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HYPERCYCLIC COMPOSITION OPERATORS ON HILBERT SPACES OF ANALYTIC FUNCTIONS

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Abstract. In the paper we consider composition operators on Hilbert spaces of analytic functions of infinitely many variables. In particular, we establish some conditions under which composition operators are hypercyclic and construct some examples of Hilbert spaces of analytic functions which do not admit hypercyclic operators of composition with linear operators.

Introduction

A sequence $\{T_n : E \to E : n \in \mathbb{N}\}$ of operators on a Fréchet space E is called *universal* provided there exists some vector $x \in E$, called a universal vector, for which

$$\overline{\{T_n x : n \in \mathbb{N}\}} = E.$$

An operator T on E is said to be *hypercyclic* if the sequence of its iterates $\{T^n : n \in \mathbb{N}\}$ is universal.

The study of hypercyclic operators started after Birkhoff's result [5] that the operator of composition with translation $x \mapsto x + a$, $a \neq 0$, $T_a : f(x) \mapsto f(x + a)$ is hypercyclic in the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$. Godefroy and Shapiro in [11] generalized this result for the translation operator on a Fréchet space $H(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$, endowed with the topology of uniform convergence on compact subsets. Aron and Bès in [2] proved that the operator of composition with translation T_a is hypercyclic in the space of entire functions which are weakly continuous on all bounded subsets of a separable Banach space X. In [19] the authors considered hypercyclic operators on $H(\mathbb{C}^n)$ which are composition with more complicated analytic maps on $\mathbb{C}^n$ and proved the hypercyclicity of a "symmetric" translation operator on the space of symmetric analytic functions of bounded type on ℓ_1. Detailed information about hypercyclic operators is given in [3, 12].

Let $\{T_n : E \to E : n \in \mathbb{N}\}$ be a sequence of (continuous and linear) operators on a separable Fréchet space E. There is a general sufficient condition for universality. This condition is inspired in the so-called Hypercyclicity Criterion given by Kitai [14] in her unpublished Ph.D. thesis and rediscovered by Gethner and Shapiro [10]. We use the general form of this Criterion as given in [4].

Definition 0.1. We say that $\{T_n : n \in \mathbb{N}\}$ satisfies the Universality Criterion (UC) provided there exist X and Y dense subsets of E and maps $S_n : Y \to E$, $n \in \mathbb{N}$, such that

(i) $T_n x \to 0$, $n \to \infty$ for all $x \in X$,
(ii) $S_n y \to 0$, $n \to \infty$ for all $y \in Y$,
(iii) $(T_n \circ S_n)y \to y$, $n \to \infty$ for all $y \in Y$.

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An operator T on E is said to satisfy the *Hypercyclicity Criterion* (HC) with respect to an increasing sequence of positive integers (n_k), if the sequence of iterates $\{T^{n_k} : k \in \mathbb{N}\}$ satisfies the Universality Criterion.

Note that according to [8] there are hypercyclic operators which do not satisfy the HC.

Shapiro [21] gave an another useful condition, known as the *Hypercyclicity Comparison Principle* (HCP):

If $T : (E, \tau) \to (E, \tau)$ is such that there is a dense subspace $F \subset E$ and a finer topology τ' on F such that $T|_F : (F, \tau') \to (F, \tau')$ is hypercyclic, then T is hypercyclic. In [17] Martìnez-Gimènez and Peris generalized this result in the following form.

Lemma 0.2. *Let $T_{i,n} : E_i \to E_i : n \in \mathbb{N}$ be a sequence of operators on a separable Fréchet space E_i, $i = 1, 2$, let $\phi : E_1 \to E_2$ be a continuous map with dense range such that $T_{2,n} \circ \phi = \phi \circ T_{1,n}$, for all $n \in \mathbb{N}$. That is, the diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{T_{1,n}} & E_1 \\ \phi \downarrow & & \downarrow \phi \\ E_2 & \xrightarrow{T_{2,n}} & E_2 \end{array}$$

commutes for all $n \in \mathbb{N}$. If $\{T_{1,n} : n \in \mathbb{N}\}$ is universal sequence (satisfies the UC), then $\{T_{2,n} : n \in \mathbb{N}\}$ is also a universal sequence (satisfies the UC). For single operators $T_i : E_i \to E_i$, $i = 1, 2$, such that $T_2 \circ \phi = \phi \circ T_1$ we have

(1) *If T_1 is hypercyclic, then T_2 is also hypercyclic.*
(2) *If T_1 satisfies the HC, then T_2 also satisfies the HC.*

Now we consider conditions under which tensor products of operators are hypercyclic.

