Methods of Functional Analysis and Topology
Vol. 20 (2014), no. 4, pp. 301–320

DARBOUX TRANSFORMATION OF GENERALIZED JACOBI MATRICES

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Dedicated to Professor V. M. Adamyan on His 75-th Birthday

ABSTRACT. Let $\mathfrak{J}$ be a monic generalized Jacobi matrix, i.e. a three-diagonal block matrix of special form, introduced by M. Derevyagin and V. Derkach in 2004. We find conditions for a monic generalized Jacobi matrix $\mathfrak{J}$ to admit a factorization $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ with $\mathfrak{L}$ and $\mathfrak{U}$ being lower and upper triangular two-diagonal block matrices of special form. In this case the Darboux transformation of $\mathfrak{J}$ defined by $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$ is shown to be also a monic generalized Jacobi matrix. Analogues of Christoffel formulas for polynomials of the first and the second kind, corresponding to the Darboux transformation $\mathfrak{J}^{(p)}$ are found.

1. INTRODUCTION

Let $\{\mathfrak{s}_n\}_{n=0}^{\infty}$ be a sequence of real moments and let a functional $\mathfrak{S}$ be defined on the linear space $\mathcal{P} = \operatorname{span}\{\lambda^n : n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}\}$ by the equality

$$\mathfrak{S}(\lambda^n) = \mathfrak{s}_n, \quad n \in \mathbb{Z}_+. \tag{1.1}$$

The functional $\mathfrak{S}$ is called *quasi − definite* if all the principal submatrices of the Hankel matrix $(\mathfrak{s}_{i+k})_{i,k=0}^{n}$ are nonsingular for every $n \in \mathbb{Z}_+$. Associated with such functional is a sequence of monic polynomials $\{P_n\}_{n=0}^{\infty}$ which are orthogonal with respect to $\mathfrak{S}$ and satisfy a three-term recurrence equations

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + c_n P_n(\lambda) + b_n P_{n-1}(\lambda), \quad n \in \mathbb{Z}_+, \tag{1.2}$$

where b_n, $c_n \in \mathbb{R}$, $b_n \neq 0$, $b_0 = 1$ and initial conditions $P_{-1}(\lambda) = 0$ and $P_0(\lambda) = 1$.

The matrix

$$J = \begin{pmatrix} c_0 & 1 & & \\ b_1 & c_1 & 1 & \\ & b_2 & c_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{1.3}$$

is called a monic Jacobi matrix associated with the functional $\mathfrak{S}$.

Let $\mathbb{C}[\lambda]$ be the set of all complex polynomials and let $\widetilde{\mathfrak{S}}_1 = \lambda\mathfrak{S}$ be a perturbed functional defined by

$$(\lambda\mathfrak{S})(p) = \mathfrak{S}(\lambda p(\lambda)), \quad p \in \mathbb{C}[\lambda]. \tag{1.4}$$

As is known (see [4]) the functional $\widetilde{\mathfrak{S}}_1$ is *quasi − definite* if and only if

$$P_n(0) \neq 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{1.5}$$

2010 *Mathematics Subject Classification.* Primary 47B36; Secondary 47B50, 42C05, 15A23.

Key words and phrases. Darboux transformation, indefinite inner product, m-function, monic generalized Jacobi matrix, triangular factorization.

A sequence of monic polynomials $\left\{\widetilde{P}_n\right\}_{n=0}^{\infty}$ associated with the functional $\widetilde{\mathfrak{S}}_1$ is called *the Christoffel transform* of $\{P_n\}_{n=0}^{\infty}$ (see [4], [21]). Relations between J and the monic Jacobi matrix $J^{(p)}$ associated with $\widetilde{\mathfrak{S}}_1$ were studied in the quasi-definite case (see [3]). As was shown in [3], every monic Jacobi matrix which satisfies (1.5) admits an $LU-$factorization $J = LU$ (see [3]), where L and U are lower-triangular and upper-triangular, respectively, two-diagonal matrices and $J^{(p)}$ admits the representation $J^{(p)} = UL$. The monic Jacobi matrix $J^{(p)}$ is called *the Darboux transformation of* J *without parameter*.

Darboux transformations of monic Jacobi matrices which do not meet the condition (1.5) were studied in [10]. In this case it may happen that the perturbed functional $\widetilde{\mathfrak{S}}_1 = \lambda\mathfrak{S}$ defined by (1.4) is not quasi-definite and as was shown in [10] the natural candidate for the *Darboux transformation* $\mathfrak{J}^{(p)}$ (without parameter) of such a matrix J can be found in a class of generalized Jacobi matrices studied in [5], [9]).

In the present paper *Darboux transformation* of generalized Jacobi matrices associated with not quasi-define functionals $\mathfrak{S}$ are studied. It is shown that every generalized Jacobi matrix $\mathfrak{J}$, which satisfies conditions similar to (1.5), admits an $\mathfrak{L}\mathfrak{U}-$factorization $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$, with lower-triangular and upper-triangular two-diagonal block matrices $\mathfrak{L}$ and $\mathfrak{U}$. It turns out that the monic generalized Jacobi matrix $\mathfrak{J}^{(p)}$, associated with the functional $\widetilde{\mathfrak{S}}_1$, can be represented as $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$. This monic generalized Jacobi matrix $\mathfrak{J}^{(p)}$ is called *the Darboux transformation of* $\mathfrak{J}$ *(without parameter).*

The Darboux transformations for generalized Jacobi matrices considered in the present paper turns out to be useful in the investigation of special Stieltjes type continued fractions associated with *non-quasi-definite functionals* and the corresponding moment problem studied in [8]. The results related to this topic will be published elsewhere.

The paper is organized as follows. In Section 2 we expose some material from [5] and [9] concerning generalized Jacobi matrices associated with non-quasi-definite functionals. In Section 3 we study *the Darboux transformation* of generalized Jacobi matrices (without parameter). Analogues of Christoffel transforms of orthogonal polynomials corresponding to generalized Jacobi matrices are found. In Section 4 the results of Section 3 are generalized to the case of Darboux transformation of generalized Jacobi matrices with a shift. In Section 5 an example of Darboux transformation of the monic generalized Jacobi matrix is considered.

2. Monic generalized Jacobi matrices associated with non-quasi-definite functional

Let $\{\mathfrak{s}_j\}_{j=0}^{\infty}$ be a sequence of real moments and let $\mathfrak{S}$ be a linear functional defined on the linear space $\mathcal{P} = \operatorname{span}\left\{\lambda^j : j \in \mathbb{Z}_+\right\}$ by the formula (1.1).

Definition 2.1. ([10]). Define a set $\mathcal{N}(\mathfrak{s})$ of normal indices of the sequence $\mathfrak{s} = \{\mathfrak{s}_i\}_{i=0}^{\infty}$ by

$$\mathcal{N}(\mathfrak{s}) = \{\mathfrak{n}_j : \mathbf{d}_{\mathfrak{n}_j} \neq 0, j = 1, 2, \dots\}, \quad \mathbf{d}_{n_j} = \det(\mathfrak{s}_{i+k})_{i,k=0}^{n_j-1}. \tag{2.1}$$

As follows from (2.1) $\mathfrak{n}_j$ is a normal index of $\mathfrak{s}$ if and only if

$$\det\begin{pmatrix} \mathfrak{s}_0 & \cdots & \mathfrak{s}_{\mathfrak{n}_j-1} \\ \cdots & \cdots & \cdots \\ \mathfrak{s}_{\mathfrak{n}_j-1} & \cdots & \mathfrak{s}_{2\mathfrak{n}_j-2} \end{pmatrix} \neq 0. \tag{2.2}$$

We denote the first nontrivial moment $\varepsilon_0 := \mathfrak{s}_{\mathfrak{n}_1-1}$, i.e., $\mathfrak{s}_k = 0$ for all $k < \mathfrak{n}_1 - 1$. For example, if $\mathfrak{n}_1 = 1$, then $\mathfrak{s}_0 \neq 0$.

Using moment sequence $\{\mathfrak{s}_j\}_{j=0}^{\infty}$, we can construct the polynomials of the first and the second kind (see [1], [6]), defined by for all $j \in \mathbb{N}$

$$(2.3)\quad P_{\mathfrak{n}_j}(\lambda)=\frac{1}{\mathbf{d}_{\mathfrak{n}_j}}\det\begin{pmatrix}\mathfrak{s}_0 & \mathfrak{s}_1 & \cdots & \mathfrak{s}_{\mathfrak{n}_j}\\ \cdots & \cdots & \cdots & \cdots\\ \mathfrak{s}_{\mathfrak{n}_j-1} & \mathfrak{s}_{\mathfrak{n}_j} & \cdots & \mathfrak{s}_{2\mathfrak{n}_j-1}\\ 1 & \lambda & \cdots & \lambda^{\mathfrak{n}_j}\end{pmatrix},\qquad \varepsilon_0 Q_{\mathfrak{n}_j}(\lambda)=\mathfrak{S}_t\left(\frac{P_{\mathfrak{n}_j}(\lambda)-P_{\mathfrak{n}_j}(t)}{\lambda-t}\right).$$

The polynomials $P_{\mathfrak{n}_j}(\lambda)$ and $Q_{\mathfrak{n}_j}(\lambda)$ are solutions of a system of difference equations (see [9], [18])

$$(2.4)\qquad \mathfrak{b}_j y_{\mathfrak{n}_{j-1}}(\lambda)-\mathfrak{p}_j(\lambda)y_{\mathfrak{n}_j}(\lambda)+y_{\mathfrak{n}_{j+1}}(\lambda)=0\quad(\mathfrak{b}_0=\varepsilon_0)$$

subject to the initial conditions

$$(2.5)\qquad P_{\mathfrak{n}_{-1}}(\lambda)\equiv 0,\quad P_{\mathfrak{n}_0}(\lambda)\equiv 1,\quad Q_{\mathfrak{n}_{-1}}(\lambda)\equiv -\frac{1}{\mathfrak{b}_0},\quad Q_{\mathfrak{n}_0}(\lambda)\equiv 0,$$

where $\mathfrak{b}_j \in \mathbb{R}\setminus\{0\}$, $\mathfrak{p}_j(\lambda)=\lambda^{\ell_j}+\mathfrak{p}_{\ell_j-1}^{(j)}\lambda^{\ell_j-1}+\ldots+\mathfrak{p}_1^{(j)}\lambda+\mathfrak{p}_0^{(j)}$ are monic polynomials of degree $\ell_j=\mathfrak{n}_{j+1}-\mathfrak{n}_j$ and generating polynomials of the following generalized Jacobi matrix $\mathfrak{J}$, $j\in\mathbb{Z}_+$.

One can associate with the system (2.4) the so-called monic generalized Jacobi matrix (GJM) (see [9], [10])

$$(2.6)\qquad \mathfrak{J}=\begin{pmatrix}\mathfrak{C}_{\mathfrak{p}_0} & \mathfrak{D}_0 & & \\ \mathfrak{B}_1 & \mathfrak{C}_{\mathfrak{p}_1} & \mathfrak{D}_1 & \\ & \mathfrak{B}_2 & \mathfrak{C}_{\mathfrak{p}_2} & \ddots\\ & & \ddots & \ddots\end{pmatrix},$$

where the diagonal entries are companion matrices associated with some real polynomials $\mathfrak{p}_j(\lambda)$ (see [16])

$$(2.7)\qquad \mathfrak{C}_{\mathfrak{p}_j}=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 0 & 1\\ -\mathfrak{p}_0^{(j)} & -\mathfrak{p}_1^{(j)} & \cdots & -\mathfrak{p}_{\ell_j-2}^{(j)} & -\mathfrak{p}_{\ell_j-1}^{(j)}\end{pmatrix}\quad\text{are } \ell_j\times\ell_j \text{ matrices,}$$

$\mathfrak{D}_j$ and $\mathfrak{B}_{j+1}$ are $\ell_j\times\ell_{j+1}$ and $\ell_{j+1}\times\ell_j$ matrices, respectively, determined by

$$(2.8)\quad \mathfrak{D}_j=\begin{pmatrix}0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0\\ 1 & 0 & \cdots & 0\end{pmatrix}\ \text{and}\ \mathfrak{B}_{j+1}=\begin{pmatrix}0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0\\ \mathfrak{b}_{j+1} & 0 & \cdots & 0\end{pmatrix},\quad \mathfrak{b}_{j+1}\in\mathbb{R}\setminus\{0\},\ j\in\mathbb{Z}_+.$$

The matrix $\mathfrak{J}$ defined by (2.6)– (2.8) is called a GJM associated with the functional $\mathfrak{S}$. Sometimes $\mathfrak{J}$ is called a GJM associated with the sequence $\{\mathfrak{s}_j\}_{j=0}^{\infty}$ or the system (2.4) to emphasize connection with polynomials $\mathfrak{p}_j(\lambda)$ and numbers $\mathfrak{b}_{j+1}$, $j\in\mathbb{Z}_+$.

