Let us recall that a vector $\Omega \in \mathcal{D}$ is called a strong cyclic vector for operators A and B if for any $p, q \in \mathbb{N}$ we have $\Omega \in \mathrm{Dom}(A^p) \cap \mathrm{Dom}(B^q)$, $A^p B^q \Omega \in \mathcal{D}$ and the set of all these vectors and Ω, as $p, q = \mathbb{N}_0$, is total in the space $\mathcal{H}_+$ (and, hence, also in $\mathcal{H}$).

Assuming that the strong cyclic vector exists we formulate a short version of the projection spectral theorem. A complete version and a corresponding proof one can found in [4], Ch. 3, Theorem 2.7, or [3], Ch. 5, [5], Ch. 15; [31].

Theorem 1. *For commuting selfadjoint operators A and B with a strong cyclic vector in a separable Hilbert space $\mathcal{H}$ there exists a nonnegative finite Borel measure $d\rho(x,y)$ such that for ρ-almost every point $(x,y) \in \mathbb{R}^2$ there exists a generalized joint eigenvector $\xi_{x,y} \in \mathcal{H}_-$, i.e.,*

$$(\xi_{x,y}, Af)_{\mathcal{H}} = x(\xi_{x,y}, f)_{\mathcal{H}}, \quad (\xi_{x,y}, Bf)_{\mathcal{H}} = y(\xi_{x,y}, f)_{\mathcal{H}}, \quad f \in \mathcal{D}, \quad \xi_{x,y} \neq 0. \tag{23}$$

The corresponding Fourier transform F given by the rule

$$\mathcal{H} \supset \mathcal{H}_+ \ni f \mapsto (Ff)(x,y) = \hat{f}(x,y) = (f, \xi_{x,y})_{\mathcal{H}} \in L_2(\mathbb{R}^2, d\rho(x,y)) := L_2 \tag{24}$$

is a unitary operator (after taking the closure) acting from $\mathcal{H}$ into L_2. The image of the operators A and B under F are the operators of multiplication by x and y respectively in L_2.

Let us also recall that for a selfadjoint operator T defined on $\mathrm{Dom}(T)$ in $\mathcal{H}$, a vector $f \in \bigcap_{n=0}^{\infty} \mathrm{Dom}(T^n)$ is called quasianalytic [29, 30] if the class $\mathrm{C}\{m_n\}$ where in our case $m_n = \sqrt{\|T^n f\|_{\mathcal{H}}}$ is quasianalytic. We recall that this class of functions on $[a,b] \subset \mathbb{R}$ is defined by

$$\mathrm{C}\{m_n\} = \{f \in C^\infty([a,b]) \ \exists K = K_f > 0, \ |f^{(n)}(t)| \leq K^n m_n, \ t \in [a,b], \ n \in \mathbb{N}_0\}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\|T^n f\|_{\mathcal{H}}}} = \infty. \tag{25}$$

We use the condition of quasianalytic via the criterion of selfadjointness and commutativity [3, 29, 30, 4, 5] (see also [34]). In the next section we also use the following two theorems from [4], Ch. 5, §1, or [5], Ch. 13, §9, and from [28] (see also [32]).

Theorem 2. *A closed Hermitian operator T on a Hilbert space $\mathcal{H}$ is selfadjoint iff there exists a set, total in $\mathcal{H}$, of quasianalytic vectors.*

The next theorem gives a useful criterion for two operators that are essentially selfadjoint and have commuting closure.

Theorem 3. *Let A and B be two Hermitian operators defined on $\operatorname{Dom}(A)$ and $\operatorname{Dom}(B)$ on a Hilbert space $\mathcal{H}$ and a dense in $\mathcal{H}$ linear set $\mathcal{D}$ is contained in the domains of the operators A, B, A^2, AB, BA, and B^2 so that $ABf = BAf$, for all $f \in \mathcal{D}$.*

If the restriction $A^2 + B^2$ on $\mathcal{D}$ is essentially selfadjoint, then A and B are selfadjoint and commute in the strong resolvent sense.

3. Two-dimensional real power moment problem

The two-dimensional real power moment problem consists of finding condition on the sequence $\{s_{m,n}\}$, $m, n \in \mathbb{N}_0$, of real numbers that would imply existence of a nonnegative Borel measure $d\rho(x, y)$ on the real plane $\mathbb{R}^2$ for which

$$s_{m,n} = \int_{\mathbb{R}^2} x^m y^n d\rho(x, y), \quad m, n \in \mathbb{N}_0. \tag{26}$$

Theorem 4. *For existence of representation (26) for a given sequence of real numbers $\{s_{m,n}\}_{m,n=0}^{\infty}$ it is necessary that it be positive definite, that is, i.e.*

$$\sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{f}_{m,n}s_{j+m,k+n} \geq 0 \tag{27}$$

for all finite sequences of complex numbers $(f_{j,k})_{j,k=0}^{\infty}$, $f_{j,k} \in \mathbb{C}$.

The representation (26) for a given sequence of real numbers $\{s_{m,n}\}_{m,n=0}^{\infty}$ exists and is unique if it is positive definite and

$$\sum_{p=1}^{\infty} \frac{1}{\sqrt[p]{\sum_{k=0}^{p} C_p^k \sqrt{s_{4p-4k,4k}}}} = \infty. \tag{28}$$

Recall that the condition (27) is necessary to have representation (26). The result states that the conditions (27) and (28) both guarantee not only existence but also uniqueness of representation (26) for a given sequence $\{s_{m,n}\}_{m,n=0}^{\infty}$.

Proof. Necessity of condition (27) is obvious. Indeed, if the sequence $\{s_{m,n}\}_{m,n=0}^{\infty}$ has representation (26), then for an arbitrary finite sequence $f = (f_{m,n})_{m,n=0}^{\infty}$, $f_{m,n} \in \mathbb{C}$, we have

$$\sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{f}_{m,n}s_{j+m,k+n} = \int_{\mathbb{R}^2} \left| \sum_{m,n=0}^{\infty} f_{m,n}x^m y^n \right|^2 d\rho(x, y) \geq 0. \tag{29}$$

Let us denote by l the linear space $\mathbb{C}^{\infty}$ of sequences $f = (f_{m,n})_{m,n=0}^{\infty}$, $f_{m,n} \in \mathbb{C}$, and by l_{fin} its linear subspace consisting of finite sequences $f = (f_{m,n})_{m,n=0}^{\infty}$, i.e., the sequences such that $f_{m,n} \neq 0$ for only a finite number of indices n and m. Let $\delta_{m,n}$, $m, n \in \mathbb{N}_0$, be the δ-sequence such that each $f \in l_{\text{fin}}$ has a representation $f = \sum_{n,m=0}^{\infty} f_{m,n}\delta_{m,n}$.

Let us consider linear operators on l_{fin}

$$(J_A f)_{j,k} = f_{j-1,k}, \quad (J_B f)_{j,k} = f_{j,k-1}, \quad j, k \in \mathbb{N}_0, \tag{30}$$

where it is always assumed that $f_{j,-1} = f_{-1,k} \equiv 0$. The operators J_A and J_B are the "creation" type operators. For the δ-sequence we get

$$J_A\delta_{j,k} = \delta_{j+1,k}, \quad J_B\delta_{j,k} = \delta_{j,k+1}. \tag{31}$$

The operators J_A and J_B are symmetric on finite vector with respect to the (quasi)scalar product

$$(f,g)_S = \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m,n}s_{j+m,k+n}, \quad f,g \in l_{\text{fin}}. \tag{32}$$

Indeed,

$$\begin{aligned}
(J_Af,g)_S &= \sum_{j,k,m,n=0}^{\infty} (J_Af)_{j,k}\bar{g}_{m,n}s_{j+m,k+n} = \sum_{j,k,m,n=0}^{\infty} f_{j-1,k}\bar{g}_{m,n}s_{j+m,k+n}\\
&= \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m,n}s_{j+m+1,k+n} = \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m-1,n}s_{j+m,k+n}\\
&= \sum_{j,k,m,n=0}^{\infty} f_{j,k}\overline{(J_Ag)}_{m,n}s_{j+m,k+n} = (f,J_Ag)_S,
\end{aligned}$$

$$\begin{aligned}
(J_Bf,g)_S &= \sum_{j,k,m,n=0}^{\infty} (J_Bf)_{j,k}\bar{g}_{m,n}s_{j+m,k+n} = \sum_{j,k,m,n=0}^{\infty} f_{j,k-1}\bar{g}_{m,n}s_{j+m,k+n}\\
&= \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m,n}s_{j+n+1,k+m} = \sum_{j,k,m,n=0}^{\infty} f_{j,k}\bar{g}_{m-1,n}s_{j+n,k+m}\\
&= \sum_{j,k,m,n=0}^{\infty} f_{j,k}\overline{(J_Bg)}_{m,n}s_{j+m,k+n} = (f,J_Bg)_S.
\end{aligned}$$

The operator J_A commute with J_B on l_{fin}

$$(J_BJ_Af)_{j,k} = f_{j-1,k-1} = (J_AJ_Bf)_{j,k}.$$

Let S be the Hilbert space obtained as a completion of the factor space

$$\dot{l}_{\text{fin}} = l_{\text{fin}}/\{h \in l_{\text{fin}} \mid (h,h)_S = 0\}.$$

An element f of S is a representative of the equivalence class $\dot{f}$ in $\dot{l}_{\text{fin}}$. Hence, the operators $\dot{J}_A$ and $\dot{J}_B$ are correctly defined on S. This fact is described in detail in [3], Ch. 8, §1, Subsect. 4 and [4], Ch. 5, §5, Subsect. 2. Analogously to this case we get

$$\dot{J}_A\dot{f} = (J_Af)^{\cdot}, \quad f \in \operatorname{Dom}(\dot{J}_A) = \dot{l}_{\text{fin}}; \quad \dot{J}_B\dot{f} = (J_Bf)^{\cdot}, \quad f \in \operatorname{Dom}(\dot{J}_B) = \dot{l}_{\text{fin}}. \tag{33}$$

Let us denote for the next considerations, by A and B, the closure $\sim$ of $\dot{J}_A$ and $\dot{J}_B$ in S.

For simplicity, we suppose that the given sequence $\{s_{m,n}\}$ is nondegenerate, i.e., if $(f,f)_S = 0$ for $f \in l_{\text{fin}}$, then $f = 0$, and now $\dot{f} = f$ and $\tilde{\dot{J}}_A = A$ and $\tilde{\dot{J}}_A = B$. The investigation in the general case is more complicated, see in [3], Ch. 8, §1, Subsect. 4 and [4], Ch. 5, §5, Subsect. 1–3. The general case will be considered in a next publication.

Let us also assume for the moment that the operators A and B are selfadjoint. Later we will prove that A and B are selfadjoint and commuting under the condition (28). In general, the condition for Hermitian operators that have commuting extensions are described in [16].

Let us construct the rigging of spaces,

$$(l_2(p))_{-,S} \supset S \supset l_2(p) \supset l_{\text{fin}}, \tag{34}$$

where $l_2(p)$ is a weighted l_2-space with the weight $p = (p_{m,n})_{m,n=0}^{\infty}$, $p_n \geq 1$. The norm in $l_2(p)$ is given by $\|f\|_{l_2(p)}^2 = \sum_{m,n=0}^{\infty} |f_{m,n}|^2 p_{m,n}$; $(l_2(p))_{-,S} = \mathcal{H}_-$ is a negative space with respect to the positive space $l_2(p) = \mathcal{H}_+$ and the zero space $S = \mathcal{H}$.

Lemma 1. *For the space S there exists a sufficiently rapidly increasing sequence $p_{m,n}$ such that the embedding $l_2(p) \hookrightarrow S$ is quasinuclear.*

Proof. The inequality (27) also means that the multimatrix $(K_{j,k;m,n})_{j,k,m,n=0}^{\infty}$, with the coefficients $K_{j,k;m,n} = s_{j+m,k+n}$ is nonnegative definite and, therefore,

(35) $|s_{j+n,k+m}|^2 = |K_{j,k;m,n}|^2 \leq K_{j,k;j,k} K_{m,n;m,n} = s_{j+k,j+k} s_{n+m,n+m}, \quad j,k,m,n \in \mathbb{N}_0.$

Let the weight $q = (q_{j,k})_{j,k=0}^{\infty}$, $q_{j,k} \geq 1$, be such that $\sum_{j,k=0}^{\infty} s_{j+k,j+k} q_{j,k}^{-1} < \infty$. Then from (35) we obtain that

$$\|f\|_S^2 = \sum_{j,k,m,n=0}^{\infty} f_{j,k} \bar{f}_{m,n} s_{j+m,k+n} \leq \left(\sum_{j,k=0}^{\infty} \frac{s_{j+k,j+k}}{q_{j,k}} \right) \|f\|_{l_2(q)}^2, \quad f \in l_{\mathrm{fin}}.$$

Therefore, $l_2(q) \hookrightarrow S$ is topological. And if $\sum_{j,k=0}^{\infty} q_{j,k} p_{j,k}^{-1} < \infty$, then $l_2(p) \hookrightarrow l_2(q)$ is quasinuclear. The composition $l_2(p) \hookrightarrow S$ of the quasinuclear and topological embeddings is also quasinuclear. □

In the next step we use the rigging (34) to construct generalized eigenvectors. The inner structure of the space $(l_2(p))_{-,S}$ is complicated, because of the complicated structure of S. This is a reason to introduce new auxiliary rigging

(36) $$l = (l_{\mathrm{fin}})' \supset (l_2(p^{-1})) \supset l_2 \supset l_2(p) \supset l_{\mathrm{fin}},$$

where $l_2(p^{-1})$, $p^{-1} = (p_{m,n}^{-1})_{m,n=0}^{\infty}$ is a negative space with respect to the positive space $l_2(p)$ and the zero space l_2. Chains (34) and (36) have the same positive space $l_2(p)$. The next general Lemma [7] establishes that the space $(l_2(p))_{-,S}$ is isometric to the space $l_2(p^{-1})$.

Lemma 2. *Consider two riggings*

(37) $$\mathcal{K}_- \supset \mathcal{K} \supset \mathcal{K}_+, \quad \mathcal{F}_- \supset \mathcal{F} \supset \mathcal{F}_+ = \mathcal{K}_+,$$

with the equal positive spaces. Then there exist a unitary operator $U : \mathcal{K}_- \to \mathcal{F}_-$, $U\mathcal{K}_- = \mathcal{F}_-$, such that

(38) $$(U\xi, f)_{\mathcal{F}} = (\xi, f)_{\mathcal{K}}, \quad \xi \in \mathcal{K}_-, \quad f \in \mathcal{K}_+ = \mathcal{F}_+.$$

This operator can be given as follows: $U = \mathbb{I}_{\mathcal{F}}^{-1} \mathbb{I}_{\mathcal{K}}$, where $\mathbb{I}_{\mathcal{F}}$ and $\mathbb{I}_{\mathcal{K}}$ are two standard canonical Berezansky isomorphisms in the corresponding chains ($\mathbb{I}_{\mathcal{F}} \mathcal{F}_- = \mathcal{F}_+$, $\mathbb{I}_{\mathcal{K}} \mathcal{K}_- = \mathcal{K}_+$).

