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EXISTENCE THEOREMS OF THE ω-LIMIT STATES FOR CONFLICT DYNAMICAL SYSTEMS

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Abstract. We introduce a notion of the conflict dynamical system in terms of probability measures, study the behavior of trajectories of such systems, and prove the existence theorems of the ω-limit states.

1. Introduction

The notion of a conflict dynamical system takes its beginning in [16, 17] and was used in [18, 19] for construction of abstract models describing the behavior of complex systems with internal conflict interactions. In fact a big collection of dynamical systems with the conflict phenomenon have been investigated by a number of authors both on abstract level (see, for example, [15, 27, 28]) and in various applications [3, 7, 11, 24, 14, 23] (see also [9, 10, 26] and references therein). The mostly known dynamical system with various variants of conflict interactions are named as the pray-predator model. Its study has a long and rich history (for more details see, for example, [10, 26]). In the mathematical setting this model is described by a system of Lotka–Volttera equations.

In our approach presented in a series of publications [1, 2, 6, 16, 17, 18, 19, 20] we pass to the probability interpretation of the conflict interaction for complex dynamical systems with arbitrary many regions. We assume that opposite sides of a conflict are alternative and non-annihilating. Our main result states the existence of a limit fixed point (a limit compromise, equilibrium state) for each trajectory of the system.

The simplest version of our model of the conflict dynamical system may be written in terms of coordinates of the stochastic vectors $p, r \in \mathbb{R}^d_+, d \geq 2$, corresponding to the opponent sides:

$$\frac{d}{dt}p_i = p_i\Theta - \tau_i, \quad \frac{d}{dt}r_i = r_i\Theta - \tau_i, \quad i = 1, \dots, d,$$

where $\Theta = \Theta(t)$ describes a power of conflict interaction and τ_i corresponds to the local confrontation. In the standard sample we set $\theta = (p, r)$ be the inner product between vectors p, r, and $\tau_i = p_i r_i$. Then it was proved in [16, 17] that each trajectory $\{p(t), r(t)\}$ starting with any couple of stochastic vectors $\{p, r\}$ converges with $t \to \infty$ to an ω-limit fixed point $\{p^\infty, r^\infty\}$. That is, the limit state $\{p^\infty, r^\infty\}$ is a compromise in the sense that $p^\infty \perp r^\infty$. This state is uniquely determined by the starting couple $\{p, r\}$ and has an explicit representation in its terms. In the subsequent publications [12, 18, 19, 21, 22] we generalized our construction to the cases of piece-wise uniformly distributed measures and structural-similar, in particular, self-similar measures.

The problem of getting an appropriate general formulae of conflict interaction in terms of arbitrary probability measures was open up to now.

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In this paper we propose a variant of the required formulae without any additional restrictions on measures. The general structure of these formulae is actually the same and, in the case of discrete time $N = 0, 1, \dots,$ has the form

$$\mu_{N+1}(A) = \mu_N(A) + \mu_N(A)\Theta_N - \tau_N(A),$$
$$\nu_{N+1}(A) = \nu_N(A) + \nu_N(A)\Theta_N - \tau_N(A),$$

where we omit a normalization denominator and where A stands for a measurable subset. It is important to make the right choice of forms for a power of the global conflict interaction $\Theta_N = \Theta(\mu_N, \nu_N)$ and for the local confrontation $\tau_N(A)$.

In the present paper we find general appropriate requirements on Θ and τ which allows us to prove the existence theorems for ω-limit points for trajectories starting with any couple of probability measures μ, ν. Besides we are able to describe the limiting measures in terms of the Hahn–Jordan decomposition of the starting signed measure $\omega = \mu - \nu$.

To be more specific, let the evolution of the dynamical system in terms of measures be governed by a nonlinear law of the conflict dynamic as follows:

$$\frac{d}{dt}\mu = \frac{\mu\Theta - \tau}{z}, \quad \frac{d}{dt}\nu = \frac{\nu\Theta - \tau}{z},$$

(cf. with [7, 11]) where μ, ν denote an initial couple of probability measures on a compact Ω. Here $\Theta = \Theta(\mu, \nu)$ is a positive quadratic form which fixes a so-called conflict exponent of the global conflict interaction, the measure $\tau = \tau(\mu, \nu)$ has a sense of the occupation exponent, and finally z stands for a normalization. Then our main result states the existence of the fixed points $\mu^\infty = \lim_{t\to\infty} \mu(t)$, $\nu^\infty = \lim_{t\to\infty} \nu(t)$ which coincide with normalized components of the classical Hanh–Jordan decomposition of the signed measure $\omega = \mu - \nu = \omega_+ - \omega_-$, i.e.,

$$\mu^\infty = \frac{\omega_+}{\omega_+(\Omega)}, \quad \nu^\infty = \frac{\omega_-}{\omega_-(\Omega)}.$$

2. Notations and definitions

Let Ω be some metric space and $\mathcal{R}$ be a σ-algebra of Borel subsets of Ω. And let λ be a fixed σ-additive measure on $\mathcal{R}$. One can think that $\Omega \subset \mathbb{R}$ is compact and λ is the usual Lebesgue measure.

