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COMMENT ON "A UNIFORM BOUNDEDNESS THEOREM FOR LOCALLY CONVEX CONES" [W. ROTH, PROC. AMER. MATH. SOC. 126 (1998), 1973–1982]

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Abstract. In page 1975 of [W. Roth, *A uniform boundedness theorem for locally convex cones*, Proc. Amer. Math. Soc. **126** (1998), no. 7, 1973–1982] we can see: In a locally convex vector space E a barrel is defined to be an absolutely convex closed and absorbing subset A of E. The set $U = \{(a,b) \in E^2,\ a - b \in A\}$ then is seen to be a barrel in the sense of Roth's definition. With a counterexample, we show that it is not enough for U to be a barrel in the sense of Roth's definition. Then we correct this error with providing its converse and an application.

1. Introduction and Preliminaries

An ordered cone (cf. [1] and [2]) is a set $\mathcal{P}$ endowed with an addition $(a,b) \longmapsto a+b$ and a scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold. Also, $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. We will denote $\mathcal{P}^+$ the subcone of positive elements of $\mathcal{P}$.

Let $\mathcal{P}$ be an ordered cone. A subset $\mathcal{V}$ of $\mathcal{P}$ is called an (abstract) 0-neighborhood system, if the following properties hold:

(1) $0 < v$ for all $v \in \mathcal{V}$;
(2) for all $u, v \in \mathcal{V}$ there is $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;
(3) $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we define

$$v(a) = \{b \in \mathcal{P} : b \leq a + v\}, \quad \text{respectively} \quad (a)v = \{b \in \mathcal{P} : a \leq b + v\},$$

to be a neighborhood of a in the upper, respectively lower topologies on $\mathcal{P}$. Their common refinement is called symmetric topology which we show the neighborhoods in this topology as $v(a) \cap (a)v$ or $v(a)v$ for $a \in \mathcal{P}$ and $v \in \mathcal{V}$.

We call $(\mathcal{P}, \mathcal{V})$ a full locally convex cone, and each subcone of $\mathcal{P}$, not necessarily containing $\mathcal{V}$, is called a locally convex cone (L.C.C.). For technical reasons we require the elements of a locally convex cone to be bounded below, i.e. for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. An element a of $(\mathcal{P}, \mathcal{V})$ is called bounded if it is also upper bounded, i.e. for every $v \in \mathcal{V}$ there is $\rho > 0$ such that $a \leq \rho v$.

The extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers with the usual order is an example of an ordered cone. A functional on a L.C.C. $\mathcal{P}$ is a linear mapping $\mu : \mathcal{P} \longrightarrow \overline{\mathbb{R}}$. μ is u-continuous if there is a $v \in \mathcal{V}$ such that $\mu(a) \leq \mu(b) + 1$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. The u-continuous linear functionals on a locally convex cone $(\mathcal{P}, \mathcal{V})$ (into $\overline{\mathbb{R}}$)

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form a cone with the usual addition and scalar multiplication of functions. This cone is called the dual cone of $\mathcal{P}$ and is denoted by $\mathcal{P}^*$.

2. The relation between barrels of a L.C.S. and a L.C.C.

In this section, we give a counterexample to show that the set $U = \{(a,b) \in E^2 : a - b \in A\}$ is not a barrel where E is a L.C.S. and A is a barrel in E.

Definition 2.1. Let E be a locally convex vector space (L.C.S.). A barrel in E is an absolutely convex closed and absorbing subset A of E.

Definition 2.2. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. A barrel in $\mathcal{P}$ is a convex subset U of $\mathcal{P}^2$ with the following properties:

(U1) For every $b \in \mathcal{P}$ there is a $v \in \mathcal{V}$ such that for every $a \in v(b)v$ there is a $\lambda > 0$ such that $(a,b) \in \lambda U$.