The role of the riggings (37) will be played by (34) and (36) in what follows.

Obviously, the operators A and B are standardly connected with the chain (36), and the vector $\Omega = \delta_{0,0} \in l_{\mathrm{fin}}$ is strong cyclic for the operators A and B. Therefore we can apply Theorem 1. Let $\xi_{x,y} \in (l_2(p))_{-,S}$ be a generalized eigenvector of the operators A and B. So, in this case due to Theorem 1 we have

(39) $$(\xi_{x,y}, Af)_S = x(\xi_{x,y}, f)_S, \quad (\xi_{x,y}, Bf)_S = y(\xi_{x,y}, f)_S, \quad (x,y) \in \mathbb{R}^2, \quad f \in l_{\mathrm{fin}}.$$

Denote

$$P(x,y) = U\xi_{x,y} \in l_2(p^{-1}) \subset l, \quad P(x,y) = (P_{m,n}(x,y))_{m,n=0}^{\infty}, \quad P_{m,n}(x,y) \in \mathbb{R}^2.$$

Using (38) we can rewrite (39) in the form

(40) $$(P(x,y), Af)_{l_2} = x(P(x,y), f)_{l_2}, \quad (P(x,y), Bf)_{l_2} = y(P(x,y), f)_{l_2}, \quad (x,y) \in \mathbb{R}^2, \quad f \in l_{\mathrm{fin}}.$$

The corresponding Fourier transform has the form

$$S \supset l_{\text{fin}} \ni f \to (Ff)(x,y) = \hat{f}(x,y) = (f, P(x,y))_{l_2} \in L_2(\mathbb{R}^2, d\rho(x,y)). \tag{41}$$

Let us calculate $P(x,y)$. The operator A is generated by the rule (30) and therefore (40) gives

$$\begin{aligned}\sum_{m,n=0}^{\infty} xP_{m,n}(x,y)\bar{f}_{m,n} &= x(P(x,y),f)_{l_2}\\ &= (P(x,y),Af)_{l_2} = \sum_{m,n=0}^{\infty} P_{m+1,n}(x,y)\bar{f}_{m,n}, \quad \forall f \in l_{\text{fin}}.\end{aligned} \tag{42}$$

Analogously, using (30) and (40), we have

$$\begin{aligned}\sum_{m,n=0}^{\infty} yP_{m,n}(x,y)\bar{f}_{m,n} &= y(P(x,y),f)_{l_2}\\ &= (P(x,y),Bf)_{l_2} = \sum_{m,n=0}^{\infty} P_{m,n+1}(x,y)\bar{f}_{m,n}, \quad \forall f \in l_{\text{fin}}.\end{aligned} \tag{43}$$

Hence we have

$$xP_{m,n}(x,y) = P_{m+1,n}(x,y), \quad yP_{m,n}(x,y) = P_{m,n+1}(x,y), \quad m,n \in \mathbb{N}_0.$$

Without loss of generality, we can take $P_{0,0}(x,y) = 1, \quad (x,y) \in \mathbb{R}^2$. Then the last two equalities give

$$P_{m,n}(x,y) = x^m y^n, \quad m,n \in \mathbb{N}_0. \tag{44}$$

Thus the Fourier transform (41) has the form

$$S \supset l_{\text{fin}} \ni f \to (Ff)(x,y) = \hat{f}(x,y) = \sum_{m,n=0}^{\infty} f_{m,n}x^m y^n \in L_2(\mathbb{R}^2, d\rho(x,y)), \tag{45}$$

and

$$(f,g)_S = \int_{\mathbb{R}^2} \hat{f}(x,y)\overline{\hat{g}(x,y)}d\rho(x,y), \quad f,g \in l_{\text{fin}}. \tag{46}$$

To construct the Fourier transform (41) and verify the formulas (42)–(46) it is still necessary to check that, for our operators A and B, the vector $\Omega = \delta_{0,0} \in l_{\text{fin}}$ is strong cyclic in the sense of the chain (34). But it is evidently true, since by (31), $A^p B^q \Omega = J_A^p J_B^q \delta_{0,0} = \delta_{p,q}$.

The Parseval equality (46) immediately leads to representation (26), according to (44), (45) $\hat{\delta}_{m,n} = x^m y^n$ and $\hat{\delta}_{0,0} = 1$; by (32) we get

$$s_{m,n} = (\delta_{m,n}, \delta_{0,0})_S = (\hat{\delta}_{m,n}, \hat{\delta}_{0,0})_{L_2(\mathbb{R}^2, d\rho(x,y))} = \int_{\mathbb{R}^2} x^m y^n d\rho(x,y), \quad m,n \in \mathbb{N}_0. \tag{47}$$

Uniqueness of representation (26) follows from selfadjointness and commutativity of the operators A and B (compare with [3], Ch.8). So, to finish the proof of Theorem 4 it is only necessary to check that condition (28) provides selfadjointness and commutativity of A and B. In the next step we use Theorem 3. But for this (see Theorems 2 and 3) we only must check that the operator $A^2 + B^2$ has a total set $\mathcal{D}$ of quasianalytic vectors.

Due to (30), the operator $\mathcal{A} = A^2 + B^2$ acts on $\delta_{m,n} \in \mathcal{D}$ as follows:

$$\mathcal{A}\delta_{m,n} = (A^2 + B^2)\delta_{m,n} = \delta_{m+2,n} + \delta_{m,n+2}. \tag{48}$$

Obviously $\mathcal{A} \geq 0$. For $p \geq 1$ we have

$$\mathcal{A}^p \delta_{m,n} = \sum_{k=0}^{p} C_p^k \delta_{m+2p-k,n+2k}.$$

According to (32) we have the norm $\|f\|_S = \sqrt{(f,f)_S}$ in S. Since $\forall \delta_{m,n} \in \mathcal{D}$ we obtain

$$\begin{aligned} \|\mathcal{A}^p \delta_{m,n}\|_S = \|\sum_{k=0}^{p} C_p^k \delta_{m+2p-k,n+2k}\|_S &\leq \sum C_p^k \|\delta_{m+2p-k,n+2k}\| \\ &= \sum C_p^k \sqrt{s_{2m+4p-2k,2m+4p-2k}}. \end{aligned} \tag{49}$$

Since

$$\sum_{p=1}^{\infty} \frac{1}{\sqrt[p]{\|\mathcal{A}^p \delta_{m,n}\|}} \geq \sum_{p=1}^{\infty} \frac{1}{\sqrt[p]{\sum_{k=0}^{p} C_p^k \sqrt{s_{2m+4p-4k,2n+4k}}}} = \infty, \quad m,n \in \mathbb{N}_0,$$

we conclude that quasianalyticity of the class $\mathrm{C}\{\|\mathcal{A}^p \delta_{m,n}\|\}$ follows from quasianalyticity of the class $\mathrm{C}\{\sqrt{\sum_{k=0}^{p} C_p^k \sqrt{s_{2m+4p-4k,2n+4k}}}\}$ due to quasianalyticity properties from [13, 27] it is equivalent to quasianalyticity of the class $\mathrm{C}\{\sqrt{\sum_{k=0}^{p} C_p^k \sqrt{s_{4p-4k,4k}}}\}$. But this quasianalyticity gives the condition (28), taking into account (49). This completes the proof of Theorem 4. □

Remark 4. It is not sufficient to have condition (27) for representation (26). Condition (27) is only necessary in Theorem 4. Let us consider a simple counterexample. Let A and B be two selfadjoint operators commuting on the linear set $\mathcal{D}$ dense in the Hilbert space $\mathcal{H}$, where $\mathcal{D}$ is invariant under the action of A and B. Suppose A and B are essentially selfadjoint on $\mathcal{D}$, i.e., $A = (A \upharpoonright \mathcal{D})^{\sim}$, $B = (B \upharpoonright \mathcal{D})^{\sim}$, but A and B do not commute in the strong resolvent sense. Existence of such operators guarantees the Nelson's example [28] (see also [5], Ch. 13, §9).

Remark 5. The formula–the condition (28) is new, but there are many other known conditions like (28) that guarantee unfitness of the measure in the Theorem 4, for example [4], see also [15, 38, 21, 22, 17, 18].

Remark 6. For obtaining representation (26) we use the Yu. M. Beresansky method of generalized eigenfunctions expansion. One says that we recovered the measure $d\rho(x,y)$. But we don't write any distribution function that all would like to have. In fact, we need the measure for calculations of corresponding integrals. And the important expression (47) gives us such a possibility. If we have to calculate the integral as in (46) for some $\hat{f}(x,y)$ and $\hat{g}(x,y)$ then we must to decompose $\hat{f}(x,y)$ and $\hat{g}(x,y)$ into a series of polynomials of $x^n y^m$ and using given $s_{m,n}$ and (47) to write (46) as corresponding series. And what is more with this connection the answer can by given with an arbitrary precision. Correctness of the last conclusion is guaranteed by Yu. M. Beresansky method (see [3, 4, 5, 7, 2, 6] and other publications).

4. Construction of a three-diagonal block Jacobi type matrices

Let $d\rho(x,y)$ be a Borel probability measure with compact support on the real plane $\mathbb{R}^2$ and $L_2 = L_2(\mathbb{R}^2, d\rho(x,y))$ the space of square integrable complex valued functions defined on $\mathbb{R}^2$. We suppose that the support of this measure is compact and assume that the functions $\mathbb{R}^2 \ni (x,y) \longmapsto x^m y^n,\ m,n \in \mathbb{N}_0$, are linear independent and form a total set in L_2.

Let us consider the operators of multiplications

$$\hat{A}f(x,y) = xf(x,y), \quad \hat{B}f(x,y) = yf(x,y)$$

on the space L_2. Obviously that these operators are bounded and selfadjoint. For finding Jacobi type matrices of operators $\hat{A}$ and $\hat{B}$ we choose some order of orthogonalization in L_2 for the following family of functions:

$$\{x^m y^n\}, \quad m, n \in \mathbb{N}_0. \tag{50}$$

We use the following linear order [35] for the orthogonalization according to Schmidt procedure (see for example [5], Ch. 7):

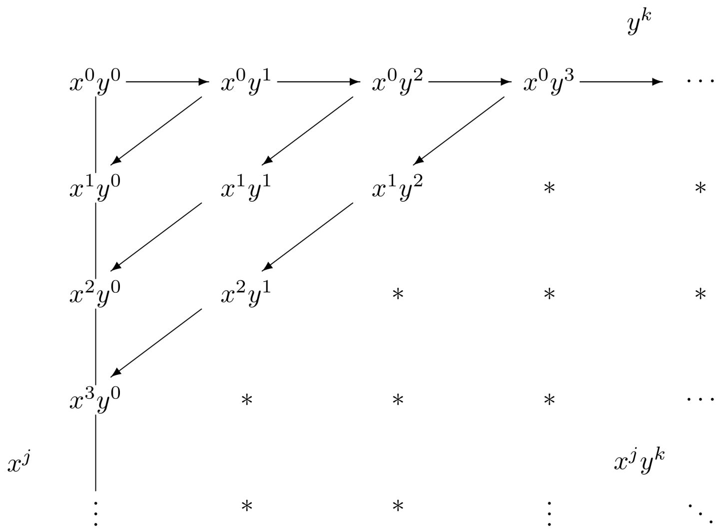

Figure 1. The orthogonalization order.

According to Figure 1 we get:

$$x^0y^0; \quad x^0y^1, x^1y^0; \quad x^0y^2, x^1y^1, x^2y^0; \quad \ldots ; \quad x^0y^n, x^1y^{n-1}, \ldots, x^ny^0; \quad \ldots \tag{51}$$

Considering the sequence of functions (51) we start the orthogonalization according to the Schmidt procedure. As a result we obtain the orthonormal system of polynomials (each polynomial is of $x^m y^n$, $m, n \in \mathbb{N}_0$) which we denote in the following way:

$$\begin{array}{lllll} P_{0;0}(x,y); & P_{1;0}(x,y), & P_{2;0}(x,y), & \ldots; & P_{n;0}(x,y), \quad \ldots \\ & P_{1;1}(x,y); & P_{2;1}(x,y), & & P_{n;1}(x,y), \\ & & P_{2;2}(x,y); & & P_{n;2}(x,y), \\ & & & & \quad\cdots \\ & & & & P_{n;n}(x,y), \end{array} \tag{52}$$

where each polynomial has a form $P_{n;\alpha}(x,y) = k_{n;\alpha}x^\alpha y^{n-\alpha} + \cdots$, $n \in \mathbb{N}_0$, $\alpha = 0, 1, \ldots, n$, $k_{n;\alpha} > 0$; here $+\cdots$ denotes the next part of the corresponding polynomial; we also put $P_{0;0}(x,y) = 1$. In such a way, $P_{n;\alpha}$ is some linear combination of

$$\{1;\ x^0y^1, x^1y^0;\ \ldots;\ x^0y^n, x^1y^{n-1}, \ldots, x^\alpha y^{n-\alpha}\}. \tag{53}$$

Due to totality of the family (50) in the space L_2, the sequence (52) is an orthonormal basis in this space.

Let us denote the subspace $\mathcal{P}_{n;\alpha}$, $\forall n \in \mathbb{N}$ spanned by (53). It is clear that $\forall n \in \mathbb{N}$ we have

$$\begin{aligned} &\mathcal{P}_{0;0} \subset \mathcal{P}_{1;0} \subset \mathcal{P}_{1;1} \subset \mathcal{P}_{2;0} \subset \mathcal{P}_{2;1} \subset \mathcal{P}_{2;2} \subset \cdots \subset \mathcal{P}_{n;0} \subset \mathcal{P}_{n;1} \subset \cdots \subset \mathcal{P}_{n;n} \subset \cdots, \\ &\mathcal{P}_{n;\alpha} = \{P_{0;0}(x,y)\} \oplus \{P_{1;0}(x,y)\} \oplus \{P_{1;1}(x,y)\} \\ &\qquad \oplus \{P_{2;0}(x,y)\} \oplus \{P_{2;1}(x,y)\} \oplus \{P_{2;2}(x,y)\} \\ &\qquad \oplus \cdots \oplus \{P_{n;0}(x,y)\} \oplus \{P_{n;1}(x,y)\} \oplus \cdots \oplus \{P_{n;\alpha}(x,y)\}, \end{aligned} \tag{54}$$

where $\{P_{n;\alpha}(x,y)\}$, $n \in \mathbb{N}$, $\alpha = 0, 1, \ldots, n$, denote the one dimensional space spanned by $P_{n;\alpha}(x,y)$; $\mathcal{P}_{0;0} = \mathbb{R}$.