Definition 0.3. ([18]). We say that a sequence $\{T_n : n \in \mathbb{N}\}$ of operators on E satisfies the Tensor Universality Criterion (TUC) if there exist dense subsets X and Y of E, and maps $S_n : Y \to E$, $n \in \mathbb{N}$, such that

(i) $\{T_n x\}_{n=1}^{\infty}$ is bounded for each $x \in X$,
(ii) $\{S_n y\}_{n=1}^{\infty}$ is bounded for each $y \in Y$,
(iii) $(T_n \circ S_n)y \to y$, $n \to \infty$ for all $y \in Y$.

An operator T on E satisfies the *Tensor Hypercyclicity Criterion* (THC) with respect to an increasing sequence of positive integers (n_k), provided the sequence of iterates $\{T^{n_k} : k \in \mathbb{N}\}$ satisfies the TUC.

Full presentation of tensor products of normed spaces we can find in [9]. A tensor norm "α" on the algebraic tensor product $E \otimes F$ of normed spaces E and F (shorthand: $E \otimes_\alpha F$ and $E \widetilde{\otimes}_\alpha F$ for the completion) is defined such that the following two conditions are satisfied:

(1) α is a crossnorm, that is $\|x \otimes y\|_\alpha = \|x\|\|y\|$, $x \in E$, $y \in F$.
(2) α satisfies the metric mapping property: If $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ are continuous linear operators, then

$$\|T_1 \otimes T_2 : E_1 \otimes_\alpha E_2 \to F_1 \otimes_\alpha F_2\| \leq \|T_1\|\|T_2\|.$$

This definition goes back to Schatten [20] in 1943 (who called such as α a uniform crossnorm).

Since every Fréchet space is a projective limit of a countable family seminormed spaces, for given a tensor norm "α", we have the corresponding locally convex topology on the tensor product $E \otimes F$ of two Fréchet spaces E and F. The metric mapping property for tensor norms yields that the operator $T_1 \otimes T_2 : E_1 \otimes_\alpha E_2 \to F_1 \otimes_\alpha F_2$ is continuous whenever $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ are continuous operators between Fréchet spaces (see [9, 35.2]).

We will need a theorem, which was proved by Martìnez-Gimènez and Peris in [18].

Theorem 0.4. *Let E and F be separable Fréchet spaces. If the sequence of linear operators $\{T^1_n : E \to E : n \in \mathbb{N}\}$ satisfies the UC, and the sequence of operators $\{T^2_n : F \to F : n \in \mathbb{N}\}$ satisfies the TUC, then $\{T^1_n \widetilde{\otimes} T^2_n : E\widetilde{\otimes}_\alpha F \to E\widetilde{\otimes}_\alpha F : n \in \mathbb{N}\}$ satisfies the UC, and therefore is universal, for any tensor norm α.*

In the paper we prove the hypercyclicity of composition operator C_T on a direct topological sum of tensor powers of a Hilbert space E^* providing C_T is continuous, where T is a linear operator on E such that T^* is hypercyclic. Also, we consider some special examples of Hilbert spaces of analytic functions which can be presented by dual to abstract symmetric Fock space over E and do not admit hypercyclic operators of composition with a linear operator and with a translation.

1. The case of abstract Hilbert spaces

Let E be a Hilbert space and E^* be its Hermitian adjoint space. We denote by $\otimes^n_s E$ the n-fold symmetric algebraic tensor power of space E. Every element from $\otimes^n_s E$ can be approached by a linear span of elements

$$x_1 \otimes_s \cdots \otimes_s x_n := \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

where $x_1, \dots, x_n \in E$ and S_n is the group of permutations on the set $\{1, \dots, n\}$.

