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ON BEHAVIOR AT INFINITY OF SOLUTIONS OF PARABOLIC DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract. For a differential equation of the form $y'(t) + Ay(t) = 0,\ t \in (0,\infty)$, where A is the generating operator of a C_0-semigroup of linear operators on a Banach space $\mathfrak{B}$, we give conditions on the operator A, under which this equation is uniformly (uniformly exponentially) stable, that is, every its weak solution defined on the open semiaxis $(0,\infty)$ tends (tends exponentially) to 0 as $t \to \infty$. As distinguished from the previous works dealing only with solutions continuous at 0, in this paper no conditions on the behavior of a solution near 0 are imposed. In the case where the equation is parabolic, there always exist weak solutions which have singularities of any order. The criterions below not only generalize, but make more precise a number of earlier results in this direction.

1. On extensions of differentiable semigroups of linear operators on a Banach space

Let $\mathfrak{F}$ be a locally convex Hausdorff space. Recall (see [1]) that a one-parameter family $\{U(t)\}_{t\geq 0}$ of continuous linear operators from $\mathfrak{F}$ into $\mathfrak{F}$ forms a semigroup in $\mathfrak{F}$ if:
(i) $U(0) = I$ (I is the identity operator in $\mathfrak{F}$) ;
(ii) $\forall t, s > 0 : U(t+s) = U(t)U(s)$.

In the sequel we consider only strongly continuous at the point 0 semigroups, that is, C_0-semigroups. A C_0-semigroup is called equicontinuous if for any continuous semi-norm $p(x)$ on $\mathfrak{F}$, there exists another continuous semi-norm $q(x)$ such that $p(U(t)x) \leq q(x)$ $(\forall t \geq 0,\ \forall x \in \mathfrak{F})$. The linear operator A defined as

$$Ax = \lim_{t\to 0} \frac{x - U(t)x}{t}, \quad \mathcal{D}(A) = \left\{ x \in \mathfrak{F} \,\middle|\, \exists \lim_{t\to 0} \frac{x - U(t)x}{t} \right\},$$

($\mathcal{D}(\cdot)$ denotes the domain of an operator) is called the generating operator or, simply, the generator of $\{U(t)\}_{t\geq 0}$. The fact that A is the generator of a semigroup $\{U(t)\}_{t\geq 0}$ is written as $U(t) = e^{-At}$.

As a rule we shall deal with C_0-semigroups on a Banach space $\mathfrak{B}$. For any such a semigroup $\{U(t)\}_{t\geq 0}$, the value

$$\omega_0 = \lim_{t\to\infty} \frac{\ln \|U(t)\|}{t}$$

is finite ($\|\cdot\|$ is the norm in $\mathfrak{B}$); it is called the type of $\{U(t)\}_{t\geq 0}$. The resolvent set of the operator $-A$ contains the half-plane $\mathrm{Re}\lambda > \omega_0$.

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A C_0-semigroup $\{U(t) = e^{-At}\}_{t\geq 0}$ on $\mathfrak{B}$ is called (strongly) differentiable if for any $x \in \mathfrak{B}$, the $\mathfrak{B}$-valued function $U(t)x$ is strongly differentiable on $(0, \infty)$. As is known (see [2]), for such a semigroup

$$\forall x \in \mathfrak{B}, \forall t > 0 : U(t)x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n),$$

the vector-valued function $U(t)x$ is infinitely differentiable on $(0, \infty)$, and

$$\forall x \in \mathfrak{B}, \forall t > 0, \forall n \in \mathbb{N} : \frac{d^n U(t)x}{dt^n} = (-1)^n A^n U(t)x.$$

Let now $\theta \in \left(0, \frac{\pi}{2}\right]$. A C_0-semigroup $\{U(t)\}_{t\geq 0}$ on $\mathfrak{B}$ is called analytic with angle θ if the operator-valued function $U(\cdot)$ is defined in the sector $S_\theta = \{z : |\arg z| < \theta\}$ and possesses the following properties:

1) $\forall z_1, z_2 \in S_\theta : U(z_1 + z_2) = U(z_1)U(z_2)$;
2) $\forall x \in \mathfrak{B} : U(z)x$ is analytic in S_θ;
3) $\forall x \in \mathfrak{B} : \|U(z)x - x\| \to 0$ as $z \to 0$ in any closed subsector of S_θ.

If in addition the family $U(z)$ is bounded on every sector S_ψ with $\psi < \theta$, then $U(t)$ is called a bounded analytic semigroup with angle θ.