For the next investigation we need, instead of the usual space l_2, the Hilbert space

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0. \tag{55}$$

Each vector $f \in \mathbf{l}_2$ has the form $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$, and, consequently,

$$\|f\|^2_{\mathbf{l}_2} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} < \infty, \quad (f,g)_{\mathbf{l}_2} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n}, \quad \forall f, g \in \mathbf{l}_2.$$

Coordinates of the vector $f_n \in \mathcal{H}_n$, $n \in \mathbb{N}_0$ in the some orthonormal basis $\{e_{n;0},\ e_{n;1},\ e_{n;2},\ \ldots,\ e_{n;n}\}$ in the space $\mathbb{C}^{n+1}$ are denoted by $(f_{n;0}, f_{n;1}, f_{n;2}, \ldots, f_{n;n})$ and, hence, we have $f_n = (f_{n;0}, f_{n;1}, f_{n;2}, \ldots, f_{n;n})$.

Using the orthonormal system (52) one can define a mapping of $\mathbf{l}_2$ into L_2. We put $P_n(x,y) = (P_{n;0}, P_{n;1}(x,y), P_{n;2}(x,y), \ldots, P_{n;n}) \in \mathcal{H}_n$, $\forall (x,y) \in \mathbb{R}^2$, $\forall n \in \mathbb{N}_0$. Then

$$\mathbf{l_2} \ni f = (f_n)_{n=0}^{\infty} \longmapsto (If)(x,y) := \hat{f}(x,y) = \sum_{n=0}^{\infty} (f_n, P_n(x,y))_{\mathcal{H}_n} \in L_2. \tag{56}$$

Since for $n \in \mathbb{N}_0$ we have

$$(f_n, P_n(x,y))_{\mathcal{H}_n} = f_{n;0}\overline{P_{n;0}(x,y)} + f_{n;1}\overline{P_{n;1}(x,y)} + f_{n;2}\overline{P_{n;2}(x,y)} + \cdots + f_{n;n}\overline{P_{n;n}(x,y)}$$

and

$$\|f\|^2_{\mathbf{l}_2} = \|(f_{0;0}, f_{1;0}, f_{1;1}, f_{2;0}, f_{2;1}, f_{2;2}, \ldots, f_{n;0}, , f_{n;1}, \ldots, f_{n;n}, \ldots)\|^2_{l_2},$$

(56) is a mapping of the space $\mathbf{l}_2$ into L_2 by the using of the orthonormal system (52) and, hence, this mapping is isometric. The image of $\mathbf{l}_2$ by the mapping (56) is equal to the space L_2, because under our assumption the system (52) is an orthonormal basis in L_2. Therefore the mapping (56) is a unitary transformation (denoted by I) that acts from $\mathbf{l}_2$ onto L_2.

Let T be a bounded linear operator defined on the space $\mathbf{l}_2$. Then there exists a unique $(\tau_{j,k})_{j,k=0}^{\infty}$, where for each $j, k \in \mathbb{N}_0$ the element $\tau_{j,k}$ is an operator from $\mathcal{H}_k$ into $\mathcal{H}_j$, so that

$$(Tf)_j = \sum_{k=0}^{\infty} \tau_{j,k} f_k, \quad j \in \mathbb{N}_0, \quad (Tf, g)_{\mathbf{l}_2} = \sum_{j,k=0}^{\infty} (\tau_{j,k} f_k, g_j)_{\mathcal{H}_j}, \quad f, g \in \mathbf{l}_2. \tag{57}$$

For a proof of (57) we only need to write the usual matrix of the operator T in the space $\mathbf{l}_2$ using the basis

$$(e_{0;0};\ e_{1;0}, e_{1;1};\ e_{2;0}, e_{2;1}, e_{2;2};\ \ldots;\ e_{n;0}, e_{n;1}, \ldots, e_{n;n};\ \ldots), \quad e_{0;0} = 1. \tag{58}$$

Then $\tau_{j,k}$ is an operator $\mathcal{H}_k \longrightarrow \mathcal{H}_j$ for each $j, k \in \mathbb{N}_0$. The operator has a matrix representation,

$$\tau_{j,k;\alpha,\beta} = (Te_{k;\beta}, e_{j;\alpha})_{\mathbf{l}_2}, \tag{59}$$

where $\alpha = 0, 1, \ldots, j$ and $\beta = 0, 1, \ldots, k$. We will write $\tau_{j,k} = (\tau_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ including cases $\tau_{0,k} = (\tau_{0,k;\alpha,\beta})_{\alpha,\beta=0}^{0,k}$, $\tau_{j,0} = (\tau_{j,0;\alpha,\beta})_{\alpha,\beta=0}^{j,0}$, and $\tau_{0,0} = (\tau_{0,0;\alpha,\beta})_{\alpha,\beta=0}^{0,0} = \tau_{0,0;0,0}$.

Note that the representation (57) is also valid for a general operator T on the space $\mathbf{l}_2$ with the domain $\mathrm{Dom}(T) = \mathbf{l}_{\mathrm{fin}} \subset \mathbf{l}_2$, where $\mathbf{l}_{\mathrm{fin}}$ denotes the set of finite vectors from $\mathbf{l}_2$. In this case the first formula in (57) takes place for $f \in \mathbf{l}_{\mathrm{fin}}$; in the second formula, $f \in \mathbf{l}_{\mathrm{fin}}$, $g \in \mathbf{l}_2$.

Let us consider the image $\hat{T} = ITI^{-1} : L_2 \longrightarrow L_2$ of the above mentioned bounded operator $T : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$ by the mapping (56). Its matrix in the basis (52),

$$P_{0;0}(x,y);\ P_{1;0}(x,y), P_{1;1}(x,y);\ P_{2;0}(x,y), P_{2,1}(x,y), P_{2,2}(x,y);\ \ldots;\\ P_{n;0}(x,y), P_{n;1}(x,y), \ldots, P_{n;n}(x,y); \ldots,$$

is equal to the usual matrix of operator T understanding as an operator $\mathbf{l}_2 \longrightarrow \mathbf{l}_2$ in the corresponding basis (58). Using (59) and the above mentioned procedure, we get an operator matrix $(\tau_{j,k})_{j,k=0}^{\infty}$ of $T : \mathbf{l}_2 \longrightarrow \mathbf{l}_2$. *By the definition this matrix is also an operator matrix of* $\hat{T} : L_2 \longrightarrow L_2$.

It is clear that $\hat{T}$ can be an arbitrary linear bounded operator in L_2. In the next text we consider T instead of $\hat{T}$, and the role of T will be played by A and B, corresponding to our matrices J_A and J_B.

Lemma 3. *For the polynomials $P_{n;\alpha}(x,y)$ and the subspaces $\mathcal{P}_{m,\beta}$, $n, m \in \mathbb{N}_0$, $\alpha = 0, 1, \ldots, n$, $\beta = 0, 1, \ldots, m$, the following relations hold:*

$$xP_{n;\alpha}(x,y) \in \mathcal{P}_{n+1;\alpha+1}, \quad yP_{n;\alpha}(x,y) \in \mathcal{P}_{n+1;\alpha}. \tag{60}$$

Proof. According to (52) the polynomial $P_{n;\alpha}(x,y)$, $n \in \mathbb{N}_0$, is equal to some linear combination of $\{1;\ x^0y^1,\ x^1y^0;\ \ldots;\ x^0y^n,\ x^1y^{n-1},\ \ldots,\ x^\alpha y^{n-\alpha}\}$. Hence, by multiplying by x we obtain a linear combination of $\{x;\ x^1y^1,\ x^2y^0;\ \ldots;\ x^1y^n,\ x^2y^{n-1},\ \ldots,\ x^{\alpha+1}y^{n-\alpha}\}$ and such a linear combination belongs to $\mathcal{P}_{n+1;\alpha+1}$. Analogously using multiplication by y we obtain a linear combination of $\{y^1;\ x^0y^2,\ x^1y^1;\ \ldots;\ x^0y^{n+1},\ x^1y^n,\ \ldots,\ x^\alpha y^{n-\alpha+1}\}$ and such a linear combination belongs to $\mathcal{P}_{n+1;\alpha}$. $\square$

Lemma 4. *Let $\hat{A}$ be an operator of multiplication by x in the space L_2,*

$$L_2 \ni \varphi(x,y) \longmapsto (\hat{A}\varphi)(x,y) = x\varphi(x,y) \in L_2.$$

(It is clear that $\hat{A}$ is selfadjoint and bounded.) The matrix $(a_{j,k})_{j,k=0}^{\infty}$ of $\hat{A}$ in basis (52) (i.e. of $A = I^{-1}\hat{A}I$) has a three-diagonal structure, $a_{j,k} = 0$ for $|j-k| > 1$.

Proof. Using (59) for $e_{n;\gamma} = I^{-1}P_{n;\gamma}(x,y)$, $n \in \mathbb{N}_0$; $\gamma = 0, 1, \ldots, n$, we have $\forall j, k \in \mathbb{N}_0$

$$a_{j,k;\alpha,\beta} = (Ae_{k;\beta}, e_{j;\alpha})_{l_2} = \int_{\mathbb{R}^2} xP_{k;\beta}(x,y)\overline{P_{j;\alpha}(x,y)}\,d\rho(x,y), \tag{61}$$

where $\alpha = 0, 1, \ldots, j$, $\beta = 0, 1, \ldots, k$. From (60) we have $xP_{k;\beta}(x,y) \in \mathcal{P}_{k+1;\beta+1}$. According to the expressions in (54), the integral in (61) is equal to zero for $j > k+1$ and for each $\beta = 0, 1, \ldots, j$.

On the other hand, the integral in (61) has the form

$$\begin{aligned}(a^*)_{j,k;\alpha,\beta} &= \overline{\int_{\mathbb{R}^2} xP_{k;\beta}(x,y)\overline{P_{j;\alpha}(x,y)}\,d\rho(x,y)}\\ &= \int_{\mathbb{R}^2} xP_{j;\alpha}(x,y)\overline{P_{k;\beta}(x,y)}\,d\rho(x,y) = \overline{a_{k,j;\beta,\alpha}},\end{aligned} \tag{62}$$

where $\alpha = 0, 1, \ldots, j$ and $\beta = 0, 1, \ldots, k$, since the operator $\hat{A}$ is symmetric. From (60) we have now that $xP_{j;\alpha}(x,y) \in \mathcal{P}_{j+1;\alpha+1}$. According to (54) the last integral is equal to zero for $k > j+1$ and for each $\alpha = 0, 1, \ldots, k$.

As a result, the integral in (61), i.e., the coefficients $a_{j,k;\alpha,\beta}$, $j, k \in \mathbb{N}_0$, are equal to zero for $|j-k| > 1$; $\alpha = 0, 1, \ldots, j$, $\beta = 0, 1, \ldots, k$. (In the previous considerations it was necessary to take into account that $e_{0;0} = I^{-1}P_{0;0}(x,y) = 1$). $\square$

In such a way the matrix $(a_{j,k})_{j,k=0}^{\infty}$ of the operator $\hat{A}$ has a three-diagonal block structure,

$$\begin{bmatrix} a_{0,0} & a_{0,1} & 0 & 0 & 0 & \dots \\ a_{1,0} & a_{1,1} & a_{1,2} & 0 & 0 & \dots \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} & 0 & \dots \\ 0 & 0 & a_{3,2} & a_{3,3} & a_{3,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{63}$$

A more careful analysis of expressions (61) gives a possibility to know about the zero and non zero elements of the matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ in each case for $|j-k|\leq 1$. We use also the permutation properties of matrix indexes j,k, and α,β.

Lemma 5. *Let $(a_{j,k})_{j,k=0}^{\infty}$ be an operator matrix of operator of multiplication by x in L_2, where $a_{j,k}:\mathcal{H}_k\longrightarrow\mathcal{H}_j$; $a_{j,k}=(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ are matrices of the operators $a_{j,k}$ in a corresponding orthonormal basis. Then $\forall j\in\mathbb{N}_0$,*

$$\begin{aligned} &\forall\alpha=0,1,\dots j-1 \quad a_{j,j+1;\alpha,\alpha+2}=a_{j,j+1;\alpha,\alpha+3}=\cdots=a_{j,j+1;\alpha,j+1}=0,\\ &\forall\beta=0,1,\dots j-1 \quad a_{j+1,j;\beta+2,\beta}=a_{j+1,j;\beta+3,\beta}=\cdots=a_{j+1,j;j+1,\beta}=0. \end{aligned} \tag{64}$$

If we choose inside of each diagonal $\{x^0y^n, x^ny^{n-1}, x^0y^{n-2},\dots,x^ny^0\}$, in Figure 1 another order (preserving the order of the diagonals) then Lemma 5 is not true but it is also possible to describe zeros of the matrices $(a_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$. Such matrices $(a_{j,k})_{j,k=0}^{\infty}$ have also three-diagonal block structure but have another elements equal to zero.

Proof. According to (61) and (60) for $j\in\mathbb{N}_0$, $\forall\alpha=0,1,\dots,j$ and $\forall\beta=0,1,\dots,j+1$, we have

$$a_{j,j+1;\alpha,\beta}=\int_{\mathbb{R}^2} xP_{j+1,\beta}(x,y)\overline{P_{j;\alpha}(x,y)}\,d\rho(x,y)=\overline{\int_{\mathbb{R}^2} xP_{j,\alpha}(x,y)\overline{P_{j+1;\beta}(x,y)}\,d\rho(x,y)},$$

where $xP_{j;\alpha}(x,y)\in\mathcal{P}_{j+1;\alpha+1}$. But according to (54) $P_{j+1;\beta}(x,y)$ is orthogonal to $\mathcal{P}_{j+1;\alpha+1}$ for $\beta>\alpha+1$ and, hence, the last integral is equal to zero. This gives the first equalities in (64).