Let $\otimes^n_{s,h} E$ be the completion $\otimes^n_s E$ with respect to a Hilbert norm h. Then space $\otimes^n_{s,h} E^*$ consists of n-homogeneous polynomials on E endowed with the Hilbert norm h.

We denote by

$$W = \Big\{ f = \bigoplus_{n=1}^{\infty} P_n : \quad P_n \in \otimes^n_{s,h} E^* \quad \forall n \in \mathbb{N} \Big\}$$

the completion of space of the direct sums $\oplus_n \otimes^n_{s,h} E^*$ with respect to some Hilbert topology τ_h such that all subspaces $\otimes^n_{s,h} E^*$ are closed in W. For a given $f \in W$, $\mathbf{P}_n(f)$ could be a projection from W onto $\otimes^n_{s,h} E^*$.

Let $T : E \to E$ be a linear operator on Hilbert space E. Let us introduce the composition operator $C_T(P) := P \circ T$ on the subspace of all polynomials in W. If this operator is continuous, we can extend it to the whole space (preserving the same notation).

Theorem 1.1. *Let E be a separable Hilbert space and $T : E \to E$ an operator such that its adjoint $T^* : E^* \to E^*$ satisfies the Hypercyclicity Criterion. Then, if composition operator*

$$C_T : (W, \tau_h) \to (W, \tau_h), \quad f \mapsto f \circ T$$

is continuous on W, then C_T is hypercyclic on W.

Proof. Since T is continuous, then we note that $f \circ T \in \otimes^n_s E^*$ if $f \in \otimes^n_s E^*$. So C_T maps $\otimes^n_s E^*$ to itself.

For every $f \in W$, $\mathbf{P}_n(C_T(f)) = \mathbf{P}_n(f) \circ T = (T^* \otimes \cdots \otimes T^*)(\mathbf{P}_n(f)) \in \otimes^n_s E^*$, $\forall n \in \mathbb{N}$. By Theorem 0.4 the extension $\widetilde{\otimes}^n_{s,h} T^* =: T^*_n$ from $E^* \otimes \cdots \otimes E^*$ to $\widetilde{\otimes}^n_{s,h} E^* = \overline{\otimes^n_s E^*}^{\tau_h}$ satisfies the HC with respect to a sequence (m_k) (independent on n). Consequently there are dense subspaces $X_n, Y_n \subset \widetilde{\otimes}^n_{s,h} E^*$ and $S_{n,m_k} : Y_n \to \widetilde{\otimes}^n_{s,h} E^*$, $k \in \mathbb{N}$ such that T^*_n, X_n, Y_n and $\{S_{n,m_k}\}_{k=1}^{\infty}$ satisfy the UC, $n \in \mathbb{N}$. Define now

$$X := \bigcup_{n \in \mathbb{N}} (\oplus_{k=1}^n X_k), \quad Y := \bigcup_{n \in \mathbb{N}} (\oplus_{k=1}^n Y_k)$$

dense subspaces of W and map

$$S_{m_k} : Y \to W, \quad S_{m_k} := \oplus_{n \in \mathbb{N}} S_{n,m_k}, \quad k \in \mathbb{N}.$$

It easily follows that C_T, X, Y and $\{S_{m_k}\}_{k=1}^{\infty}$ satisfy the conditions of the HC, therefore C_T is hypercyclic on W. □

2. The case of special Hilbert spaces

Let E be a Hilbert space with an orthonormal basis $(e_i)_{i=1}^{\infty}$ and the inner product $(\cdot \mid \cdot)$. We denote $e_i^{\otimes k} = \underbrace{e_i \otimes \cdots \otimes e_i}_{k}$ for any $k \in \mathbb{N}$ and $i = 1, 2, \ldots$. For a fixed $m \in \mathbb{N}$ we denote by $[i]$ a multi-index $(i_1, \ldots, i_m) \in \mathbb{N}^m$ such that $i_1 < \cdots < i_m$ and by (k) an arbitrary multi-index $(k_1, \ldots, k_m) \in \mathbb{Z}_+^m$, $|(k)| = k_1 + k_2 + \cdots + k_m$. The vectors