We shall say (see [3]) that an infinitely differentiable vector x of the operator A, that is, $x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$, is analytic (entire) for this operator if for some number $\alpha > 0$ (for any $\alpha > 0$) there exists a constant $c = c(x) > 0$ ($c = c(x, \alpha) > 0$) such that

$$\forall n \in \mathbb{N} : \|A^n x\| \leq c\alpha^n n!.$$

Denote by $\mathfrak{A}(A)$ and $\mathfrak{A}_c(A)$ the sets of all analytic and entire vectors of the operator A, respectively. If the C_0-semigroup $\{e^{-At}\}_{t\geq 0}$ is analytic, then (see [3]) $\overline{\mathfrak{A}_c(A)} = \mathfrak{B}$, $\forall x \in \mathfrak{B}, \forall t > 0 : e^{-At}x \in \mathfrak{A}(A)$, and

$$\mathfrak{A}(A) = \bigcup_{t>0} e^{-At}\mathfrak{B}, \quad \mathfrak{A}_c(A) = \bigcap_{t\geq 0} e^{-At}\mathfrak{B}.$$

In what follows we may assume, without loss of generality, that A is the generator of a contraction C_0-semigroup on $\mathfrak{B}$ and $\ker e^{-At} = \{0\}$ as $t > 0$.

Let $\mathfrak{B}_{-t}(A), t > 0$, be the completion of $\mathfrak{B}$ in the norm

$$\|x\|_{\mathfrak{B}_{-t}(A)} = \|e^{-At}x\|.$$

Since the norms $\|\cdot\|_{\mathfrak{B}_{-t}(A)}$, $t \in (0, \infty)$, are coordinated and comparable on $\mathfrak{B}$, we have for $t < t'$ the dense and continuous embedding $\mathfrak{B}_{-t}(A) \subseteq \mathfrak{B}_{-t'}(A)$. Set

$$\mathfrak{B}_{-}(A) = \operatorname*{proj\,lim}_{t\to 0} \mathfrak{B}_{-t}(A).$$

It should be noted that for obtaining $\mathfrak{B}_{-}(A)$, it suffices to be restricted to the spaces $\mathfrak{B}_{-\frac{1}{n}}(A), n \in \mathbb{N}$. So, $\mathfrak{B}_{-}(A)$ is a complete countably normed space.

The operator e^{-At} admits a continuous extension $\widetilde{U}(t)$ to the space $\mathfrak{B}_{-t}(A)$. By virtue of continuity of the embedding $\mathfrak{B}_{-t}(A) \subseteq \mathfrak{B}_{-t'}(A)$ as $t < t'$, we have $\widetilde{U}(t') \restriction_{\mathfrak{B}_{-t}(A)} = \widetilde{U}(t)$.

On the space $\mathfrak{B}_{-}(A)$ we define the operators $U(t), t \geq 0$, in the following way:

$$\forall x \in \mathfrak{B}_{-}(A) : U(t)x = \widetilde{U}(t)x \quad \text{if} \quad t > 0; \quad U(0)x = x. \tag{1}$$

The following assertion was proved in [3].

Proposition 1. *The family $\{U(t)\}_{t\geq 0}$ forms an equicontinuous C_0-semigroup on the space $\mathfrak{B}_{-}(A)$, possessing such properties:*

1) $U(t)\mathfrak{B}_{-}(A) \subseteq \mathfrak{B}$ as $t > 0$;
2) $\forall x \in \mathfrak{B} : U(t)x = e^{-At}x$;
3) $\forall x \in \mathfrak{B}_{-}(A),\ \forall t, s > 0 : U(t + s)x = e^{-At}U(s)x = e^{-As}U(t)x.$

Denote by $\widehat{A}$ the generator of the semigroup $\{U(t)\}_{t\geq 0}$.

Proposition 2. *If the semigroup* $\left\{e^{-At}\right\}_{t\geq 0}$ *is differentiable on* $(0,\infty)$, *then* $\mathfrak{B}\subset\mathfrak{B}_-(A)$ *strictly, the operator* $\widehat{A}$ *is continuous in* $\mathfrak{B}_-(A)$, *and the semigroup* $\left\{U(t)=e^{-\widehat{A}t}\right\}_{t\geq 0}$ *is infinitely differentiable in* $\mathfrak{B}_-(A)$ *on* $[0,\infty)$. *Moreover, for arbitrary fixed* $x\in\mathfrak{B}_-(A)$ *and* $t>0$, $U(t)x\in\bigcap\limits_{n\in\mathbb{N}}\mathcal{D}(A^n)$, *the vector-valued function* $U(t)x, t\in(0,\infty)$, *is infinitely differentiable in* $\mathfrak{B}$, *and if the semigroup* $\{e^{-At}\}_{t\geq 0}$ *is analytic on* $(0,\infty)$, *then* $U(t)x\in\mathfrak{A}(A)$ *and the function* $U(t)x$ *is analytic on* $(0,\infty)$.