We denote by $\mathcal{M}(\Omega)$ the family of all σ-additive finite signed measures on Ω. The subset of positive measures on Ω we denote by $\mathcal{M}^+(\Omega)$. If μ is probability then we write $\mu \in \mathcal{M}_1^+(\Omega)$.

Let us fix a couple of measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$, $\mu \neq \nu$, and consider the signed measure $\omega := \mu - \nu \in \mathcal{M}(\Omega)$, which we call sometimes a charge.

According to classical results of measure theory (see, for example, [29, 8]), each charge ω determines the Hahn decomposition of Ω onto two subsets

$$\Omega = \Omega_+ \bigcup \Omega_-, \quad \Omega_+ \bigcap \Omega_- = \emptyset, \quad \Omega_+, \Omega_- \in \mathcal{R}, \tag{1}$$

such that

$$\omega(A_+) \geq 0, \quad \forall A_+ \subseteq \Omega_+; \quad \omega(A_-) \leq 0, \quad \forall A_- \subseteq \Omega_-, \quad (A_+, A_- \in \mathcal{R}) \tag{2}$$

or, equivalently,

$$\mu(A_+) \geq \nu(A_+), \quad \mu(A_-) \leq \nu(A_-), \quad A_+ \subseteq \Omega_+, \quad A_- \subseteq \Omega_-. \tag{3}$$

Thus, one can define a new couple of positive measures on $\mathcal{R}$ setting

$$\omega_+ = \omega \restriction \Omega_+, \quad \omega_- = -\omega \restriction \Omega_- \tag{4}$$

with $\omega_+(\Omega_-) = \omega_-(\Omega_+) = 0$. Clearly the Hahn decomposition (1) is non-unique, if there exists a nontrivial set A_0 such that $\mu(A_0) = \nu(A_0) = 0$. However the measures

$\omega_+,\ \omega_-$ in (4) are uniquely defined and their sum provides the Jordan decomposition of the signed measure ω

$$\omega = \omega_+ + \omega_-.$$

In the sequel instead of ω_+, ω_- we use the normalized probability measures

$$\mu_+ := \frac{\omega_+}{\omega_+(\Omega)}, \quad \nu_- := \frac{\omega_-}{\omega_-(\Omega)}. \tag{5}$$

We want to show that this couple of measures, μ_+, ν_-, appears in another natural way, namely, as the ω-limit state (a fixed point) of the conflict dynamical system. Before our constructions we need in some definitions and preparations.

At first we introduce the *conflict exponent* $\Theta = \Theta(\mu, \nu)$ for a couple of measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$. It is a non-negative real-valued function which characterizes the power of the conflict interaction between μ and ν. There are several variants for definition of Θ (see [2, 6]). Here we present one of them.

Assume the measures μ, ν are absolutely continuous with respect to λ

$$\mu(A) = \int_A \rho(x)\, d\lambda(x), \quad \nu(A) = \int_A \sigma(x)\, d\lambda(x), \quad A \in \mathcal{R},$$

where the densities $\rho(x), \sigma(x) \geq 0$ are defined as the Radon–Nikodym derivatives of μ, ν with respect to λ. Then Θ may be defined as

$$\Theta(\mu, \nu) := \int_\Omega \sqrt{\rho(x)\sigma(x)}\, d\lambda(x) = \int_\Omega \varphi(x)\psi(x)\, d\lambda(x) = (\varphi, \psi)_{L^2(\Omega,\ d\lambda)}. \tag{6}$$

Here the positive functions $\varphi(x) = \sqrt{\rho(x)},\ \psi(x) = \sqrt{\sigma(x)}$ belong to $L^2(\Omega,\ d\lambda)$ since obviously

$$\|\varphi\|^2_{L^2(\Omega,\ d\lambda)} = \mu(\Omega) = 1 = \nu(\Omega) = \|\psi\|^2_{L^2(\Omega,\ d\lambda)}. \tag{7}$$

Besides we introduce a notion of (local) *occupation exponent* τ. It is a suitable positive measure, such that $\tau(A) \leq \inf\{\mu(A), \nu(A)\},\ A \in \mathcal{R}$. Below we define

$$\tau(A) := \nu(F) + \mu(G), \quad F, G \in \mathcal{R}, \tag{8}$$

where F, G are such that $\omega(F) = \sup_{E \subseteq A} \omega(E), \ \ -\omega(G) = \sup_{E \subseteq A}(-\omega(E))$. In fact,

$$\tau(A) := \nu(A_+) + \mu(A_-), \quad A_+ = A \cap \Omega_+, \quad A_- = A \cap \Omega_-. \tag{9}$$

In any case we suppose that

$$0 \leq \tau(A) < 1, \quad A \in \mathcal{R}.$$

In particular, the global meaning of the occupation exponent, $W := \tau(\Omega)$, satisfies the inequalities

$$0 < W < 1, \quad W := \tau(\Omega) = \nu(\Omega_+) + \mu(\Omega_-). \tag{10}$$

We also recall a well-known notion of the total variation distance between μ, ν

$$D = D(\mu, \nu) := \sup_{A \in \mathcal{R}} |\mu(A) - \nu(A)|. \tag{11}$$

It admits the representation by the densities ρ, σ in the form

$$D = 1/2 \int_\Omega |\rho(\lambda) - \sigma(\lambda)| d\lambda = 1/2 \int_\Omega |h(\lambda)| d\lambda, \quad h = \rho - \sigma. \tag{12}$$