$$\left\{ e_{[i]}^{\otimes_s (k)} := e_{i_1}^{\otimes k_1} \otimes_s \cdots \otimes_s e_{i_n}^{\otimes k_n} : \quad |(k)| = n \right\}$$

form a topological basis in $\otimes_s^n E$. Here we assume that zero tensor power of a basis vector is equal to 1. We say that a Hilbert space $\mathcal{F} = \mathcal{F}(E)$ with a norm $\|\cdot\|_\eta$ is an *(abstract) symmetric Fock space over a given Hilbert space E* if vectors $1, e_{[i]}^{\otimes_s (k)} = e_{i_1}^{\otimes k_1} \otimes_s \cdots \otimes_s e_{i_n}^{\otimes k_n}$, ($k_1 + \cdots + k_n = n$, $n \in \mathbb{N}$, $i_1 < \cdots < i_n$) form an orthogonal basis in $\mathcal{F}$.

Evidently, the norm $\|\cdot\|_\eta$ is completely defined by its value on the basis vectors. Hence, setting $\left\| e_{[i]}^{\otimes_s (k)} \right\|_\eta$ by arbitrary positive numbers, we can get various symmetric Fock spaces over E. In other words, $\mathcal{F}$ is a completion of

$$\mathbb{C} \oplus E \oplus \otimes_s^2 E \oplus \cdots \oplus \otimes_s^n E \oplus \cdots$$

by $\|\cdot\|_\eta$. We will use notation $\mathcal{F}_\eta$ for $(\mathcal{F}, \|\cdot\|_\eta)$. Let $\langle \cdot \mid \cdot \rangle$ be the inner product in $\mathcal{F}_\eta$.

Put $c_{[i]}^{(k)} := \left\| e_{[i]}^{\otimes_s (k)} \right\|_\eta^{-2}$ and $c_0 = 1$. Let us consider the following power series:

$$\eta(x) = \sum_{k_1 + \cdots + k_n = 0}^{\infty} \sum_{i_1 < \cdots < i_n} c_{i_1 \ldots i_n}^{k_1 \ldots k_n} x_{i_1}^{k_1} \ldots x_{i_n}^{k_n} e_{i_1}^{\otimes k_1} \otimes_s \cdots \otimes_s e_{i_n}^{\otimes k_n}$$
$$= \sum_{|(k)|=0}^{\infty} \sum_{[i]} c_{[i]}^{(k)} x_{[i]}^{(k)} e_{[i]}^{\otimes_s (k)}$$

for any $x = \sum_{i=1}^{\infty} x_i e_i \in E$. It is known [15], [16, p. 130] that under some conditions on $\{c_{[i]}^{(k)}\}$ the series converges in $\mathcal{F}_\eta$ for every x in an open set $U \in E$ and the Hermitian dual $\mathcal{F}_\eta^*$ is isomorphic to a Hilbert space of analytic functions on U (which we denote by $\mathcal{H}_\eta$) of the form $f_\phi = \langle \eta(\cdot) \mid \phi \rangle$, $\phi \in \mathcal{F}_\eta$. In this paper we consider two special cases of $\mathcal{H}_\eta$.

Hilbert space $\mathcal{H}_{\eta_E}(E)$. Let us consider the case when a map $\eta = \eta_E : E \to \mathcal{F}_\eta = \mathcal{F}_{\eta_E}$ is defined by

$$\eta_E(x) = 1 + x + \frac{1}{2!} x^{\otimes 2} + \cdots + \frac{1}{n!} x^{\otimes n} + \cdots,$$

$x \in E$ and $x^{\otimes n}$ the tensor power $\underbrace{x \otimes \cdots \otimes x}_{n}$. Clearly, $U = E$ and so $\mathcal{H}_\eta = \mathcal{H}_{\eta_E}(E)$ consists of analytic functions on E. Let $\widetilde{\otimes}_s^n E$ be the completion of $\otimes_s^n E$ with respect to $\|\cdot\|_{\eta_E}$. Then $\mathcal{F}_\eta = \mathcal{F}_{\eta_E}$ is the direct topological sum of $\widetilde{\otimes}_s^n E$ and