2. On solutions of an abstract parabolic equation on $(0,\infty)$ in a Banach space

Consider an equation of the form

$$\frac{dy(t)}{dt}+Ay(t)=0,\quad t\in(0,\infty), \tag{2}$$

where A is a closed linear operator on $\mathfrak{B}$. A vector-valued function $y(t):[0,\infty)\mapsto\mathcal{D}(A)$ is called a strong solution of equation (2) if it is continuously differentiable on $[0,\infty)$ and satisfies this equation on $[0,\infty)$. By a weak solution of equation (2) on $(0,\infty)$ we mean a vector-valued function $y(t)\in C((0,\infty),\mathfrak{B})$ such that

$$\forall s,t\in(0,\infty):\int\limits_s^t y(r)\,dr\in\mathcal{D}(A)$$

and

$$y(t)-y(s)=-A\int\limits_s^t y(r)\,dr.$$

Since A is closed, a weak solution $y(t)$ of equation (2) on $(0,\infty)$ is its strong one if and only if $y(t)\in C^1([0,\infty),\mathfrak{B})$. If $\mathcal{D}(A)=\mathfrak{B}$, then every weak solution is strong, and it admits a continuation to an entire vector-valued function of exponential type.

In the case, where A is the generator of a C_0-semigroup $\{e^{-At}\}_{t\geq 0}$ on $\mathfrak{B}$, the set of all strong solutions of (2) is described (see [4]) by the formula

$$y(t)=e^{-At}x,\quad x\in\mathcal{D}(A).$$

It is not hard to see that the vector-valued function

$$y(t)=e^{-At}x,\quad x\in\mathfrak{B}, \tag{3}$$

is a continuous at 0 weak solution of (2). As was shown in [5], formula (3) gives all such solutions when x runs over the whole $\mathfrak{B}$.

Our most urgent problem is to characterize all weak solutions of equation (2) on $(0,\infty)$ and investigate their behavior near 0. Note that no condition on a weak solution at the point 0 are preassigned. The following assertion holds true (see [3]).

Proposition 3. *Let* A *be the generator of a differentiable semigroup* $\{e^{-At}\}_{t\geq 0}$ *on* $\mathfrak{B}$. *Then each weak solution* $y(t)$ *of equation* (2) *on* $(0,\infty)$ *has a boundary value* y_0 *at the point 0 in the space* $\mathfrak{B}_-(A)$ ($y(t)\to y_0$ *in the* $\mathfrak{B}_-(A)$*-topology*), *and*

$$y(t)=U(t)y_0=e^{-\widehat{A}t}y_0, \tag{4}$$

where $U(t)$ *is determined in* (1).

Conversely, for any element $y_0\in\mathfrak{B}_-(A)$, *the vector-valued function* (4) *is a weak solution of* (2) *on* $(0,\infty)$.

The next statement follows directly from Propositions 1 and 3.

Corollary 1. *Let A be the generator of a differentiable (analytic) C_0-semigroup on $\mathfrak{B}$. Then every weak solution of equation* (2) *on* $(0, \infty)$ *is infinitely differentiable (analytic) on* $(0, \infty)$.

Corollary 1 implies a number of classical theorems on smoothness inside a domain of weak solutions for parabolic partial differential equations.

Note also that sometimes $\mathfrak{B}_-(A) = \mathfrak{B}$. For example, this will be the case if A is the generator of a C_0-group (partial differential equations of hyperbolic type). The Proposition 3 shows that in this situation any weak solution of (2) is continuous at 0. If A is the generator of a differentiable semigroup, then the space $\mathfrak{B}_-(A)$ is larger than $\mathfrak{B}$.

By the weak Cauchy problem for equation (2) we mean the problem of finding a weak solution of (2) on $(0, \infty)$, which satisfies the condition

$$\lim_{t \to 0} y(t) = y_0 \in \mathfrak{B}_-(A). \tag{5}$$

(The limit is taken in the $\mathfrak{B}_-(A)$-topology).

The Propositions 1–3 imply

Corollary 2. *Let a C_0-semigroup $\{e^{-At}\}_{t \geq 0}$ on $\mathfrak{B}$ is differentiable. Then whatever vector $y_0 \in \mathfrak{B}_-(A)$, the weak Cauchy problem* (2), (5) *is uniquely solvable. The solution is represented in form* (4).

As it follows from Proposition 3, every weak solution $y(t)$ of equation (2) has a boundary value at 0 in the space $\mathfrak{B}_-(A)$, that is, $\lim_{t \to 0} y(t)$ exists in the $\mathfrak{B}_-(A)$-topology. The question arises of finding the weak solutions whose boundary values at 0 belong to the initial space $\mathfrak{B}$. The following assertion is valid (see [3]).

