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# SPECTRAL GAPS OF THE HILL–SCHRÖDINGER OPERATORS WITH DISTRIBUTIONAL POTENTIALS

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*Dedicated to Professor V. M. Adamyan on the occasion of his 75 birthday*

ABSTRACT. The paper studies the Hill–Schrödinger operators with potentials in the space $H^{\omega} \subset H^{-1}(\mathbb{T}, \mathbb{R})$. The main results completely describe the sequences that arise as lengths of spectral gaps of these operators. The space $H^{\omega}$ coincides with the Hörmander space $H_2^{\omega}(\mathbb{T}, \mathbb{R})$ with the weight function $\omega(\sqrt{1+\xi^2})$ if $\omega$ belongs to Avakumovich's class OR. In particular, if the functions $\omega$ are power, then these spaces coincide with the Sobolev spaces. The functions $\omega$ may be nonmonotonic.

## 1. INTRODUCTION

Let us consider the Hill–Schrödinger operator

$$\mathrm{S}(q)u := -u'' + q(x)u, \quad x \in \mathbb{R}, \tag{1}$$

with the 1-periodic real-valued potential

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k)e^{ik2\pi x} \in L^2(\mathbb{T}, \mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}.$$

This condition means that

$$\sum_{k\in\mathbb{Z}} |\widehat{q}(k)|^2 < \infty \quad \text{and} \quad \widehat{q}(k) = \overline{\widehat{q}(-k)}, \quad k \in \mathbb{Z}.$$

It is well known that the operator $\mathrm{S}(q)$ is lower semibounded and self-adjoint in the Hilbert space $L^2(\mathbb{R})$. Its spectrum is absolutely continuous and has a zone structure [22].

Spectrum of the operator $\mathrm{S}(q)$ is completely defined by locations of the endpoints of the spectral gaps $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$, which satisfy the inequalities

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \leq \lambda_1^+(q) < \lambda_2^-(q) \leq \lambda_2^+(q) < \cdots \tag{2}$$

Some gaps may be degenerate, then the corresponding bands merge. For even/odd numbers $n \in \mathbb{Z}_+$, the endpoints of the spectral gaps $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ are eigenvalues of the periodic/semiperiodic problems on the interval $(0,1)$.

The interiors of the spectral bands (the stability zones),

$$\mathcal{B}_0(q) := (\lambda_0(q), \lambda_1^-(q)), \quad \mathcal{B}_n(q) := (\lambda_n^+(q), \lambda_{n+1}^-(q)), \quad n \in \mathbb{N},$$

together with the collapsed gaps,

$$\lambda = \lambda_{n_i}^- = \lambda_{n_i}^+,$$

are characterized as the set of those $\lambda \in \mathbb{R}$ for which all solutions of the equation

$$-u'' + q(x)u = \lambda u, \quad x \in \mathbb{R}, \tag{3}$$

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are bounded on $\mathbb{R}$. The open spectral gaps (the instability zones),

$$\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \quad \mathcal{G}_n(q) := (\lambda_n^-(q), \lambda_n^+(q)) \neq \emptyset, \quad n \in \mathbb{N},$$

form a set of those $\lambda \in \mathbb{R}$ for which any nontrivial solution of the equation (3) is unbounded on $\mathbb{R}$.

We study the behavior of the lengths of the spectral gaps,

$$\gamma_q(n) := \lambda_n^+(q) - \lambda_n^-(q), \quad n \in \mathbb{N},$$

of the operator $\mathrm{S}(q)$ in terms of behavior of the Fourier coefficients $\{\widehat{q}(n)\}_{n\in\mathbb{N}}$ of the potential $q$ with respect to test sequence spaces, that is in terms of potential regularity.