Proposition 1.

$$1 - D \leq \Theta \leq \sqrt{1 - D^2}. \tag{13}$$

Proof. Let H denotes the Hellinger distance [13] between μ, ν

$$H = H(\mu,\nu) := \frac{1}{\sqrt{2}}\Big(\int_\Omega |\sqrt{\rho(\lambda)} - \sqrt{\sigma(\lambda)}\,|^2 d\lambda\Big)^{1/2} = \frac{1}{\sqrt{2}}\|\varphi - \psi\|_{L_2}.$$

Obviously

$$H^2(\mu,\nu) = 1 - \Theta(\mu,\nu), \tag{14}$$

where we used (7). Besides we have

$$\begin{aligned} H^2 &= \frac{1}{2}\int_\Omega |\varphi(\lambda) - \psi(\lambda)|^2 d\lambda \le \frac{1}{2}\int_\Omega |(\varphi(\lambda) - \psi(\lambda))(\varphi(\lambda) + \psi(\lambda))|\, d\lambda \\ &= \frac{1}{2}\int_\Omega |(\varphi^2(\lambda) - \psi^2(\lambda))|\, d\lambda = 1/2 \int_\Omega |\rho(\lambda) - \sigma(\lambda)|\, d\lambda = D. \end{aligned}$$

Thus, by (14) we get $H^2 = 1 - \Theta \le D$. That proves the left part of (13). Further, according to (12),

$$D = 1/2 \int_\Omega |\varphi^2(\lambda) - \psi^2(\lambda)|\, d\lambda = 1/2 (f,g)_{L_2}, \quad f = |\,\varphi - \psi\,|, \quad g = \varphi + \psi.$$

Now by the Cauchy–Schwarz inequality

$$D^2 \le 1/4 \parallel f \parallel^2_{L_2} \parallel g \parallel^2_{L_2} = H^2(1+\Theta) = (1-\Theta)(1+\Theta) = 1 - \Theta^2,$$

where we take into account that $H^2 = 1/2 \parallel f \parallel^2_{L_2}$ and $\parallel g \parallel^2_{L_2} = 2 + 2\Theta$. This proves the right part of (13). □

Finally we define the local difference between μ, ν,

$$\Delta(A) = \Delta(\mu,\nu;A) := \mu(A) - \nu(A) \equiv \omega(A), \quad A \in \mathcal{R}.$$

Proposition 2.

$$D = \Delta(\Omega_+) = -\Delta(\Omega_-) = \omega(\Omega_+) = -\omega(\Omega_-). \tag{15}$$

Proof. Due to (12) we have

$$\begin{aligned} D &= \frac{1}{2}\Big[\int_{\Omega_+} (\rho(\lambda) - \sigma(\lambda))\, d\lambda - \int_{\Omega_-} (\rho(\lambda) - \sigma(\lambda))\, d\lambda\Big] \\ &= \frac{1}{2}\Big[\mu(\Omega_+) - \nu(\Omega_+) - \mu(\Omega_-) + \nu(\Omega_-)\Big]. \end{aligned}$$

Now using $\mu(\Omega_+) + \mu(\Omega_-) = 1 = \nu(\Omega_+) + \nu(\Omega_-)$ we obtain

$$D = \mu(\Omega_+) - \nu(\Omega_+) = \Delta(\Omega_+) = \nu(\Omega_-) - \mu(\Omega_-) = -\Delta(\Omega_-).$$

□

3. CDS in terms of measures

Now we are ready to introduce in $\mathcal{M}_1^+(\Omega)$ the non-commutative composition $\circledast$

$$\mu_1 = \mu \circledast \nu, \quad \nu_1 = \nu \circledast \mu,$$

which determines a rule of the conflict interactions between μ, ν and which we call the *law of conflict dynamic.* We define

$$\mu_1(A) = \frac{1}{z}[\mu(A)\big(\Theta + 1\big) - \tau(A)], \tag{16}$$

$$\nu_1(A) = \frac{1}{z}[\nu(A)\big(\Theta + 1\big) - \tau(A)], \quad A \in \mathcal{R} \tag{17}$$

with $z = \Theta + 1 - W$ where Θ, W are given by (6) and (10) resp. Recall our assumption that μ, ν are absolutely continuous with respect to λ. By (16), (17) it follows that

μ_1, ν_1 are also absolutely continuous and probability, i.e., $\mu_1, \nu_1 \in \mathcal{M}_1^+(\Omega)$. The triple $\{\Omega, \mathcal{M}_1^+(\Omega), \circledast\}$ we call the *Conflict Dynamical System* (CDS).

