## COMMENT ON "A UNIFORM BOUNDEDNESS THEOREM FOR LOCALLY CONVEX CONES" [W. ROTH, PROC. AMER. MATH. SOC. 126 (1998), 1973–1982]

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ABSTRACT. In page 1975 of [W. Roth, A uniform boundedness theorem for locally convex cones, Proc. Amer. Math. Soc. **126** (1998), no. 7, 1973–1982] we can see: In a locally convex vector space E a barrel is defined to be an absolutely convex closed and absorbing subset A of E. The set  $U = \{(a, b) \in E^2, a - b \in A\}$  then is seen to be a barrel in the sense of Roth's definition. With a counterexample, we show that it is not enough for U to be a barrel in the sense of Roth's definition. Then we correct this error with providing its converse and an application.

## 1. INTRODUCTION AND PRELIMINARIES

An ordered cone (cf. [1] and [2]) is a set  $\mathcal{P}$  endowed with an addition  $(a, b) \mapsto a + b$ and a scalar multiplication  $(\alpha, a) \mapsto \alpha a$  for real numbers  $\alpha \geq 0$ . The addition is supposed to be associative and commutative, and there is a neutral  $0 \in \mathcal{P}$ . For the scalar multiplication the usual associative and distributive properties hold. Also,  $\mathcal{P}$  carries a (partial) order, i.e., a reflexive transitive relation  $\leq$  such that  $a \leq b$  implies  $a + c \leq b + c$ and  $\alpha a \leq \alpha b$  for all  $a, b, c \in \mathcal{P}$  and  $\alpha \geq 0$ . We will denote  $\mathcal{P}^+$  the subcone of positive elements of  $\mathcal{P}$ .

Let  $\mathcal{P}$  be an ordered cone. A subset  $\mathcal{V}$  of  $\mathcal{P}$  is called an (abstract) 0-neighborhood system, if the following properties hold:

(1) 0 < v for all  $v \in \mathcal{V}$ ;

- (2) for all  $u, v \in \mathcal{V}$  there is  $w \in \mathcal{V}$  with  $w \leq u$  and  $w \leq v$ ;
- (3)  $u + v \in \mathcal{V}$  and  $\alpha v \in \mathcal{V}$  whenever  $u, v \in \mathcal{V}$  and  $\alpha > 0$ .

For every  $a \in \mathcal{P}$  and  $v \in \mathcal{V}$  we define

 $v(a) = \{b \in \mathcal{P} : b \le a + v\}, \quad \text{respectively} \quad (a)v = \{b \in \mathcal{P} : a \le b + v\},\$ 

to be a neighborhood of a in the upper, respectively lower topologies on  $\mathcal{P}$ . Their common refinement is called symmetric topology which we show the neighborhoods in this topology as  $v(a) \cap (a)v$  or v(a)v for  $a \in \mathcal{P}$  and  $v \in \mathcal{V}$ .

We call  $(\mathcal{P}, \mathcal{V})$  a full locally convex cone, and each subcone of  $\mathcal{P}$ , not necessarily containing  $\mathcal{V}$ , is called a locally convex cone (L.C.C.). For technical reasons we require the elements of a locally convex cone to be bounded below, i.e. for every  $a \in \mathcal{P}$  and  $v \in \mathcal{V}$  we have  $0 \leq a + \rho v$  for some  $\rho > 0$ . An element a of  $(\mathcal{P}, \mathcal{V})$  is called bounded if it is also upper bounded, i.e. for every  $v \in \mathcal{V}$  there is  $\rho > 0$  such that  $a \leq \rho v$ .

The extended scalar field  $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  of real numbers with the usual order is an example of an ordered cone. A functional on a L.C.C.  $\mathcal{P}$  is a linear mapping  $\mu : \mathcal{P} \longrightarrow \overline{\mathbb{R}}$ .  $\mu$  is u-continuous if there is a  $v \in \mathcal{V}$  such that  $\mu(a) \leq \mu(b) + 1$  whenever  $a \leq b + v$  for  $a, b \in \mathcal{P}$ . The u-continuous linear functionals on a locally convex cone  $(\mathcal{P}, \mathcal{V})$  (into  $\overline{\mathbb{R}}$ )

<sup>2000</sup> Mathematics Subject Classification. 46A03.

Key words and phrases. Locally convex cone, barrel.

The work was supported by a grant from Payame Noor University, Iran.

form a cone with the usual addition and scalar multiplication of functions. This cone is called the dual cone of  $\mathcal{P}$  and is denoted by  $\mathcal{P}^*$ .