Hochstadt [6, 7], Marchenko and Ostrovskii [14], McKean and Trubowitz [12, 23] proved that the potential $q$ is an infinitely differentiable function if and only if the lengths of spectral gaps $\{\gamma_q(n)\}_{n=1}^{\infty}$ decrease faster than an arbitrary power of $1/n$,

$$q \in C^{\infty}(\mathbb{T}) \Leftrightarrow \gamma_q(n) = O(n^{-k}), \quad n \to \infty, \quad k \in \mathbb{Z}_+.$$

However, the scale of spaces $\left\{C^k(\mathbb{T})\right\}_{k\in\mathbb{N}}$ turned out to be unsuitable to obtain precise quantitative results. Marchenko and Ostrovskii [14] (see also [13, 11]) found that

$$q \in H^s(\mathbb{T}) \Leftrightarrow \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^s(\mathbb{N}), \quad s \in \mathbb{Z}_+. \tag{4}$$

The Sobolev spaces $H^s(\mathbb{T})$, $s \in \mathbb{R}$, of 1-periodic functions/generalized functions may also be defined by means of their Fourier coefficients

$$H^s(\mathbb{T}) = \Big\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ik2\pi x} \in \mathfrak{D}(\mathbb{T}) \,\Big|\, \|f\|^2_{H^s(\mathbb{T})} := \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |\widehat{f}(k)|^2 < \infty \Big\}.$$

Here by $\mathfrak{D}(\mathbb{T})$ we denote the space of 1-periodic generalized functions on $\mathbb{T}$.

We define the weighted sequence spaces $h^s(\mathbb{N})$, $s \in \mathbb{R}$, in the following way:

$$h^s(\mathbb{N}) := \Big\{ a = \{a(k)\}_{k\in\mathbb{N}} \,\Big|\, \|a\|^2_{h^s(\mathbb{N})} := \sum_{k\in\mathbb{N}} (1+|k|)^{2s} |a(k)|^2 < \infty \Big\}.$$

Marchenko–Ostrovskii theorem (4) can be extended to a more general scale of Hörmander spaces $\{H^{\omega}(\mathbb{T})\}_{\omega}$ [1, 2, 21, 17, 18], where $\omega = \{\omega(k)\}_{k\in\mathbb{Z}}$ is a weighted sequence. Recall that a sequence $a = \{a(k)\}_{k\in\mathbb{Z}}$ is called a weight or weighted sequence if it is positive and even, i. e., $a(k) \geq 0$ $\ a(-k) = a(k)$ for $k \in \mathbb{Z}_+$.

However, a complete description of the sequences that form lengths of the gaps with potentials from a given functional class, in particular the Hörmander space or the Sobolev space, remained an open question. This paper deals with this issue in a more general situation of distributional potentials.

## 2. Main results

Let us start with necessary notations. The spaces $H^{\omega}(\mathbb{T})$ and $h^{\omega}(\mathbb{N})$ are defined similarly to the spaces $H^s(\mathbb{T})$ and $h^s(\mathbb{N})$

$$H^{\omega}(\mathbb{T}) := \Big\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ik2\pi x} \in \mathfrak{D}(\mathbb{T}) \,\Big|\, \|f\|^2_{H^{\omega}(\mathbb{T})} := \sum_{k\in\mathbb{Z}} \omega^2(k) |\widehat{f}(k)|^2 < \infty \Big\},$$

$$h^{\omega}(\mathbb{N}) := \Big\{ a = \{a(k)\}_{k\in\mathbb{N}} \,\Big|\, \|a\|^2_{h^{\omega}(\mathbb{N})} := \sum_{k\in\mathbb{N}} \omega^2(k) |a(k)|^2 < \infty \Big\}.$$

We say that a weighted sequence $\omega = \{\omega(k)\}_{k\in\mathbb{Z}}$ belongs to the class $\mathrm{I}_0$, if it satisfies the following condition:

$$|k|^s \ll \omega(k) \ll |k|^{1+s}, \quad s \in [0, \infty).$$

The notation

$$b(k) \ll a(k) \ll c(k), \quad k \in \mathbb{N},$$

means that there are positive constants $C_1$ and $C_2$ such that the following inequalities hold:

$$C_1 b(k) \le a(k) \le C_2 c(k), \quad k \in \mathbb{N}.$$

We say that a weighted sequence $\omega = \{\omega(k)\}_{k\in\mathbb{Z}}$ belongs to the class $\mathrm{M}_0$, if it satisfies the following conditions:

$$\begin{aligned} &(\mathtt{i}) \;\; \omega(k) \uparrow \infty,\; k \in \mathbb{N} \quad \text{(monotonicity)};\\ &(\mathtt{ii}) \;\; \omega(k+m) \le \omega(k)\omega(m),\; k, m \in \mathbb{N} \quad \text{(submultiplicity)};\\ &(\mathtt{iii}) \;\; \frac{\log \omega(k)}{k} \downarrow 0,\; k \to \infty \quad \text{(subexponentiality)}. \end{aligned}$$

Suppose that a weighted sequence $\omega = \{\omega(k)\}_{k\in\mathbb{Z}}$ belongs either to the class $\mathrm{I}_0$ or to the class $\mathrm{M}_0$. Then

$$q \in H^\omega(\mathbb{T}) \Leftrightarrow \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^\omega(\mathbb{N}). \tag{5}$$

The statement (5) for the case where $\omega \in \mathrm{I}_0$ was proved by the authors [18], and the case $\mathrm{M}_0$ was closely studied in [1, 21].

The statement (5) may be strengthened. It is well-known that the sequence of lengths of spectral gaps $\{\gamma_q(n)\}_{n\in\mathbb{N}}$ of the Hill–Schrödinger operator $\mathrm{S}(q)$ with an $L^2(\mathbb{T})$-potential $q$ belongs to the space

$$h^0_+(\mathbb{N}) := \left\{a = \{a(k)\}_{k\in\mathbb{N}} \in l^2(\mathbb{N}) \mid a(k) \ge 0,\; k \in \mathbb{N}\right\}.$$

Let us consider the map

$$\gamma : L^2(\mathbb{T}) \ni q \mapsto \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^0_+(\mathbb{N}).$$

Then, due to Garnett and Trubowitz [4, 5],

$$\gamma\left(L^2(\mathbb{T})\right) = h^0_+(\mathbb{N}).$$

We introduce the following notations:

$$h^\omega_+(\mathbb{N}) := \{a = \{a(k)\}_{k\in\mathbb{N}} \in h^\omega(\mathbb{N}) \mid a(k) \ge 0,\; k \in \mathbb{N}\}.$$

**Theorem 1.** *Suppose that $q \in L^2(\mathbb{T})$ and that either $\omega \in \mathrm{I}_0$ or $\omega \in \mathrm{M}_0$. Then the map*

$$\gamma : L^2(\mathbb{T}) \ni q \mapsto \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^0_+(\mathbb{N})$$

*satisfies the relations*

$$\begin{aligned} &(\mathtt{i}) \quad \gamma\left(H^\omega(\mathbb{T})\right) = h^\omega_+(\mathbb{N}),\\ &(\mathtt{ii}) \quad \gamma^{-1}\left(h^\omega_+(\mathbb{N})\right) = H^\omega(\mathbb{T}). \end{aligned}$$

Now let us consider the Hill–Schrödinger operator $\mathrm{S}(q)$ with a 1-periodic real-valued distribution potential $q$ that belongs to the negative Sobolev space,

$$q = \sum_{k\in\mathbb{Z}} \widehat{q}(k) e^{ik2\pi x} \in H^{-1}(\mathbb{T}). \tag{6}$$

All real-valued pseudo-functions, measures, pseudo-measures and some more singular distributions on the circle satisfy this condition. For a more detailed discussion of operators with strongly singular potentials see [8] and references therein.

Under the assumption (6) the operator (1) may be well defined on the complex Hilbert space $L^2(\mathbb{R})$ in the following basic ways:

- as form-sum operator;
- as quasi-differential operators;
- as limit of operators with smooth 1-periodic potentials in the norm resolvent sense.

Equivalence of all these definitions was proved in the paper [15], a more general case was treated in [19].