The shortened *GJM* $\mathfrak{J}_{[i,j]}$ is defined by

$$(2.9)\qquad \mathfrak{J}_{[i,j]}=\begin{pmatrix}\mathfrak{C}_{\mathfrak{p}_i} & \mathfrak{D}_i & & \\ \mathfrak{B}_{i+1} & \mathfrak{C}_{i+1} & \ddots & \\ & \ddots & \ddots & \mathfrak{D}_{j-1}\\ & & \mathfrak{B}_j & \mathfrak{C}_{\mathfrak{p}_j}\end{pmatrix},\quad i\le j\quad\text{and}\quad i,j\in\mathbb{Z}_+.$$

The following connection between the polynomials of the first and the second kind and the shortened GJM's can be found in [9]

$$P_{\mathfrak{n}_j}(\lambda) = \det(\lambda - \mathfrak{J}_{[0,j-1]}) \quad \text{and} \quad Q_{\mathfrak{n}_j}(\lambda) = \varepsilon_0 \det(\lambda - \mathfrak{J}_{[1,j-1]}). \tag{2.10}$$

Next, we introduce the inner product in the space $\ell^2_{[0,\mathfrak{n}_j-1]}$, by the formula

$$[x, y] = (Gx, y)_{\ell^2_{[0,\mathfrak{n}_j-1]}}\,, \tag{2.11}$$

where $x, y \in \ell^2_{[0,\mathfrak{n}_j-1]}$ and the matrix $G_{[0,j-1]}$ is defined by the equality

$$G_{[0,j-1]} = \operatorname{diag}(G_0, G_1, \dots G_{j-1}),\; G_i = \begin{pmatrix} \mathfrak{p}_1^{(i)} & \cdots & \mathfrak{p}_{\ell_i-1}^{(i)} & 1 \\ \vdots & \iddots & \iddots & \\ \mathfrak{p}_{\ell_i-1}^{(i)} & \iddots & & \\ 1 & & & 0 \end{pmatrix}^{-1}, \; i = \overline{0, j-1}. \tag{2.12}$$

Let us set

$$\begin{aligned} \mathbf{P}(\lambda) &= \left(P_0(\lambda), P_1(\lambda), \dots, P_{\mathfrak{n}_j}(\lambda), \dots\right)^T, \\ \mathbf{Q}(\lambda) &= \left(Q_0(\lambda), Q_1(\lambda), \dots, Q_{\mathfrak{n}_j}(\lambda), \dots\right)^T, \end{aligned} \tag{2.13}$$

where $P_{\mathfrak{n}_j+k}(\lambda) = \lambda^k P_{\mathfrak{n}_j}(\lambda)$ and $Q_{\mathfrak{n}_j+k}(\lambda) = \lambda^k Q_{\mathfrak{n}_j}(\lambda)$, where $0 \le k < \mathfrak{n}_{j+1} - \mathfrak{n}_j$. Then it follows from (2.4), (2.5) and (2.6)–(2.8), that

$$(\mathfrak{J} - \lambda I)\mathbf{P}(\lambda) = 0 \quad \text{and} \quad (\mathfrak{J} - \lambda I)\mathbf{Q}(\lambda) = (\underbrace{0, \dots, 0, 1}_{\ell_0}, 0, \dots)^T. \tag{2.14}$$

Definition 2.2. Let us define the $m-$function of the matrix $\mathfrak{J}$ by equality

$$m_{[0,j-1]}(\lambda) = [(\mathfrak{J}^T_{[0,j-1]} - \lambda)^{-1} e_0, e_0], \tag{2.15}$$

where $e_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$ is $\mathfrak{n}_j \times 1$ vector.

As was shown in [9, Proposition 6.1]

$$m_{[0,j-1]}(\lambda) = -\varepsilon_0 \frac{\det(\lambda - \mathfrak{J}_{[1,j-1]})}{\det(\lambda - \mathfrak{J}_{[0,j-1]})} = -\frac{Q_{\mathfrak{n}_j}(\lambda)}{P_{\mathfrak{n}_j}(\lambda)} \tag{2.16}$$

and $m_{[0,j-1]}(\lambda)$ admits the following asymptotic expansion:

$$m_{[0,j-1]}(\lambda) = -\frac{\mathfrak{s}_0}{\lambda} - \frac{\mathfrak{s}_1}{\lambda^2} - \dots - \frac{\mathfrak{s}_{2\mathfrak{n}_j-2}}{\lambda^{2\mathfrak{n}_j-1}} + o\left(\frac{1}{\lambda^{2\mathfrak{n}_j-1}}\right), \tag{2.17}$$

where

$$\mathfrak{s}_k = \left[\left(\mathfrak{J}^T_{[0,j-1]}\right)^k e_0, e_0\right], \quad k \le 2\mathfrak{n}_j - 2. \tag{2.18}$$

Lemma 2.3. *Let $\mathfrak{J}$ be a GJM and let $P_{\mathfrak{n}_j}(\lambda)$, $Q_{\mathfrak{n}_j}(\lambda)$ be the corresponding polynomials of the first and the second kind. Then there exists a monic Jacobi matrix J, such that*

$$P_{\mathfrak{n}_j}(0) = \widehat{P}_j(0) \quad \textit{and} \quad Q_{\mathfrak{n}_j}(0) = \widehat{Q}_j(0), \tag{2.19}$$

where $\widehat{P}_j(\lambda)$ and $\widehat{Q}_j(\lambda)$ are polynomials of the first and the second kind, respectively, associated with J for all $j \in \mathbb{N}$.

Proof. First of all, we compute $P_{\mathfrak{n}_j}(0) = \det(-\mathfrak{J}_{[0,j-1]})$ and expand it along the rows, which have only one element equal to -1 and others equal to 0. Then we get

$$P_{\mathfrak{n}_j}(0) = \begin{vmatrix} -\mathfrak{C}_{\mathfrak{p}_0} & -\mathfrak{D}_0 & & \\ -\mathfrak{B}_1 & -\mathfrak{C}_{\mathfrak{p}_1} & \ddots & \\ & \ddots & \ddots & -\mathfrak{D}_{j-2} \\ & & -\mathfrak{B}_{j-1} & -\mathfrak{C}_{\mathfrak{p}_{j-1}} \end{vmatrix} = \begin{vmatrix} \mathfrak{p}_0^{(0)} & -1 & & \\ -\mathfrak{b}_1 & \mathfrak{p}_0^{(1)} & \ddots & \\ & \ddots & \ddots & -1 \\ & & -\mathfrak{b}_{j-1} & \mathfrak{p}_0^{(j-1)} \end{vmatrix}. \tag{2.20}$$

We use this equality for the following construction of the Jacobi matrix, as a hint

$$J = \begin{pmatrix} -\mathfrak{p}_0^{(0)} & 1 & & \\ \mathfrak{b}_1 & -\mathfrak{p}_0^{(1)} & 1 & \\ & \mathfrak{b}_2 & -\mathfrak{p}_0^{(2)} & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{2.21}$$

It follows from (2.10) and (2.20) that $P_{\mathfrak{n}_j}(0) = \widehat{P}_j(0)$ $(j \in \mathbb{N})$. The proof of the second equality in (2.19) is analogous. □

Corollary 2.4. *Let $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$ be GJM's associated with systems (2.4) and*

$$\widetilde{\mathfrak{b}}_j \widetilde{y}_{\widetilde{\mathfrak{n}}_{j-1}}(\lambda) - \widetilde{\mathfrak{p}}_j(\lambda)\widetilde{y}_{\widetilde{\mathfrak{n}}_j}(\lambda) + \widetilde{y}_{\widetilde{\mathfrak{n}}_{j+1}}(\lambda) = 0 \quad (\widetilde{\mathfrak{b}}_0 = \widetilde{\varepsilon}_0), \tag{2.22}$$

respectively and let $P_{\mathfrak{n}_j}(\lambda)$, $\widetilde{P}_{\widetilde{\mathfrak{n}}_j}(\lambda)$ and $Q_{\mathfrak{n}_j}(\lambda)$, $\widetilde{Q}_{\widetilde{\mathfrak{n}}_j}(\lambda)$ be the corresponding polynomials of the first and the second kind, respectively, associated with the matrices $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$, for all $j \in \mathbb{N}$. If $\mathfrak{p}_0^{(j-1)} = \widetilde{\mathfrak{p}}_0^{(j-1)}$ and $\mathfrak{b}_j = \widetilde{\mathfrak{b}}_j$, then

$$P_{\mathfrak{n}_j}(0) = \widetilde{P}_{\widetilde{\mathfrak{n}}_j}(0) \quad \text{and} \quad Q_{\mathfrak{n}_j}(0) = \widetilde{Q}_{\widetilde{\mathfrak{n}}_j}(0), \quad j \in \mathbb{N}. \tag{2.23}$$

Proof. The proof is immediate from Lemma 2.3, due to (2.20) since $P_{\mathfrak{n}_j}(0)$, $\widetilde{P}_{\widetilde{\mathfrak{n}}_j}(0)$ are completely determined by $\mathfrak{p}_0^{(j-1)} = \widetilde{\mathfrak{p}}_0^{(j-1)}$ and $\mathfrak{b}_j = \widetilde{\mathfrak{b}}_j$, for all $j \in \mathbb{N}$. Then we have $P_{\mathfrak{n}_j}(0) = \widetilde{P}_{\widetilde{\mathfrak{n}}_j}(0)$, for all $j \in \mathbb{N}$. Similarly, the polynomials $Q_{\mathfrak{n}_j}(0)$, $\widetilde{Q}_{\widetilde{\mathfrak{n}}_j}(0)$ are determined by $\mathfrak{p}_0^{(j-1)} = \widetilde{\mathfrak{p}}_0^{(j-1)}$ and $\mathfrak{b}_j = \widetilde{\mathfrak{b}}_j$, for all $j \in \mathbb{N}$. Then $Q_{\mathfrak{n}_j}(0) = \widetilde{Q}_{\widetilde{\mathfrak{n}}_j}(0)$, for all $j \in \mathbb{N}$. □

3. The Darboux transformation of monic generalized Jacobi matrices

In this section, we study the Darboux transformation of GJM $\mathfrak{J}$ and prove some properties for polynomials of the first kind associated with matrix $\mathfrak{J}$. We use the factorization matrices $\mathfrak{L}$ and $\mathfrak{U}$, where $\mathfrak{L}$ and $\mathfrak{U}$ are lower and upper triangular block matrices, respectively, having the forms

$$\mathfrak{L} = \begin{pmatrix} \mathfrak{A}_0 & 0 & & \\ \mathfrak{L}_1 & \mathfrak{A}_1 & 0 & \\ & \mathfrak{L}_2 & \mathfrak{A}_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad \mathfrak{U} = \begin{pmatrix} \mathfrak{U}_0 & \mathfrak{D}_0 & & \\ 0 & \mathfrak{U}_1 & \mathfrak{D}_1 & \\ & 0 & \mathfrak{U}_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{3.1}$$

the diagonal blocks $\mathfrak{A}_j$ and $\mathfrak{U}_j$ are $\ell_j \times \ell_j$ matrices

$$\mathfrak{A}_j = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ -\mathfrak{p}_1^{(j)} & \cdots & -\mathfrak{p}_{\ell_j-2}^{(j)} & -\mathfrak{p}_{\ell_j-1}^{(j)} & 1 \end{pmatrix} \quad \text{and} \quad \mathfrak{U}_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -\mathfrak{u}_j & 0 & \cdots & 0 & 0 \end{pmatrix}, \ \mathfrak{u}_j \neq 0, \tag{3.2}$$

the blocks $\mathfrak{L}_{j+1}$ and $\mathfrak{D}_j$ are $\ell_{j+1} \times \ell_j$ and $\ell_j \times \ell_{j+1}$ matrices, respectively

$$\mathfrak{L}_{j+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \mathfrak{l}_{j+1} \end{pmatrix}, \quad \mathfrak{l}_{j+1} \neq 0, \quad \mathfrak{D}_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.3}$$

However, if $\ell_j = \ell_{j+1} = 1$, then we suppose

$$\mathfrak{U}_j = (-\mathfrak{u}_j), \quad \mathfrak{L}_{j+1} = (\mathfrak{l}_{j+1}), \quad \mathfrak{D}_j = (1) \quad \text{and} \quad \mathfrak{A}_j = (1). \tag{3.4}$$

Let us say that the GJM $\mathfrak{J}$ admits $\mathfrak{L}\mathfrak{U}-$factorization if $\mathfrak{J}$ can be represented in the form $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$, where $\mathfrak{L}$ and $\mathfrak{U}$ are given by (3.1)–(3.3).

Definition 3.1. Let a monic generalized Jacobi matrix $\mathfrak{J}$ admit the $\mathfrak{L}\mathfrak{U}-$factorization of the form (3.1)–(3.3). Then the transformation

$$\mathfrak{J} = \mathfrak{L}\mathfrak{U} \to \mathfrak{U}\mathfrak{L} = \mathfrak{J}^{(p)} \tag{3.5}$$

is called *the Darboux transformation of matrix* $\mathfrak{J}$, where the matrix $\mathfrak{J}^{(p)}$ is a GJM.

3.1. **$\mathfrak{L}\mathfrak{U}-$factorization of generalized Jacobi matrices.**

Lemma 3.2. *Let $\mathfrak{J}$ be a monic generalized Jacobi matrix associated with the functional $\mathfrak{S}$ and let $\ell_j := \mathfrak{n}_{j+1} - \mathfrak{n}_j \geq 1$, $j \in \mathbb{Z}_+$, where $\mathfrak{n}_0 = 0$ and $\{\mathfrak{n}_j\}_{j=1}^{\infty}$ is the set of normal indices of the sequence $\mathfrak{s} = \{\mathfrak{s}_j\}_{j=0}^{\infty}$. Let $\mathfrak{L}$ and $\mathfrak{U}$ be defined by (3.1)–(3.3). Then $\mathfrak{J}$ admits $\mathfrak{L}\mathfrak{U}-$factorization of the form (3.1)–(3.3) if and only if the system of equations*

$$\mathfrak{u}_0 = \mathfrak{p}_0^{(0)}, \quad -\mathfrak{u}_j + \mathfrak{l}_j = -\mathfrak{p}_0^{(j)}, \quad j \in \mathbb{N}; \quad -\mathfrak{u}_j\mathfrak{l}_{j+1} = \mathfrak{b}_{j+1}, \quad j \in \mathbb{Z}_+ \tag{3.6}$$

is solvable.