Analogously from (61) and (60) for $j\in\mathbb{N}_0$ $\forall\alpha=0,1,\dots,j+1$ and $\forall\beta=0,1,\dots,j$ we have

$$a_{j+1,j;\alpha,\beta}=\int_{\mathbb{R}^2} xP_{j,\beta}(x,y)\overline{P_{j+1;\alpha}(x,y)}\,d\rho(x,y),$$

where $xP_{j;\beta}(x,y)\in\mathcal{P}_{j+1;\beta+1}$. But according to (54) $P_{j+1;\alpha}(x,y)$ is orthogonal to $\mathcal{P}_{j+1;\beta+1}$ if $\alpha>\beta+1$ and, hence, the last integral is equal to zero. This gives the second equalities in (64). □

We conclude from the above that in (63) for $\forall j\in\mathbb{N}$ the upper right corner of the every $((j+1)\times(j+2))$-matrix $a_{j,j+1}$ (starting from the third diagonal) and the lower left corner of the every $((j+2)\times(j+1))$-matrix $a_{j+1,j}$ (starting from the third diagonal) consist of zero elements. Taking into account (63) we can conclude that the matrix of the operator of multiplication by x is a multi-diagonal usual scalar matrix, i.e., in the usual basis of the space $\mathbf{l}_2$.

Lemma 6. *The elements*

$$a_{0,1;0,1}, a_{1,0;1,0}, \quad a_{j,j+1;\alpha,\alpha+1}, a_{j+1,j;\alpha+1,\alpha}, \quad j\in\mathbb{N}, \quad \alpha=0,1,\dots j, \tag{65}$$

of the matrix $(a_{j,k})_{j,k=0}^{\infty}$ from Lemma 8 are positive.

Proof. We start by looking at $a_{0,1;0,1}$. Denote by $P'_{1;1}(x,y)$ the non normalized vector $P_{1;1}(x,y)$. According to (51) and (52) we have

$$P'_{1;1}(x,y) = x - (x, P_{1;0}(x,y))_{L_2} P_{1;0}(x,y) - (x,1)_{L_2}.$$

Therefore using (79) we get

$$\begin{aligned} a_{0,1;0,1} &= \int_{\mathbb{R}^2} x P_{0;0} P_{1;1}(x,y)\, d\rho(x,y) = \|P'_{1;1}(x,y)\|^{-1}_{L_2} \int_{\mathbb{R}^2} x P'_{1;1}(x,y)\, d\rho(x,y) \\ &= \|P'_{1;1}(x,y)\|^{-1}_{L_2} \int_{\mathbb{R}^2} x(x - (x, P_{1;0}(x,y))_{L_2} P_{1;0}(x,y) - (x,1)_{L_2})\, d\rho(x,y) \\ &= \|P'_{1;1}(x,y)\|^{-1}_{L_2} (\|x\|^2_{L_2} - |(x, P_{1;0}(x,y))_{L_2}|^2 - |(x,1)_{L_2}|^2), \end{aligned} \tag{66}$$

where we took into account that $(1 = P_{0;0}(x,y))$.

Also using (85) we conclude that the last expression is positive and hence $a0,1;0,1 > 0$. Since the operator A is symmetric, $a_{0,1;0,1} = a_{1,0;1,0}$.

Positiveness in (66) follows from the Parseval equality for the decomposition of the function $y \in L_2$ with respect to the orthonormal basis (52) in the space L_2,

$$|(x,1)_{L_2}|^2 + |(x, P_{1;0}(x,y))_{L_2}|^2 + |(x, P_{1;1}(x,y))_{L_2}|^2 + \cdots = \|x\|^2_{L_2}. \tag{67}$$

Consider the elements $a_{j,j+1;\alpha,\alpha+1}$ where $j \in \mathbb{N}$, $\alpha = 0,1,\ldots,j$. From (79) we get

$$\begin{aligned} a_{j,j+1;\alpha,\alpha+1} &= \int_{\mathbb{R}^2} x P_{j+1,\alpha+1}(x,y) \overline{P_{j;\alpha}(x,y)}\, d\rho(x,y) \\ &\overline{= \int_{\mathbb{R}^2} x P_{j,\alpha}(x,y) \overline{P_{j+1;\alpha+1}(x,y)}\, d\rho(x,y).} \end{aligned} \tag{68}$$

For $P_{j;\alpha}(x,y)$ we have, according to (52) and (54), that

$$P_{j;\alpha}(x,y) = k_{j;\alpha} x^{\alpha} y^{j-\alpha} + R_{j;\alpha}(x,y), \tag{69}$$

where $R_{j;\alpha}(x,y)$ is some polynomial from $\mathcal{P}_{j;\alpha-1}$ if $\alpha > 0$ or from $\mathcal{P}_{j-1;j-1}$ if $\alpha = 0$. Therefore $xR_{j;\alpha}(x,y)$ is some polynomial from $\mathcal{P}_{j+1;\alpha}$ or from $\mathcal{P}_{j;j}$ (see (60) and (54)). Multiplying it by x we get

$$x P_{j;\alpha}(x,y) = k_{j;\alpha} x^{\alpha+1} y^{j-\alpha} + x R_{j;\alpha}(x,y), \tag{70}$$

where $xR_{j;\alpha}(x,y) \in \mathcal{P}_{j+1;\alpha}$ or to $\mathcal{P}_{j;j}$. On the other hand equality (69) gives

$$P_{j+1;\alpha}(x,y) = k_{j+1;\alpha} x^{\alpha+1} y^{j-\alpha} + R_{j+1;\alpha}(x,y), \tag{71}$$

where $R_{j+1;\alpha}(x,y) \in \mathcal{P}_{j+1;\alpha}$ if $\alpha > 0$ or belongs to $\mathcal{P}_{j;j}$, if $\alpha = 0$.

Find $x^{\alpha+1} y^{j-\alpha}$ from (71) and substitute it into (70). We get

$$\begin{aligned} x P_{j;\alpha}(x,y) &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}} (P_{j+1;\alpha}(x,y) - R_{j+1;\alpha}(x,y)) + x R_{j;\alpha}(x,y) \\ &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}} P_{j+1;\alpha}(x,y) - \frac{k_{j;\alpha}}{k_{j+1;\alpha}} R_{j+1;\alpha}(x,y) + x R_{j;\alpha}(x,y). \end{aligned} \tag{72}$$

The second two terms in (72) belong to $\mathcal{P}_{j+1;\alpha}$ or to $\mathcal{P}_{j;j}$ and are in any case orthogonal to $P_{j+1;\alpha+1}(x,y)$.

Therefore, substitution of the expression (72) into (68) gives $a_{j,j+1;\alpha,\alpha+1} = \frac{k_{j;\alpha}}{k_{j+1;\alpha+1}} > 0$. Since the matrix (81) is symmetric, the elements $a_{j,j+1;\alpha,\alpha+1} = a_{j,j+1;\alpha,\alpha+1}$ are also positive $j \in \mathbb{N}$, $\alpha = 0,1,\ldots,j$. □

In what follows we will use the following convenient notations for elements $a_{j,k}$ of the Jacobi matrix (63):

$$
\begin{array}{lcll}
a_n = a_{n+1,n} & : & \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, & \\
b_n = a_{n,n} & : & \mathcal{H}_n \longrightarrow \mathcal{H}_n, & \\
c_n = a_{n,n+1} & : & \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, & n \in \mathbb{N}_0.
\end{array} \tag{73}
$$

All the previous investigation are summarized in the following theorem.

Theorem 5. *The bounded selfadjoint operator $\hat{A}$ of multiplication by x in the space L_2 in the orthonormal basis (52) of polynomials has the form of a three-diagonal block Jacobi type symmetric matrix $J_A = (a_{j,k})_{j,k=0}^{\infty}$ which acts on the space (55)*

$$
\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0. \tag{74}
$$

The norms of all operators $a_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$ are uniformly bounded with respect to $j, k \in \mathbb{N}_0$. In notations (73), this matrix has the form

$$
J_A = \left[\begin{array}{cccccc}
b_0 & c_0 & 0 & 0 & 0 & \dots \\
a_0 & b_1 & c_1 & 0 & 0 & \dots \\
0 & a_1 & b_2 & c_2 & 0 & \dots \\
0 & 0 & a_2 & b_3 & c_3 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
$$

$$
= \left[\begin{array}{c|cc|ccc|cccc|ccccc}
* & * & + & & & & & & & & & & & & \\
\hline
* & * & * & * & + & 0 & & & & & & & & & \\
+ & * & * & * & * & + & & & & & & & 0 & & \\
\hline
 & * & * & * & * & * & * & + & 0 & 0 & & & & & \\
 & + & * & * & * & * & * & * & + & 0 & & & & & \\
 & 0 & + & * & * & * & * & * & * & + & & & & & \\
\hline
 & & & * & * & * & * & * & * & * & * & + & 0 & 0 & 0 \\
 & & & + & * & * & * & * & * & * & * & * & + & 0 & 0 \\
 & 0 & & 0 & + & * & * & * & * & * & * & * & * & + & 0 \\
 & & & 0 & 0 & + & * & * & * & * & * & * & * & * & + \\
\hline
 & & & & \ddots & & & & \ddots & & & & \ddots & &
\end{array}\right]. \tag{75}
$$

In (75) $\forall n \in \mathbb{N}_0$ b_n is an $((n+1)\times(n+1))$-matrix $b_n = (b_{n;\alpha,\beta})_{\alpha,\beta=0}^{n,n}$, ($b_0 = b_{0;0,0}$ is a scalar); a_n is an $((n+2)\times(n+1))$-matrix $a_n = (a_{n;\alpha,\beta})_{\alpha,\beta=0}^{n+1,n}$; c_n is an $((n+1)\times(n+2))$-matrix $c_n = (c_{n;\alpha,\beta})_{\alpha,\beta=0}^{n,n+1}$. In these matrices a_n and c_n, some elements are always equal to zero, $\forall n \in \mathbb{N}$

$$
\begin{array}{ll}
a_{n;\beta+2,\beta} = a_{n;\beta+3,\beta} = \cdots = a_{n;n+1,\beta} = 0, & \beta = 0, 1, \dots, n-1, \\
c_{n;\alpha,\alpha+2} = c_{n;\alpha,\alpha+3} = \cdots = c_{n;\alpha,n+1} = 0, & \alpha = 0, 1, \dots, n-1.
\end{array} \tag{76}
$$

Some other their elements are positive, namely $\forall n \in \mathbb{N}_0$

$$
a_{n;\alpha+1,\alpha}; c_{n;\alpha,\alpha+1} > 0, \quad \alpha = 0, 1, \dots, n, \quad \forall n \in \mathbb{N}_0. \tag{77}
$$

Thus, it is possible to say that $\forall n \in \mathbb{N}_0$ every lower left corner of the matrices a_n (starting from the third diagonal) and every upper right corner of the matrices c_n (starting from the third diagonal) consist of zero elements. All positive elements in (75) are denoted by "+".

So, the matrix (75) in the scalar form is multi-diagonals of the indicated structure.

The symmetry of the operator $(\hat{A})^* = \hat{A}$ gives

$$
a_{n;\alpha,\beta} = c_{n;\beta,\alpha}, \quad \beta = 0, 1, 2, \dots, n, \quad \alpha = 0, 1, \dots, \beta, \beta+1, \quad n \in \mathbb{N}_0.
$$

The matrix J_A acts as follows:

$$(78)\quad (J_A f)_n = a_{n-1}f_{n-1} + b_n f_n + c_n f_{n+1}, \quad n \in \mathbb{N}_0, \quad f_{-1} = 0, \quad \forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2.$$

Pass now to investigate the matrix J_B related to the variable y. For this reason we need a lemma like 3.

Lemma 7. *Let $\hat{B}$ be an operator of multiplication by y in the space L_2,*

$$L_2 \ni \varphi(x,y) \longmapsto (\hat{B}\varphi)(x,y) = y\varphi(x,y) \in L_2.$$

(It is clear that $\hat{B}$ is selfadjoint and bounded.) The operator matrix $(b_{j,k})_{j,k=0}^{\infty}$ of $\hat{B}$ in basis (52) (i.e. of $B = I^{-1}\hat{B}I$) has a three-diagonal structure, $b_{j,k} = 0$ for $|j-k| > 1$.

Proof. Using (59) for $e_{n;\gamma} = I^{-1}P_{n;\gamma}(x,y)$, $n \in \mathbb{N}_0$; $\gamma = 0,1,\dots,n$, we have $\forall j,k \in \mathbb{N}_0$

$$(79)\qquad b_{j,k;\alpha,\beta} = (Be_{k;\beta}, e_{j;\alpha})_{l_2} = \int_{\mathbb{R}^2} yP_{k;\beta}(x,y)\overline{P_{j;\alpha}(x,y)}d\rho(x,y),$$

where $\alpha = 0,1,\dots,j$, $\beta = 0,1,\dots,k$. From (60) we have $yP_{k;\alpha}(x,y) \in \mathcal{P}_{k+1;\alpha}$. According to (54) the integral in (79) is equal to zero for $j > k+1$ and for each $\alpha = 0,1,\dots,j$.

On the other hand, the integral in (79) has the form

$$(80)\qquad \begin{aligned} (b^*)_{j,k;\alpha,\beta} &= \overline{\int_{\mathbb{R}^2} yP_{k;\beta}(x,y)\overline{P_{j;\alpha}(x,y)}\,d\rho(x,y)} \\ &= \int_{\mathbb{R}^2} yP_{j;\alpha}(x,y)\overline{P_{k;\beta}(x,y)}\,d\rho(x,y) = \overline{b_{k,j;\beta,\alpha}}, \end{aligned}$$

where $\alpha = 0,1,\dots,j$ and $\beta = 0,1,\dots,k$. From (60) we have $yP_{j;\alpha}(x,y) \subset \mathcal{P}_{j+1,\alpha}$. According to (54) the integral (80) is equal to zero for $k > j+1$ for each $\alpha = 0,1,\dots,j$.

As a result, the integral in (79), i.e., the coefficients $b_{j,k;\alpha,\beta}$, $j,k \in \mathbb{N}_0$, are equal to zero for $|j-k| > 1$; $\alpha = 0,1,\dots,j$, $\beta = 0,1,\dots,k$. (In the previous considerations it is necessary to take into account that $e_{0;0} = I^{-1}P_{0;0}(x,y) = 1$). □

In such a way the matrix $(b_{j,k})_{j,k=0}^{\infty}$ of the operator $\hat{B}$ has a three-diagonal block structure,

$$(81)\qquad \begin{bmatrix} b_{0,0} & b_{0,1} & 0 & 0 & 0 & \dots \\ b_{1,0} & b_{1,1} & b_{1,2} & 0 & 0 & \dots \\ 0 & b_{2,1} & b_{2,2} & b_{2,3} & 0 & \dots \\ 0 & 0 & b_{3,2} & b_{3,3} & b_{3,4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

A more careful analysis of expressions (79) gives a possibility to know about the zero and non zero elements of the matrices $(b_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ in each case for $|j-k| \le 1$. We use also the permutation properties of the matrix indexes j,k, and α,β.