The Hill–Schrödinger operator $\mathrm{S}(q)$ with strongly singular potential $q$ is lower semibounded and self-adjoint, its spectrum is absolutely continuous and has a band and gap structure as in the classical case [9, 10, 15, 3, 20]. The endpoints of the spectral gaps satisfy inequalities (2). For even/odd numbers $n \in \mathbb{Z}_+$, they are eigenvalues of periodic/semiperiodic problems on the interval $[0,1]$ [15, Theorem C].

We say that a weighted sequence $\omega = \{\omega(k)\}_{k\in\mathbb{Z}}$ belongs to $\mathrm{I}_{-1}$, if it satisfies the following conditions:

(i) $\omega(k) = (1+|k|)^{-1}, \qquad s = 1,$

(ii) $|k|^s \ll \omega(k) \ll |k|^{1+2s-\delta} \quad \forall\delta > 0, \qquad s \in (-1,0),$

(iii) $|k|^s \ll \omega(k) \ll |k|^{1+s}, \qquad s \in [0,\infty).$

We say that a weighted sequence $\omega = \{\omega(k)\}_{k\in\mathbb{Z}}$ belongs to $\mathrm{M}_{-1}$, if it can be represented as

$$\omega(k) = \frac{\omega^*(k)}{1+|k|}, \quad k \in \mathbb{Z}, \quad \omega^* = \{\omega^*(k)\}_{k\in\mathbb{Z}} \in \mathrm{M}_0.$$

Suppose that a weighted sequence $\omega = \{\omega(k)\}_{k\in\mathbb{Z}}$ belongs either to the class $\mathrm{I}_{-1}$, or to the class $\mathrm{M}_{-1}$. Then

$$q \in H^\omega(\mathbb{T}) \Leftrightarrow \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^\omega(\mathbb{N}). \tag{7}$$

The statement (7) for the case $\omega \in \mathrm{I}_{-1}$ is proved below (in a weaker form, this assertion was proved earlier by the authors [17]), also for the case $\omega \in \mathrm{M}_{-1}$ the statement (7) was proved in [2]. Note that $\mathrm{I}_0$ and $\mathrm{M}_0$, as well as $\mathrm{I}_{-1}$ and $\mathrm{M}_{-1}$, intersect, but do not cover each other.

Let us consider the map $\gamma : q \mapsto \{\gamma_q(n)\}_{n\in\mathbb{N}}$. Then, according to Korotyaev [10, Theorem 1.1], the map $\gamma$ maps $H^{-1}(\mathbb{T})$ onto $h_+^{-1}(\mathbb{N})$,

$$\gamma(H^{-1}(\mathbb{T})) = h_+^{-1}(\mathbb{N}). \tag{8}$$

**Theorem 2.** *Suppose that $q \in H^{-1}(\mathbb{T})$ and that either $\omega \in \mathrm{I}_{-1}$ or $\omega \in \mathrm{M}_{-1}$. Then the map*

$$\gamma : H^{-1}(\mathbb{T}) \ni q \mapsto \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h_+^{-1}(\mathbb{N})$$

*satisfies following:*

(i) $\gamma\left(H^\omega(\mathbb{T})\right) = h_+^\omega(\mathbb{N}),$

(ii) $\gamma^{-1}\left(h_+^\omega(\mathbb{N})\right) = H^\omega(\mathbb{T}).$

## 3. Proofs

*Proof of Theorem 1.* Due to Garnett and Trubowitz [4, 5] it occurs that for any sequence $\{\gamma(n)\}_{n\in\mathbb{N}} \in h_+^0(\mathbb{N})$ we can place open intervals $I_n$ of lengths $\gamma(n)$ (to length 0 there corresponds a point) on the positive semi-axis $(0,\infty)$ in such a single way that there exists a potential $q \in L^2(\mathbb{T})$ for which the sequence $\{\gamma(n)\}_{n\in\mathbb{N}}$ is a sequence of the lengths of spectral gaps of the Hill–Schrödinger operator $S(q)$, i.e., the map $\gamma$ maps the space $L^2(\mathbb{T})$ *onto* the sequence space $h_+^0(\mathbb{N})$,

$$\gamma(L^2(\mathbb{T})) = h_+^0(\mathbb{N}). \tag{9}$$

And, as a consequence, we also have that

$$\gamma^{-1}\left(h_+^0(\mathbb{N})\right) = L^2(\mathbb{T}). \tag{10}$$

The case $\omega \in \mathrm{I}_0$ was investigated by the authors in [18].