Proof. Consider the product $\mathfrak{L}\mathfrak{U}$ of the matrices $\mathfrak{L}$ and $\mathfrak{U}$

$$\mathfrak{L}\mathfrak{U} = \begin{pmatrix} \mathfrak{A}_0\mathfrak{U}_0 & \mathfrak{A}_0\mathfrak{D}_0 & & \\ \mathfrak{L}_1\mathfrak{U}_0 & \mathfrak{L}_1\mathfrak{D}_0 + \mathfrak{A}_1\mathfrak{U}_1 & \mathfrak{A}_1\mathfrak{D}_1 & \\ & \mathfrak{L}_2\mathfrak{U}_1 & \mathfrak{L}_2\mathfrak{D}_1 + \mathfrak{A}_2\mathfrak{U}_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{3.7}$$

where the blocks $\mathfrak{A}_j\mathfrak{U}_j$ and $\mathfrak{L}_{j+1}\mathfrak{D}_j$ are $\ell_j \times \ell_j$ and $\ell_{j+1} \times \ell_{j+1}$ matrices, respectively

$$\mathfrak{A}_j\mathfrak{U}_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -\mathfrak{u}_j & -\mathfrak{p}_1^{(j)} & \cdots & -\mathfrak{p}_{\ell_j-2}^{(j)} & -\mathfrak{p}_{\ell_j-1}^{(j)} \end{pmatrix}, \quad \mathfrak{L}_j\mathfrak{D}_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \mathfrak{l}_{j+1} & 0 & \cdots & 0 \end{pmatrix}, \tag{3.8}$$

the blocks $\mathfrak{L}_{j+1}\mathfrak{U}_j$ and $\mathfrak{A}_j\mathfrak{D}_j$ are $\ell_{j+1} \times \ell_j$ and $\ell_j \times \ell_{j+1}$ matrices, respectively

$$\mathfrak{L}_{j+1}\mathfrak{U}_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ -\mathfrak{l}_{j+1}\mathfrak{u}_j & 0 & \cdots & 0 \end{pmatrix}, \quad \mathfrak{A}_j\mathfrak{D}_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} = \mathfrak{D}_j. \tag{3.9}$$

Then $\mathfrak{L}_{j+1}\mathfrak{D}_j + \mathfrak{A}_{j+1}\mathfrak{U}_{j+1}$ has the following form:

$$\mathfrak{L}_{j+1}\mathfrak{D}_j + \mathfrak{A}_{j+1}\mathfrak{U}_{j+1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -\mathfrak{u}_j + \mathfrak{l}_j & -\mathfrak{p}_1^{(j)} & \cdots & -\mathfrak{p}_{\ell_j-2}^{(j)} & -\mathfrak{p}_{\ell_j-1}^{(j)} \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{3.10}$$

Comparing the product $\mathfrak{L}\mathfrak{U}$ with the matrix $\mathfrak{J}$ in (2.6), we obtain the system (3.6).

If the system (3.6) is solvable, then $\mathfrak{J}$ admits the factorization $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ of the form (3.1)–(3.3), where $\mathfrak{L}$ and $\mathfrak{U}$ are found uniquely. Conversely, if $\mathfrak{J}$ admit $\mathfrak{L}\mathfrak{U}-$factorization then the system of equations (3.6) is solvable. □

Lemma 3.3. *Let $\mathfrak{J}$ be a GJM associated with the functional $\mathfrak{S}$ and let $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3) and let $P_{\mathfrak{n}_{j+1}}(\lambda)$ be polynomials of the first kind associated with $\mathfrak{J}$. Then*

$$P_{\mathfrak{n}_{j+1}}(0) = \prod_{k=0}^{j} \mathfrak{u}_k, \quad \textit{for all} \quad j \in \mathbb{Z}_+ \ . \tag{3.11}$$

Proof. By Lemma 2.3 and Lemma 3.2 we obtain
(3.12)

$$P_{\mathfrak{n}_{j+1}}(0) = \det\left(-\mathfrak{J}_{[0,j]}\right) = \begin{vmatrix} -\mathfrak{C}_{\mathfrak{p}_0} & -\mathfrak{D}_0 & & \\ -\mathfrak{B}_1 & -\mathfrak{C}_{\mathfrak{p}_1} & \ddots & \\ & \ddots & \ddots & -\mathfrak{D}_{j-1} \\ & & -\mathfrak{B}_j & -\mathfrak{C}_{\mathfrak{p}_j} \end{vmatrix} = \begin{vmatrix} \mathfrak{p}_0^{(0)} & -1 & & \\ -\mathfrak{b}_1 & \mathfrak{p}_0^{(1)} & \ddots & \\ & \ddots & \ddots & -1 \\ & & -\mathfrak{b}_j & \mathfrak{p}_0^{(j)} \end{vmatrix}$$

$$= \begin{vmatrix} \mathfrak{u}_0 & -1 & & \\ \mathfrak{u}_0\mathfrak{l}_1 & \mathfrak{u}_1 - \mathfrak{l}_1 & \ddots & \\ & \ddots & \ddots & -1 \\ & & \mathfrak{u}_{j-1}\mathfrak{l}_j & \mathfrak{u}_j - \mathfrak{l}_j \end{vmatrix} = \begin{vmatrix} \mathfrak{u}_0 & -1 & & \\ 0 & \mathfrak{u}_1 & \ddots & \\ & \ddots & \ddots & -1 \\ & & 0 & \mathfrak{u}_j \end{vmatrix} = \prod_{k=0}^{j} \mathfrak{u}_k.$$

This completes the proof. □

Corollary 3.4. *Let $\mathfrak{J}$ be a GJM associated with the functional $\mathfrak{S}$ and let $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3) and let $P_{\mathfrak{n}_{j+1}}(\lambda)$ be polynomials of the first kind associated with $\mathfrak{J}$. Then we have*

$$P_{\mathfrak{n}_{j+1}}(0) = \mathfrak{u}_j\mathfrak{u}_{j-1}\dots\mathfrak{u}_{j-k}P_{\mathfrak{n}_{j-k}}(0), \quad k \le j \quad \textit{and} \quad j,k \in \mathbb{Z}_+. \tag{3.13}$$

Theorem 3.5. *Let $\mathfrak{J}$ be a monic generalized Jacobi matrix associated with the functional $\mathfrak{S}$ and let $\ell_j := \mathfrak{n}_{j+1} - \mathfrak{n}_j \ge 1$, $j \in \mathbb{Z}_+$, where $\mathfrak{n}_0 = 0$ and $\{\mathfrak{n}_j\}_{j=1}^{\infty}$ is the set of normal indices of the sequence $\mathfrak{s} = \{\mathfrak{s}_j\}_{j=0}^{\infty}$ and let $P_{\mathfrak{n}_j}(\lambda)$ be polynomials of the first kind associated with the sequence $\mathfrak{s} = \{\mathfrak{s}_j\}_{j=0}^{\infty}$. Then $\mathfrak{J}$ admits the $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3) if and only if*

$$P_{\mathfrak{n}_j}(0) \ne 0, \quad \textit{for all} \quad j \in \mathbb{Z}_+. \tag{3.14}$$

Furthermore

$$\mathfrak{l}_{j+1} = -\frac{\mathfrak{b}_{j+1}}{\mathfrak{u}_j}, \quad \mathfrak{u}_j = \frac{P_{\mathfrak{n}_{j+1}}(0)}{P_{\mathfrak{n}_j}(0)}, \quad \mathfrak{u}_0 = \mathfrak{p}_0^{(0)}, \quad \textit{for all} \quad j \in \mathbb{Z}_+ \ . \tag{3.15}$$

Proof. Let $P_{\mathfrak{n}_j}(0) \ne 0$ for all $j \in \mathbb{Z}_+$ then by Lemma 3.3 the equalities (3.15) are equivalent to the system (3.6). Consequently, by Lemma 3.2 the matrix $\mathfrak{J}$ admits the $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3). Conversely, let $\mathfrak{J}$ admit the $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3). Then by Lemma 3.3 $P_{\mathfrak{n}_j}(0) \ne 0$ for all $j \in \mathbb{Z}_+$. □

Remark 3.6. If $\ell_j = 1$ for each $j \in \mathbb{Z}_+$, then the factorization (3.1)–(3.3) coincides with the factorization in [3], (see [3], section 2).

Remark 3.7. If $\ell_j = 1$ or $\ell_j = 2$ for each $j \in \mathbb{Z}_+$, then factorization (3.1)–(3.3) coincides with the $\mathfrak{L}\mathfrak{U}$–factorization in [10], (see [10], section 4).

Remark 3.8. If $\mathfrak{n}_1 = 1$ (i.e. $\ell_0 = 1$), then $P_{\mathfrak{n}_1}(\lambda) = \det(\lambda - \mathfrak{J}_{[0,0]}) = \mathfrak{p}^{(0)}(\lambda) = \lambda + \mathfrak{p}_0^{(0)}$ and by (2.3)

$$P_{\mathfrak{n}_1}(\lambda) = \frac{1}{\mathfrak{s}_0}\begin{vmatrix} \mathfrak{s}_0 & \mathfrak{s}_1 \\ 1 & \lambda \end{vmatrix} = \lambda - \frac{\mathfrak{s}_1}{\mathfrak{s}_0}. \tag{3.16}$$

Due to $P_{\mathfrak{n}_1}(0) \ne 0$ see (3.14), we have $\mathfrak{p}_0^{(0)} = -\frac{\mathfrak{s}_1}{\mathfrak{s}_0} \ne 0$ and by Lemma 3.3 $\mathfrak{u}_0 = -\frac{\mathfrak{s}_1}{\mathfrak{s}_0}$.

Proposition 3.9. *Let $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$ be GJM's associated with the difference systems (2.4) and (2.22), respectively. If $\mathfrak{J}$ admits $\mathfrak{LU}$–factorization of the form (3.1)–(3.3) and $\mathfrak{p}_0^{(j)} = \widetilde{\mathfrak{p}}_0^{(j)}$, $\mathfrak{b}_{j+1} = \widetilde{\mathfrak{b}}_{j+1}$ for all $j \in \mathbb{Z}_+$. Then the matrix $\widetilde{\mathfrak{J}}$ also admits $\mathfrak{LU}$–factorization of the form (3.1)–(3.3).*

Proof. This proof is clear, because by Theorem 3.5, we know $P_{\mathfrak{n}_j}(0) \neq 0$ for all $j \in \mathbb{Z}_+$ and $\mathfrak{n}_0 = 0$, where $P_{\mathfrak{n}_j}(\lambda)$ are polynomials of the first kind associated with the $\mathfrak{J}$. Using Corollary 2.4, we obtain $\widetilde{P}_{\widetilde{\mathfrak{n}}_j}(0) \neq 0$ for all $j \in \mathbb{Z}_+$, where $\widetilde{P}_{\widetilde{\mathfrak{n}}_j}(\lambda)$ are polynomials of the first kind associated with the GJM $\widetilde{\mathfrak{J}}$. From here the matrix $\widetilde{\mathfrak{J}}$ satisfies Theorem 3.5, i.e. $\widetilde{\mathfrak{J}} = \widetilde{\mathfrak{L}}\widetilde{\mathfrak{U}}$, where the matrices $\widetilde{\mathfrak{L}}$ and $\widetilde{\mathfrak{U}}$ are defined by (3.1)–(3.3). □

3.2. Some properties of the Darboux transformation.

Theorem 3.10. *Let $\mathfrak{J}$ be a GJM associated with the functional $\mathfrak{S}$ and let $\mathfrak{J} = \mathfrak{LU}$ be its $\mathfrak{LU}$–factorization of the form (3.1)–(3.3). Then the matrix $\mathfrak{J}^{(p)} = \mathfrak{UL}$ is a monic generalized Jacobi matrix.*

Proof. Consider the product $\mathfrak{UL}$ of the matrices $\mathfrak{U}$ and $\mathfrak{L}$

$$\mathfrak{UL} = \begin{pmatrix} \mathfrak{U}_0\mathfrak{A}_0 + \mathfrak{D}_0\mathfrak{L}_1 & \mathfrak{D}_0\mathfrak{A}_1 & & \\ \mathfrak{U}_1\mathfrak{L}_1 & \mathfrak{U}_1\mathfrak{A}_1 + \mathfrak{D}_1\mathfrak{L}_2 & \mathfrak{D}_1\mathfrak{A}_2 & \\ & \mathfrak{U}_2\mathfrak{L}_2 & \mathfrak{U}_2\mathfrak{A}_2 + \mathfrak{D}_2\mathfrak{L}_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{3.17}$$

(i) In this part, we consider the case, when $\ell_j \geq 2$ for all $j \in \mathbb{Z}_+$. And we have the following:

$$\mathfrak{U}_{j+1}\mathfrak{L}_{j+1} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \mathfrak{l}_{j+1} \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \mathfrak{D}_j\mathfrak{A}_{j+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \tag{3.18}$$

where $\mathfrak{U}_{j+1}\mathfrak{L}_{j+1}$ and $\mathfrak{D}_j\mathfrak{A}_{j+1}$ are $\ell_{j+1} \times \ell_j$ and $\ell_j \times \ell_{j+1}$ matrices, respectively. The blocks $\mathfrak{U}_j\mathfrak{A}_j$ and $\mathfrak{D}_j\mathfrak{L}_{j+1}$ are $\ell_j \times \ell_j$ matrices, such that

$$\mathfrak{U}_j\mathfrak{A}_j = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ -\mathfrak{p}_1^{(j)} & \cdots & -\mathfrak{p}_{\ell_j-2}^{(j)} & -\mathfrak{p}_{\ell_j-1}^{(j)} & 1 \\ -\mathfrak{u}_j & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{D}_j\mathfrak{L}_{j+1} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}. \tag{3.19}$$