$$\left\| e_{[i]}^{\otimes_s (k)} \right\|_{\eta_E}^2 = k_1! \ldots k_n!.$$

Let $w \in \mathcal{F}_{\eta_E}$, $w = \sum_{n=0}^{\infty} w_n$, where $w_n \in \widetilde{\otimes}_s^n E$. Then

$$P_n(x) = \langle \eta_E(x) \mid w_n \rangle = \frac{1}{n!}\langle x^{\otimes n} \mid w_n \rangle$$

is an n-homogeneous polynomial on E and the series

$$f(x) = \sum_{n=0}^{\infty} P_n(x)$$

is the Taylor series expansion of f. So

$$\|f\|^2 = \sum_{n=0}^{\infty} \|P_n\,\|^2 := \sum_{n=0}^{\infty} \|w_n\|^2_{\eta_E} < \infty.$$

We need the following definition.

Definition 2.1. Let Z be an abstract set and $\mathcal{H}$ a Hilbert space of complex valued functions f on Z equipped with inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$. A function $K(x \mid z)$ defined on $Z \times Z$ is called *reproducing kernel* of a closed subspace $\mathcal{H}_K \subset \mathcal{H}$ if

(i) for any fixed $z \in Z$, $K(x \mid z)$ belongs to $\mathcal{H}_K$ as a function in $x \in Z$;

(ii) for any $f \in \mathcal{H}_K$ and for any $z \in Z$, $f(z) = \langle f \mid K(\cdot \mid z)\rangle_{\mathcal{H}}$.

The space $\mathcal{H}_K$ is called a Hilbert space with reproducing kernel.

From the definition of $\mathcal{H}_{\eta_E}(E)$ it follows that the reproducing kernel of $\mathcal{H}_{\eta_E}(E)$ is

$$K(x \mid z) = e^{(x|z)} = \sum_{n=0}^{\infty} \frac{(x \mid z)^n}{n!} = \sum_{n=0}^{\infty} \frac{\langle x^{\otimes n} \mid z^{\otimes n}\rangle}{n!}.$$

The following theorem is an infinite-dimensional analogue of the result in [6].

Theorem 2.2. *Let $T : E \to E$ be an analytical map. If the operator C_T is continuous on $\mathcal{H}_{\eta_E}(E)$, then $T(x) = Ax + Q$, where $A : E \to E$ is a linear operator, $Q \in E$. Moreover, $\| A \| \leq 1$ and, if $\| A\zeta \| = \| \zeta \|$ for some $\zeta \in E$ then $(A\zeta \mid Q) = 0$.*

Proof. If C_T is bounded on $\mathcal{H}_{\eta_E}(E)$, then

$$\text{(1)} \quad \sup_{z\in E} \frac{\| C_T^*(K(\cdot \mid z)) \|}{\| K(\cdot \mid z) \|} = \sup_{z\in E} \frac{\| K(\cdot \mid T(z)) \|}{\| K(\cdot \mid z) \|} = \sup_{z\in E} \exp(\| T(z) \|^2 - \| z \|^2) < \infty,$$

where the first equality follows from easily verified property (see, for example [13]) $C_T^*(K(\cdot \mid z)) = K(\cdot \mid T(z))$. From (1) it follows that

$$\text{(2)} \quad \limsup_{\|z\|\to\infty} \frac{\| T(z) \|}{\| z \|} \leq 1.$$

For each $\zeta \in \partial B$, where ∂B is boundary of unit ball B, we define the analytic function F^j_ζ by the equation $F^j_\zeta(\lambda) = (T(\lambda\zeta) \mid e_j)$, $\lambda \in \mathbb{C}$. By (2) we must have

$$\limsup_{|\lambda|\to\infty} \frac{\| F^j_\zeta(\lambda) \|}{| \lambda |} \leq 1.$$

Let $F^j_\zeta(x) = \sum_{n=0}^{\infty} P_n(x)$ be the homogeneous expansion of F^j_ζ. Since

$$F^j_\zeta(\lambda) = \sum_{n=0}^{\infty} \lambda^n P_n(\zeta),$$

we must have $P_n(\zeta) = 0$ for all $n \geq 2$ and $\zeta \in \partial B$; that is $P_n \equiv 0$ for $n \geq 2$ and each coordinate function F^j_ζ is linear. This proves that $T(x) = Ax + Q$ as desired.