2. The relation between barrels of a L.C.S. and a L.C.C.

In this section, we give a counterexample to show that the set  $U = \{(a, b) \in E^2 : a - b \in A\}$  is not a barrel where E is a L.C.S. and A is a barrel in E.

**Definition 2.1.** Let E be a locally convex vector space (L.C.S.). A barrel in E is an absolutely convex closed and absorbing subset A of E.

**Definition 2.2.** Let  $(\mathcal{P}, \mathcal{V})$  be a locally convex cone. A barrel in  $\mathcal{P}$  is a convex subset U of  $\mathcal{P}^2$  with the following properties:

- (U1) For every  $b \in \mathcal{P}$  there is a  $v \in \mathcal{V}$  such that for every  $a \in v(b)v$  there is a  $\lambda > 0$  such that  $(a, b) \in \lambda U$ .
- (U2) For all  $a, b \in \mathcal{P}$  such that  $(a, b) \notin U$  there is a  $\mu \in \mathcal{P}^*$  such that  $\mu(c) \leq \mu(d) + 1$  for all  $(c, d) \in U$  and  $\mu(a) > \mu(b) + 1$ .

**Lemma 2.3.** Let  $(\mathcal{P}, \mathcal{V})$  be a locally convex cone and  $U \subset \mathcal{P}^2$  be a barrel.

- (a) If  $a \leq b$  for  $a, b \in \mathcal{P}$ , then  $(a, b) \in U$ .
- (b) If  $(a + \epsilon b, b + \epsilon b) \in U$  for  $a, b \in \mathcal{P}$  and some  $\epsilon \ge 0$ , then  $(a, b) \in U$ .
- (c) If  $(a,b) \in U$ , if  $a' + c \leq a + d$  and  $b' + c \geq b + d$  for  $a',b',c,d \in \mathcal{P}$  and if c is bounded, then  $(a',b') \in U$ .

*Proof.* ([3], Lemma 2.1).

In [3] we can see the following statement:

In a locally convex vector space E a barrel is defined to be an absolutely convex closed and absorbing subset A of E (cf. [4], II.7). The set  $U = \{(a, b) \in E^2 : a - b \in A\}$  is seen to be a barrel in the sense of definition 2.2, where E is a locally convex vector space and A is a barrel in E.

We now show that it is not enough for U to be a barrel in the sense of definition 2.2, by giving some counterexamples. Then we correct this error by adding an extra condition to U.

**Counterexample 2.4.** Let  $(\mathbb{R}, \mathcal{V})$  be the full locally convex cone with the usual order, where  $\mathcal{V} = \{\epsilon > 0 : \epsilon \in \mathbb{R}\}$  is an (abstract) 0-neighborhood system, then  $(\mathbb{R}, \mathcal{V})$  will be a locally convex cone. Let  $A = [-\epsilon, \epsilon], \epsilon > 0$ , then A is a barrel in the locally convex vector space  $\mathbb{R}$ . Suppose  $U = \{(x, y) \in \mathbb{R}^2 : x - y \in A\}$  then by letting  $x = -2\epsilon$  and  $y = 2\epsilon$ , we have  $x \leq y$  but  $(x, y) \notin U$ . This contradicts with Lemma 2.3(a), hence U is not a barrel in the sense of definition 2.2.

Now we are ready to correct this error and verify its converse,

**Theorem 2.5.** Suppose E is an ordered locally convex vector space.

(a) If A is a barrel in E, then

 $U = \{(a, b) \in E^2 : a - b \in A \text{ or } b - a \in E^+\}$ 

- is a barrel in the sense of definition 2.2. Conversely,
- (b) If U is a barrel in E in the sense of definition 2.2, then  $U \cap U^{-1} \cap E$  is a barrel in E.

*Proof.* (a) We first show that U is convex. Suppose  $(a, b), (c, d) \in U, 0 \leq \lambda \leq 1$  and  $\lambda(a, b) + (1 - \lambda)(c, d) \notin U$ , then  $\lambda a + (1 - \lambda)c - \lambda b - (1 - \lambda)d \notin A$ . Since A is absolutely convex and closed set in the locally convex vector space E, by Hahn-Banach theorem, there exists a  $\mu \in E^*$  such that  $\mu(x) \leq 1$  for all  $x \in A$ . Hence,

$$\lambda\mu(a) + (1-\lambda)\mu(c) > \lambda\mu(b) + (1-\lambda)\mu(d) + 1.$$

Assume that  $a - b \in A$  and  $d - c \in E^+$ . In the other cases the convexity of U is clear. We have  $\lambda \mu(a) \leq \lambda \mu(b) + 1$  and from the monotonicity of u-continuous functional  $\mu$  it follows that  $(1 - \lambda)\mu(c) \leq (1 - \lambda)\mu(d)$ . Therefore,

$$\lambda \mu(a) + (1 - \lambda)\mu(c) \le \lambda \mu(b) + (1 - \lambda)\mu(d) + 1.$$

This is a contradiction and so U is convex. Now we show U is a barrel for E:

(U1) It is obvious, because A is absorbing.