Proposition 4. *Suppose the space $\mathfrak{B}$ to be reflexive. If A is the generator of a C_0-semigroup $\{e^{-At}\}_{t \geq 0}$ on $\mathfrak{B}$ and $y(t)$ is a weak solution of equation* (2) *on* $(0, \infty)$, *then*

$$y(t) \to y_0 \ \textit{ in } \ \mathfrak{B} \ \textit{ as } \ t \to 0 \iff \|y(t)\| \leq c < \infty, \quad 0 < c = \text{const}.$$

So, the boundedness in the norm of $\mathfrak{B}$ of a weak solution of equation (2) in a neighborhood of 0 is equivalent to its continuity at 0 in $\mathfrak{B}$ (analog of the Fatou and Riesz theorems for functions analytic in a disk or half-plane). It should be noted that the reflexivity of $\mathfrak{B}$ plays an essential role. There are examples of nonreflexive $\mathfrak{B}$ for which Proposition 4 is not correct. For instance, the L_1-boundedness of a harmonic in a disk or half-plane function on concentric circles or lines parallel to the real axis, respectively, does not yet imply the existence of the L_1-limit of such a function when approaching the boundary of a domain (see [6]). Observe also that Propositions 3, 4 contain a number of well-known results from the boundary values theory for solutions of partial differential equations in various classical function spaces (see e.g. [3]).

3. On behavior at infinity of solutions of parabolic differential equations in a Banach space

We say that equation (2) is:

1) *uniformly stable* if

$$\lim_{t \to \infty} y(t) = 0 \tag{6}$$

for any its weak solution $y(t)$;

2) *uniformly exponentially stable* if

$$\exists \omega > 0 : \lim_{t \to \infty} e^{\omega t} y(t) = 0 \tag{7}$$

for all weak solutions of this equation.

If $\dim \mathfrak{B} < \infty$, both the definitions are equivalent. But this is, in general, not the case if $\dim \mathfrak{B} = \infty$.

Since no condition on behavior near 0 of a weak solution is imposed, it is possible for such a solution to have a singularity when approaching to 0, that is, $\lim\limits_{t\to 0} y(t) = \infty$; moreover, the order of growth of $y(t)$ as $t \to 0$ may be arbitrary.

In the case where the Cauchy problem for equation (2) is well-posed (A is the generator of a C_0-semigroup), it suffices in the definitions 1), 2) to require for equalities (6),(7) to be fulfilled at least for all continuous weak solutions. More exactly, the following theorem takes place.

Theorem 1. *Let A be the generator of a contraction C_0-semigroup $\{e^{-At}\}_{t\geq 0}$ on $\mathfrak{B}$ such that $\ker e^{-At} = \{0\}$ for any $t > 0$. In order that equation* (2) *be uniformly (uniformly exponentially) stable, it is necessary and sufficient that equality* (6) *(equality* (7)*) hold true for all its weak solutions $y(\cdot) \in C([0,\infty), \mathfrak{B})$. If the semigroup $\{e^{-At}\}_{t\geq 0}$ is differentiable (analytic) on $(0,\infty)$, it suffices for conditions* (6) *and* (7) *to be valid only for all differentiable (analytic) on $[0,\infty)$ solutions of* (2).

Proof. As has been noted above, under the conditions of the theorem on the operator A, the set of all continuous at 0 weak solutions $y(t)$ of equation (2) is described by formula (3) where x goes through the whole space $\mathfrak{B}$. If the semigroup $\{e^{-At}\}_{t\geq 0}$ is differentiable (analytic) and x passes through the set of all infinitely differentiable (analytic) vectors of the operator A, that is, $x \in \bigcap\limits_{n\in\mathbb{N}} \mathcal{D}(A^n)$ ($x \in \mathfrak{A}(A)$), then formula (3) gives all differentiable (analytic) on $[0,\infty)$ solutions of (2).

Let $y(t)$ be an arbitrary weak solution of (2) on $(0,\infty)$. Then, because of Propositions 1, 3,

$$\exists y_0 \in \mathfrak{B}_-(A) : y(t) = U(t)y_0 = e^{-A(t-t_0)}U(t_0)y_0, \quad t > t_0. \tag{8}$$

Since $U(t_0)y_0 \in \mathfrak{B}$ and for an arbitrary fixed $t_0 > 0$, $t - t_0 \to \infty$ as $t \to \infty$, formula (8) implies that if relation (6) is fulfilled for all continuous at 0 weak solutions of equation (2), then $y(t) \to 0$ for any weak solution of this equation on $(0,\infty)$. The equality

$$e^{\omega t}\|y(t)\| = e^{\omega t_0}e^{\omega(t-t_0)}\|e^{-A(t-t_0)}U(t_0)y_0\| \tag{9}$$

shows that if formula (7) is fulfilled for any continuous at 0 weak solution of (2), then it is valid for an arbitrary weak solution on $(0,\infty)$.

Suppose now that the semigroup $\{e^{-At}\}_{t\geq 0}$ is differentiable (analytic). By Proposition 2,

$$\forall t > 0, \forall y_0 \in \mathfrak{B}_-(A) : e^{-At}y_0 \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n) \quad (e^{-At}y_0 \in \mathfrak{A}(A)).$$

It follows from (8) and (9) that if relations (6) and (7) are fulfilled for all infinitely differentiable (analytic) on $[0,\infty)$ weak solution of (2), then they are valid for any weak one on $(0,\infty)$. □

In accordance with [7], a C_0-semigroup $\{U(t)\}_{t\geq 0}$ on $\mathfrak{B}$ is:

(i) uniformly stable if

$$\forall x \in \mathfrak{B} : \lim_{t\to\infty} \|U(t)x\| = 0;$$

(ii) uniformly exponentially stable if

$$\exists M > 0, \exists \omega > 0, \forall t \geq 0 : \|U(t)\| \leq Me^{-\omega t}$$

(M and ω are constants).