If $\| A\zeta \| > \| \zeta \|$ for some ζ, $\| \zeta \| = 1$, then setting $z = t\zeta$, $t > 0$ in (2) and letting $t \to \infty$, we obtain a contradiction. Thus we must have $\| A \| \leq 1$.

Next we show that if $\| A\zeta \| = \| \zeta \|$, then $(A\zeta \mid Q) = 0$. As a special case of this, suppose $A\zeta = \lambda\zeta$, where λ is a complex number of modulus 1. If $(A\zeta \mid Q) \neq 0$, we may choose $\rho \in \mathbb{C}$, $| \rho | = 1$, so that $\rho(A\zeta \mid Q) > 0$. Considering $z = t\rho\zeta$ as $t \to \infty$, we obtain a contradiction to (1). Now suppose $A\zeta = \beta$, where $\| \zeta \| = \| \beta \| = 1$. Let U be a unitary map of E such that $U(\beta) = \zeta$. Then for $\tau(x) \equiv T \circ U(x) = A \circ U(x) + Q$ we have C_τ bounded on $\mathcal{H}_{\eta_E}(E)$ and, since $A(U(\beta)) = \beta$, by the special case just considered, we have $(AU\beta \mid Q) = 0$ or $(A\zeta \mid Q) = 0$ as desired. □

Note that if $T(x) = Ax + Q$ and A is a linear operator with $\|A\| < 1$, then T has a fixed point $x_0 = (I - A)^{-1}Q$ and so C_T can not to be hypercyclic on $\mathcal{H}_{\eta_E}(E)$ (see e.g. [12, p. 112]). Hence, we have the following corollary.

Corollary 2.3. *If C_T is a composition hypercyclic operator on $\mathcal{H}_{\eta_E}(E)$, then $T(x) = Ax + Q$, where A is linear, $\|A\| = 1$ and if $\|A\zeta\| = \|\zeta\|$, $\zeta \in E$, then $(A\zeta \mid Q) = 0$.*

In particular, we have that the translation operator $f(x) \mapsto f(x+a)$ is not hypercyclic in $\mathcal{H}_{\eta_E}(E)$ because it is discontinuous by Theorem 2.2. Note that in [7] it is constructed a class of Hilbert spaces of analytic functions of the complex variable for which the translation operator is hypercyclic.

Let $\mathcal{H}^0_{\eta_E}(E)$ be the subspace of functions $f \in \mathcal{H}_{\eta_E}(E)$ with $f(0) = 0$.

Proposition 2.4. *There is no hypercyclic composition operator with a linear operator on the space $\mathcal{H}^0_{\eta_E}(E)$.*

Proof. If there is such a hypercyclic operator C_T with a linear operator T, then by Theorem 2.2 $\|T\| \leq 1$. From this it follows that $\|C_T\| \leq 1$. Indeed, if $f \in \mathcal{H}^0_{\eta_E}(E)$ and $f = \sum_{n=1}^{\infty} P_n$, then

$$\|C_T(f)\|^2 = \sum_{n=1}^{\infty} \|P_n \circ T\|^2 \leq \sum_{n=1}^{\infty} \|P_n\|^2 \|T\|^2 \leq \sum_{n=1}^{\infty} \|P_n\|^2 = \|f\|^2.$$

But it is impossible because the norm of a hypercyclic operator must be greater than 1. □

Hilbert space $\mathcal{H}_{\eta_B}(B)$. Now let us consider a Hilbert space of analytic functions on the unit ball $B \subset E$.

Let us define a norm on $\otimes_s^n E$ by its value on the basis vectors $e_{[i]}^{\otimes_s(k)}$ setting

$$\left\| e_{[i]}^{\otimes_s(k)} \right\|^2_{\eta_B} = \frac{k_1! \dots k_n!}{(k_1 + \cdots + k_n)!}.$$

Hence, (see [15]) for any $w \in \mathcal{F}_{\eta_B}$ there is an analytic function f on the unit ball $B \subset E$ such that

$$f(x) = \langle \eta_B(x) \mid w \rangle, \tag{3}$$

where

$$\eta_B(x) = 1 + x + \cdots + x^{\otimes n} + \cdots ,$$

$\|x\| < 1$. We denote the space of all such functions by $\mathcal{H}_{\eta_B}(B)$.