So, the matrix $\mathfrak{J}^{(p)} = \mathfrak{UL}$ has the following form:

$$\mathfrak{J}^{(p)} = \mathfrak{UL} = \begin{pmatrix} \mathfrak{C}_{\mathfrak{p}_0}^0 & \mathfrak{D}_{0,0} & & & \\ \mathfrak{B}_{1,0} & \mathfrak{C}_{\mathfrak{p}_0}^1 & \mathfrak{D}_{0,1} & & \\ & \mathfrak{B}_{1,1} & \mathfrak{C}_{\mathfrak{p}_1}^0 & \mathfrak{D}_{1,0} & \\ & & \mathfrak{B}_{2,0} & \mathfrak{C}_{\mathfrak{p}_1}^1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \tag{3.20}$$

where the blocks $\mathfrak{C}^0_{\mathfrak{p}_j}$ are $(\ell_j-1)\times(\ell_j-1)$ matrices, such that

$$\mathfrak{C}^0_{\mathfrak{p}_j}=\begin{pmatrix} 0 & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\\ -\mathfrak{p}^{(j)}_1 & \cdots & -\mathfrak{p}^{(j)}_{\ell_j-2} & -\mathfrak{p}^{(j)}_{\ell_j-1}\end{pmatrix}, \tag{3.21}$$

and the blocks $\mathfrak{D}_{j,0}$, $\mathfrak{B}_{j+1,0}$ and $\mathfrak{B}_{j+1,1}$ are $(\ell_j-1)\times 1$, $1\times(\ell_j-1)$ and $(\ell_j-1)\times 1$ matrices, respectively

$$\mathfrak{D}_{j,0}=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix},\quad \mathfrak{B}_{j+1,0}=\begin{pmatrix}-\mathfrak{u}_j & 0 & \cdots & 0\end{pmatrix},\quad \mathfrak{B}_{j+1,1}=\begin{pmatrix}0\\ \vdots\\ 0\\ \mathfrak{l}_{j+1}\end{pmatrix}, \tag{3.22}$$

$$\mathfrak{C}^1_{\mathfrak{p}_j}=\begin{pmatrix}0\end{pmatrix},\quad \mathfrak{D}_{j,1}=\begin{pmatrix}1 & 0 & 0 & \cdots & 0\end{pmatrix}\quad \text{are } 1\times(\ell_j-1) \text{ matrices},\quad j\in\mathbb{Z}_+. \tag{3.23}$$

(ii) In this part, we consider the case, when $\ell_{k-1}\geq 2$, $\ell_k=1$ and $\ell_{k+1}\geq 2$, $k\in\mathbb{N}$. Then matrix $\mathfrak{J}^{(p)}$ has the following representation:

$$\mathfrak{J}^{(p)}=\mathfrak{U}\mathfrak{L}=\begin{pmatrix} \mathfrak{C}^0_{\mathfrak{p}_0} & \mathfrak{D}_{0,0} & & & & & \\ \mathfrak{B}_{1,0} & \mathfrak{C}^1_{\mathfrak{p}_0} & \mathfrak{D}_{0,1} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \mathfrak{B}_{k,0} & \mathfrak{C}^1_{\mathfrak{p}_{k-1}} & \mathfrak{D}_{k-1,1} & & \\ & & & \mathfrak{B}_{k+1,1} & \mathfrak{C}^0_{\mathfrak{p}_k} & \mathfrak{D}_{k,0} & \\ & & & & \mathfrak{B}_{k+2,1} & \mathfrak{C}^0_{\mathfrak{p}_{k+1}} & \mathfrak{D}_{k+1,0}\\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3.24}$$

where

$$\begin{pmatrix}\mathfrak{C}^1_{\mathfrak{p}_{k-1}} & \mathfrak{D}_{k-1,1}\\ \mathfrak{B}_{k+1,1} & \mathfrak{C}^0_{\mathfrak{p}_k}\end{pmatrix}=\begin{pmatrix}\mathfrak{l}_k & 1\\ -\mathfrak{u}_k\mathfrak{l}_k & -\mathfrak{u}_k\end{pmatrix}. \tag{3.25}$$

(iii) Next, we consider the case, when $\ell_{k-1}\geq 2$, $\ell_k=\ldots=\ell_{k+h}=1$ and $\ell_{k+h+1}\geq 2$, $h,k\in\mathbb{N}$. Then we have

$$\begin{pmatrix}\mathfrak{C}^1_{\mathfrak{p}_{k-1}} & \mathfrak{D}_{k-1,1} & & \\ \mathfrak{B}_{k+1,1} & \mathfrak{C}^0_{\mathfrak{p}_k} & \ddots & \\ & & \ddots & \mathfrak{D}_{k+h-1,1}\\ & & \mathfrak{B}_{k+h+1,1} & \mathfrak{C}^0_{\mathfrak{p}_{k+h}}\end{pmatrix}=\begin{pmatrix}\mathfrak{l}_k & 1 & & & \\ -\mathfrak{u}_k\mathfrak{l}_k & \mathfrak{l}_{k+1}-\mathfrak{u}_k & 1 & & \\ & -\mathfrak{u}_{k+1}\mathfrak{l}_{k+1} & -\mathfrak{u}_{k+1} & \ddots & \\ & & \ddots & \ddots & 1\\ & & & -\mathfrak{u}_{k+h}\mathfrak{l}_{k+h} & -\mathfrak{u}_{k+h}\end{pmatrix}.$$

(iv) In this case, we suppose $\ell_0=\cdots=\ell_k=1$ and $\ell_{k+1}\geq 2$, $k\in\mathbb{Z}_+$. We obtain

$$\begin{pmatrix}\mathfrak{C}^0_{\mathfrak{p}_0} & \mathfrak{D}_{0,0} & & \\ \mathfrak{B}_{1,0} & \mathfrak{C}^0_{\mathfrak{p}_1} & \ddots & \\ & & \ddots & \mathfrak{D}_{k-1,0}\\ & & \mathfrak{B}_{k,0} & \mathfrak{C}^0_{\mathfrak{p}_k}\end{pmatrix}=\begin{pmatrix}\mathfrak{l}_1-\mathfrak{u}_0 & 1 & & & \\ -\mathfrak{u}_1\mathfrak{l}_1 & \mathfrak{l}_2-\mathfrak{u}_1 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & -\mathfrak{u}_{k-1}\mathfrak{l}_{k-1} & \mathfrak{l}_k-\mathfrak{u}_{k-1} & 1\\ & & & -\mathfrak{u}_k\mathfrak{l}_k & -\mathfrak{u}_k\end{pmatrix}.$$

So, we have shown $\mathfrak{J}^{(p)}=\mathfrak{U}\mathfrak{L}$ is a monic generalized Jacobi matrix. This completes the proof. $\square$

Remark 3.11. Let the moment sequence $\mathfrak{s}^{(p)} = \left\{\mathfrak{s}_j^{(p)}\right\}_{j=0}^{\infty}$ be associated with the matrix $\mathfrak{J}^{(p)}$. Let $\mathfrak{n}_1 > 1$ and $\mathfrak{n}_1^{(p)}$ be the first nontrivial normal indexes of the moment sequences $\mathfrak{s}$ and $\mathfrak{s}^{(p)}$, respectively. Then

$$\mathfrak{n}_1^{(p)} = \mathfrak{n}_1 - 1. \tag{3.26}$$

Definition 3.12. Define a functional $\lambda\mathfrak{S}$ by the formula

$$(\lambda\mathfrak{S})(p) := \mathfrak{S}(\lambda p(\lambda)), \quad p(\lambda) \text{ is a polynomial .} \tag{3.27}$$

Theorem 3.13. *Let $\mathfrak{J}$ be a monic generalized Jacobi matrix associated with the functional $\mathfrak{S}$, such that (3.14) holds and let $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3). Then the matrix $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$ is associated with the functional*

$$\mathfrak{S}^{(p)} = \begin{cases} \lambda\mathfrak{S}, & \mathfrak{n}_1 > 1, \\ \frac{\mathfrak{s}_0}{\mathfrak{s}_1}\lambda\mathfrak{S}, & \mathfrak{n}_1 = 1. \end{cases} \tag{3.28}$$

Proof. In this proof we follow the relations from [10] (see Section 4, Theorem 4.2). Note that $\mathfrak{s}_1 \neq 0$ if $\mathfrak{n}_1 = 1$, see Remark 3.8. We divide the proof into two cases

(i) First of all, we consider the case, when $\mathfrak{n}_1 > 1$. We note that

$$\mathfrak{L}_{[0,j-1]}^T e_0 = e_0, \quad \mathfrak{U}_{[0,j-1]} G_{[0,j-1]} e_0 = e_{\ell_0 - 2}, \quad j \in \mathbb{N}, \tag{3.29}$$

where the shortened matrices $\mathfrak{L}_{[0,j-1]}$, $\mathfrak{U}_{[0,j-1]}$ and $G_{[0,j-1]}$ are defined analogously to (3.1) and (2.12). Calculating $\mathfrak{s}_k$, we get for j large enough

$$\begin{aligned} \mathfrak{S}(\lambda^k) = \mathfrak{s}_k &= \left[\left(\mathfrak{J}_{[0,j-1]}^k\right)^T e_0, e_0\right]_{\ell^2_{[0,\mathfrak{n}_j - 1]}} = \left(G_{[0,j-1]}\left(\mathfrak{J}_{[0,j-1]}^k\right)^T e_0, e_0\right) \\ &= (e_0, \underbrace{\mathfrak{L}_{[0,j-1]}\mathfrak{U}_{[0,j-1]} \dots \mathfrak{L}_{[0,j-1]}\mathfrak{U}_{[0,j-1]}}_{k \text{ times}} G_{[0,j-1]} e_0) \\ &= (\mathfrak{L}_{[0,j-1]}^T e_0, \underbrace{\mathfrak{U}_{[0,j-1]}\mathfrak{L}_{[0,j-1]} \dots \mathfrak{U}_{[0,j-1]}\mathfrak{L}_{[0,j-1]}}_{k-1 \text{ times}} \mathfrak{U}_{[0,j-1]} G_{[0,j-1]} e_0). \end{aligned} \tag{3.30}$$

Let $\widetilde{G}_{[0,j+n-1]}$ be associated with the matrix $\mathfrak{J}^{(p)}_{[0,j+n-1]}$, where n is the number of ℓ_h, such that $\ell_h \geq 2$, $0 \leq h \leq j-1$, as is defined by (2.12). Then $\widetilde{G}_{[0,j+n-1]} e_0 = e_{\ell_0 - 2}$. Substituting (3.29) into (3.30), we obtain

$$\begin{aligned} \mathfrak{s}_k &= \left(e_0, \left(\mathfrak{U}_{[0,j-1]}\mathfrak{L}_{[0,j-1]}\right)^{k-1} e_{\ell_0-2}\right) = \left(\left(\left(\mathfrak{J}^{(p)}_{[0,j+n-1]}\right)^{k-1}\right)^T e_0, \widetilde{G}_{[0,j+n-1]} e_0\right) \\ &= \left[\left(\left(\mathfrak{J}^{(p)}_{[0,j+n-1]}\right)^{k-1}\right)^T e_0, e_0\right]_{\ell^2_{[0,\tilde{\mathfrak{n}}_j - 1]}} = \mathfrak{s}^{(p)}_{k-1} = (\lambda\mathfrak{S})(\lambda^{k-1}) = \mathfrak{S}^{(p)}(\lambda^{k-1}), \end{aligned}$$

the moment sequence $\left\{\mathfrak{s}_j^{(p)}\right\}_{j=0}^{\infty}$ is associated with the matrix $\mathfrak{J}^{(p)}$. By definition (3.27), we obtain that functional $\lambda\mathfrak{S}$ is associated with matrix $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$.

(ii) Now we consider the case when $\mathfrak{n}_1 = 1$. We note that

$$\mathfrak{L}_{[0,j-1]}^T e_0 = e_0, \quad \mathfrak{U}_{[0,j-1]} G_{[0,j-1]} e_0 = -\mathfrak{u}_0 e_0, \quad j \in \mathbb{N}, \tag{3.31}$$

Let $\widetilde{G}_{[0,j+n-1]}$ be associated with the matrix $\mathfrak{J}^{(p)}_{[0,j+n-1]}$, where n is the number of ℓ_h, such that $\ell_h \geq 2$, $0 < h \leq j-1$. The matrix $\widetilde{G}_{[0,j+n-1]}$ is defined by (2.12). Then $\widetilde{G}_{[0,j+n-1]} e_0 = e_0$. Calculating $\mathfrak{s}_k$, from (3.16), (3.29) and (3.30), we get

$$\begin{aligned}\mathfrak{S}(\lambda^k) &= \frac{\mathfrak{s}_1}{\mathfrak{s}_0}\left(\left(\left(\mathfrak{J}^{(p)}_{[0,j+n-1]}\right)^{k-1}\right)^T e_0, \widetilde{G}_{[0,j+n-1]}e_0\right)\\ &= \frac{\mathfrak{s}_1}{\mathfrak{s}_0}\left[\left(\left(\mathfrak{J}^{(p)}_{[0,j+n-1]}\right)^{k-1}\right)^T e_0, e_0\right]_{\ell^2_{[0,\widetilde{\mathfrak{n}}_j-1]}} = \frac{\mathfrak{s}_1}{\mathfrak{s}_0}\mathfrak{s}^{(p)}_{k-1} = \frac{\mathfrak{s}_1}{\mathfrak{s}_0}\mathfrak{S}^{(p)}(\lambda^{k-1}),\end{aligned}\tag{3.32}$$

the moment sequence $\left\{\mathfrak{s}^{(p)}_j\right\}_{j=0}^{\infty}$ is associated with the matrix $\mathfrak{J}^{(p)}$.