As all continuous at 0 weak solutions $y(t)$ of equation (2) are described by formula (3) where x runs through the whole $\mathfrak{B}$, Theorem 1 may be reformulated in terms of stability of a C_0-semigroup. Namely, the following assertion holds.

Corollary 3. *Let A be the generator of a contraction C_0-semigroup on $\mathfrak{B}$ such that* $\ker e^{-At} = \{0\}$ *for any $t > 0$. Then for equation* (2) *to be uniformly (uniformly exponentially) stable, it is sufficient that the semigroup $\{e^{-At}\}_{t\geq 0}$ be uniformly (uniformly exponentially) stable. If the semigroup $\{e^{-At}\}_{t\geq 0}$ is differentiable (analytic), it is sufficient in the relations*

$$\forall x \in \mathfrak{B} : e^{-At}x \to 0 \text{ as } t \to 0 \quad (e^{\omega t}e^{-At}x \to 0 \text{ as } t \to 0) \tag{10}$$

confine ourselves to $x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$ $(x \in \mathfrak{A}(A))$.

Note also that a number of works of various mathematicians were devoted to searching uniform and uniform exponential stability criterions for C_0-semigroups (see, for instance, [7–9]). In what follows some new ones are given.

Denote by $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_c(\cdot), \sigma_r(\cdot)$, and $\rho(\cdot)$ the spectrum, the point, continuous, residual spectra, and the resolvent set of an operator, respectively.

Theorem 2. *In order that a C_0-semigroup $\{e^{-At}\}_{t\geq 0}$ be uniformly stable, it is necessary that $0 \in \sigma_c(A) \cup \rho(A)$. If the semigroup $\{e^{-At}\}_{t\geq 0}$ is uniformly exponentially stable, then $0 \in \rho(A)$. In order that $\{e^{-At}\}_{t\geq 0}$ be uniformly but not uniformly exponentially stable, it is necessary that $0 \in \sigma_c(A)$. In the case where $\{e^{-At}\}_{t\geq 0}$ is bounded analytic, all conditions mentioned above are sufficient, too.*

Proof. Let the semigroup $\{e^{-At}\}_{t\geq 0}$ be uniformly stable. Assume that $0 \in \sigma_p(A)$. Then there exists $x \in \mathcal{D}(A)$, $x \neq 0$, such that $Ax = 0$. It follows from this that $\lim_{t\to\infty} e^{-At}x = x \neq 0$ contrary to the uniform stability of $\{e^{-At}\}_{t\geq 0}$.

Suppose now $0 \in \sigma_r(A)$. Then $\overline{\mathcal{R}(A)} \neq \mathfrak{B}$ ($\mathcal{R}(\cdot)$ is the range of an operator). This implies that

$$\exists f \in \mathfrak{B}^* \ (f \neq 0), \ \forall x \in \mathcal{D}(A) : f(Ax) = 0. \tag{11}$$

Consider the function $\varphi_x(t) = f(e^{-At}x)$. Since $e^{-At}\mathcal{D}(A) \subset \mathcal{D}(A)$ as $t > 0$, the function $\varphi_x(t)$ is continuously differentiable on $[0, \infty)$ and $\varphi_x'(t) = -f(Ae^{-At}x) \equiv 0$. So $\varphi_x(t) = c_x = \text{const}$ on $[0, \infty)$. Because of $\varphi_x(0) = f(x) = \lim_{t\to 0} f(e^{-At}x) = 0$, we have $f(x) = 0$ for any $x \in \mathcal{D}(A)$. Taking into account that $\overline{\mathcal{D}(A)} = \mathfrak{B}$ and the continuity of f, we may conclude that $f = 0$ which contradicts to (11). Thus, in the case of uniform stability of $\{e^{-At}\}_{t\geq 0}$, $0 \in \sigma_c(A) \cup \rho(A)$.

Next, suppose $\{e^{-At}\}_{t\geq 0}$ to be uniformly exponentially stable. Then $\{\lambda \in \mathbb{C} : \text{Re}\lambda > \omega\} \subset \rho(A)$. As $\omega < 0$, we have $0 \in \rho(A)$. It follows from this that if $\{e^{-At}\}_{t\geq 0}$ is uniformly but not uniformly exponentially stable, then $0 \in \sigma_c(A)$.