If $B \subset \mathbb{C}^n$, then this space is called Drury-Arveson-Hardy space [1]. Also this space has an alternative description as a Besov-Sobolev space $B_2^{\frac{1}{2}}$ of analytic functions on open unit ball in $\mathbb{C}^n$. Note that if $\dim E = 1$, the space $\mathcal{H}_{\eta_B}(B)$ coincides with the classical Hardy space.

Since $\mathcal{H}_{\eta_E}(E)$ and $\mathcal{H}_{\eta_B}(B)$ are the spaces with orthonormal bases

$$\Big\{\Big(\sqrt{k_1!\ldots k_n!}\Big)^{-1} e^{*\otimes_s(k)}_{[i]}\Big\} \quad \text{and} \quad \Big\{\Big(\sqrt{\frac{k_1!\ldots k_n!}{(k_1+\cdots+k_n)!}}\Big)^{-1} e^{*\otimes_s(k)}_{[i]}\Big\}$$

respectively, where $e^{*\otimes_s(k)}_{[i]}(x) = (x \mid e^{\otimes k_1}_{i_1})\cdots(x \mid e^{\otimes k_n}_{i_n})$, $x \in E$, the operator $\mathcal{A}$ defined on $\mathcal{H}_{\eta_E}(E)$ by

$$\mathcal{A}\Big((k_1!\ldots k_n!)^{-1/2} e^{*\otimes_s(k)}_{[i]}\Big) = \sqrt{\frac{(k_1+\cdots+k_n)!}{k_1!\ldots k_n!}}\, e^{*\otimes_s(k)}_{[i]}$$

is an isometric isomorphism onto $\mathcal{H}_{\eta_B}(B)$ and

$$\mathcal{A}(e^{*\otimes_s(k)}_{[i]}) = \sqrt{(k_1+\cdots+k_n)!}\, e^{*\otimes_s(k)}_{[i]} = \sqrt{n!}\, e^{*\otimes_s(k)}_{[i]}.$$

So, for an arbitrary function $f \in \mathcal{H}_{\eta_E}(E)$ with the Taylor series expansion $f = \sum_{n=0}^{\infty} f_n$:

$$\mathcal{A}(f) = \sum_{n=0}^{\infty} \sqrt{n!} f_n.$$

Let $T : E \to E$ be a linear operator. We denote by $C_T^B : \mathcal{H}_{\eta_B}(B) \to \mathcal{H}_{\eta_B}(B)$ the operator which makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{H}_{\eta_E}(E) & \xrightarrow{C_T} & \mathcal{H}_{\eta_E}(E) \\ \mathcal{A}\downarrow & & \downarrow\mathcal{A} \\ \mathcal{H}_{\eta_B}(B) & \xrightarrow{C_T^B} & \mathcal{H}_{\eta_B}(B). \end{array}$$

So, $C_T^B = \mathcal{A} C_T \mathcal{A}^{-1}$.

Let $g \in \mathcal{H}_{\eta_B}(B)$, $g(x) = \sum_{n=0}^{\infty} g_n(x)$, $x \in B$. Then

$$\mathcal{A}^{-1}(g(x)) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} g_n(x),$$

$$C_T \mathcal{A}^{-1}(g(x)) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} g_n(T(x)),$$

$$C_T^B = \mathcal{A} C_T \mathcal{A}^{-1}(g(x)) = \sum_{n=0}^{\infty} \sqrt{n!} \cdot \frac{1}{\sqrt{n!}} g_n(T(x)) = \sum_{n=0}^{\infty} g_n(T(x)).$$

We can see that C_T^B is a composition operator.

From Proposition 2.4 and Lemma 0.2 it easily follows the next corollary.

Corollary 2.5. *There is no hypercyclic composition operator with a linear operator on the space $\mathcal{H}_{\eta_B}(B)$.*

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