Lemma 8. *Let $(b_{j,k})_{j,k=0}^{\infty}$ be an operator matrix of multiplication by y in L_2, where $b_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$; $b_{j,k} = (b_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$ are matrices of the operators $b_{j,k}$ in a corresponding orthonormal basis. Then $\forall j \in \mathbb{N}_0$ and*

$$(82)\qquad \begin{aligned} &\forall\alpha = 0,1,\dots j \quad b_{j,j+1;\alpha,\alpha+1} = b_{j,j+1;\alpha,\alpha+2} = \dots = b_{j,j+1;\alpha,j+1} = 0, \\ &\forall\beta = 0,1,\dots j \quad b_{j+1,j;\beta+1,\beta} = b_{j+1,j;\beta+2,\beta} = \dots = b_{j+1,j;j+1,\beta} = 0. \end{aligned}$$

If we choose inside of each diagonal$\{x^0y^n, x^1y^{n-1}, x^2y^{n-2}, \dots, x^ny^0\}$, in Figure 1 another order (preserving the order of the diagonals) then Lemma 8 is not true but it is also possible to describe zeros of the matrices $(b_{j,k;\alpha,\beta})_{\alpha,\beta=0}^{j,k}$. Such matrices $(b_{j,k})_{j,k=0}^{\infty}$ also have a three-diagonal block structure and have zeros although in other places.

Proof. According to (79) and (60) for $j \in \mathbb{N}_0$, $\forall\alpha = 0, 1, \ldots, j$ and $\forall\beta = 0, 1, \ldots, j$ we have

$$b_{j,j+1;\alpha,\beta} = \int_{\mathbb{R}^2} yP_{j+1,\beta}(x,y)\overline{P_{j;\alpha}(x,y)}\,d\rho(x,y) = \overline{\int_{\mathbb{R}^2} yP_{j,\alpha}(x,y)\overline{P_{j+1;\beta}(x,y)}\,d\rho(x,y)},$$

where $yP_{j;\alpha}(x,y) \in \mathcal{P}_{j+1;\alpha}$. But according to (54) $P_{j+1;\beta}(x,y)$ is orthogonal to $\mathcal{P}_{j+1;\alpha}$ for $\beta > \alpha$ and, hence, the last integral is equal to zero. This gives the first equalities in (82).

Analogously from (79) and (60) for $j \in \mathbb{N}_0$, $\forall\alpha = 0, 1, \ldots, j+1$ and $\forall\beta = 0, 1, \ldots, j$ we have

$$b_{j+1,j;\alpha,\beta} = \int_{\mathbb{R}^2} yP_{j,\beta}(x,y)\overline{P_{j+1;\alpha}(x,y)}\,d\rho(x,y),$$

where $yP_{j;\beta}(x,y) \in \mathcal{P}_{j+1;\beta}$. But according to (54) $P_{j+1;\alpha}(x,y)$ is orthogonal to $\mathcal{P}_{j+1;\beta}$ if $\alpha > \beta$ and, hence, the last integral is equal to zero. This gives the second equalities in (64). $\square$

So, after these investigations we conclude that in (81) for $\forall j \in \mathbb{N}$ the upper right corner of the every $((j+1) \times (j+2))$-matrix $b_{j,j+1}$ (starting from the second diagonal) and the lower left corner of every $((j+2) \times (j+1))$-matrix $b_{j+1,j}$ (starting from the second diagonal) consist of zero elements. Taking into account (63) we can conclude that the symmetric matrix of the operator of multiplication by y is a multi-diagonal usual scalar matrix, i.e., in the usual basis of the space $\mathbf{l}_2$.

Lemma 9. *The elements*

$$b_{0,1;0,0}, b_{1,0;0,0}, \quad b_{j,j+1;\alpha,\alpha}, b_{j+1,j;\alpha,\alpha}, \quad j \in \mathbb{N}, \quad \alpha = 0, 1, \ldots j \tag{83}$$

of the matrix $(b_{j,k})_{j,k=0}^{\infty}$ *from Lemma 5 are positive.*

Proof. We start by looking at $b_{1,0;0,0}$. Using (61) and denoting by $P'_{1;0}(x,y) = y - (y,1)_{L_2}$ the non normalized vector $P_{1;0}(x,y)$ we get

(84)
$$\begin{aligned} b_{1,0;0,0} &= \int_{\mathbb{R}^2} yP_{0;0}(x,y)\overline{P_{1;0}(x,y)}\,d\rho(x,y) = \|P'_{1;0}(x,y)\|_{L_2}^{-1}\int_{\mathbb{R}^2} y\overline{(y-(y,1)_{L_2})}\,d\rho(x,y) \\ &= \|P'_{1;0}(x,y)\|_{L_2}^{-1}(\|y\|_{L_2}^2 - |(y,1)_{L_2}|^2), \end{aligned}$$

where we took into account $P_{0;0}(x,y) = 1$. The last difference is positive (see below, (85)), therefore $b_{1,0;0,0} > 0$.

The element $b_{0,1;0,0}$ is also positive since the matrix B is symmetric, i.e., $b_{1,0;0,0} = b_{0,1;0,0}$.

Positiveness in (84) follows from the Parseval equality for the decomposition of the function $y \in L_2$ with respect to the orthonormal basis (52) in the space L_2,

$$|(y,1)_{L_2}|^2 + |(y, P_{1;0}(x,y))_{L_2}|^2 + |(y, P_{1;1}(x,y))_{L_2}|^2 + \cdots = \|y\|_{L_2}^2. \tag{85}$$

Let us pass to a proof of positiveness of $b_{j+1,j;\alpha,\alpha}$, where $j \in \mathbb{N}$, $\alpha = 0, 1, \ldots, j$. From (61) we have

$$b_{j+1,j;\alpha,\alpha} = \int_{\mathbb{R}^2} yP_{j;\alpha}(x,y)\overline{P_{j+1;\alpha}(x,y)}\,d\rho(x,y). \tag{86}$$

According to (52) and (54)

$$P_{j;\alpha}(x,y) = k_{j;\alpha} x^{\alpha} y^{j-\alpha} + R_{j;\alpha}(x,y), \tag{87}$$

where $R_{j;\alpha}(x,y)$ is some polynomial from $\mathcal{P}_{j;\alpha-1}$ if $\alpha > 0$ or from $\mathcal{P}_{j-1;j-1}$ if $\alpha = 0$. Therefore $yR_{j;\alpha}(x,y)$ is some polynomial from $\mathcal{P}_{j+1;\alpha}$ or from $\mathcal{P}_{j;j-1}$ (see (60) and (54)). Multiplying (87) by x we conclude that

$$yP_{j;\alpha}(x,y) = k_{j;\alpha} x^{\alpha} y^{j-\alpha+1} + xR_{j;\alpha}(x,y), \tag{88}$$

where $yR_{j;\alpha}(x,y) \in \mathcal{P}_{j+1;\alpha-1}$ or to $\mathcal{P}_{j;j-1} \subset \mathcal{P}_{j;j}$.

On the other hand, equality (87) for $P_{j+1;\alpha}(x,y)$ gives

$$P_{j+1;\alpha}(x,y) = k_{j+1;\alpha} x^{\alpha} y^{j-\alpha+1} + R_{j+1;\alpha}(x,y), \tag{89}$$

where $R_{j+1;\alpha}(x,y) \in \mathcal{P}_{j+1;\alpha-1}$ or to $\mathcal{P}_{j;j}$.

Find $x^{\alpha} y^{j-\alpha+1}$ from (89) and substitute it into (88). We get

$$\begin{aligned} yP_{j;\alpha}(x,y) &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}} (P_{j+1;\alpha}(x,y) - R_{j+1;\alpha}(x,y)) + yR_{j;\alpha}(x,y) \\ &= \frac{k_{j;\alpha}}{k_{j+1;\alpha}} P_{j+1;\alpha}(x,y) - \frac{k_{j;\alpha}}{k_{j+1;\alpha}} R_{j+1;\alpha}(x,y) + yR_{j;\alpha(x,y)}(x,y), \end{aligned} \tag{90}$$

where second two terms belong to $\mathcal{P}_{j+1;\alpha-1}$ or to $\mathcal{P}_{j;j}$ and are in any case orthogonal to $P_{j+1;\alpha}(x,y)$.

Therefore after substituting the expression (90) into (86) we get that $b_{j+1,j;\alpha,\alpha} = \frac{k_{j;\alpha}}{k_{j+1;\alpha}} > 0$.

Since the matrix (63) is symmetric, the elements $b_{j,j+1;\alpha,\alpha+1} = b_{j+1,j;\alpha+1,\alpha}$ are also positive where $j \in \mathbb{N}$, $\alpha = 0, 1, \ldots, j$. ☐

In what follows it will be convenient to use the following notations for elements $b_{j,k}$ of the Jacobi matrix

$$\begin{array}{lcll} u_n = b_{n+1,n} & : & \mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, & \\ w_n = b_{n,n} & : & \mathcal{H}_n \longrightarrow \mathcal{H}_n, & \\ v_n = b_{n,n+1} & : & \mathcal{H}_{n+1} \longrightarrow \mathcal{H}_n, & n \in \mathbb{N}_0. \end{array} \tag{91}$$

All previous results are summarized in the theorem.

Theorem 6. *The bounded selfadjoint operator $\hat{B}$ of multiplication by y in the space L_2 in the orthonormal basis (52) of polynomials has the form of a three-diagonal block Jacobi type symmetric matrix $J_B = (b_{j,k})_{j,k=0}^{\infty}$ which acts in the space (55)*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0. \tag{92}$$

The norms of all operators $b_{j,k} : \mathcal{H}_k \longrightarrow \mathcal{H}_j$ are uniformly bounded with respect to $j, k \in \mathbb{N}_0$. In notations (73), this matrix has the form

$$J_B = \begin{bmatrix} w_0 & v_0 & 0 & 0 & 0 & \dots \\ u_0 & w_1 & v_1 & 0 & 0 & \dots \\ 0 & u_1 & w_2 & v_2 & 0 & \dots \\ 0 & 0 & u_2 & w_3 & v_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \left[\begin{array}{c|cc|ccc|cccc|ccccc} * & + & 0 & & & & & & & & & & & & \\ \hline + & * & * & + & 0 & 0 & & & & & & & & & \\ 0 & * & * & * & + & 0 & & & & & & 0 & & & \\ \hline & + & * & * & * & * & + & 0 & 0 & 0 & & & & & \\ & 0 & + & * & * & * & * & + & 0 & 0 & & & & & \\ & 0 & 0 & * & * & * & * & * & + & 0 & & & & & \\ \hline & & & + & * & * & * & * & * & * & + & 0 & 0 & 0 & 0 \\ & & & 0 & + & * & * & * & * & * & * & + & 0 & 0 & 0 \\ & 0 & & 0 & 0 & + & * & * & * & * & * & * & + & 0 & 0 \\ & & & 0 & 0 & 0 & * & * & * & * & * & * & * & + & 0 \\ \hline & & & & \ddots & & & & \ddots & & & & \ddots & & \end{array}\right]. \tag{93}$$

In (93) $\forall n \in \mathbb{N}_0$ w_n is an $((n+1)\times(n+1))$-matrix $w_n = (w_{n;\alpha,\beta})_{\alpha,\beta=0}^{n,n}$, ($w_0 = w_{0;0,0}$ is a scalar); u_n is an $((n+2)\times(n+1))$-matrix $u_n = (a_{n;\alpha,\beta})_{\alpha,\beta=0}^{n+1,n}$; v_n is an $((n+1)\times(n+2))$-matrix $v_n = (c_{n;\alpha,\beta})_{\alpha,\beta=0}^{n,n+1}$. In these matrices u_n and v_n some elements are always equal to zero: $\forall n \in \mathbb{N}_0$

$$\begin{aligned} u_{n;\beta+1,\beta} &= u_{n;\beta+2,\beta} = \cdots = u_{n;n+1,\beta} = 0, \quad \beta = 0, 1, \dots, n, \\ v_{n;\alpha,\alpha+1} &= v_{n;\alpha,\alpha+2} = \cdots = v_{n;\alpha,n+1} = 0, \quad \alpha = 0, 1, \dots, n. \end{aligned} \tag{94}$$

Some other their elements are positive, namely,

$$u_{n;\alpha,\alpha}; v_{n;\alpha,\alpha+1} > 0, \quad \alpha = 0, 1, \dots, n, \quad \forall n \in \mathbb{N}_0. \tag{95}$$

Thus, it is possible to say, that $\forall n \in \mathbb{N}_0$ every lower left corner of the matrices u_n (starting from the second diagonal) and every upper right corner of the matrices v_n (starting from the second diagonal) consist of zero elements. All positive elements in (93) are denoted by "+".

So, the matrix (93) in the scalar form is multi-diagonal of the indicated structure.

The the symmetry of the operator $(\hat{B})^* = \hat{B}$ gives

$$u_{n;\alpha,\beta} = v_{n;\beta,\alpha}, \quad \alpha = 0, 1, 2, \dots, n, \quad \beta = \alpha, \dots, n, \quad n \in \mathbb{N}.$$

The matrix J_B acts as follows:

$$(J_B f)_n = u_{n-1} f_{n-1} + w_n f_n + v_n f_{n+1}, \quad n \in \mathbb{N}_0, \quad f_{-1} = 0, \quad \forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2. \tag{96}$$

5. The direct spectral problems for two commuting three-diagonal block Jacobi type bounded symmetric operators

As it was mentioned above the main result of the previous section is the solution of the inverse problem for the corresponding direct one appearing in the title of this section.

We consider operators in the space $\mathbf{l}_2$ of the form (55). Additionally to the space $\mathbf{l}_2$ we consider its rigging

$$(\mathbf{l}_{\text{fin}})' \supset \mathbf{l}_2(p^{-1}) \supset \mathbf{l}_2 \supset \mathbf{l}_2(p) \supset \mathbf{l}_{\text{fin}}, \tag{97}$$

where $\mathbf{l}_2(p)$ is a weighted $\mathbf{l}_2$-space with a weight $p = (p_n)_{n=0}^{\infty}$, $p_n \geq 1$, $(p^{-1} = (p_n^{-1})_{n=0}^{\infty})$. In our case $\mathbf{l}_2(p)$ is the Hilbert space of sequences $f = (f_n)_{n=0}^{\infty}$, $f_n \in \mathcal{H}_n$ for which we have the norm and the scalar product

$$\|f\|^2_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} \|f_n\|^2_{\mathcal{H}_n} p_n, \quad (f, g)_{\mathbf{l}_2(p)} = \sum_{n=0}^{\infty} (f_n, g_n)_{\mathcal{H}_n} p_n. \tag{98}$$

The space $\mathbf{l}_2(p^{-1})$ is defined analogously; recall that $\mathbf{l}_{\text{fin}}$ is the space of finite sequences and $(\mathbf{l}_{\text{fin}})'$ is the space conjugate to $\mathbf{l}_{\text{fin}}$. It is easy to show that the embedding $\mathbf{l}_2(p) \hookrightarrow \mathbf{l}_2$ is quasinuclear if $\sum_{n=0}^{\infty} n p_n^{-1} < \infty$ (see, for example, [3], Ch. 7; [5], Ch. 15).