A consecutive iteration of the non-linear transformation $\circledast$ generates a trajectory of CDS in terms of couples of probability measures

$$\left\{ \begin{array}{c} \mu_N \\ \nu_N \end{array} \right\} \xrightarrow{\circledast} \left\{ \begin{array}{c} \mu_{N+1} \\ \nu_{N+1} \end{array} \right\}, \quad N = 0, 1, \ldots, \tag{18}$$

where $\mu^0 = \mu,\ \nu^0 = \nu$ and

$$\begin{aligned} \mu_{N+1}(A) &= \frac{1}{z_N}[\mu_N(A)\big(\Theta_N + 1\big) - \tau_N(A)], \\ \nu_{N+1}(A) &= \frac{1}{z_N}[\nu_N(A)\big(\Theta_N + 1\big) - \tau_N(A)], \quad A \in \mathcal{R}, \end{aligned} \tag{19}$$

where, in according with (6), (9),

$$\tau_N(A) = \nu_N(A_+) + \mu_N(A_-), \tag{20}$$

$$\Theta_N = \int_\Omega \sqrt{\rho_N(x)\sigma_N(x)}\, d\lambda(x) \tag{21}$$

and

$$z_N = \Theta_N + 1 - W_N, \quad W_N = \mu_N(\Omega_-) + \nu_N(\Omega_+).$$

Here $\rho_N(x), \sigma_N(x)$ denote as above the Radon-Nikodym derivatives of μ_N, ν_N with respect to λ. Therefore

$$\Theta_N = \int_\Omega \varphi_N(x)\psi_N(x)\, d\lambda(x) = (\varphi_N, \psi_N)_{L^2(\Omega,\ d\lambda)}, \tag{22}$$

if we denote

$$\varphi_N(x) = \sqrt{\rho_N(x)}, \quad \psi_N(x) = \sqrt{\sigma_N(x)}.$$

We remark that CDS defined by (19) has two separate sets of fixed points. The first set contains all couples of measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$ which is identical, $\mu = \nu$. Then $\Theta = 1,\ \ D = 0$ and therefore $\mu_N = \mu = \nu_N = \nu$ for all N. The second set is composed by measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$ which are orthogonal, $\mu \perp \nu$. In the later case $\Theta = 0,\ D = 1$ and it is easy to see that $\mu_N = \mu = \nu_N = \nu$ too.

Theorem 1. *Each trajectory* (18) *of the conflict dynamical system starting with a couple of probability measures* $\mu_0 = \mu, \nu_0 = \nu \in \mathcal{M}_1^+(\Omega)$ $(\mu \neq \nu)$ *converges to the ω-limit state* $\left\{ \begin{array}{c} \mu_\infty \\ \nu_\infty \end{array} \right\}$

$$\mu_\infty(A) = \lim_{N\to\infty} \mu_N(A), \quad \nu_\infty(A) = \lim_{N\to\infty} \nu_N(A), \quad A \in \mathcal{R}. \tag{23}$$

That is the measures $\mu_\infty, \nu_\infty \in \mathcal{M}_1^+(\Omega)$ coincide with the normalized components of the Hanh-Jordan decomposition of the charge $\omega = \mu - \nu$ (see (5)). It means that for each $A \in \mathcal{R}$

$$\mu_\infty(A) = \frac{(\mu - \nu)(A \cap \Omega_+)}{D} = \mu_+(A), \quad \nu_\infty(A) = -\frac{(\mu - \nu)(A \cap \Omega_-)}{D} = \nu_-(A), \tag{24}$$

where D stands for the total difference for an initial couple of measures μ, ν.

Before proving of the theorem we state several propositions.

Naturally we assume that the measures μ, ν are not orthogonal. Then clearly

$$0 < \tau(\Omega) = W = \mu(\Omega_-) + \nu(\Omega_+) < 1, \quad 0 < \Theta < 1.$$

Proposition 3. *Assume that for some $A \in \mathcal{R}$ the local difference $\Delta(A)$ is strongly positive, $\mu(A)-\nu(A)>0$. Then the sequence $\Delta_N(A)=\mu_N(A)-\nu_N(A)$ is monotonically increasing*

$$\Delta_{N+1}(A) > \Delta_N(A), \quad N \geq 0.$$

Proof. By (19)

$$\Delta_{N+1}(A) = \Delta_N(A)\frac{\Theta_N+1}{\Theta_N+1-W_N} > \Delta_N(A),$$

since $0<W_N<1$ for all N. □

Corollary 1. *Under the same condition, $\mu(A)-\nu(A)>0$, there exist the limit*

$$\Delta_\infty(A) = \lim_{N\to\infty}\Delta_N(A) \leq 1.$$

In particular, for $A=\Omega_+$ and for $A=\Omega_-$ we obtain due to $\mu(\Omega_+)>\nu(\Omega_+)$ and $\mu(\Omega_-)<\nu(\Omega_-)$ the existence of the limits

$$\Delta_\infty(\Omega_+) = \lim_{N\to\infty}\Delta_N(\Omega_+) \leq 1,\ -\Delta_\infty(\Omega_-) = \lim_{N\to\infty}-\Delta_N(\Omega_-) \leq 1.$$

Moreover it is easy to see that

$$\Delta_\infty(\Omega_+) = 1 = -\Delta_\infty(\Omega_-).$$

Proposition 4.