Hence $\mathfrak{s}^{(p)}_{k-1} = \mathfrak{S}^{(p)}(\lambda^{k-1}) = \frac{\mathfrak{s}_0}{\mathfrak{s}_1}\lambda\mathfrak{S}(\lambda^{k-1})$ and consequently, the functional $\frac{\mathfrak{s}_0}{\mathfrak{s}_1}\lambda\mathfrak{S}$ is associated with the matrix $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$. This completes the proof. □

Remark 3.14. The transformation $\mathfrak{S} \to \mathfrak{S}^{(p)} = \lambda\mathfrak{S}$ is called *the Christoffel transformation* of the functional $\mathfrak{S}$.

By Theorem 3.13 we have that the matrix $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$ is associated with the moment sequence $\mathfrak{s}^{(p)} = \{\mathfrak{s}_{j+1}\}_{j=0}^{\infty}$. Define a set $\mathcal{N}(\mathfrak{s}^{(p)})$ of normal indices of the sequence $\mathfrak{s}^{(p)}$ by

$$\mathcal{N}(\mathfrak{s}^{(p)}) = \left\{\mathfrak{n}^{(p)}_j : \mathbf{d}^{(p)}_{\mathfrak{n}^{(p)}_j} \neq 0\right\}, \quad \text{where} \quad \mathbf{d}^{(p)}_{\mathfrak{n}^{(p)}_j} = \det\begin{pmatrix}\mathfrak{s}_1 & \cdots & \mathfrak{s}_{\mathfrak{n}^{(p)}_j}\\ \cdots & \cdots & \cdots\\ \mathfrak{s}_{\mathfrak{n}^{(p)}_j} & \cdots & \mathfrak{s}_{2\mathfrak{n}^{(p)}_j-1}\end{pmatrix}.\tag{3.33}$$

Proposition 3.15. *Let $\mathcal{N}(\mathfrak{s})$ be a set of normal indices associated with the monic generalized Jacobi matrix $\mathfrak{J}$ and let $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3). Let*

$$\mathfrak{P}_j(0) = \frac{(-1)^{\mathfrak{n}_j+2}}{\mathbf{d}_{\mathfrak{n}_j}}\begin{vmatrix}\mathfrak{s}_1 & \cdots & \mathfrak{s}_{\mathfrak{n}_j}\\ \cdots & \cdots & \cdots\\ \mathfrak{s}_{\mathfrak{n}_j} & \cdots & \mathfrak{s}_{2\mathfrak{n}_j-1}\end{vmatrix} \neq 0, \quad \textit{for each} \quad j \in \mathbb{Z}_+.\tag{3.34}$$

Then

$$\mathcal{N}(\mathfrak{s}^{(p)}) = \mathcal{N}(\mathfrak{s}) \cup \{\mathfrak{n}_j - 1 : j \in \mathbb{N}\ \ell_{j-1} \geq 2\}.\tag{3.35}$$

Proof. (*i*) If $\mathfrak{n} = \mathfrak{n}_j$ for some $j \in \mathbb{N}$, i.e. $\mathfrak{n} \in \mathcal{N}(\mathfrak{s})$, then by (3.33) and (3.34) $\mathbf{d}^{(p)}_{\mathfrak{n}} \neq 0$. Therefore

$$\mathcal{N}(\mathfrak{s}) \subseteq \mathcal{N}(\mathfrak{s}^{(p)}).\tag{3.36}$$

(*ii*) Assume that $\mathfrak{n} \in \mathcal{N}(\mathfrak{s}^{(p)})\backslash\mathcal{N}(\mathfrak{s})$. Then $\mathbf{d}^{(p)}_{\mathfrak{n}} \neq 0$ and $\mathbf{d}_{\mathfrak{n}} = 0$ and by ([8], see Lemma 5.1 [item 1]) $\mathbf{d}_{\mathfrak{n}+1} \neq 0$. Therefore $\mathfrak{n}+1 = \mathfrak{n}_j$ for some $j \in \mathbb{N}$ and thus $\mathfrak{n} = \mathfrak{n}_j - 1$. Moreover, $\ell_{j-1} = \mathfrak{n}_j - \mathfrak{n}_{j-1} \geq 2$. This proves that

$$\mathfrak{n} \in \mathcal{N}(\mathfrak{s}^{(p)})\backslash\mathcal{N}(\mathfrak{s}) = \{\mathfrak{n}_j - 1 : j \in \mathbb{N},\ \ell_{j-1} \geq 2\}.\tag{3.37}$$

Conversely, if $\mathfrak{n} = \mathfrak{n}_j - 1$ and $\ell_{j-1} \geq 2$, then

$$\mathbf{d}_{\mathfrak{n}_{j-1}} \neq 0, \quad \mathbf{d}_{\mathfrak{n}_{j-1}+1} = 0, \quad \cdots \quad \mathbf{d}_{\mathfrak{n}_j-1} = 0, \quad \mathbf{d}_{\mathfrak{n}_j} \neq 0$$

and hence $\mathfrak{n}_j - 1 \notin \mathcal{N}(\mathfrak{s})$. Assuming that $\mathbf{d}^{(p)}_{\mathfrak{n}_j-1} = 0$ one obtain from ([8], see Lemma 5.1 [item 2]) that $\mathbf{d}^{(p)}_{\mathfrak{n}_{j-1}} = 0$, which contradicts to the inclusion (3.36). This completes the proof. □

Remark 3.16. Let $\mathcal{N}(\mathfrak{s})$ and $\mathcal{N}(\mathfrak{s}^{(p)})$ be sets of normal indices associated with the matrices $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ and $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$, respectively. If $\ell_{j-1} = \mathfrak{n}_j - \mathfrak{n}_{j-1} \geq 2$, $\mathfrak{n}_0 = 0$ and $j \in \mathbb{N}$, then

$$\mathcal{N}(\mathfrak{s}^{(p)}) = \{\mathfrak{n}_1 - 1, \mathfrak{n}_1, \mathfrak{n}_2 - 1, \mathfrak{n}_2, \ldots\}.\tag{3.38}$$

Remark 3.17. Let $\mathcal{N}(\mathfrak{s})$ and $\mathcal{N}(\mathfrak{s}^{(p)})$ be sets of normal indices associated with the matrices $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ and $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L}$, respectively. If $\ell_{j-1} = \mathfrak{n}_j - \mathfrak{n}_{j-1} = 1$, $\mathfrak{n}_0 = 0$ and $j \in \mathbb{N}$, then

$$\mathcal{N}(\mathfrak{s}) = \mathcal{N}(\mathfrak{s}^{(p)}). \tag{3.39}$$

Proposition 3.18. *Let $\mathfrak{J}$ be a monic generalized Jacobi matrix satisfying (3.14) and let $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}-$factorization of the form (3.1)–(3.3). Let $m(\lambda)$ and $m^{(p)}(\lambda)$ be the $m-$functions of matrices $\mathfrak{J}$ and $\mathfrak{J}^{(p)}$, respectively. Then*

$$m^{(p)}_{[0,j+n-1]}(\lambda) = \begin{cases} \lambda m_{[0,j-1]}(\lambda), & \mathfrak{n}_1 > 1, \\ \frac{\mathfrak{s}_0}{\mathfrak{s}_1}\left(\lambda m_{[0,j-1]}(\lambda) + \mathfrak{s}_0\right), & \mathfrak{n}_1 = 1, \end{cases} \tag{3.40}$$

where n is the number of ℓ_i of matrix $\mathfrak{J}$, such that $\ell_i \geq 2$ and $i = \overline{0, j-1}$.

Proof. Let n be the number of $\ell_i \geq 2$, where $i = \overline{0, j-1}$ and let $\widetilde{G}_{[0,j+n-1]}$ be associated with the matrix $\mathfrak{J}^{(p)}_{[0,j+n-1]}$. It is defined by (2.12).

(i) Let $\mathfrak{n}_1 > 1$. Then $\mathfrak{s}_0 = 0$, $\widetilde{G}_{[0,j+n-1]}e_0 = e_{\ell_0-2}$ and the equalities (3.29) hold. Calculating

$$\begin{aligned}
m_{[0,j-1]}(\lambda) &= \lambda\left[\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, e_0\right] = -\left[(\mathfrak{J}^T_{[0,j-1]} - \lambda)\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, e_0\right] \\
&\quad + \left[\mathfrak{J}^T_{[0,j-1]}\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, e_0\right] = -\mathfrak{s}_0 + \left[\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, \mathfrak{J}_{[0,j-1]}e_0\right] \\
&= \left(\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, \mathfrak{L}_{[0,j-1]}\mathfrak{U}_{[0,j-1]}G_{[0,j-1]}e_0\right) \\
\lambda \qquad &= \left(e_0, \left(\mathfrak{L}_{[0,j-1]}\mathfrak{U}_{[0,j-1]} - \overline{\lambda}\right)^{-1}\mathfrak{L}_{[0,j-1]}e_{\ell_0-2}\right) \\
&= \left(e_0, \mathfrak{L}_{[0,j-1]}\left(\mathfrak{U}_{[0,j-1]}\mathfrak{L}_{[0,j-1]} - \overline{\lambda}\right)^{-1} e_{\ell_0-2}\right) \\
&= \left(\left(\left(\mathfrak{J}^{(p)}_{[0,j+n-1]}\right)^T - \lambda\right)^{-1} e_0, \widetilde{G}_{[0,j+n-1]}e_0\right) = m^{(p)}_{[0,j+n-1]}(\lambda).
\end{aligned}$$

(ii) Now we consider the case when $\mathfrak{n}_1 = 1$. Then $\widetilde{G}_{[0,j+n-1]}e_0 = e_0$ and the equalities (3.31) hold. Computing

$$\begin{aligned}
\lambda m_{[0,j-1]}(\lambda) &= \lambda\left[\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, e_0\right] = -\mathfrak{s}_0 + \left[\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, \mathfrak{J}_{[0,j-1]}e_0\right] \\
&= -\mathfrak{s}_0 + \left(\left(\mathfrak{J}^T_{[0,j-1]} - \lambda\right)^{-1} e_0, \mathfrak{L}_{[0,j-1]}\mathfrak{U}_{[0,j-1]}G_{[0,j-1]}e_0\right) \\
&= -\mathfrak{s}_0 + \frac{\mathfrak{s}_1}{\mathfrak{s}_0}\left(e_0, \left(\mathfrak{L}_{[0,j-1]}\mathfrak{U}_{[0,j-1]} - \overline{\lambda}\right)^{-1}\mathfrak{L}_{[0,j-1]}e_0\right) \\
&= -\mathfrak{s}_0 + \frac{\mathfrak{s}_1}{\mathfrak{s}_0}\left(e_0, \mathfrak{L}_{[0,j-1]}\left(\mathfrak{U}_{[0,j-1]}\mathfrak{L}_{[0,j-1]} - \overline{\lambda}\right)^{-1} e_0\right) \\
&= -\mathfrak{s}_0 + \frac{\mathfrak{s}_1}{\mathfrak{s}_0}\left(\left(\left(\mathfrak{J}^{(p)}_{[0,j+n-1]}\right)^T - \lambda\right)^{-1} e_0, \widetilde{G}_{[0,j+n-1]}e_0\right) \\
&= -\mathfrak{s}_0 + \frac{\mathfrak{s}_1}{\mathfrak{s}_0}m^{(p)}_{[0,j+n-1]}(\lambda).
\end{aligned}$$

Thus, we have

$$m^{(p)}_{[0,j+n-1]}(\lambda) = \frac{\mathfrak{s}_0}{\mathfrak{s}_1}\left(\lambda m_{[0,j-1]}(\lambda) + \mathfrak{s}_0\right).$$

So, the formula (3.40) is proved. This competes the proof. □

Theorem 3.19. *Let $\mathfrak{J}$ be a monic generalized Jacobi matrix satisfying (3.14) and let $\mathfrak{J}=\mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3). Let $\mathfrak{J}^{(p)}=\mathfrak{U}\mathfrak{L}$ be its Darboux transformation and let*

$$\mathfrak{J}^{(p)}\mathbf{P}^{(p)}(\lambda)=\lambda\mathbf{P}^{(p)}(\lambda), \tag{3.41}$$

where $\mathbf{P}^{(p)}(\lambda)=\begin{pmatrix}P_0^{(p)}(\lambda), & P_1^{(p)}(\lambda), & \dots\end{pmatrix}^T$. *Then*

$$\begin{aligned}
&P^{(p)}_{\mathfrak{n}_j-1}(\lambda)=\frac{1}{\lambda}\left(P_{\mathfrak{n}_j}(\lambda)-\frac{P_{\mathfrak{n}_j}(0)}{P_{\mathfrak{n}_{j-1}}(0)}P_{\mathfrak{n}_{j-1}}(\lambda)\right),\quad j\in\mathbb{N},\\
&P^{(p)}_{\mathfrak{n}_j+k}(\lambda)=\lambda^kP_{\mathfrak{n}_j}(\lambda),\quad 0\le k\le \ell_j-2\quad \textit{and}\quad j\in\mathbb{Z}_+.
\end{aligned}\tag{3.42}$$