(U2) For all $a, b \in \mathcal{P}$ such that $(a,b) \notin U$ there is a $\mu \in \mathcal{P}^*$ such that $\mu(c) \leq \mu(d) + 1$ for all $(c,d) \in U$ and $\mu(a) > \mu(b) + 1$.

Lemma 2.3. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and $U \subset \mathcal{P}^2$ be a barrel.*

(a) *If $a \leq b$ for $a, b \in \mathcal{P}$, then $(a,b) \in U$.*

(b) *If $(a + \epsilon b, b + \epsilon b) \in U$ for $a, b \in \mathcal{P}$ and some $\epsilon \geq 0$, then $(a,b) \in U$.*

(c) *If $(a,b) \in U$, if $a' + c \leq a + d$ and $b' + c \geq b + d$ for $a', b', c, d \in \mathcal{P}$ and if c is bounded, then $(a', b') \in U$.*

Proof. ([3], Lemma 2.1). □

In [3] we can see the following statement:

In a locally convex vector space E a barrel is defined to be an absolutely convex closed and absorbing subset A of E (cf. [4], II.7). The set $U = \{(a,b) \in E^2 : a - b \in A\}$ is seen to be a barrel in the sense of definition 2.2, where E is a locally convex vector space and A is a barrel in E.

We now show that it is not enough for U to be a barrel in the sense of definition 2.2, by giving some counterexamples. Then we correct this error by adding an extra condition to U.

Counterexample 2.4. Let $(\overline{\mathbb{R}}, \mathcal{V})$ be the full locally convex cone with the usual order, where $\mathcal{V} = \{\epsilon > 0 : \epsilon \in \mathbb{R}\}$ is an (abstract) 0-neighborhood system, then $(\mathbb{R}, \mathcal{V})$ will be a locally convex cone. Let $A = [-\epsilon, \epsilon]$, $\epsilon > 0$, then A is a barrel in the locally convex vector space $\mathbb{R}$. Suppose $U = \{(x,y) \in \mathbb{R}^2 : x - y \in A\}$ then by letting $x = -2\epsilon$ and $y = 2\epsilon$, we have $x \leq y$ but $(x,y) \notin U$. This contradicts with Lemma 2.3(a), hence U is not a barrel in the sense of definition 2.2.

Now we are ready to correct this error and verify its converse,

Theorem 2.5. *Suppose E is an ordered locally convex vector space.*

(a) *If A is a barrel in E, then*

$$U = \{(a,b) \in E^2 : a - b \in A \text{ or } b - a \in E^+\}$$

is a barrel in the sense of definition 2.2. Conversely,

(b) *If U is a barrel in E in the sense of definition 2.2, then $U \cap U^{-1} \cap E$ is a barrel in E.*

Proof. (a) We first show that U is convex. Suppose $(a,b), (c,d) \in U$, $0 \leq \lambda \leq 1$ and $\lambda(a,b) + (1-\lambda)(c,d) \notin U$, then $\lambda a + (1-\lambda)c - \lambda b - (1-\lambda)d \notin A$. Since A is absolutely convex and closed set in the locally convex vector space E, by Hahn-Banach theorem, there exists a $\mu \in E^*$ such that $\mu(x) \leq 1$ for all $x \in A$. Hence,

$$\lambda\mu(a) + (1-\lambda)\mu(c) > \lambda\mu(b) + (1-\lambda)\mu(d) + 1.$$

Assume that $a - b \in A$ and $d - c \in E^+$. In the other cases the convexity of U is clear. We have $\lambda\mu(a) \leq \lambda\mu(b) + 1$ and from the monotonicity of u-continuous functional μ it follows that $(1 - \lambda)\mu(c) \leq (1 - \lambda)\mu(d)$. Therefore,

$$\lambda\mu(a) + (1 - \lambda)\mu(c) \leq \lambda\mu(b) + (1 - \lambda)\mu(d) + 1.$$

This is a contradiction and so U is convex. Now we show U is a barrel for E:

(U1) It is obvious, because A is absorbing.