Let now $\{e^{-At}\}_{t\geq 0}$ be bounded analytic and $0 \in \sigma_c(A) \cup \rho(A)$, hence, $\overline{\mathcal{R}(A)} = \mathfrak{B}$. Then for every $g \in \mathcal{R}(A)$, there exists $x \in \mathcal{D}(A)$ such that $g = Ax$. The boundedness and analyticity of $\{e^{-At}\}_{t\geq 0}$ imply the relation

$$\left\|e^{-At}g\right\| = \left\|e^{-At}Ax\right\| \leq \frac{c_x\|x\|}{t} \to 0 \quad \text{as} \quad t \to \infty \quad (0 < c_x = \text{const}).$$

Since $\overline{\mathcal{R}(A)} = \mathfrak{B}$, we make sure, on the basis of the principle of uniform boundedness (Banach-Steinhaus theorem), that $e^{-At}g \to 0$ for any $g \in \mathfrak{B}$, that is, the semigroup $\{e^{-At}\}_{t\geq 0}$ is uniformly stable. If $\{e^{-At}\}_{t\geq 0}$ is bounded analytic and $0 \in \rho(A)$, then the spectrum $\sigma(-A)$ of the operator $-A$ lies in the sector $S(\varphi, \delta) = \{\lambda \in \mathbb{C} : |\arg(\lambda + \delta)| < \pi - \varphi\}$ with some $\delta > 0$ and $\varphi \in (0, \pi]$. For this reason, $S(A) = \inf_{\lambda\in\sigma(A)} \text{Re}\lambda < 0$. Taking into account that, by virtue of analyticity of the semigroup $\{e^{-At}\}_{t\geq 0}$, $S(A) = \omega(A)$, we arrive at the conclusion that this semigroup is uniformly exponentially stable. This implies also that if a bounded analytic semigroup $\{e^{-At}\}_{t\geq 0}$ is uniformly but not uniformly exponentially stable, then $0 \in \sigma_c(A)$. □

Theorem 3. *Let $\{e^{-At}\}_{t\geq 0}$ be a C_0-semigroup on $\mathfrak{B}$, and $\gamma(t) > 0$ a continuous on $[0,\infty)$ function such that $\gamma(t) \to 0$ as $t \to \infty$. If*

$$\forall x \in \mathfrak{B}, \exists c = c(x) > 0 : \left\|e^{-At}x\right\| \leq c\gamma(t), \quad t \in [0,\infty), \tag{12}$$

then $\{e^{-At}\}_{t\geq 0}$ is uniformly exponentially stable. In the case where the semigroup $\{e^{-At}\}_{t\geq 0}$ is differentiable (analytic) on $(0,\infty)$, it suffices for inequality (12) *to fulfilled at least for $x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$ $(x \in \mathfrak{A}(A))$.*

Proof. Denote by $C_\gamma([0,\infty),\mathfrak{B})$ the Banach space of all continuous on $[0,\infty)$ vector-valued functions $y(t)$ for which

$$\|y\|_\gamma = \sup_{t\geq 0} \frac{\|y(t)\|}{\gamma(t)} < \infty.$$

The operator

$$C : \mathfrak{B} \mapsto C_\gamma([0,\infty),\mathfrak{B}), \quad Cx = e^{-At}x,$$

admits a closure. Really, suppose $x_n \to 0$ in $\mathfrak{B}$ and $e^{-At}x_n \to y(t)$ in $C_\gamma([0,\infty),\mathfrak{B})$. As $e^{-At}x_n \to 0$ uniformly on each compact set from $[0,\infty)$, we have $y(t) \equiv 0$. Since the operator C is defined on the whole $\mathfrak{B}$ we make sure, in view of Closed Graph Theorem, that C is continuous. So,

$$\exists d > 0 : \left\|e^{-At}x\right\|_\gamma \leq d\|x\|,$$

whence

$$\left\|e^{-At}x\right\| \leq d\gamma(t).$$

Taking into account that

$$\omega_0 = \inf_{t>0} \frac{\ln\left\|e^{-At}\right\|}{t} < 0$$

(see [7]), we obtain

$$\left\|e^{-At}\right\| \leq c_{\omega_0-\varepsilon}e^{-(\omega_0-\varepsilon)t}, \quad 0 < c_{\omega_0-\varepsilon} = \text{const}, \quad 0 < \varepsilon < \omega_0,$$

which means that the semigroup $\{e^{-At}\}_{t\geq 0}$ is uniformly exponentially stable.