(U2) For  $a, b \in E$  such that  $(a, b) \notin U$ , we have  $a - b \notin A$ . By Hahn-Banach theorem, there exists a  $\mu \in E^*$  such that  $\mu(x) \leq 1$  for all  $x \in A$ . Thus  $\mu(a) > \mu(b) + 1$ . For  $(c, d) \in U$ , we have  $c - d \in A$  or  $d - c \in E^+$ . If  $c - d \in A$ , then  $\mu(c) \leq \mu(d) + 1$ . If  $d - c \in E^+$ , then  $\mu(d - c) \geq \mu(0)$  and therefore  $1 + \mu(d) \geq \mu(c)$ . From (U1) and (U2) we conclude U is a barrel.

(b) Let  $A = U \cap U^{-1} \cap E$ . It is clear that A is convex. We show that A is balanced. Suppose that  $b \in A$ ,  $|\lambda| \leq 1$  and  $\lambda b \notin U$ . Since U is a barrel for E, from (U2) it follows that there exists a  $\mu \in E^*$  such that  $\mu(\lambda b) > 1$ . If  $\lambda = 0$ , then we obtain the contradiction  $\mu(0) > 1$ . If  $-1 \leq \lambda < 0$  or  $0 < \lambda \leq 1$ , we obtain the contradiction  $\mu(\lambda b) \leq 1$ . In the same manner we can see that  $\lambda b \in U^{-1}$ . We now show that A is absorbing. Suppose that  $b \in E$ . Assume for every  $\lambda > 0$ ,  $b \notin \lambda A$ . Then  $\frac{b}{\lambda} \notin U$  or  $\frac{b}{\lambda} \notin U^{-1}$ . If  $\frac{b}{\lambda} \notin U$ , then from (U2), there exists a  $\mu \in E^*$  such that  $\mu(b) > \lambda$ . This is a contradiction, because  $\mu(b)$  is finite. Similar arguments apply to the case  $\frac{b}{\lambda} \notin U^{-1}$ . At the end, we prove that A is closed. Suppose  $x \in A^c$ . Then  $x \notin U$  or  $x \notin U^{-1}$ . Let  $x \notin U$ . From (U2), there exists a  $\mu \in E^*$  such that  $\mu(x) > 1$ . Choose  $\epsilon = \frac{\mu(x)-1}{3}$  and  $V = \mu^{-1}((\mu(x) - \epsilon, \mu(x) + \epsilon))$ . Hence  $x \in V \subseteq A^c$ . Similar arguments apply to the case  $x \in U^{-1}$ .

**Definition 2.6.** A locally convex vector space is said to be barreled if every barrel is a neighborhood of the origin.

**Definition 2.7.** A locally convex cone  $(\mathcal{P}, \mathcal{V})$  is said to be barreled if for every barrel  $U \subset \mathcal{P}^2$  and every element  $b \in \mathcal{P}$  there are a neighborhood  $v \in \mathcal{V}$  and a  $\lambda > 0$  such that  $(a, b) \in \lambda U$  for all  $a \in v(b)v$ .

In the following corollary, we give an application of our result.

**Corollary 2.8.** Suppose that the locally convex cone  $(\mathcal{P}, \mathcal{V})$  is also a locally convex vector space. If  $\mathcal{P}$  is a barreled locally convex cone, then  $\mathcal{P}$  is also a barreled locally convex vector space.

*Proof.* Let A be a barrel for the locally convex vector space  $\mathcal{P}$ . From theorem 2.5(a) it follows that  $U = \{(a, 0) \in \mathcal{P}^2 : a \in A \text{ or } -a \in \mathcal{P}^+\}$  is a barrel for the locally convex cone  $\mathcal{P}$ . Since the locally convex cone  $\mathcal{P}$  is barreled, there are a neighborhood  $v \in \mathcal{V}$  and a  $\lambda > 0$  such that  $(a, 0) \in \lambda U$  for all  $a \in v(0)v$ . It is impossible that  $a \notin \lambda A$  and  $-a \in \mathcal{P}^+$ . Thus  $a \in \lambda A$  for all  $a \in v(0)v$ . This means A is a neighborhood of the origin.  $\Box$ 

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Received 02/06/2013