$$\mu_N(\Omega_+)\to 1, \quad \mu_N(\Omega_-)\to 0, \quad \nu_N(\Omega_+)\to 0, \quad \nu_N(\Omega_-)\to 1, \quad N\to\infty, \tag{25}$$

and therefore

$$W_N \to 0, \quad \tau_N(\Omega)\to 0, \quad N\to\infty. \tag{26}$$

Proof. By the Hahn–Jordan decomposition, $\mu(\Omega_+)>\nu(\Omega_+)$. Moreover, the similar inequality $\mu_N(\Omega_+)>\nu_N(\Omega_+)$ is true for all N since from (19) by induction we have

$$\Delta_{N+1}(\Omega_+) = \Delta_N(\Omega_+)\frac{\Theta_N+1}{\Theta_N+1-W_N} > \Delta_N(\Omega_+) \tag{27}$$

due to $0<W=\tau_N=\mu_N(\Omega_-)+\nu_N(\Omega_+)<1$. Similarly,

$$\begin{aligned}\mu_{N+1}(\Omega_-) &= \frac{1}{z_N}[\mu_N(\Omega_-)(\Theta_N+1)-\tau_N(\Omega_-)] \\ &= \mu_N(\Omega_-)\frac{\Theta_N+1-\tau_N(\Omega_-)/\mu_N(\Omega_-)}{\Theta_N+1-W_N} = \mu_N(\Omega_-)\frac{\Theta_N}{\Theta_N+1-W_N} < \mu_N(\Omega_-),\end{aligned}$$

since $\tau_N(\Omega_-)=\mu_N(\Omega_-)$ and $0<W_N<1$. In the same way we find that $\nu_{N+1}(\Omega_+)<\nu_N(\Omega_+)$. It follows that W_N decreases as $N\to\infty$. Therefore $\mu_N(\Omega_-),\ \nu_N(\Omega_+)$ converge to zero, and $\mu_N(\Omega_+),\ \nu_N(\Omega_-)$ converge to 1. So we prove the convergence of all terms in (25). Now (25) implies (26). □

Proposition 5.

$$D_{N+1} \geq D_N \tag{28}$$

for all N.

Proof. Recall that by definition $\Delta(A)=\Delta(\mu,\nu;A)=\mu(A)-\nu(A)=\omega(A)$. Therefore

$$\Delta_N(\Omega_+) = \mu_N(\Omega_+)-\nu_N(\Omega_+) = D_N, \quad N\geq 0.$$

Now the inequality (28) follows from (27). □

Proposition 6. *There is at last a subsequence of time moments such that*

$$\Theta_{N''} \leq \Theta_{N'}, \quad N''>N'. \tag{29}$$

Proof. By (28) and (25) $D_N\to 1$. Therefore (29) follows from (13). □

In what follows without lost of generality we assume that (29) is fulfilled for all N.

Proof of Theorem. Let $A \in \Omega_+ \vee \Omega_-$. In the general case we decompose each A onto the union of $A_+ = A \cap \Omega_+$ and $A_- = A \cap \Omega_-$ and take into account the additive property of measures. So in what follows we shall consider A_+ and A_- separately.

Let $A = A_+ \subseteq \Omega_+$. Then $\mu(A_+) \geq \nu(A_+)$.

If $\mu(A_+) = \nu(A_+) \neq 0$, then $\mu_N(A_+) = \nu_N(A_+) \to 0$ as $N \to \infty$. Indeed by induction for all $N \geq 0$ we have

$$\begin{aligned}\mu_{N+1}(A_+) &= \nu_{N+1}(A_+) \\ &= \frac{1}{z_N}[\nu_N(A_+)(\Theta_N + 1) - \tau(A_+)] = \nu_N(A_+)\frac{\Theta_N + 1 - \tau(A_+)/\nu_N(A_+)}{\Theta_N + 1 - W_N} \\ &= \nu_N(A_+)\frac{\Theta_N}{\Theta_N + 1 - W_N} < \mu_N(A_+),\end{aligned}$$

since $\tau_N(A_+) = \nu_N(A_+)$ and $0 < W_N < 1$. Thus $\mu_N(A_+) = \nu_N(A_+) \to 0$, because of $\frac{\Theta_N}{\Theta_N+1-W_N} \to 0$ due to $W_N \to 0$ by (26).

Let now $\mu(A_+) > \nu(A_+)$. Then from (19) it easily follows for all $N \geq 0$:

$$\Delta_{N+1}(A_+) = \frac{1}{z_N}[\mu_N(A_+) - \nu_N(A_+)] = \Delta_N(A_+)\frac{\Theta_N + 1}{\Theta_N + 1 - W_N} > \Delta_N(A_+), \tag{30}$$

due to $W_N > 0$. It means that the difference $\Delta_N(A_+)$ monotonically increases with $N \to \infty$. Obviously this sequence is bounded, $\Delta_N(A_+) \leq 1$. Therefore it is convergent. Thus there exist a limit

$$\Delta_\infty(A_+) = \lim_{N\to\infty} \Delta_N(A_+) \leq 1. \tag{31}$$

Further, by $\mu(A_+) > \nu(A_+) > 0$ it follows that the relation $R_N(A_+) = \mu_N(A_+)/\nu_N(A_+)$ creates a monotonically increasing sequence, $R_{N+1}(A_+) > R_N(A_+)$. Indeed,

$$\begin{aligned}R_{N+1}(A_+) = \frac{\mu_{N+1}(A_+)}{\nu_{N+1}(A_+)} &= \frac{\mu_N(A_+)(\Theta_N + 1 - \nu_N(A_+)/\mu_N(A_+))}{\nu_N(A_+)(\Theta_N + 1) - \nu_N(A_+)} \\ &= \frac{\mu_N(A_+)(\Theta_N + 1 - \nu_N(A_+)/\mu_N(A_+))}{\nu_N(A_+)\Theta_N} \\ &= R_N(A_+)k_N = R(A_+) \cdot k \cdot k_1 \cdots k_N,\end{aligned} \tag{32}$$

where

$$k_N = \frac{1 + \Theta_N - \nu_N(A_+)/\mu_N(A_+)}{\Theta_N} > 1,$$

since $\nu_N(A_+) < \mu_N(A_+)$ and therefore $\nu_N(A_+)/\mu_N(A_+) < 1$.