Proof. First of all, we introduce the following polynomials:

$$\mathbf{P}^{(p)}(\lambda)=\frac{1}{\lambda}\mathfrak{U}\mathbf{P}(\lambda)=\frac{1}{\lambda}\begin{pmatrix}P_1(\lambda)\\ \vdots\\ P_{\mathfrak{n}_1-1}(\lambda)\\ P_{\mathfrak{n}_1}(\lambda)-\mathfrak{u}_0P_{\mathfrak{n}_0}(\lambda)\\ P_{\mathfrak{n}_1+1}(\lambda)\\ \vdots\\ P_{\mathfrak{n}_2-1}(\lambda)\\ P_{\mathfrak{n}_2}(\lambda)-\mathfrak{u}_1P_{\mathfrak{n}_1}(\lambda)\\ \vdots\end{pmatrix}=\frac{1}{\lambda}\begin{pmatrix}\lambda P_0(\lambda)\\ \vdots\\ \lambda^{\ell_0-1}P_0(\lambda)\\ P_{\mathfrak{n}_1}(\lambda)-\frac{P_{\mathfrak{n}_1}(0)}{P_{\mathfrak{n}_0}(0)}P_{\mathfrak{n}_0}(\lambda)\\ \lambda P_{\mathfrak{n}_1}(\lambda)\\ \vdots\\ \lambda^{\ell_1-1}P_{\mathfrak{n}_1}(\lambda)\\ P_{\mathfrak{n}_2}(\lambda)-\frac{P_{\mathfrak{n}_2}(0)}{P_{\mathfrak{n}_1}(0)}P_{\mathfrak{n}_1}(\lambda)\\ \vdots\end{pmatrix}.$$

Therefore

$$\mathfrak{J}^{(p)}\mathbf{P}^{(p)}(\lambda)=\lambda\mathbf{P}^{(p)}(\lambda),$$

because

$$\mathfrak{J}^{(p)}\mathbf{P}^{(p)}(\lambda)=\mathfrak{U}\mathfrak{L}\mathfrak{U}\frac{1}{\lambda}\mathbf{P}(\lambda)=\frac{1}{\lambda}\mathfrak{U}\mathfrak{J}\mathbf{P}(\lambda)=\lambda(\frac{1}{\lambda}\mathfrak{U}\mathbf{P}(\lambda))=\lambda\mathbf{P}^{(p)}(\lambda).$$

From here, we obtain that the polynomials $P_i^{(p)}(\lambda)$ can be represented by the formula (3.42), for all $i\in\mathbb{Z}_+$. This completes the proof. □

Remark 3.20. If $\ell_j=1$ for all $j\in\mathbb{Z}_+$, then

$$P^{(p)}_{\mathfrak{n}_j}(\lambda)=\frac{1}{\lambda}\left(P_{\mathfrak{n}_{j+1}}(\lambda)-\frac{P_{\mathfrak{n}_{j+1}}(0)}{P_{\mathfrak{n}_j}(0)}P_{\mathfrak{n}_j}(\lambda)\right)\tag{3.43}$$

is a Christoffel formula (see [22]).

Remark 3.21. If at least one $\ell_j\ge 2$, then the formula (3.42) is a special case of Christoffel formula (see [22]).

Theorem 3.22. *Let $\mathfrak{J}$ be a monic generalized Jacobi matrix satisfying (3.14) and let $\mathfrak{J}=\mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3). Let $\mathfrak{J}^{(p)}=\mathfrak{U}\mathfrak{L}$ be its Darboux transformation and let*

$$(\mathfrak{J}^{(p)}-\lambda)\mathbf{Q}^{(p)}(\lambda)=\Theta_{\ell_0-1},\tag{3.44}$$

where $\mathbf{Q}^{(p)}(\lambda)=\begin{pmatrix}Q_0^{(p)}(\lambda), & Q_1^{(p)}(\lambda), & \dots\end{pmatrix}^T$, $\Theta_{\ell_0-1}=(\underbrace{0,\dots,0,1}_{\ell_0-1},0\dots)$. *Then*

$$\begin{aligned}
&Q^{(p)}_{\mathfrak{n}_j-1}(\lambda)=Q_{\mathfrak{n}_j}(\lambda)-\frac{P_{\mathfrak{n}_j}(0)}{P_{\mathfrak{n}_{j-1}}(0)}Q_{\mathfrak{n}_{j-1}}(\lambda),\quad j\in\mathbb{N},\\
&Q^{(p)}_{\mathfrak{n}_j+k}(\lambda)=\lambda^{k+1}Q_{\mathfrak{n}_j}(\lambda),\quad 0\le k\le \ell_j-2\quad \textit{and}\quad j\in\mathbb{Z}_+.
\end{aligned}\tag{3.45}$$

Proof. Setting

$$\mathbf{Q}^{(p)}(\lambda) = \mathfrak{U}\mathbf{Q}(\lambda) = \begin{pmatrix} Q_1(\lambda) \\ \vdots \\ \lambda^{\ell_0-1}Q_0(\lambda)(\lambda) \\ Q_{\mathfrak{n}_1}(\lambda) - \frac{P_{\mathfrak{n}_1}(0)}{P_{\mathfrak{n}_0}(0)}Q_{\mathfrak{n}_0}(\lambda) \\ \lambda Q_{\mathfrak{n}_1}(\lambda) \\ \vdots \\ \lambda^{\ell_1-1}Q_{\mathfrak{n}_1}(\lambda) \\ Q_{\mathfrak{n}_2}(\lambda) - \frac{P_{\mathfrak{n}_2}(0)}{P_{\mathfrak{n}_1}(0)}Q_{\mathfrak{n}_1}(\lambda) \\ \vdots \end{pmatrix}.$$

Using $(\mathfrak{J} - \lambda)\mathbf{Q}(\lambda) = \Theta_{\ell_0}$, we obtain

$$\begin{aligned}\mathfrak{U}(\mathfrak{L}\mathfrak{U} - \lambda)\mathbf{Q}(\lambda) &= (\mathfrak{U}\mathfrak{L}\mathfrak{U} - \lambda\mathfrak{U})\mathbf{Q}(\lambda) = (\mathfrak{U}\mathfrak{L} - \lambda)\mathfrak{U}\mathbf{Q}(\lambda) \\ &= (\mathfrak{J}^{(p)} - \lambda)\mathbf{Q}^{(p)}(\lambda) = \mathfrak{U}\Theta_{\ell_0} = \Theta_{\ell_0-1}.\end{aligned}$$

So, the formula (3.45) is proved. This completes the proof. □

Definition 3.23. In the next theorem we use index $\kappa(a)$, $a \in \mathbb{N}$. It is defined by

$$\kappa(a) = \begin{cases} 1, & a = 1, \\ 2, & a \geq 2. \end{cases} \tag{3.46}$$

Proposition 3.24. *Let $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$ be monic generalized Jacobi matrices associated with the functionals $\mathfrak{S}$ and $\widetilde{\mathfrak{S}}$, respectively. Let $\mathfrak{J} = \mathfrak{L}\mathfrak{U}$ be its $\mathfrak{L}\mathfrak{U}$−factorization of the form (3.1)–(3.3). If $\kappa(\ell_j) = \kappa(\widetilde{\ell}_j)$, $\mathfrak{p}_0^{(j)} = \widetilde{\mathfrak{p}}_0^{(j)}$ and $\mathfrak{b}_{j+1} = \widetilde{\mathfrak{b}}_{j+1}$, where $\mathfrak{p}_0^{(j)}$, $\mathfrak{b}_{j+1}$ and $\widetilde{\mathfrak{p}}_0^{(j)}$, $\widetilde{\mathfrak{b}}_{j+1}$ are elements of matrices $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$, respectively, for all $j \in \mathbb{Z}_+$. Then $P_{\mathfrak{n}_j}^{(p)}(0) = \widetilde{P}_{\widetilde{\mathfrak{n}}_j}^{(p)}(0)$, where $P_{\mathfrak{n}_j}^{(p)}(\lambda)$ and $\widetilde{P}_{\widetilde{\mathfrak{n}}_j}^{(p)}(\lambda)$ are polynomials of the first kind associated with the matrices $\mathfrak{J}^{(p)}$ and $\widetilde{\mathfrak{J}}^{(p)}$, respectively, for all $j \in \mathbb{Z}_+$ and $\widetilde{\mathfrak{n}}_0 = \mathfrak{n}_0 = 0$.*

Proof. By Proposition 3.9, $\widetilde{\mathfrak{J}}$ admits $\mathfrak{L}\mathfrak{U}$−factorization of the form (3.1)–(3.3) and by Theorem 3.10, the matrix $\widetilde{\mathfrak{J}}^{(p)}$ exists. Due to $\kappa(\ell_j) = \kappa(\widetilde{\ell}_j)$ for all $j \in \mathbb{Z}_+$ and using Corollary 2.4, we have $P_{\mathfrak{n}_j}^{(p)}(0) = \widetilde{P}_{\widetilde{\mathfrak{n}}_j}^{(p)}(0)$, where $P_{\mathfrak{n}_j}^{(p)}(\lambda)$ and $\widetilde{P}_{\widetilde{\mathfrak{n}}_j}^{(p)}(\lambda)$ are polynomials of the first kind associated with the matrices $\mathfrak{J}^{(p)}$ and $\widetilde{\mathfrak{J}}^{(p)}$, respectively, for all $j \in \mathbb{Z}_+$ and $\widetilde{\mathfrak{n}}_0 = \mathfrak{n}_0 = 0$. This completes the proof. □

4. Darboux transformation with a shift

In this section we study the Darboux transformation with shift α, which may be more comfortable for calculation. It helps us to construct factorization of GJM $\mathfrak{J}$, when $P_{\mathfrak{n}_j}(0) = 0$ for some $j \in \mathbb{N}$.

Setting $\lambda := \lambda + \alpha$ in (2.4) and (2.5), we obtain the system of difference equations for all $j \in \mathbb{Z}_+$

$$\mathfrak{b}_j y_{\mathfrak{n}_{j-1}}(\lambda + \alpha) - \mathfrak{p}_j(\lambda + \alpha) y_{\mathfrak{n}_j}(\lambda + \alpha) + y_{\mathfrak{n}_{j+1}}(\lambda + \alpha) = 0 \quad (\mathfrak{b}_0 = \varepsilon_0). \tag{4.1}$$

The solutions of the system (4.1) are polynomials $P_{\mathfrak{n}_j}(\lambda+\alpha)$ and $Q_{\mathfrak{n}_j}(\lambda+\alpha)$. The system (4.1) is associated with the following initial conditions:

$$P_{\mathfrak{n}_{-1}}(\lambda + \alpha) \equiv 0, \quad P_{\mathfrak{n}_0}(\lambda + \alpha) \equiv 1, \quad Q_{\mathfrak{n}_{-1}}(\lambda + \alpha) \equiv -\frac{1}{\mathfrak{b}_0}, \quad Q_{\mathfrak{n}_0}(\lambda + \alpha) \equiv 0. \tag{4.2}$$

Denote $\widetilde{P}_{\mathfrak{n}_j}(\lambda) := P_{\mathfrak{n}_j}(\lambda + \alpha)$, $\widetilde{Q}_{\mathfrak{n}_j}(\lambda) := Q_{\mathfrak{n}_j}(\lambda + \alpha)$ and $\widetilde{\mathfrak{p}}_j(\lambda) := \mathfrak{p}_j(\lambda + \alpha)$.

Lemma 4.1. *Let $\mathfrak{J}$ be a GJM corresponding to the functional $\mathfrak{S}$ and let*

$$A_\alpha = \operatorname{diag}\left(\mathfrak{C}_{\mathfrak{p}_0} - \mathfrak{C}_{\widetilde{\mathfrak{p}}_0}, \mathfrak{C}_{\mathfrak{p}_1} - \mathfrak{C}_{\widetilde{\mathfrak{p}}_1}, \dots\right), \tag{4.3}$$

where $\mathfrak{C}_{\mathfrak{p}_j}$ and $\mathfrak{C}_{\widetilde{\mathfrak{p}}_j}$ are companion matrices associated with the polynomials $\mathfrak{p}_j(\lambda)$ and $\widetilde{\mathfrak{p}}_j(\lambda)$, respectively. Then the GJM $\mathfrak{J} - A_\alpha$ corresponds to the functional

$$\widetilde{\mathfrak{S}}(p(\lambda)) := \mathfrak{S}(p(\lambda - \alpha)). \tag{4.4}$$

If $\widetilde{\mathfrak{s}}$ is a moment sequence associated with $\widetilde{\mathfrak{S}}$ via (1.1), then the corresponding set $\mathcal{N}(\widetilde{\mathfrak{s}})$ of normal indices coincides with $\mathcal{N}(\mathfrak{s})$ and $\left\{\widetilde{P}_{\mathfrak{n}_j}(\lambda)\right\}_{j=0}^{\infty}$ is the sequence of quasi - orthogonal polynomials with respect to $\widetilde{\mathfrak{S}}$.