Let A and B be a pair of commuting bounded selfadjoint operators standardly connected with the chain (97). According to the projection spectral theorem (see [4], Ch. 3, Theorem 2.7; [3], Ch. 5; [5], Ch. 15; [31]) such an operator has the representation

$$Af = \int_{\mathbb{R}^2} x \Phi(x, y)\, d\sigma(x, y) f, \quad Bf = \int_{\mathbb{R}^2} y \Phi(x, y)\, d\sigma(x, y) f, \quad f \in \mathbf{l}_2, \tag{99}$$

where $\Phi(x, y) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is the operator of generalized projection and $d\sigma(x, y)$ is a spectral measure. For every $f \in \mathbf{l}_{\text{fin}}$ the projection $\Phi(x, y) f \in \mathbf{l}_2(p^{-1})$ is a generalized eigenvector of the operators A and B with corresponding eigenvalues x and y. For all $f, g \in \mathbf{l}_{\text{fin}}$ we have the Parseval equality

$$(f, g)_{\mathbf{l}_2} = \int_{\mathbb{R}^2} (\Phi(x, y) f, g)_{\mathbf{l}_2} d\sigma(x, y); \tag{100}$$

after closure by continuity the equality (100) is true $\forall f, g \in \mathbf{l}_2$.

Let us denote by π_n the operator of orthogonal projection in $\mathbf{l}_2$ on $\mathcal{H}_n$, $n \in \mathbb{N}_0$. Hence $\forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2$ we have $f_n = \pi_n f$. This operator acts analogously on the spaces $\mathbf{l}_2(p)$ and $\mathbf{l}_2(p^{-1})$.

Let us consider the operator matrix $(\Phi_{j,k}(x, y))_{j,k=0}^{\infty}$, where

$$\Phi_{j,k}(x, y) = \pi_j \Phi(x, y) \pi_k \ : \ \mathbf{l}_2 \longrightarrow \mathcal{H}_j, \quad (\mathcal{H}_k \longrightarrow \mathcal{H}_j). \tag{101}$$

The Parseval equality (100) can be rewritten as follows:

(102)

$$\begin{aligned}(f, g)_{\mathbf{l}_2} &= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}^2} (\Phi(x, y) \pi_k f, \pi_j g)_{\mathbf{l}_2} d\sigma(x, y) = \sum_{j,k=0}^{\infty} \int_{\mathbb{R}^2} (\pi_j \Phi(x, y) \pi_k f, g)_{\mathbf{l}_2} d\sigma(x, y) \\ &= \sum_{j,k=0}^{\infty} \int_{\mathbb{R}^2} (\Phi_{j,k}(x, y) f_k, g_j)_{\mathbf{l}_2} d\sigma(x, y), \quad \forall f, g \in \mathbf{l}_2.\end{aligned}$$

Let us now pass to a study of more special bounded operators A and B that act on the space $\mathbf{l}_2$. Namely, let they be given by matrices J_A and J_B which have a three-diagonal block structure of the form (75). So, these operators A and B are defined by the expressions in (96) and (78). Recall that the norm of all elements a_n, b_n, c_n and u_n, w_n, v_n are uniformly bounded with respect to $n \in \mathbb{N}_0$.

For further investigations *we suppose that conditions (94), (95) and (76), (77) are fulfilled and, additionally, the operators A and B given by (93) and (75) are bounded commuting selfadjoint on* $\mathbf{l}_2$. The conditions that such operators A and B to be bounded commuting will be investigated in Section 6.

In the next step we will rewrite the Parseval equality (102) in terms of generalized eigenvectors of the commuting selfadjoint operators A and B. At first we prove the following lemma.

Lemma 10. *Let $\varphi(x,y) = (\varphi_n(x,y))_{n=0}^{\infty}$, $\varphi_n(x,y) \in \mathcal{H}_n$, $(x,y) \in \mathbb{R}^2$, be a generalized eigenvector from $(\mathbf{l}_{\mathrm{fin}})'$ of the operator A with an eigenvalue x and also a generalized eigenvector of B with an eigenvalue y. By multiplying $\varphi(x,y)$ by a scalar constant (depending on x,y) we can obtain that $\varphi_0(x,y) = \varphi_0$ is independent of x,y. Thus $\varphi(x,y)$ is a solution in $(\mathbf{l}_{\mathrm{fin}})'$ of the two difference equations (see (96))*

$$\begin{gathered}(J_A\varphi(x,y))_n = a_{n-1}\varphi_{n-1}(x,y) + b_n\varphi_n(x,y) + c_n\varphi_{n+1}(x,y) = x\varphi_n(x,y),\\ (J_B\varphi(x,y))_n = u_{n-1}\varphi_{n-1}(x,y) + w_n\varphi_n(x,y) + v_n\varphi_{n+1}(x,y) = y\varphi_n(x,y),\\ n \in \mathbb{N}_0, \quad \varphi_{-1}(x,y) =: 0,\end{gathered} \tag{103}$$

with the initial condition $\varphi_0 \in \mathbb{R}$.

We assert that this solution is the following: $\forall n \in \mathbb{N}$

$$\varphi_n(x,y) = Q_n(x,y)\varphi_0 = (Q_{n;0}, Q_{n;1}, \ldots, Q_{n;n},)\varphi_0. \tag{104}$$

Here $Q_{n;\alpha}$, $\alpha = 0,1,\ldots,n$ are polynomials of x and y and these polynomials have the form

$$Q_{n;\alpha}(x,y) = l_{n;\alpha}y^{n-\alpha}x^{\alpha} + q_{n;\alpha}(x,y), \quad \alpha = 1,\ldots,n, \tag{105}$$

where $l_{n;\alpha} > 0$ and $q_{n;\alpha}(x,y)$ is a previous part due to (51) and it is some linear combinations of $y^j x^k$, $0 \le j+k \le n-1$, $y^{n-(\alpha-1)}x^{\alpha-1}$. The last expressions are described in the case $\alpha = 1,\ldots n$. We have $y^{n-1}x^{n-1}$ if $\alpha = 0$.

Proof. For $n = 0$, system (103) has the form

$$\begin{array}{lcll} w_0\varphi_0 + v_0\varphi_1 = y\varphi_0, & \text{or} & v_{0;0,0}\varphi_{1;0} & = (y - w_{0;0,0})\varphi_0,\\ b_0\varphi_0 + c_0\varphi_1 = x\varphi_0, & & c_{0;0,0}\varphi_{1;0} + c_{0;0,1}\varphi_{1;1} & = (x - b_{0;0,0})\varphi_0.\end{array} \tag{106}$$

Here and in what follows we denote

$$\varphi_n(x,y) = (\varphi_{n;0}(x,y), \varphi_{n;1}(x,y), \ldots, \varphi_{n;n}(x,y)) \in \mathcal{H}_n, \quad \forall n \in \mathbb{N}; \quad \varphi_0 = \varphi_{0;0}.$$

Using the assumption (94), (95) and (76), (77) we rewrite the last two equalities in (106) in the form

$$\begin{gathered}\Delta_0\varphi_1(x,y) = ((y - w_{0;0,0})\varphi_0, (x - b_{0;0,0})\varphi_0);\\ \Delta_0 = \begin{pmatrix} v_{0;0,0} & 0\\ c_{0;0,0} & c_{0;0,1}\end{pmatrix}, \quad v_{0;0,0} > 0, \quad c_{0;0,1} > 0.\end{gathered} \tag{107}$$

Therefore

$$\begin{aligned}\varphi_{1;0}(x,y) &= \frac{1}{v_{0;0,0}}(y - w_{0;0,0})\varphi_0 = Q_{1;0}(x,y)\varphi_0,\\ \varphi_{1;1}(x,y) &= \left(\frac{(x - b_{0;0,0})}{c_{0;0,1}} - \frac{c_{0;0,0}(y - w_{0;0,0})}{c_{0;0,1}v_{0;0,0}}\right)\varphi_0 = Q_{1;1}(x,y)\varphi_0.\end{aligned} \tag{108}$$

In other words, the solution $\varphi_n(x,y)$ of (103) for $n = 0$ has the form (104) and (105).

Suppose, using induction, that for $n \in \mathbb{N}$ the coordinates $\varphi_{n-1}(x,y)$ and $\varphi(x,y)$ of our generalized eigenvector $\varphi(x,y) = (\varphi_n(x,y))_{n=0}^{\infty}$ have the form (104) and (105) and prove that $\varphi_{n+1}(x,y)$ is also of the form (104) and (105).

Our eigenvector $\varphi(x,y)$ satisfies system (103) of two equations. But this system is overdetermined, — it consists of $2(n+1)$ scalar equations from which it is necessary to find only $n+2$ unknowns $\varphi_{n+1;0}$, $\varphi_{n+1;1}$, …, $\varphi_{n+1;n+1}$ using as an initial data the previous $n+1$ values $\varphi_{n;0}$, $\varphi_{n;1}$, …, $\varphi_{n;n}$ of coordinates of the vector $\varphi_n(x,y)$.

According to Theorems 6 and 5, especially to (94), (95) and (76), (77) the $((n+1)\times(n+2))$-matrices c_n, v_n and their application to $\varphi_{n+1}\in\mathcal{H}_n$ have the form

(109)
$$v_n\varphi_{n+1}(x,y)=\begin{bmatrix} v_{n;0,0} & 0 & 0 & \dots & 0 & 0\\ v_{n;1,0} & v_{n;1,1} & 0 & \dots & 0 & 0\\ v_{n;2,0} & v_{n;2,1} & v_{n;2,2} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ v_{n;n-1,0} & v_{n;n-1,1} & v_{n;n-1,2} & \dots & 0 & 0\\ v_{n;n,0} & v_{n;n,1} & v_{n;n,2} & \dots & v_{n;n,n} & 0\end{bmatrix}\varphi_{n+1}(x,y),$$
$$c_n\varphi_{n+1}(x,y)=\begin{bmatrix} c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0\\ c_{n;1,0} & c_{n;1,1} & c_{n;1,2} & \dots & 0 & 0\\ c_{n;2,0} & c_{n;2,1} & c_{n;2,2} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{n;n-1,0} & c_{n;n-1,1} & c_{n;n-1,2} & \dots & c_{n;n-1,n} & 0\\ c_{n;n,0} & c_{n;n,1} & c_{n;n,2} & \dots & c_{n;n,n} & c_{n;n,n+1}\end{bmatrix}\varphi_{n+1}(x,y),$$

where $\varphi_{n+1}(x,y)=(\varphi_{n+1;0}(x,y),\varphi_{n+1;1}(x,y),\dots,\varphi_{n+1;n+1}(x,y))$.

Construct similarly to (107) the following combination from the matrices (109): the $((n+2)\times(n+2))$-matrix

(110)
$$\Delta_n\varphi_{n+1}(x,y)=\begin{bmatrix} v_{n;0,0} & 0 & 0 & \dots & 0 & 0\\ c_{n;0,0} & c_{n;0,1} & 0 & \dots & 0 & 0\\ c_{n;1,0} & c_{n;1,1} & c_{n;1,2} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ c_{n;n-1,0} & c_{n;n-1,1} & c_{n;n-1,2} & \dots & c_{n;n-1,n} & 0\\ c_{n;n,0} & c_{n;n,1} & c_{n;n,2} & \dots & c_{n;n,n} & c_{n;n,n+1}\end{bmatrix}\varphi_{n+1}(x,y),$$

where $\varphi_{n+1}(x,y)=(\varphi_{n+1;0}(x,y),\varphi_{n+1;1}(x,y),\dots,\varphi_{n+1;n+1}(x,y))$.

The matrix (110) is invertible because its elements on the main diagonal are positive (see (95) and (77)). Rewrite the equalities (103) as follows:

(111)
$$\begin{aligned} c_n\varphi_{n+1}(x,y)&=x\varphi_n(x,y)-a_{n-1}\varphi_{n-1}(x,y)-b_n\varphi_n(x,y),\\ v_n\varphi_{n+1}(x,y)&=y\varphi_n(x,y)-u_{n-1}\varphi_{n-1}(x,y)-v_n\varphi_n(x,y),\quad n\in\mathbb{N}.\end{aligned}$$

We see that the first $n+2$ scalar equations (from $2(n+1)$ scalar equations (111)) have the form

(112)
$$\begin{aligned}\Delta_n\varphi_{n+1}(x,y)=\Big(&xQ_{n;0}(x,y)-(u_{n-1}Q_{n-1}(x,y)-(w_nQ_n(x,y))_{n;0},\\ &yQ_{n;0}(x,y)-(a_{n-1}Q_{n-1}(x,y))_{n;0}-(b_nQ_n(x,y))_{n;0},\dots,\\ &yQ_{n;n}(x,y)-(a_{n-1}Q_{n-1}(x,y))_{n;n}-(b_nQ_n(x,y))_{n;n}\Big)\varphi_0.\end{aligned}$$

The construction of matrix Δ_n and the form of vector in the right hand size in (112) and (104), (105) shows that

(113)
$$\begin{aligned}\varphi_{n+1;0}(x,y)&=Q_{n+1;0}(x,y)\varphi_0\\ &=\frac{1}{a_{n;0,0}}(xQ_{n;0}(x,y)-(u_{n-1}Q_{n-1}(x,y))_{n;0})-(w_nQ_n(x,y))_{n;0})\varphi_0\\ &=\frac{1}{a_{n;0,0}}(x(l_{n;0}x^n+q_{n;0}(x,y))-(u_{n-1}Q_{n-1}(x,y))_{n;0}-(w_nQ_n(x,y))_{n;0})\varphi_0,\end{aligned}$$

i.e., the main summand in the right hand side of (113) is equal to $\frac{l_{n;0}}{a_{n;0,0}}x^{n+1}y^0$, so it has the form (105).

An analogous calculation gives the same result for $\varphi_{n+1;1}(x,y)$, …, $\varphi_{n+1;n+1}(x,y)$. It is necessary to take into account that the next diagonal elements $v_{n;0,0}$, $c_{n;0,1}$, $c_{n;1,2}$, …, $c_{n;n,n+1}$ of the matrix Δ_n are positive due to (95) and (77). This completes the induction and finishes the proof. $\square$

Remark 7. Note, that we did not assert that a solution of the overdetermined system (103) exists for an arbitrary initial data $\varphi_0 \in \mathbb{R}$: we prove only that the generalized eigenvector from $(\mathbf{l}_{\mathrm{fin}})'$ of operators A and B is a solution of (103) and has the form (104) and (105).