Assume now that the semigroup $\{e^{-At}\}_{t\geq 0}$ is differentiable and inequality (12) holds true only for $x \in \bigcap\limits_{n\in\mathbb{N}} \mathcal{D}(A^n)$. Fix $t_0 > 0$. By Proposition 2

$$\forall x \in \mathfrak{B} : g = e^{-At_0}x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n).$$

So,

$$\forall t \geq t_0 : \left\|e^{-At}x\right\| = \left\|e^{-A(t-t_0)}e^{-At_0}x\right\| \leq c_g\gamma(t-t_0).$$

Putting

$$\gamma_1(t) = \left\{ \begin{array}{rll} \gamma(0) & \text{as} & 0 \leq t \leq t_0 \\ \gamma(t-t_0) & \text{as} & t > t_0 \end{array} \right., \quad \widetilde{c_x} = \max\left\{\frac{1}{\gamma(0)} \max_{t\in[0,t_0]} \left\|e^{-At}x\right\|, c_g\right\},$$

we obtain

$$\forall x \in \mathfrak{B}, \forall t \in [0,\infty) : \left\|e^{-At}x\right\| \leq \widetilde{c_x}\gamma_1(t),$$

that is, $\{e^{-At}\}_{t\geq 0}$ is uniformly exponentially stable.

In the case when $\{e^{-At}\}_{t\geq 0}$ is analytic, the proof scheme is the same. □

Theorem 3 shows that if the semigroup $\{e^{-At}\}_{t\geq 0}$ is uniformly but not uniformly exponentially stable, then its orbits $e^{-At}x$ may tend to 0 anyhow slowly when approaching to infinity. But it is impossible for such a semigroup to have an exponential decrease for all its orbits. Indeed, suppose that

$$\forall x \in \mathfrak{B}, \ \exists c = c(x) > 0, \ \exists \omega_x > 0 : \left\|e^{-At}x\right\| \leq ce^{-\omega_x t}.$$

Then

$$\forall x \in \mathfrak{B} : \left\| e^{-At}x \right\| \le c_1 \frac{1}{1+t}, \quad 0 < c_1 = c \sup_{t\in[0,\infty)} \left\{ (1+t)e^{-\omega_x t} \right\}.$$

Setting in Theorem 3 $\gamma(t) = \frac{1}{1+t}$, we conclude that $\{e^{-At}\}_{t\ge 0}$ is uniformly exponentially stable contrary to the above assumption.

The next theorem gives one more criterion of uniform exponential stability.

Theorem 4. *Let $\{e^{-At}\}_{t\ge 0}$ be a C_0-semigroup on $\mathfrak{B}$. If*

$$\forall x \in \mathfrak{B},\ \exists p_x > 0 : \int_0^\infty \left\| e^{-At}x \right\|^{p_x} dt < \infty, \tag{13}$$

then this semigroup is uniformly exponentially stable. If $\{e^{-At}\}_{t\ge 0}$ is differentiable (analytic), it is sufficient that inequality (13) be valid at least for infinitely differentiable (analytic) vectors of the operator A.

Proof. Consider first the case where the semigroup $\{e^{-At}\}_{t\ge 0}$ is bounded: $\left\| e^{-At} \right\| \le c =$ const, $t \in (0,\infty)$. We may assume, without restriction of generality, that $c = 1$ because we can introduce in $\mathfrak{B}$ the equivalent to $\|\cdot\|$ norm

$$\|x\|_1 = \sup_{t\in[0,\infty)} \left\| e^{-At}x \right\|,$$

with respect to which $\{e^{-At}\}_{t\ge 0}$ is a contraction semigroup. Then $\left\| e^{-At}x \right\|$ does not increase for any $x \in \mathfrak{B}$ and, therefore, condition (13) implies

$$\forall x \in \mathfrak{B},\ \exists c_x > 0 : \left\| e^{-At}x \right\| \le c_x (1+t)^{-\frac{1}{p_x}},$$

whence

$$\forall x \in \mathfrak{B},\ \exists \widetilde{c_x} > 0 : \left\| e^{-At}x \right\| \le \widetilde{c_x} \frac{1}{\ln(2+t)},$$

where

$$\widetilde{c_x} = \sup_{t\in[0,\infty)} \frac{\ln(2+t)}{(1+t)^{\frac{1}{p_x}}} c_x.$$

By Theorem 3, the semigroup $\{e^{-At}\}_{t\ge 0}$ is uniformly exponentially stable.

Let now $\{e^{-At}\}_{t\ge 0}$ be not bounded on $[0,\infty)$. Since the growth of $\{e^{-At}\}_{t\ge 0}$ at infinity is not higher than exponential, we have

$$\exists \omega > 0,\ \exists c > 0 : \left\| e^{-At} \right\| \le c e^{\omega t}.$$

Suppose that for some $x \in \mathfrak{B}$ relation (13) is fulfilled, but $\left\| e^{-At}x \right\|$ does not tend to 0 at infinity. Then there exists a sequence $t_i \to \infty$ such that $\left\| e^{-At_i}x \right\| > \delta$ with some $\delta > 0$. Choose this sequence so that $t_{i+1} - t_i > \omega^{-1}$. Then for $s \in \triangle_i = [t_i - \omega^{-1}, t_i]$, we obtain

$$\delta \le \left\| e^{-At_i}x \right\| \le \left\| e^{-A(t_i - s)} \right\| \left\| e^{-As}x \right\| \le c e^{\omega\omega^{-1}} \left\| e^{-As}x \right\| = ce \left\| e^{-As}x \right\|.$$

It follows from this that

$$\forall s \in \triangle_i : \left\| e^{-As}x \right\| \ge (ce)^{-1}\delta.$$

So,

$$\int_0^\infty \left\| e^{-At}x \right\|^{p_x} dt \ge \sum_{i\in\mathbb{N}} \int_{\triangle_i} \left\| e^{-At}x \right\|^{p_x} dt = \infty$$

contrary to (13). Thus, for an arbitrary $x \in \mathfrak{B}$, $\left\| e^{-At}x \right\|$ is nonincreasing and the investigation amounts to the considered above case of a bounded semigroup.