Proof. It follows from (2.7) and (4.3) that the matrix $\mathfrak{J} - A_\alpha$ is the GJM,

$$\mathfrak{J} - A_\alpha = \begin{pmatrix} \mathfrak{C}_{\widetilde{\mathfrak{p}}_0} & \mathfrak{D}_0 & & \\ \mathfrak{B}_1 & \mathfrak{C}_{\widetilde{\mathfrak{p}}_1} & \mathfrak{D}_1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{4.5}$$

is associated with the sequence of polynomials $\widetilde{\mathfrak{p}}_j$ and numbers $\mathfrak{b}_j$. Thus, the system (4.1) is associated with the matrix $\mathfrak{J} - A_\alpha$. Consequently, (4.4) holds and $\mathcal{N}(\mathfrak{s}) = \mathcal{N}(\widetilde{\mathfrak{s}})$. Due to $\widetilde{P}_{\mathfrak{n}_j}(\lambda) = P_{\mathfrak{n}_j}(\lambda+\alpha)$ and (4.4), $\left\{\widetilde{P}_{\mathfrak{n}_j}(\lambda)\right\}_{j=0}^{\infty}$ is the sequence of quasi - orthogonal polynomials with respect to $\widetilde{\mathfrak{S}}$. This completes the proof. □

Note, if $\mathfrak{n}_1 = 1$, then $\mathfrak{s}_0 = \mathfrak{S}(1) = \widetilde{\mathfrak{S}}(1) = \widetilde{\mathfrak{s}}_0$ and $\widetilde{P}_{\mathfrak{n}_1}(\lambda) = P_{\mathfrak{n}_1}(\lambda+\alpha) = \lambda + \alpha - \frac{\mathfrak{s}_1}{\mathfrak{s}_0}$, see Remark 3.8. On the other hand

$$\widetilde{P}_{\mathfrak{n}_1}(\lambda) = \frac{1}{\mathfrak{s}_0}\begin{vmatrix} \mathfrak{s}_0 & \widetilde{\mathfrak{s}}_1 \\ 1 & \lambda \end{vmatrix} = \lambda - \frac{\widetilde{\mathfrak{s}}_1}{\mathfrak{s}_0}, \tag{4.6}$$

therefore $\widetilde{P}_{\mathfrak{n}_1}(0) = -\frac{\widetilde{\mathfrak{s}}_1}{\mathfrak{s}_0} = \alpha - \frac{\mathfrak{s}_1}{\mathfrak{s}_0}$. This implies $\widetilde{\mathfrak{s}}_1 = \mathfrak{s}_1 - \alpha\mathfrak{s}_0$.

Theorem 4.2. *Let $\alpha \in \mathbb{R}$ be such that*

$$P_{\mathfrak{n}_j}(\alpha) \neq 0, \quad j \in \mathbb{Z}_+. \tag{4.7}$$

Then the GJM $\mathfrak{J} - A_\alpha$ admits the $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3)

$$T = \mathfrak{J} - A_\alpha = \mathfrak{L}\mathfrak{U} \tag{4.8}$$

and the corresponding Darboux transform $T^{(p)} = \mathfrak{U}\mathfrak{L}$ corresponds to the functional

$$\widetilde{\mathfrak{S}}^{(p)} = \begin{cases} \lambda\widetilde{\mathfrak{S}}, & \mathfrak{n}_1 > 1, \\ \frac{\mathfrak{s}_0}{\mathfrak{s}_1 - \alpha\mathfrak{s}_0}\lambda\widetilde{\mathfrak{S}}, & \mathfrak{n}_1 = 1. \end{cases} \tag{4.9}$$

Furthermore, if $\widetilde{\mathfrak{n}}_j^{(p)}$ and $\widetilde{\mathfrak{p}}_j^{(p)}(\lambda)$ are normal indices and generating polynomials of $T^{(p)}$ and

$$A_\alpha^{(p)} = \operatorname{diag}\left(\mathfrak{C}_{\mathfrak{p}_0^{(p)}} - \mathfrak{C}_{\widetilde{\mathfrak{p}}_0^{(p)}}, \mathfrak{C}_{\mathfrak{p}_1^{(p)}} - \mathfrak{C}_{\widetilde{\mathfrak{p}}_1^{(p)}}, \dots\right), \tag{4.10}$$

where $\mathfrak{C}_{\widetilde{\mathfrak{p}}_j^{(p)}}$ and $\mathfrak{C}_{\mathfrak{p}_j^{(p)}}$ are companion matrices associated with the polynomials $\widetilde{\mathfrak{p}}_j^{(p)}(\lambda)$ and $\mathfrak{p}_j^{(p)}(\lambda) := \widetilde{\mathfrak{p}}_j^{(p)}(\lambda - \alpha)$, respectively. Then

$$\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L} + A_\alpha^{(p)} \tag{4.11}$$

is a GJM corresponding to the functional

$$\mathfrak{S}^{(p)}(p(\lambda)) = \begin{cases} \mathfrak{S}((\lambda-\alpha)p(\lambda)), & \mathfrak{n}_1 > 1, \\ \frac{\mathfrak{s}_0}{\mathfrak{s}_1 - \alpha\mathfrak{s}_0}\mathfrak{S}((\lambda-\alpha)p(\lambda)), & \mathfrak{n}_1 = 1. \end{cases} \tag{4.12}$$

Proof. By Theorem 3.5 $T = \mathfrak{J} - A_\alpha$ admits $\mathfrak{L}\mathfrak{U}$–factorization of the form (3.1)–(3.3) and by Theorem 3.10 $T^{(p)} = \mathfrak{U}\mathfrak{L}$ is the GJM. The relation between functionals in (4.9) follows from Theorem 3.13 and relation (4.12) follows from Lemma 4.1 and (4.9). □

Theorem 4.3. *Suppose that the assumptions of Theorem 4.1 hold. Let* $P_{\mathfrak{n}_j}(\lambda)$, $P^{(p)}_{\mathfrak{n}_j^{(p)}}(\lambda)$ *and* $Q_{\mathfrak{n}_j}(\lambda)$, $Q^{(p)}_{\mathfrak{n}_j^{(p)}}(\lambda)$ *be polynomials of the first and the second kind of matrices* $\mathfrak{J}$ *and* $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L} + A^{(p)}_\alpha$, *respectively. Then*

$$
\begin{aligned}
&P^{(p)}_{\mathfrak{n}_j-1}(\lambda) = \frac{1}{\lambda-\alpha}\left(P_{\mathfrak{n}_j}(\lambda) - \frac{P_{\mathfrak{n}_j}(\alpha)}{P_{\mathfrak{n}_{j-1}}(\alpha)}P_{\mathfrak{n}_{j-1}}(\lambda)\right), \quad j \in \mathbb{N},\\
&P^{(p)}_{\mathfrak{n}_j+k}(\lambda) = \lambda^k P_{\mathfrak{n}_j}(\lambda), \quad 0 \le k \le \ell_j - 2 \quad \textit{and} \quad j \in \mathbb{Z}_+.
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
&Q^{(p)}_{\mathfrak{n}_j-1}(\lambda) = Q_{\mathfrak{n}_j}(\lambda) - \frac{P_{\mathfrak{n}_j}(\alpha)}{P_{\mathfrak{n}_{j-1}}(\alpha)}Q_{\mathfrak{n}_{j-1}}(\lambda), \quad j \in \mathbb{N},\\
&Q^{(p)}_{\mathfrak{n}_j+k}(\lambda) = \lambda^{k+1} Q_{\mathfrak{n}_j}(\lambda), \quad 0 \le k \le \ell_j - 2 \quad \textit{and} \quad j \in \mathbb{Z}_+.
\end{aligned}
\tag{4.14}
$$

Proof. The matrix $\mathfrak{J}$ is associated with the system of difference equations (2.4). By Lemma 4.1 $T = \mathfrak{J} - A_\alpha = \mathfrak{L}\mathfrak{U}$ is associated with the system of difference equations (4.1) and by Theorem 4.2 $T^{(p)} = \mathfrak{U}\mathfrak{L}$ is associated with the following system of difference equations for all $j \in \mathbb{Z}_+$

$$
\mathfrak{c}^{(p)}_j y_{\mathfrak{n}^{(p)}_{j-1}}(\lambda) - \widetilde{\mathfrak{p}}^{(p)}_j(\lambda) y_{\mathfrak{n}^{(p)}_j}(\lambda) + y_{\mathfrak{n}^{(p)}_{j+1}}(\lambda) = 0 \quad (\mathfrak{c}^{(p)}_0 = \varepsilon^{(p)}_0).
\tag{4.15}
$$

The solutions of the system (4.15) are polynomials $\widetilde{P}^{(p)}_{\mathfrak{n}^{(p)}_{j-1}}(\lambda)$ and $\widetilde{Q}^{(p)}_{\mathfrak{n}^{(p)}_{j-1}}(\lambda)$. By Theorem 3.19 and Theorem 3.22

$$
\begin{aligned}
&\widetilde{P}^{(p)}_{\mathfrak{n}_j-1}(\lambda) = \frac{1}{\lambda}\left(P_{\mathfrak{n}_j}(\lambda+\alpha) - \frac{P_{\mathfrak{n}_j}(\alpha)}{P_{\mathfrak{n}_{j-1}}(\alpha)}P_{\mathfrak{n}_{j-1}}(\lambda+\alpha)\right), \quad j \in \mathbb{N},\\
&P^{(p)}_{\mathfrak{n}_j+k}(\lambda) = \lambda^k P_{\mathfrak{n}_j}(\lambda+\alpha), \quad 0 \le k \le \ell_j - 2 \quad \text{and} \quad j \in \mathbb{Z}_+,\\
&\widetilde{Q}^{(p)}_{\mathfrak{n}_j-1}(\lambda) = Q_{\mathfrak{n}_j}(\lambda+\alpha) - \frac{P_{\mathfrak{n}_j}(\alpha)}{P_{\mathfrak{n}_{j-1}}(\alpha)}Q_{\mathfrak{n}_{j-1}}(\lambda+\alpha), \quad j \in \mathbb{N},\\
&Q^{(p)}_{\mathfrak{n}_j+k}(\lambda) = \lambda^{k+1} Q_{\mathfrak{n}_j}(\lambda+\alpha), \quad 0 \le k \le \ell_j - 2 \quad \text{and} \quad j \in \mathbb{Z}_+.
\end{aligned}
\tag{4.16}
$$

On the other hand, the matrix $\mathfrak{J} = \mathfrak{U}\mathfrak{L} + A^{(p)}_\alpha$ is associated with the system of difference equations for all $j \in \mathbb{Z}_+$

$$
\mathfrak{c}^{(p)}_j y_{\mathfrak{n}^{(p)}_{j-1}}(\lambda-\alpha) - \widetilde{\mathfrak{p}}^{(p)}_j(\lambda-\alpha) y_{\mathfrak{n}^{(p)}_j}(\lambda-\alpha) + y_{\mathfrak{n}^{(p)}_{j+1}}(\lambda-\alpha) = 0 \quad (\mathfrak{c}^{(p)}_0 = \varepsilon^{(p)}_0),
\tag{4.17}
$$

where the solutions of system (4.17) are polynomials

$$
P^{(p)}_{\mathfrak{n}^{(p)}_{j-1}}(\lambda) := \widetilde{P}^{(p)}_{\mathfrak{n}^{(p)}_{j-1}}(\lambda-\alpha) \quad \text{and} \quad Q^{(p)}_{\mathfrak{n}^{(p)}_{j-1}}(\lambda) := \widetilde{Q}^{(p)}_{\mathfrak{n}^{(p)}_{j-1}}(\lambda-\alpha).
\tag{4.18}
$$

Substituting (4.18) into (4.16) we obtain (4.13) and (4.14). This completes the proof. □

5. Example

5.1. **Example 1.** Recall that the class $\mathbf{N}_{-\infty}$ consists of holomorphic functions F on $\mathbb{C}_+$, such that $Im F(\lambda) \ge 0$ for all $\lambda \in \mathbb{C}_+$ and F admits the following asymptotic expansion:

$$
F(\lambda) = -\frac{\mathfrak{s}_0}{\lambda} - \frac{\mathfrak{s}_1}{\lambda^2} \cdots - \frac{\mathfrak{s}_{2n}}{\lambda^{2n+1}} + o\left(\frac{1}{\lambda^{2n+1}}\right), \quad \lambda \widehat{\rightarrow} \infty,
\tag{5.1}
$$

with $\mathfrak{s}_j \in \mathbb{R}$ for all $j \in \mathbb{Z}_+$, where $\lambda \widehat{\to} \infty$ means that λ tends to ∞ nontangentially, that is inside the sector $\varepsilon < \arg \lambda < \pi - \varepsilon$ for some $\varepsilon > 0$. Every function $F \in \mathbf{N}_{-\infty}$ admits the J–fraction expansion

$$F(\lambda) \sim -\cfrac{b_0}{\lambda - c_0 - \cfrac{b_1}{\lambda - c_1 - \cfrac{b_2}{\ddots}}} = -\frac{b_0}{\lambda - c_0} - \frac{b_1}{\lambda - c_1} - \frac{b_2}{\lambda - c_2} - \cdots . \tag{5.2}$$

Next, we construct the function $F(\lambda^3)$ with the following asymptotic expansion:

$$F(\lambda^3) = -\frac{\mathfrak{s}_0}{\lambda^3} - \frac{\mathfrak{s}_1}{\lambda^6} \cdots - \frac{\mathfrak{s}_{2n}}{\lambda^{2n+1}} - \cdots = -\frac{\widetilde{\mathfrak{s}}_0}{\lambda} - \frac{\widetilde{\mathfrak{s}}_1}{\lambda^2} \cdots - \frac{\widetilde{\mathfrak{s}}_{2n}}{\lambda^{2n+1}} - \cdots, \quad \lambda \widehat{\to} \infty, \tag{5.3}$$

where $\widetilde{\mathfrak{s}}_{3j-1} = \mathfrak{s}_j$ and $\widetilde{\mathfrak{s}}_j = 0$ otherwise. The expansion (5.3) can be rewritten as the $P-$fraction (see [17])

$$F(\lambda^3) \sim -\frac{b_0}{\lambda^3 - c_0} - \frac{b_1}{\lambda^3 - c_1} - \frac{b_2}{\lambda^3 - c_2} - \cdots . \tag{5.4}$$

The function $F(\lambda^3)$ is associated with the monic generalized Jacobi matrix $\mathfrak{J}$

$$\mathfrak{J} = \begin{pmatrix} \mathfrak{C}_{\mathfrak{p}_0} & \mathfrak{D}_0 & & \\ \mathfrak{B}_1 & \mathfrak{C}_{\mathfrak{p}_1} & \mathfrak{D}_1 & \\ & \mathfrak{B}_2 & \mathfrak{C}_{\mathfrak{p}_2} & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{5.5}$$

where

$$\mathfrak{C}_{\mathfrak{p}_j} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_j & 0 & 0 \end{pmatrix}, \quad \mathfrak{D}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{B}_{j+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{j+1} & 0 & 0 \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{5.6}$$

Let us assume that the polynomials of the first kind $P_{\mathfrak{n}_j}(\lambda)$ associated with the matrix $\mathfrak{J}$ do not vanish at α, i.e. $P_{\mathfrak{n}_j}(\alpha) \neq 0$ for all $j \in \mathbb{Z}_+$.