Let $\omega \in \mathrm{M}_0$. From statement (5) we get

$$\gamma\left(H^{\omega}(\mathbb{T})\right) \subset h_{+}^{\omega}(\mathbb{N}). \tag{11}$$

To establish the equality (i) in Theorem 1 it is necessary to prove the inverse inclusion in formula (11). So, let $\{\gamma(n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence in the space $h_{+}^{\omega}(\mathbb{N})$. Then $\{\gamma(n)\}_{n\in\mathbb{N}} \in h_{+}^{0}(\mathbb{N})$. Due to (9) there exists a potential $q \in L^2(\mathbb{T})$ such that the sequence $\{\gamma(n)\}_{n\in\mathbb{N}} \in h_{+}^{0}(\mathbb{N})$ is its sequence of lengths of spectral gaps. Since by assumption $\{\gamma(n)\}_{n\in\mathbb{N}} \in h_{+}^{\omega}(\mathbb{N})$ due to (5) we conclude that $q \in H^{\omega}(\mathbb{T})$ and, consequently, $\{\gamma(n)\}_{n\in\mathbb{N}} \in \gamma\left(H^{\omega}(\mathbb{T})\right)$. Therefore the inclusion

$$\gamma\left(H^{\omega}(\mathbb{T})\right) \supset h_{+}^{\omega}(\mathbb{N}) \tag{12}$$

holds.

Inclusions (11) and (12) give the equality (i).

Now, let us prove the equality (ii) in Theorem 1. Let $q$ be an arbitrary function in the space $H^{\omega}(\mathbb{T})$. Then, due to statement (5), we have $\gamma_q = \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h_{+}^{\omega}(\mathbb{N})$ and, hence, $q \in \gamma^{-1}\left(h_{+}^{\omega}(\mathbb{N})\right)$. Therefore

$$\gamma^{-1}\left(h_{+}^{\omega}(\mathbb{N})\right) \supset H^{\omega}(\mathbb{T}). \tag{13}$$

Conversely, let $\{\gamma(n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence in the space $h_{+}^{\omega}(\mathbb{N})$. Then due to (10) we have $\gamma^{-1}\left(\{\gamma(n)\}_{n\in\mathbb{N}}\right) \subset L^2(\mathbb{T})$. Taking into account (5) we conclude that $\gamma^{-1}\left(\{\gamma(n)\}_{n\in\mathbb{N}}\right) \subset H^{\omega}(\mathbb{T})$, that is,

$$\gamma^{-1}\left(h_{+}^{\omega}(\mathbb{N})\right) \subset H^{\omega}(\mathbb{T}). \tag{14}$$

Inclusions (13) and (14) give the equality (ii) in Theorem 1.

The proof of Theorem 1 is complete. □

*Proof of Formula* (7). Notice that, if $\omega(k) = (1+|k|)^s$, $s \in [-1,\infty)$, relation (7) has the form

$$q \in H^s(\mathbb{T}) \Leftrightarrow \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^s(\mathbb{N}), \quad s \in [-1,\infty). \tag{15}$$

The limiting case $s = -1$ was treated by Korotyaev [10]. The proof of statement (15) was completed in [2]. Earlier (15) was established by the authors [16] under a stronger assumption $q \in H^{-1+}(\mathbb{T})$ and $s > -1$.