The latter assertion of the theorem follows from the identity

$$\int_{t_0}^\infty \left\| e^{-At}x \right\|^q dt = \int_{t_0}^\infty \left\| e^{-A(t-t_0)} e^{-At_0}x \right\|^q dt = \int_0^\infty \left\| e^{-A\xi} e^{-At_0}x \right\|^q d\xi$$

($t_0 > 0$ and $q > 0$ are arbitrary) and the fact that $e^{-At_0}x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$ $(e^{-At_0}x \in \mathfrak{A}(A))$ if $\{e^{-At}\}_{t\geq 0}$ is differentiable (analytic). □

It should be noted that Theorem 4 is a generalization of the corresponding results of Datko [10], Pazy [11], M. Krein [12] where it was required the existence of one the same p in (13) for all $x \in \mathfrak{B}$. In Theorem 4 p may be different for different x. Moreover, if $\{e^{-At}\}_{t\geq 0}$ is infinitely differentiable (analytic), it is sufficient for (13) to be fulfilled at least for infinitely differentiable (analytic) vectors of the operator A.

Observe also that in the case where A is the generator of a uniformly exponentially stable C_0-semigroup, every solution $y(t)$ of equation (2) tends to 0 exponentially at infinity. Namely,

$$\forall a < -\omega_0 : \lim_{t\to\infty} y(t)e^{at} = 0.$$

As for uniformly but not uniformly exponentially stable semigroups, Theorem 3 shows that this is not the case. The question arises of finding a connection between the order of decrease to 0 of solutions $y(t)$ when approaching to ∞ and the properties of their initial data $y(0)$. Taking into account [13], we arrive, by virtue of Theorems 2 and 3, at the next assertion.

Theorem 5. *Let A be the generator of a bounded analytic C_0-semigroup $\{e^{-At}\}_{t\geq 0}$ on $\mathfrak{B}$ such that $0 \in \sigma_c(A)$. If $y(t)$ is a continuous at 0 solution of equation* (2)*, then the following equivalence relations take place:*

$$\forall n \in \mathbb{N} : \lim_{t\to\infty} t^n y(t) = 0 \Longleftrightarrow y(0) \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^{-n});$$

$$\exists a > 0 : \lim_{t\to\infty} e^{a\sqrt{t}} y(t) = 0 \Longleftrightarrow y(0) \in \mathfrak{A}(A^{-1});$$

$$\forall a > 0 : \lim_{t\to\infty} e^{a\sqrt{t}} y(t) = 0 \Longleftrightarrow y(0) \in \mathfrak{A}_c(A^{-1}).$$

If $y(t)$ is exponentially decreasing at ∞, then

$$\exists a > 0 : \lim_{t\to\infty} e^{at} y(t) = 0 \Longleftrightarrow y(0) \in \mathfrak{E}(A^{-1}),$$

where

$$\mathfrak{E}(A^{-1}) = \left\{ x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^{-n}) \big| \exists \alpha > 0, \exists c = c(x), \forall n \in \mathbb{N} : \|A^{-n}x\| \leq c\alpha^n \right\}$$

is the space of entire vectors of exponential type for the operator A^{-1}.

If the semigroup $\{e^{-At}\}_{t\geq 0}$ is bounded analytic, then the operator A^{-1} generates an analytic semigroup, too (see [14]), and, as was shown there, $\overline{\mathfrak{A}_c(A^{-1})} = \mathfrak{B}$; moreover, the set of solutions of equation (2) behaving like $e^{-a\sqrt{t}}$ when $t \to \infty$, is dense in the set of all its weak solutions. As for the set of weak solutions decreasing at ∞ exponentially, it may consist only of the trivial one $y(t) \equiv 0$ even in the case where the analyticity angle of $\{e^{-At}\}_{t\geq 0}$ is equal to $\frac{\pi}{2}$. But if in the latter case

$$\int_0^1 \ln\ln M(s)\, ds < \infty, \quad M(s) = \sup_{\mathrm{Im}\lambda \geq s} \left\| A(A - \lambda I)^{-1} \right\|,$$

then $\overline{\mathfrak{E}(A^{-1})} = \mathfrak{B}$, and the set of weak solutions decreasing exponentially to 0 when approaching to ∞ is wide enough.

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