In what follows, it will be convenient to look at $Q_n(x,y)$ with fixed x and y as a linear operator that acts from $\mathcal{H}_0$ into $\mathcal{H}_n$, i.e., $\mathcal{H}_0 \ni \varphi_0 \longmapsto Q_n(x,y)\varphi_0 \in \mathcal{H}_n$. We understand also $Q_n(x,y)$ as an operator-valued polynomial of $x, y \in \mathbb{R}^2$; hence, for the adjoint operator we have $Q_n^*(x,y) = (Q_n(x,y))^* : \mathcal{H}_n \longrightarrow \mathcal{H}_0$. Using these polynomials $Q_n(x,y)$ we construct the following representation for $\Phi_{j,k}(x,y)$.

Lemma 11. *The operator $\Phi_{j,k}(x,y)$, $\forall (x,y) \in \mathbb{R}^2$ has the following representation:*

$$\Phi_{j,k}(x,y) = Q_j(x,y)\Phi_{0,0}(x,y)Q_k^*(x,y) : \mathcal{H}_k \longrightarrow \mathcal{H}_j, \quad j,k \in \mathbb{N}_0, \tag{114}$$

where $\Phi_{0,0}(x,y) \geq 0$ is a scalar.

Proof. For fixed $k \in \mathbb{N}_0$, the vector $\varphi = \varphi(x,y) = (\varphi_j(x,y))_{j=0}^{\infty}$, where

$$\varphi_j(x,y) = \Phi_{j,k}(x,y) = \pi_j\Phi(x,y)\pi_k \in \mathcal{H}_j, \quad (x,y) \in \mathbb{R}^2, \tag{115}$$

is a generalized solution, in $(\mathbf{l}_{\mathrm{fin}})'$, of the equations

$$J_A\varphi(x,y) = x\varphi(x,y), \quad J_B\varphi(x,y) = y\varphi(x,y),$$

since $\Phi(x,y)$ is a projection on generalized eigenvectors of the operators A and B with correspondence generalized eigenvalues (x,y). Hence, it follows that $\varphi = \varphi(x,y) \in \mathbf{l}_2(p^{-1})$ exists as a usual solution of the equation $J_A\varphi = x\varphi$, $J_B\varphi = y\varphi$ with the initial condition $\varphi_0 = \pi_0\Phi(x,y)\pi_k \in \mathcal{H}_0$.

Using Lemma 10 and due to (104) we obtain

$$\Phi_{j,k}(x,y) = Q_j(x,y)(\Phi_{0,k}(x,y)), \quad j \in \mathbb{N}_0. \tag{116}$$

The operator $\Phi(x,y) : \mathbf{l}_2(p) \longrightarrow \mathbf{l}_2(p^{-1})$ is formally selfadjoint on $\mathbf{l}_2$, being the derivative of the resolution of identity of the operator A on $\mathbf{l}_2$ with respect to the spectral measure. Hence, according to (114) we get

$$(\Phi_{j,k}(x,y))^* = (\pi_j\Phi(x,y)\pi_k)^* = \pi_k\Phi(x,y)\pi_j = \Phi_{k,j}(x,y), \quad j,k \in \mathbb{N}_0. \tag{117}$$

For fixed $j \in \mathbb{N}_0$ from (117) and previous conversation, it follows that the vector

$$\varphi = \varphi(x,y) = (\varphi_k(x,y))_{k=0}^{\infty}, \quad \varphi_k(x,y) = \Phi_{k,j}(x,y) = (\Phi_{j,k}(x,y))^*$$

is a usual solution of the equations $J_A\varphi = x\varphi$ and $J_B\varphi = y\varphi$ with the initial condition $\varphi_0 = \Phi_{0,j}(x,y) = (\Phi_{j,0}(x,y))^*$.

Again using Lemma 10 we obtain the representation of type (116),

$$\Phi_{k,j}(x,y) = Q_k(x,y)(\Phi_{0,j}(x,y)), \quad k \in \mathbb{N}_0. \tag{118}$$

Taking into account (117) and (118) we get

$$\Phi_{0,k}(x,y) = (\Phi_{k,0}(x,y))^* = (Q_k(x,y)\Phi_{0,0}(x,y))^* = \Phi_{0,0}(x,y)(Q_k(x,y))^*, \quad k \in \mathbb{N}_0 \tag{119}$$

(here we used that $\Phi_{0,0}(x,y) \geq 0$, this inequality follows from (100) and (101)). Substituting (119) into (116) we obtain (114). $\square$

Now it is possible to rewrite the Parseval equality (102) in a more concrete form. To this end, we substitute the expression (114) for $\Phi_{j,k}(x,y)$ into (102) and get that

$$
\begin{aligned}
(f,g)_{\mathbf{l}_2} &= \sum_{j,k=0}^{\infty}\int_{\mathbb{R}^2}(\Phi_{j,k}(x,y)f_k,g_j)_{\mathbf{l}_2}d\sigma(x,y)\\
&= \sum_{j,k=0}^{\infty}\int_{\mathbb{R}^2}(Q_j(x,y)\Phi_{0,0}(x,y)Q_k^*(x,y)f_k,g_j)_{\mathbf{l}_2}d\sigma(x,y)\\
&= \sum_{j,k=0}^{\infty}\int_{\mathbb{R}^2}(Q_k^*(x,y)f_k,Q_j^*(x,y)g_j)_{\mathbf{l}_2}d\rho(x,y) \qquad (120)\\
&= \int_{\mathbb{R}^2}\Big(\sum_{k=0}^{\infty}Q_k^*(x,y)f_k\Big)\overline{\Big(\sum_{j=0}^{\infty}Q_j^*(x,y)g_j\Big)}d\rho(x,y),
\end{aligned}
$$
$$d\rho(x,y)=\Phi_{0,0}(x,y)\,d\sigma(x,y),\quad \forall f,g\in\mathbf{l}_{\mathrm{fin}}.$$

Introduce the Fourier transform $\widehat{\ }$ induced by the commuting selfadjoint operators A and B on the space $\mathbf{l}_2$,

$$\mathbf{l}_2\supset\mathbf{l}_{\mathrm{fin}}\ni f=(f_n)_{n=0}^{\infty}\longmapsto\hat f(x,y)=\sum_{n=0}^{\infty}Q_n^*(x,y)f_n\in L_2(\mathbb{R}^2,d\rho(x,y)). \qquad (121)$$

Hence, (120) gives the Parseval equality in a final form,

$$(f,g)_{\mathbf{l}_2}=\int_{\mathbb{R}^2}\hat f(x,y)\overline{\hat g(x,y)}\,d\rho(x,y),\quad\forall f,g\in\mathbf{l}_{\mathrm{fin}}. \qquad (122)$$

Extending (122) by continuity, it becomes valid $\forall f,g\in\mathbf{l}_2$.

Orthogonality of the polynomials $Q_n^*(x,y)$ follows from (121) and (122). Namely, it is sufficient only to take $f=(0,\dots,0,f_k,0,\dots)$, $f_k\in\mathcal{H}_k$, $g=(0,\dots,0,g_j,0,\dots)$, $g_j\in\mathcal{H}_j$ in (121) and (122). Then

$$\int_{\mathbb{R}^2}(Q_k^*(x,y)f_k)\overline{(Q_j^*(x,y)g_j)}\,d\rho(x,y)=\delta_{j,k}(f_j,g_j)_{\mathcal{H}_j},\quad\forall k,j\in\mathbb{N}_0. \qquad (123)$$

Using representation (104) for these polynomials we can rewrite the equality (123) in a usual classical scalar form. To do this, we remark that in general $Q_0^*(x,y)=\bar Q_0(x,y)$ and for $n\in\mathbb{N}$ according to (104), $Q_n(x,y)=(Q_{n;0}(x,y),Q_{n;1}(x,y),\dots,Q_{n;n}(x,y)):\mathcal{H}_0\longrightarrow\mathcal{H}_n$. Hence, for the adjoint operator $Q_n^*(x,y):\mathcal{H}_n\longrightarrow\mathcal{H}_0$ we have

$$
\begin{aligned}
(Q_n(x,y)q,p)_{\mathcal{H}_n} &= ((Q_{n;0}(x,y)q,Q_{n;1}(x,y)q,\dots,Q_{n;n}(x,y)q),(p_0,p_1,\dots,p_n))_{\mathcal{H}_n}\\
&= Q_{n;0}(x,y)q\bar p_0+Q_{n;1}(x,y)q\bar p_1+\dots+Q_{n;n}(x,y)q\bar p_n\\
&= q\overline{(\overline{Q_{n;0}(x,y)}p_0+\overline{Q_{n;1}(x,y)}p_1+\dots+\overline{Q_{n;n}(x,y)}p_n)}=(q,Q_n^*(x,y)p)_{\mathcal{H}_0},
\end{aligned}
$$

that is, $Q_n^*(x,y)p=\overline{Q_{n;0}(x,y)}p_0+\overline{Q_{n;1}(x,y)}p_1+\dots+\overline{Q_{n;n}(x,y)}p_n$, $\forall q\in\mathcal{H}_0$, and $p=(p_0,p_1,\dots,p_n)\in\mathcal{H}_n$.

Due to the last equality for $n\in\mathbb{N}$ and $f_n=(f_{n,0},f_{n,1},\dots,f_{n,n})\in\mathcal{H}_n$, we obtain

$$Q_n^*(x,y)f_n=\overline{Q_{n;0}(x,y)}f_{n;0}+\overline{Q_{n;1}(x,y)}f_{n;1}+\dots+\overline{Q_{n;n}(x,y)}f_{n;n},\quad Q_0^*(x,y)=1. \qquad (124)$$

Therefore (123) has the form $\forall f_{k;0},f_{k;1},\dots,f_{k;k},\ g_{j;0},g_{j;1},\dots,g_{j;j}\in\mathbb{C}$, $j,k\in\mathbb{N}_0$,

$$\int_{\mathbb{R}^2}\Big(\sum_{\alpha=0}^{k}\overline{Q_{k;\alpha}(x,y)}f_{k;\alpha}\Big)\overline{\Big(\sum_{\beta=0}^{j}\overline{Q_{j;\beta}(x,y)}f_{j;\beta}\Big)}\,d\rho(x,y)=\delta_{j,k}\sum_{\alpha=0}^{j}f_{j;\alpha}\bar g_{j;\alpha}.$$

This equality is equivalent to the following orthogonality relation in the usual classical form:

$$\int_{\mathbb{R}^2} \overline{Q^*_{k;\beta}(x,y)} Q_{j;\alpha} d\rho(x,y) = \delta_{j,k}\delta_{\alpha,\beta} \quad (Q_{0;0} = Q_0(x,y)), \tag{125}$$

$\forall j,k \in \mathbb{N}_0$, $\forall \alpha = 0,1,\ldots,j, \beta = 0,1,\ldots,k$.

Let us remark that due to (124) the Fourier transform (121) can be rewritten as

$$\hat{f}(x,y) = \sum_{n=0}^{\infty}\sum_{\alpha=0}^{n} \overline{Q_{n;\alpha}(x,y)} f_{n;\alpha}, \quad \forall f = (f_n)_{n=0}^{\infty} \in \mathbf{l}_2. \quad (x,y) \in \mathbb{R}^2, \tag{126}$$

Using the stated above results of this section, we can formulate the following spectral theorem for our bounded commuting symmetric operators A and B.

Theorem 7. *Consider the space (55)*

$$\mathbf{l}_2 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus, \cdots, \quad \mathcal{H}_n = \mathbb{C}^{n+1}, \quad n \in \mathbb{N}_0, \tag{127}$$

and linear operators A and B which are defined on finite vectors $\mathbf{l}_{\mathrm{fin}}$ by a block three-diagonal Jacobi type matrices J_A and J_B of the form (93) and (75) with the help of expressions in (96) and (78). We suppose that all the coefficients a_n, b_n, c_n, and u_n, v_n, w_n, $n \in \mathbb{N}_0$, are uniformly bounded, some elements of these matrices are equal to zero or positive according to (94), (95) and (76), (77) and the closure of A and B by continuity are bounded commuting selfadjoint operators on this space.

The eigenfunction expansion of the operators A and B has the following form. According to Lemma 10 we represent, using $\varphi_0 \in \mathbb{R}$, the solution $\varphi(x,y) = (\varphi_n(x,y))_{n=0}^{\infty}$, $\varphi_n(x,y) \in \mathcal{H}_n$, of equations (103) (which exists due to the projection spectral theorem) for $(x,y) \in \mathbb{R}^2$

$$\varphi_n(x,y) = Q_n(x,y)\varphi_0 = (Q_{n;0}(x,y), Q_{n;1}(x,y), \cdots, Q_{n;n}(x,y))\varphi_0,$$

where $Q_{n;\alpha}(x,y)$, $\alpha = 0,1,\ldots,n$ are polynomials of x and y. Then the Fourier transform has the form

$$\begin{aligned} \mathbf{l}_2 \supset \mathbf{l}_{\mathrm{fin}} \ni f = (f_n)_{n=0}^{\infty} \longmapsto \hat{f}(x,y) &= \sum_{n=0}^{\infty} Q^*_n(x,y) f_n \\ &= \sum_{n=0}^{\infty}\sum_{\alpha=0}^{n} \overline{Q_{n;\alpha}(x,y)} f_{n;\alpha} \in L_2(\mathbb{R}^2, d\rho(x,y)). \end{aligned} \tag{128}$$

Here $Q^*_n(x,y) : \mathcal{H}_n \longrightarrow \mathcal{H}_0$ is the adjoint to the operator $Q_n(x,y) : \mathcal{H}_0 \longrightarrow \mathcal{H}_n$, $d\rho(x,y)$ is the probability spectral measure of A and B.

The Parseval equality has the following form: $\forall f,g \in \mathbf{l}_{\mathrm{fin}}$

$$\begin{aligned} (f,g)_{\mathbf{l}_2} &= \int_{\mathbb{R}^2} \hat{f}(x,y)\overline{\hat{g}(x,y)}\, d\rho(x,y), \\ (J_A f,g)_{\mathbf{l}_2} &= \int_{\mathbb{R}^2} x\hat{f}(x,y)\overline{\hat{g}(x,y)}\, d\rho(x,y), \\ (J_B f,g)_{\mathbf{l}_2} &= \int_{\mathbb{R}^2} y\hat{f}(x,y)\overline{\hat{g}(x,y)}\, d\rho(x,y). \end{aligned} \tag{129}$$

Identities (128) and (129) are extended by continuity to $\forall f,g \in \mathbf{l}_2$ making the operator (128) unitary, which maps $\mathbf{l}_2$ onto the whole $L_2(\mathbb{R}^2, d\rho(x,y))$.