Let us show now that

$$1 < k < k_1 < \cdots < k_N < \cdots \tag{33}$$

At first we observe that

$$k = \frac{1 + \Theta - \nu(A_+)/\mu(A_+)}{\Theta} > 1.$$

Therefore

$$R_1(A_+) > R(A_+) > 1.$$

Let us show that at least $k < k_1$. By construction we have

$$k_1 = \frac{1 + \Theta_1 - \nu_1(A_+)/\mu_1(A_+)}{\theta_1} = \frac{\Theta_1 + \varepsilon_1}{\Theta_1}, \tag{34}$$

where $\varepsilon_1 = 1 - \nu_1(A_+)/\mu_1(A_+) = 1 - (R_1(A_+))^{-1}$ satisfies the inequality

$$1 > \varepsilon_1 > 0. \tag{35}$$

This implies

$$k_1 = 1 + \varepsilon_1/\Theta_1 > 1 + \varepsilon/\Theta_1 \geq 1 + \varepsilon/\Theta = k,$$

where $\varepsilon = 1-(R(A_+))^{-1}$, and where we used the obvious inequalities $\varepsilon_1 > \varepsilon$ and $\Theta_1 \leq \Theta$. By the way, the latter one follows from Proposition 5. Now (33) we get by induction.

Thus, we proved that $R_N(A_+) \to \infty$, as $N \to \infty$. Then, with necessity $\nu_N(A_+) \to 0$, and for the sequence $\mu_N(A_+)$ there exists a limit

$$\mu_\infty(A_+) = \lim_{N\to\infty} \mu_N(A_+) = \Delta_\infty(A_+) > 0,$$

where we take into account that $\nu_\infty(A_+) = 0$.

In a similar way we prove that $\mu_N(A_-) \to 0$ for each $A_- \subset \Omega_-$, and that

$$\nu_\infty(A_-) = \lim_{N\to\infty} \nu_N(A_-) = -\Delta_\infty(A_-) > 0.$$

We have to prove else that the limiting measures μ_∞, ν_∞ coincide with μ_+, ν_-. To this end consider a couple of subsets $A_1,\ A_2 \subseteq \Omega_+$ such that $\omega(A_1) \neq 0 \neq \omega(A_2)$. Then we observe that thanks to (19), the ratio $\Delta_N(A_1)/\Delta_N(A_2)$ does not depend on N. Indeed

$$\frac{\Delta_1(A_1)}{\Delta_1(A_2)} = \frac{\big(\mu(A_1) - \nu(A_1)\big)\big(\Theta + 1\big)}{\big(\mu(A_2) - \nu(A_2)\big)\big(\Theta + 1\big)} = \frac{\Delta(A_1)}{\Delta(A_2)}.$$

And by the same construction,

$$\frac{\Delta(A_1)}{\Delta(A_2)} = \frac{\Delta_N(A_1)}{\Delta_N(A_2)}.$$

So we can go to infinity and get for $N \longrightarrow \infty$ that

$$\frac{\Delta(A_1)}{\Delta(A_2)} = \frac{\Delta_\infty(A_1)}{\Delta_\infty(A_2)}.$$

Further, due to $\mu_\infty(A_+) = \Delta_\infty(A_+)$, we obtain

$$\frac{\mu_\infty(A_1)}{\mu_\infty(A_2)} = \frac{\Delta(A_1)}{\Delta(A_2)}. \tag{36}$$

From (36) we conclude that the values $\mu_\infty(A_+)$ for $A_+ \subseteq \Omega_+$ are proportional to $\Delta(A_+)$. Therefore $\mu_\infty(A_+) = k_\mu \Delta(A_+)$, where the coefficient k_μ is independent of $A_+ \subseteq \Omega_+$. From this and due to $\operatorname{supp}\mu_\infty \subseteq \Omega_+$ and $\mu_\infty(\Omega_+) = 1$ we find

$$k_\mu = 1/\Delta(\Omega_+) = 1/\omega_+(\Omega).$$

Thus

$$\mu_\infty(A_+) = \frac{\mu(A_+) - \nu(A_+)}{\omega_+(\Omega)} = \mu_+(A_+), \quad A_+ \subseteq \Omega_+.$$