Next, we introduce the following diagonal block matrix:

$$A_\alpha = \operatorname{diag}\left(\mathfrak{C}_{\mathfrak{p}_0} - \mathfrak{C}_{\widetilde{\mathfrak{p}}_0}, \quad \mathfrak{C}_{\mathfrak{p}_1} - \mathfrak{C}_{\widetilde{\mathfrak{p}}_1}, \ \ldots\right), \tag{5.7}$$

where $\mathfrak{C}_{\widetilde{\mathfrak{p}}_j}$ is a companion matrix of the monic polynomial

$$\widetilde{\mathfrak{p}}_j(\lambda) := \mathfrak{p}_j(\lambda + \alpha) = \lambda^3 + 3\alpha\lambda^2 + 3\alpha^2\lambda + \alpha^3 + c_j. \tag{5.8}$$

Then by Theorem 4.2 a GJM $T = \mathfrak{J} - A_\alpha$ admits the $\mathfrak{L}\mathfrak{U}$–factorization ($T = \mathfrak{L}\mathfrak{U}$), where

$$\mathfrak{L} = \begin{pmatrix} \mathfrak{A}_0 & 0 & \\ \mathfrak{L}_1 & \mathfrak{A}_1 & \ddots \\ & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad \mathfrak{U} = \begin{pmatrix} \mathfrak{U}_0 & \mathfrak{D}_0 & \\ 0 & \mathfrak{U}_1 & \ddots \\ & \ddots & \ddots \end{pmatrix}, \tag{5.9}$$

where the blocks $\mathfrak{A}_j$, $\mathfrak{D}_j$, $\mathfrak{L}_{j+1}$, $\mathfrak{U}_j$ take the form (see (3.2)–(3.3))

$$\mathfrak{A}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3\alpha^2 & -3\alpha & 1 \end{pmatrix}, \ \mathfrak{L}_{j+1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{b_{j+1}}{\mathfrak{u}_j} \end{pmatrix}, \ \mathfrak{D}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathfrak{U}_j = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mathfrak{u}_j & 0 & 0 \end{pmatrix}, \tag{5.10}$$

where $-\mathfrak{u}_0 = c_0 - \alpha^3$ and $-\mathfrak{u}_i - \frac{b_i}{\mathfrak{u}_{i-1}} = c_i - \alpha^3$, $i \in \mathbb{N}$, (see (3.6)). Then $T^{(p)} = \mathfrak{U}\mathfrak{L}$ and

$$T^{(p)} = \begin{pmatrix} \widetilde{\mathfrak{C}}_0^0 & \mathfrak{D}_{0,0} & & \\ \mathfrak{B}_{1,0} & \widetilde{\mathfrak{C}}_0^1 & \mathfrak{D}_{0,1} & \\ & \mathfrak{B}_{1,1} & \widetilde{\mathfrak{C}}_1^0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{5.11}$$

where

$$\widetilde{\mathfrak{C}}_j^0 = \begin{pmatrix} 0 & 1 \\ -3\alpha^2 & -3\alpha \end{pmatrix}, \quad \mathfrak{D}_{j,0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathfrak{B}_{j+1,1} = \begin{pmatrix} 0 \\ -\frac{b_{j+1}}{\mathfrak{u}_j} \end{pmatrix}, \tag{5.12}$$
$$\mathfrak{B}_{j+1,0} = \begin{pmatrix} -\mathfrak{u}_j & 0 \end{pmatrix}, \quad \widetilde{\mathfrak{C}}_j^1 = (0), \quad \mathfrak{D}_{j,1} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad j \in \mathbb{Z}_+.$$

Let $\widetilde{a}_j^0$, $\widetilde{a}_j^1$ be polynomials associated with the matrices $\widetilde{\mathfrak{C}}_j^0$ and $\widetilde{\mathfrak{C}}_j^1$, respectively, i.e. $\widetilde{a}_j^0(\lambda) = \lambda^2 + 3\alpha\lambda + 3\alpha^2$ and $\widetilde{a}_j^1(\lambda) = \lambda$, for all $j \in \mathbb{Z}_+$. Let us introduce the polynomials $a_j^0(\lambda) := \widetilde{a}_j^0(\lambda - \alpha) = \lambda^2 + \alpha\lambda + \alpha^2$ and $a_j^1(\lambda) := \widetilde{a}_j^1(\lambda - \alpha) = \lambda - \alpha$, $j \in \mathbb{Z}_+$. Denote the companion matrices of a_j^0, a_j^1 by

$$\mathfrak{C}_j^0 = \begin{pmatrix} 0 & 1 \\ -\alpha^2 & -\alpha \end{pmatrix} \quad \text{and} \quad \mathfrak{C}_j^1 = \begin{pmatrix} \alpha \end{pmatrix}, \quad \text{for all} \quad j \in \mathbb{Z}_+ \tag{5.13}$$

and let the matrix $A_\alpha^{(p)}$is given by

$$A_\alpha^{(p)} = \operatorname{diag}\left(\mathfrak{C}_0^0 - \widetilde{\mathfrak{C}}_0^0, \quad \mathfrak{C}_0^1 - \widetilde{\mathfrak{C}}_0^1, \ \ldots\right). \tag{5.14}$$

Then the Darboux transformation of $\mathfrak{J}$ with the shift α takes the form (see (4.11))

$$\mathfrak{J}^{(p)} = T^{(p)} + A_\alpha^{(p)} = \begin{pmatrix} \mathfrak{C}_{\mathfrak{p}_0}^0 & \mathfrak{D}_{0,0} & & \\ \mathfrak{B}_{1,0} & \mathfrak{C}_{\mathfrak{p}_0}^1 & \mathfrak{D}_{0,1} & \\ & \mathfrak{B}_{1,1} & \mathfrak{C}_{\mathfrak{p}_1}^0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{5.15}$$

By Theorem 4.2 see (4.12), the moment sequence $\mathfrak{s}^{(p)} = \left\{\mathfrak{s}_j^{(p)}\right\}_{j=0}^{\infty}$ is associated with the matrix $\mathfrak{J}^{(p)}$ and

$$\mathfrak{s}_j^{(p)} = \mathfrak{S}^{(p)}(\lambda^j) = \mathfrak{S}((\lambda - \alpha)\lambda^j) = \mathfrak{S}(\lambda^{j+1}) - \alpha\mathfrak{S}(\lambda^j) = \widetilde{\mathfrak{s}}_{j+1} - \alpha\widetilde{\mathfrak{s}}_j. \tag{5.16}$$

On the other hand, we can rewrite $\mathfrak{s}_j^{(p)}$ as follows:

$$\mathfrak{s}_{3j}^{(p)} = 0, \quad \mathfrak{s}_{3j+1}^{(p)} = \mathfrak{s}_j, \quad \mathfrak{s}_{3j+2}^{(p)} = -\alpha\mathfrak{s}_j, \quad j \in \mathbb{Z}_+. \tag{5.17}$$

Consequently, the function $F^{(p)}(\lambda)$ associated with the matrix $\mathfrak{J}^{(p)}$ has the following representation:

$$F^{(p)}(\lambda) = (\lambda - \alpha)F(\lambda^3). \tag{5.18}$$

5.2. **Example 2.** This example is a special case of Example 1. We consider the monic Chebyshev-Hermite polynomials $\{H_k(\lambda)\}_{k=0}^\infty$ and study the Darboux transformation with a shift of monic generalized Jacobi matrix $\mathfrak{J}$ associated with $\{H_k(\lambda^3)\}_{k=0}^\infty$.

Let $\mathfrak{s} = \{\mathfrak{s}_j\}_{j=0}^\infty$ be a moment sequence corresponding to the measure $e^{-t^2}dt$ on $\mathbb{R}$, i.e.

$$\mathfrak{s}_0 = \sqrt{\pi}, \quad \mathfrak{s}_{2j} = \frac{\sqrt{\pi}}{2^j}(2j-1)!! \quad \text{and} \quad \mathfrak{s}_{2j-1} = 0, \quad j \in \mathbb{N}. \tag{5.19}$$

Then the corresponding recurrence relation takes the form

$$\lambda H_j(\lambda) = H_{j+1}(\lambda) + \frac{k}{2}H_{j-1}(\lambda) \quad \text{for} \quad j \in \mathbb{Z}_+ \tag{5.20}$$

and the corresponding polynomials of the first kind coincide with the monic Chebyshev-Hermite polynomials

$$H_j(\lambda) = \frac{(-1)^j}{2^j} e^{\lambda^2} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \left(e^{-\lambda^2}\right) \quad \text{for all} \quad j \in \mathbb{Z}_+, \tag{5.21}$$

where $x \in (-\infty, +\infty)$ and these polynomials are orthogonal in $L_2(\mathbb{R}, w(\lambda))$ with the weight function $w(\lambda) = e^{-\lambda^2}$.

Consider the sequence of polynomials $\{H_j(\lambda^3)\}_{j=0}^{\infty}$ which satisfy the recurrence relation

$$\lambda^3 H_j(\lambda^3) = H_{j+1}(\lambda^3) + \frac{k}{2} H_{j-1}(\lambda^3) \quad \text{for} \quad j \in \mathbb{Z}_+. \tag{5.22}$$

The polynomials $\{H_j(\lambda^3)\}_{j=0}^{\infty}$ are polynomials of the first kind associated with the monic generalized Jacobi matrix $\mathfrak{J}$ defined by (5.5)–(5.6).

Then the moment sequence $\widetilde{\mathfrak{s}} = \{\widetilde{\mathfrak{s}}_j\}_{j=0}^{\infty}$ associated with the matrix $\mathfrak{J}$ takes the form

$$\widetilde{\mathfrak{s}}_{3j} = \widetilde{\mathfrak{s}}_{3j+1} = 0, \quad \widetilde{\mathfrak{s}}_{3j+2} = \mathfrak{s}_j, \quad j \in \mathbb{Z}_+. \tag{5.23}$$

This GJM $\mathfrak{J}$ does not admit the Darboux transformation of the form (3.1)–(3.3), since $H_1(\lambda^3) = \lambda^3$ and hence the assumption (3.14) does not hold ($H_1(0) = 0$). Let us choose $\alpha \in \mathbb{R}$ such that

$$H_j(\alpha^3) \neq 0, \quad j \in \mathbb{Z}_+ \tag{5.24}$$

and let A_α be a diagonal block matrix introduced in (5.7). Then the GJM $\mathfrak{J} - A_\alpha$ admits the factorization $\mathfrak{J} - A_\alpha = \mathfrak{L}\mathfrak{U}$ (5.9)–(5.10). Consider the Darboux transformation $\mathfrak{J}^{(p)}$ of $\mathfrak{J}$ with the shift α

$$\mathfrak{J}^{(p)} - A_\alpha^{(p)} = \mathfrak{U}\mathfrak{L} \tag{5.25}$$

determined by (5.11)–(5.15).

Using Example 1, consider the Darboux transformation of $\mathfrak{J}$ with a shift α, we obtain the matrix $\mathfrak{J}^{(p)}$ which is defined by (5.15).

By Theorem 4.3 the polynomials $\left\{P_j^{(p)}(\lambda)\right\}_{j=0}^{\infty}$ of the first kind associated with the matrix $\mathfrak{J}^{(p)}$ are given by

$$\begin{aligned} &P_{3j}^{(p)}(\lambda) = H_j(\lambda^3), \quad P_{3j+1}^{(p)}(\lambda) = \lambda H_j(\lambda^3), \\ &P_{3j+2}^{(p)}(\lambda) = \frac{1}{\lambda - \alpha}\left(H_{j+1}(\lambda^3) - \frac{H_{j+1}(\alpha)}{H_j(\alpha)} H_j(\lambda^3)\right), \quad j \in \mathbb{Z}_+. \end{aligned} \tag{5.26}$$

Acknowledgments. The author thanks V. A. Derkach for many suggestions and observations.

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Received 11/03/2014