The polynomials $\overline{Q_{n;\alpha}(x,y)}$, $n \in \mathbb{N}$, $\alpha = 0,1,\ldots,n$, and $Q_{0;0}(x,y) = 1$, form an orthonormal system in $L_2(\mathbb{R}^2, d\rho(x,y))$ in the sense of (125) and it is total in this space.

Proof. It is only necessary to show that the orthogonal polynomials $\overline{Q_{n;\alpha}(x,y)}$, $n \in \mathbb{N}$, $\alpha = 0,1,\dots,n$, and $Q_{0;0}(x,y) = 1$ form a total set in the space $L_2(\mathbb{R}^2, d\rho(x,y))$. For this reason we remark at first that due to the compactness of the support of the measure $d\rho(x,y)$ on $\mathbb{R}^2$, the elements $x^j y^k$, $j,k \in \mathbb{N}_0$, form a total set in $L_2(\mathbb{R}^2, d\rho(x,y))$.

Let us suppose the contrary, i.e., that our system of polynomials is not total. Then there exist non zero function $h(x,y) \in L_2(\mathbb{R}^2, d\rho(x,y))$ that is orthogonal to all these polynomials and hence, according to (105) to all $x^j y^k$, $j,k \in \mathbb{N}_0$. Hence $h(x,y) = 0$. □

The last theorem solves the direct problem for the bounded symmetric commuting operators A and B that are generated on the space $\mathbf{l}_2$ by the matrices J_A and J_B of the form (93) and (75).

The inverse problem consists in a construction, from a given measure $d\rho(x,y)$ on $\mathbb{R}^2$ with compact support, a bounded symmetric commuting matrices J_A and J_B of the form (93) and (75) that have their spectral measure equal to $d\rho(x,y)$. *This construction is conducted according to Theorem 6 and 5, with a use of the Schmidt orthogonalization procedure for the system (51). For matrices J_A and J_B of the form (93) and (93), which are constructed from $d\rho(x,y)$, the spectral measure of the corresponding bounded symmetric commuting operators A and B coincides with the initial measure.*

Proof. This fact is true, since the system of orthogonal polynomials, connected with A and B, $\overline{Q_{n,\alpha}(x,y)}$ $\alpha = 0,1,\dots,n$, $n \in \mathbb{N}_0$, are orthonormal in $L_2(\mathbb{R}^2, d\rho(x,y))$ and, according to Lemma 10, are constructed from $y^j x^k$, $(x,y) \in \mathbb{R}^2$, in the same way as the system (52) is constructed from $x^j y^k$, $j,k \in \mathbb{N}_0$. Hence,

$$Q_0(x,y) = P_0(x,y) = 1, \quad \overline{Q_{n,\alpha}(x,y)} = P_{n;\alpha}(x,y), \quad \alpha = 0,1,\dots,n, \ \forall n \in \mathbb{N}. \tag{130}$$

Since both systems of polynomials form a total set in $L_2(\mathbb{R}^2, d\rho(x,y))$, (130) shows that the spectral measures constructed from the operator and the given one coincide. □

Let us remark that the expressions (61) and (79) (as it was known in the classical theory of Jacobi matrices) reestablish the initial matrices (93) and (75) from the spectral measure $d\rho(x,y)$ of operators generated by J_A and J_B on $\mathbf{l}_2$.

6. On condition of commutativity of Jacobi type block matrices

We will find conditions that would guarantee that the matrix J_A commutes with the matrix J_B of type (9) and (11). Multiplying these matrices we get

(131)
$$J_A J_B = \begin{bmatrix} b_0w_0 + c_0v_0 & b_0v_0 + c_0w_1 & c_0v_1 & 0 & 0 & \cdots \\ a_0w_0 + b_1u_0 & a_0v_0 + b_1w_1 + c_1u_1 & b_1v_1 + c_1w_2 & c_1v_2 & 0 & \cdots \\ a_1u_0 & a_1w_1 + b_2u_1 & a_1v_1 + b_2w_2 + c_2u_2 & b_2v_2 + c_2w_3 & c_2v_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The expression for $J_A J_B$ is analogous to (131) if a_n, b_n and c_n are replaced with u_n, w_n and v_n, respectively and vice versa.

Comparing these expressions for $J_A J_B$ and $J_B J_A$ we conclude that the equality $J_A J_B = J_B J_A$ is equivalent to fulfillment of the following equalities (we take into account that b_0 is a scalar, $w_0 = b_0$):

$$\begin{aligned} &c_0u_0 = v_0a_0; \\ &c_nv_{n+1} = v_nc_{n+1}, \\ &b_nv_n + c_nw_{n+1} = w_nc_n + v_nb_{n+1}, \\ &a_nv_n + b_{n+1}w_{n+1} + c_{n+1}u_{n+1} = u_nc_n + w_{n+1}b_{n+1} + v_{n+1}a_{n+1}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{132}$$

Note that the necessary equalities

$$a_nw_n + b_{n+1}u_n = u_nb_n + w_{n+1}a_n, \quad a_{n+1}u_n = u_{n+1}a_n, \quad n \in \mathbb{N}_0,$$

follow from third and second equalities of (132) writing them in an adjoint form.

So, conditions (132) are necessary and sufficient for the matrix equality $J_A J_B = J_B J_A$ to hold. When the norms of the operators a_n, b_n, c_n and u_n, w_n, v_n are uniformly bounded w.r.t. $n \in \mathbb{N}_0$, the operators J_A and J_B on $\mathbf{l}_2$ are bounded, selfadjoint and conditions (132) gives commutativity of these operators.

Taking the initial matrices a_0, b_0, c_0 and finding a_1, b_1, c_1; a_2, b_2, c_2 from (132) step by step, etc. (in non-uniquely manner) we can construct some symmetric matrices J_A and J_B. But for such matrices Theorem 7 in general is not valid, because it is necessary to find these matrices in such way that a_n, c_n and u_n, v_n must be of the form (10) and (12) (i.e. (93) and (75)). Only in this case according to Lemma 10 and (119) the functions (17) are linearly independent and Theorem 7 is applicable (or the condition of Remark 2 is fulfilled for A and B).

It is a sufficiently complicated problem to find matrices a_n, b_n, c_n and u_n, w_n, v_n, $n \in \mathbb{N}_0$ which make a solution of equations (132) and a_n, c_n and u_n, v_n to have the form (10) and (12), so we investigate here only some special cases.

Namely, we assume in the first place that all the matrices $w_n = b_n$, $n \in \mathbb{N}_0$. Then the conditions (132) can be rewritten in the following form:

$$
\begin{aligned}
&c_0 u_0 = v_0 a_0;\\
&c_n v_{n+1} = v_n c_{n+1},\\
&b_n(v_n - c_n) = (v_n - c_n) b_{n+1},\\
&a_n v_n + c_{n+1} u_{n+1} = u_n c_n + v_{n+1} a_{n+1}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{133}
$$

Then we put $w_n = b_n = 0$, $n \in \mathbb{N}_0$. Hence

$$
\begin{aligned}
&c_0 u_0 = v_0 a_0;\\
&c_n v_{n+1} = v_n c_{n+1},\\
&a_n v_n + c_{n+1} u_{n+1} = u_n c_n + v_{n+1} a_{n+1}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{134}
$$

Further, we assume that all the matrices a_n, c_n and u_n, v_n, $n \in \mathbb{N}_0$, have the form (10) and (12) where $a_{n;1,0}, a_{n;2,1}, \dots, a_{n;n+1,n}$, $c_{n;0,1}, c_{n;1,2}, \dots, c_{n;n,n+1}$, $u_{n;0,0}, u_{n;1,1}, \dots, u_{n;n,n}$, $v_{n;0,0}, v_{n;1,1}, \dots, v_{n;n,n}$, $\quad \forall n \in \mathbb{N}_0$, are positive and all other elements of these matrices are equal to zero. So, our matrices are the following:

(135)

$$
u_n = \underbrace{\left.\begin{bmatrix}
u_{n;0,0} & 0 & \dots & 0\\
0 & u_{n;1,1} & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & u_{n;n,n}\\
0 & 0 & \dots & 0
\end{bmatrix}\right\}}_{n+1} n+2, \quad
a_n = \underbrace{\left.\begin{bmatrix}
0 & 0 & \dots & 0\\
a_{n;1,0} & 0 & \dots & 0\\
0 & a_{n;2,1} & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & a_{n;n+1,n}
\end{bmatrix}\right\}}_{n+2} n+1,
$$

$$
v_n = \underbrace{\left.\begin{bmatrix}
v_{n;0,0} & 0 & \dots & 0 & 0\\
0 & v_{n;1,1} & \dots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \dots & v_{n;n,n} & 0
\end{bmatrix}\right\}}_{n+2} n+1, \quad
c_n = \underbrace{\left.\begin{bmatrix}
0 & c_{n;0,1} & 0 & \dots & 0\\
0 & 0 & c_{n;1,2} & \dots & 0\\
0 & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \dots & c_{n;n,n+1}
\end{bmatrix}\right\}}_{n+1} n+2,
$$

$n \in \mathbb{N}_0$.

Multiplying the matrices of type (135) we can rewrite the first, the second and the fourth equality from (134) in the form of corresponding equalities for elements of these matrices,

(136)

$$\begin{array}{ll}
v_{0;0,0}c_{1;0,1} = v_{1;1,1}c_{0;0,1}, & v_{0;0,0}c_{0;0,1} = v_{1;1,1}c_{1;0,1}, \\
\\
v_{1;0,0}c_{2;0,1} = v_{2;1,1}c_{1;0,1}, & v_{1;0,0}c_{1;0,1} = v_{2;1,1}c_{2;0,1}, \\
v_{1;1,1}c_{2;1,2} = v_{2;2,2}c_{1;1,2}, & v_{1;1,1}c_{1;1,2} = v_{2;2,2}c_{2;1,2}, \\
\\
v_{2;0,0}c_{3;0,1} = v_{3;1,1}c_{2;0,1}, & v_{2;0,0}c_{2;0,1} = v_{3;1,1}c_{3;0,1}, \\
v_{2;1,1}c_{3;1,2} = v_{3;2,2}c_{2;1,2}, & v_{2;1,1}c_{2;1,2} = v_{3;2,2}c_{3;1,2}, \\
v_{2;2,2}c_{3;2,3} = v_{3;3,3}c_{2;2,3}, & v_{2;2,2}c_{2;2,3} = v_{3;3,3}c_{3;2,3}, \\
\\
\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots \\
\\
v_{n;0,0}c_{n+1;0,1} = v_{n+1;1,1}c_{n;0,1}, & v_{n;0,0}c_{n;0,1} = v_{n+1;1,1}c_{n+1;0,1}, \\
v_{n;1,1}c_{n+1;1,2} = v_{n+1;2,2}c_{n;1,2}, & v_{n;1,1}c_{n;1,2} = v_{n+1;2,2}c_{n+1;1,2}, \\
\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots \\
v_{n;n,n}c_{n+1;n,n+1} = v_{n+1;n+1,n+1}c_{n;n,n+1}, & v_{n;n,n}c_{n;n,n+1} = v_{n+1;n+1,n+1}c_{n+1;n,n+1},
\end{array}$$

where it is assumed that $v_{n;k,k} = u_{n;k,k}$, and $c_{n;k,k+1} = a_{n;k+1,k}$, $k = 0, 1, ..., n$, $\forall n \in \mathbb{N}_0$. The system of equalities (136) is equivalent to system (133) in the case where all $b_n = w_n = 0$, $n \in \mathbb{N}_0$.

Each system consisting of two equations in each line in (136) using positivity of $v_{n;k,k}$ and $c_{n;k,k+1}$, $k = 0, 1, \dots, n$, $\forall n \in \mathbb{N}_0$ gives

$$\begin{array}{lllll}
c_{1;0,1} = c_{0;0,1}, & c_{2;0,1} = c_{1;0,1}, & \dots, & c_{n+1;0,1} = c_{n;0,1}; \\
 & c_{2;1,2} = c_{1;1,2}, & \dots, & c_{n+1;1,2} = c_{n;1,2}; \\
 & & \dots, & \dots\dots\dots\dots \\
 & & \dots, & c_{n+1;n,n+1} = c_{n;n,n+1}.
\end{array} \tag{137}$$

We summarize the last previous investigations in the Proposition.

Proposition 1. *The matrices J_A and J_B of the form (93) and (75) with the coefficients a_n, c_n and u_n, v_n as in (135) and $b_n = w_n = 0$, $n \in \mathbb{N}_0$ are commuting symmetric if for an arbitrary $v_{n;k,k} = u_{n;k,k} < c < \infty$, $k = 0, 1, \dots, n$, the numbers $c_{n;k,k+1} = c_{n+1;k,k+1}$, $k = 0, 1, \dots, n$, $\forall n \in \mathbb{N}_0$ satisfy equalities (137).*

Proof. For a construction of J_A and J_B, it suffices to choose the bounded sequence $\{\delta_n\}$, $|\delta_n| < c_1 < \infty$, $n \in \mathbb{N}_0$ and to put $c_{n;k,k+1} := \delta_n$, $k = 0, 1, ..., n$, $\forall n \in \mathbb{N}_0$. □

The Proposition 1 gives a possibility to generate a number of simple examples.

Example 1. Let us put $b_n = w_n = 0$ and for (135) $v_{n;k,k} = u_{n;k,k} = c_{n;k,k+1} = a_{n;k+1,k} = 1$, $k = 0, 1, ..., n$, $\forall n \in \mathbb{N}_0$. Then by an obvious observation we obtain matrices of the type J_A and J_B that satisfy all the conditions described in Theorems 6 and 5. Let us denote such matrices by J_{A1} and J_{B1}).

Example 2. Using the previous example let us put $J_{A11} := J_{A1} + J_{B1}$. This matrix: 1) is obviously symmetric (as a sum of two symmetric one); 2) is a bounded operator (as a sum of two bounded operators); 3) commutes with J_{B1}, since J_{A1} commutes with J_{B1} and J_{B1} commutes with it self.

Remark 8. The matrix J_{A11} from the previous example is the adjacency matrix of the following unbounded graph

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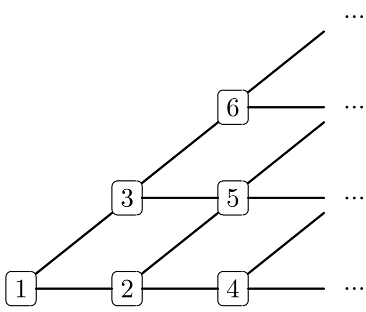

FIGURE 2. The graph of the adjacency matrix J_{A11}.

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