Similarly, for $B_1, B_2 \subseteq \Omega_-$, $\omega(B_1) \neq 0 \neq \omega(B_2)$ we prove that

$$\frac{\nu_\infty(B_1)}{\nu_\infty(B_2)} = \frac{\Delta(B_1)}{\Delta(B_2)} \tag{37}$$

and that $\nu_\infty(B) = k_\nu \Delta(B)$, where the coefficient k_ν is independent of $B \subseteq \Omega_-$. As above we show that $k_\nu = -1/\Delta(\Omega_-) = -1/\omega_-(\Omega)$. Therefore

$$\nu_\infty(B) = \frac{\nu(B) - \mu(B)}{\omega_-(\Omega)} = \nu_-(B), \quad B \subseteq \Omega_-.$$

This proves (24) since $\operatorname{supp}\mu_\infty, \operatorname{supp}\mu_+ \subseteq \Omega_+$ and $\operatorname{supp}\nu_\infty, \operatorname{supp}\nu_- \subseteq \Omega_-$. By the same reason the limiting measures μ_∞, ν_∞ compose the ω-limit state. This concludes the proof of the theorem. □

4. On axiomatic approach

In this section we study the conflict dynamical system in terms of abstract signed measures:

$$\{\omega_N\} \xrightarrow{*} \{\omega_{N+1}\}, \quad N = 0, 1, \ldots \tag{38}$$

Here the starting point $\omega_0 \equiv \omega = \mu - \nu$ is given by a couple of various measures $\mu, \nu \in \mathcal{M}_1^+(\Omega)$. The signed measures ω_{N+1} are defined as $\omega_{N+1} = \mu_{N+1} - \nu_{N+1}$, where the measures μ_{N+1}, ν_{N+1} are constructed by an iterative procedure,

$$\begin{aligned} \mu_{N+1}(A) &= \frac{1}{z_N}[\mu_N(A)(\Theta_N + 1) - \tau_N(A)], \\ \nu_{N+1}(A) &= \frac{1}{z_N}[\nu_N(A)(\Theta_N + 1) - \tau_N(A)], \quad A \in \mathcal{R}. \end{aligned} \tag{39}$$

In (39) we suppose that the measures $\tau_N(A) \in \mathcal{M}^+(\Omega)$ are constructed as some linear combination of μ_N, ν_N. For instance,

$$\tau_N(A) = c \cdot [\mu_N(A \cap \Omega_-) + \nu_N(A \cap \Omega_+)], \quad c > 0,$$

where Ω_-, Ω_+ correspond to the Hahn decomposition of Ω generated by ω. In particular, below for simplicity we put $c = 1$. Thus, in what follows

$$\tau_N(A) = \mu_N(A \cap \Omega_-) + \nu_N(A \cap \Omega_+). \tag{40}$$

Further, Θ_N in (39) is defined as $\Theta_N = \Theta(\omega_N, \omega_N)$, where Θ is a strongly positive quadratic form on $\mathcal{M}(\Omega)$. Now by our assumptions it is easy to see that $\mu^N(A), \nu^N(A)$ in (39) are $\sigma-$additive functions of $A \in \mathcal{R}$. Moreover, we require for μ_N, ν_N to be probability measures. To ensure this condition it is sufficient to put

$$z_N := \Theta_N + 1 - W_N, \quad W_N := \tau_N(\Omega). \tag{41}$$

In this section we will show that our main result is true without any additional specific conditions on Θ_N and τ_N.

Theorem 2. *Each trajectory of the conflict dynamical system (38) starting with a signed measure $\omega_0 = \omega = \mu - \nu$, where $\mu, \nu \in \mathcal{M}_1^+(\Omega)$, $\mu \neq \nu$, converges to the ω-limit state $\omega_\infty = \mu_\infty - \nu_\infty$ with $\mu_\infty, \nu_\infty \in \mathcal{M}_1^+(\Omega)$. In addition,*

$$\mu_\infty = \mu_+, \quad \nu_\infty = \nu_-, \tag{42}$$

where μ_+, ν_- are the normalized components of the Hanh-Jordan decomposition of ω.

Proof. Let $\omega_0(A) = \mu(A) - \nu(A) > 0$ for some $A \in \mathcal{R}$. Then $\omega_{N+1}(A) > \omega_N(A)$ for all $N \geq 0$. Indeed, by (39) we have

$$\omega_{N+1}(A) = k_N \omega_N(A), \quad k_N = \frac{\Theta_N + 1}{\Theta_N + 1 - W_N}.$$

By the construction (see (40), (41)), the sequence $W_N = \tau(\Omega) = \mu_N(\Omega_-) + \nu_N(\Omega_+)$ satisfies the inequality $0 < W_N < 1$. Thus, $k_N > 1$ for all $N \geq 0$. This proves that the values $\omega_N(A)$ are monotonically increasing when $N \to \infty$. Since $\omega_N(A) \leq 1$, there exists the limit

$$\lim_{N\to\infty} \omega_N(A) = \omega_\infty(A), \quad 0 < \omega_\infty(A) \leq 1.$$

Consequently, we get $k_N \to 1$, and $W_N \to 0$. That is, $W_\infty = 0$.

In a similar way we prove the existence of the limiting values $-1 \leq \omega_\infty(A) < 0$ for all $A \in \mathcal{R}$ such that $\omega(A) < 0$.