(U2) For $a, b \in E$ such that $(a, b) \notin U$, we have $a - b \notin A$. By Hahn-Banach theorem, there exists a $\mu \in E^*$ such that $\mu(x) \leq 1$ for all $x \in A$. Thus $\mu(a) > \mu(b) + 1$. For $(c, d) \in U$, we have $c - d \in A$ or $d - c \in E^+$. If $c - d \in A$, then $\mu(c) \leq \mu(d) + 1$. If $d - c \in E^+$, then $\mu(d - c) \geq \mu(0)$ and therefore $1 + \mu(d) \geq \mu(c)$. From (U1) and (U2) we conclude U is a barrel.

(b) Let $A = U \cap U^{-1} \cap E$. It is clear that A is convex. We show that A is balanced. Suppose that $b \in A$, $|\lambda| \leq 1$ and $\lambda b \notin U$. Since U is a barrel for E, from (U2) it follows that there exists a $\mu \in E^*$ such that $\mu(\lambda b) > 1$. If $\lambda = 0$, then we obtain the contradiction $\mu(0) > 1$. If $-1 \leq \lambda < 0$ or $0 < \lambda \leq 1$, we obtain the contradiction $\mu(\lambda b) \leq 1$. In the same manner we can see that $\lambda b \in U^{-1}$. We now show that A is absorbing. Suppose that $b \in E$. Assume for every $\lambda > 0$, $b \notin \lambda A$. Then $\frac{b}{\lambda} \notin U$ or $\frac{b}{\lambda} \notin U^{-1}$. If $\frac{b}{\lambda} \notin U$, then from (U2), there exists a $\mu \in E^*$ such that $\mu(b) > \lambda$. This is a contradiction, because $\mu(b)$ is finite. Similar arguments apply to the case $\frac{b}{\lambda} \notin U^{-1}$. At the end, we prove that A is closed. Suppose $x \in A^c$. Then $x \notin U$ or $x \notin U^{-1}$. Let $x \notin U$. From (U2), there exists a $\mu \in E^*$ such that $\mu(x) > 1$. Choose $\epsilon = \frac{\mu(x)-1}{3}$ and $V = \mu^{-1}((\mu(x) - \epsilon, \mu(x) + \epsilon))$. Hence $x \in V \subseteq A^c$. Similar arguments apply to the case $x \in U^{-1}$. □

Definition 2.6. A locally convex vector space is said to be barreled if every barrel is a neighborhood of the origin.

Definition 2.7. A locally convex cone $(\mathcal{P}, \mathcal{V})$ is said to be barreled if for every barrel $U \subset \mathcal{P}^2$ and every element $b \in \mathcal{P}$ there are a neighborhood $v \in \mathcal{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda U$ for all $a \in v(b)v$.

In the following corollary, we give an application of our result.

Corollary 2.8. Suppose that the locally convex cone $(\mathcal{P}, \mathcal{V})$ is also a locally convex vector space. If $\mathcal{P}$ is a barreled locally convex cone, then $\mathcal{P}$ is also a barreled locally convex vector space.

Proof. Let A be a barrel for the locally convex vector space $\mathcal{P}$. From theorem 2.5(a) it follows that $U = \{(a, 0) \in \mathcal{P}^2 : a \in A \text{ or } -a \in \mathcal{P}^+\}$ is a barrel for the locally convex cone $\mathcal{P}$. Since the locally convex cone $\mathcal{P}$ is barreled, there are a neighborhood $v \in \mathcal{V}$ and a $\lambda > 0$ such that $(a, 0) \in \lambda U$ for all $a \in v(0)v$. It is impossible that $a \notin \lambda A$ and $-a \in \mathcal{P}^+$. Thus $a \in \lambda A$ for all $a \in v(0)v$. This means A is a neighborhood of the origin. □

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