In fact, there exist all limits

$$\lim_{N\to\infty} \mu_N(A) = \mu_\infty(A), \quad \lim_{N\to\infty} \nu_N(A) = \nu_\infty(A), \quad A \in \mathcal{R}.$$

To prove this we begin with $A = \Omega_+$ or $A = \Omega_-$. So, by (39)

$$\nu_{N+1}(\Omega_+) = \nu_N(\Omega_+)\frac{\Theta_N}{\Theta_N + 1 - W_N} < \nu_N(\Omega_+), \tag{43}$$

where we used two facts: $\tau_N(\Omega_+) = \nu_N(\Omega_+)$ and $0 < W_N < 1$. Let $\nu_\infty(\Omega_+)$ denotes the limiting value of the decreasing sequence $\nu_N(\Omega_+)$. We assert that $\nu_\infty(\Omega_+) = 0$ because the assumption $\nu_\infty(\Omega_+) > 0$ leads to a contradiction. Indeed, in this case the equality $\frac{\Theta_\infty}{\Theta_\infty+1-W_\infty} = 1$ have to take place due to (43). However it is impossible since $W_\infty = 0$. Therefore $\mu_N(\Omega_-)$ converges to zero, i.e., $\mu_\infty(\Omega_-) = 0$. This implies $\lim_{N\to\infty} \mu_N(\Omega_+) = \mu_\infty(\Omega_+) = 1$ since all above measures are probability.

Similarly we get $\lim_{N\to\infty} \nu_N(\Omega_+) = 0$ and $\nu_\infty(\Omega_-) = 1$.

Let now $A_+ \subseteq \Omega_+$ and $\nu(A_+) > 0$. Then as above by (39) we have

$$\begin{aligned}\nu_{N+1}(A_+) &= \nu_N(A_+)\frac{\Theta_N + 1 - \tau_N(A_+)/\nu_N(A_+)}{\Theta_N + 1 - W_N}\\ &= \nu_N(A_+)\frac{\Theta_N}{\Theta_N + 1 - W_N} < \nu_N(A_+),\end{aligned}$$

where we used that $\tau_N(A_+) = \nu_N(A_+)$. This shows that the sequence $\nu_N(A_+)$ is monotonically decreasing. So there exists the limiting value

$$\nu_\infty(A_+) = \lim_{N\to\infty} \nu_N(A_+).$$

Obviously this value is zero since by construction ν_∞ is an additive function and we already proved that $\nu_\infty(\Omega_+) = 0$. This implies that $\mu_\infty(A_+) = \omega_\infty(A_+) > 0$ since now $\omega(A_+) > 0$ and $\omega_\infty(A_+) = \mu_\infty(A_+) + \nu_\infty(A_+)$. Repeating the same way for $A_- \subseteq \Omega_-$ with $\mu(A_-) > 0$ we get $\mu_\infty(A_-) = 0$ and the existence $\nu_\infty(A_-) = -\omega(A_-) > 0$.

In the general case we decompose each $A \in \mathcal{R}$ into $A = A_+ \cup A_+$, $A_+ \subseteq \Omega_+$, $A_- \subseteq \Omega_-$ and use the additivity property.

If for $A_0 \in \mathcal{R}$ the equality $\mu(A_0) = \nu(A_0) \neq 0$ is fulfilled, then $\mu_N(A_0) = \nu_N(A_0) \to 0$ as $N \to \infty$. Indeed

$$\begin{aligned}\mu_{N+1}(A_0) = \nu_{N+1}(A_0) &= \frac{1}{z_N}[\nu_N(A_0)(\Theta_N + 1) - \tau(A_0)]\\ &= \nu_N(A_0)\frac{\Theta_N + 1 - \tau(A_0)/\nu_N(A_0)}{\Theta_N + 1 - W_N} = \nu_N(A_0)\frac{\Theta_N}{\Theta_N + 1 - W_N} < \mu_N(A_0),\end{aligned}$$

since now $\tau_N(A_0) = \nu_N(A_0) = \mu_N(A_0)$. Thus $\mu_N(A_0) = \nu_N(A_0) \to 0$.

Let $\omega(A) \neq 0$, $\omega(A') \neq 0$ for $A, A' \in \mathcal{R}$. Then obviously for any N

$$\frac{\omega_N(A)}{\omega_N(A')} = \frac{\omega(A)}{\omega(A')}$$

and therefore

$$\frac{\omega(A)}{\omega(A')} = \frac{\omega_\infty(A)}{\omega_\infty(A')}.$$

This implies that $\omega_\infty = k \cdot \omega$ with some $k > 0$. In particular, taking into account that $\nu_\infty(\Omega_+) = 0 = \mu_\infty(\Omega_-)$ and $\omega(\Omega_+) = D = \omega(\Omega_-)$ we have

$$\omega_\infty(\Omega_+) = \mu_\infty(\Omega_+) = 1 = k \cdot \omega(\Omega_+) = k \cdot D.$$

Thus $k = 1/D$ and for any $A \in \mathcal{R}$ and $A_+ = A \cap \Omega_+$, $A_- = A \cap \Omega_-$

$$\mu_\infty(A) = \frac{\omega(A_+)}{D} = \mu_+(A), \quad \nu_\infty(A) = -\frac{\omega(A_-)}{D} = \nu_+(A).$$

□

Finally we remark, that proven theorem is valid for any strongly positive quadratic form Θ, so, it is possible that $\Theta_\infty \neq 0$.

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