Furthermore, if $q \in H^s(\mathbb{T})$, $s \in [-1,\infty)$, then for the lengths of spectral gaps the following asymptotic formula hold [10, 16]:

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{-1}(n), \qquad \text{if} \quad s = -1, \tag{16}$$

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{1+2s-\delta}(n) \quad \forall\delta > 0, \qquad \text{if} \quad s \in (-1,0), \tag{17}$$

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{1+s}(n), \qquad \text{if} \quad s \in [0,\infty). \tag{18}$$

Let us also recall that if $\omega_1 \gg \omega_2$, i. e., $\omega_1(k) \gg \omega_2(k)$, $k \in \mathbb{Z}$, then

$$H^{\omega_1}(\mathbb{T}) \hookrightarrow H^{\omega_2}(\mathbb{T}), \quad h^{\omega_1}(\mathbb{N}) \hookrightarrow h^{\omega_2}(\mathbb{N}). \tag{19}$$

Let $q \in H^{\omega}(\mathbb{T})$ and $\omega \in \mathrm{I}_{-1}$, then taking into account (19) we have $q \in H^s(\mathbb{T})$, $s \in [-1,\infty)$, as $\omega(k) \gg |k|^s$. Taking into account that $\omega \in \mathrm{I}_{-1}$, together with (19), from (16)–(18) we get

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{\omega}(n),$$

i. e., $\{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^{\omega}(\mathbb{N})$.

Now, let $\{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^{\omega}(\mathbb{N})$, then due to (19) we have $\{\gamma_q(n)\}_{n\in\mathbb{N}} \in h^s(\mathbb{N})$, $s \in [-1,\infty)$, and, consequently, from (15) we get $q \in H^s(\mathbb{T})$, $s \in [-1,\infty)$, and the asymptotics (16)–(18) hold. Taking into account that $\omega \in \mathrm{I}_{-1}$ and (19) we have

$$\gamma_q(n) = 2|\widehat{q}(n)| + h^{\omega}(n),$$

from where we get the necessary result $q \in H^{\omega}(\mathbb{T})$.

The statement (7) for the case $\omega \in \mathrm{I}_{-1}$ is completely proved. □

*Proof of Theorem 2.* From statement (7) we get

$$\gamma\left(H^{\omega}(\mathbb{T})\right) \subset h_{+}^{\omega}(\mathbb{N}). \tag{20}$$

To establish the equality (i) in Theorem 2 it is necessary to prove the inverse inclusion in formula (20). So, let $\{\gamma(n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence in the space $h_{+}^{\omega}(\mathbb{N})$. Then $\{\gamma(n)\}_{n\in\mathbb{N}} \in h_{+}^{-1}(\mathbb{N})$. Due to (8) there is a potential $q \in H^{-1}(\mathbb{T})$ such that the sequence $\{\gamma(n)\}_{n\in\mathbb{N}} \in h_{+}^{-1}(\mathbb{N})$ is its sequence of the lengths of spectral gaps. Since by assumption $\{\gamma(n)\}_{n\in\mathbb{N}} \in h_{+}^{\omega}(\mathbb{N})$ due to (7) we conclude that $q \in H^{\omega}(\mathbb{T})$ and, hence, $\{\gamma(n)\}_{n\in\mathbb{N}} \in \gamma\left(H^{\omega}(\mathbb{T})\right)$. Therefore the inclusion

$$\gamma\left(H^{\omega}(\mathbb{T})\right) \supset h_{+}^{\omega}(\mathbb{N}) \tag{21}$$

holds.

Inclusions (20) and (21) give the equality (i).

Now, let us prove the equality (ii) in Theorem 2. Let $q$ be an arbitrary function in the space $H^{\omega}(\mathbb{T})$. Then, due to statement (7), we have $\gamma_q = \{\gamma_q(n)\}_{n\in\mathbb{N}} \in h_{+}^{\omega}(\mathbb{N})$, i. e., $q \in \gamma^{-1}\left(h_{+}^{\omega}(\mathbb{N})\right)$. Therefore

$$\gamma^{-1}\left(h_{+}^{\omega}(\mathbb{N})\right) \supset H^{\omega}(\mathbb{T}). \tag{22}$$

Conversely, let $\{\gamma(n)\}_{n\in\mathbb{N}}$ be an arbitrary sequence in the space $h_{+}^{\omega}(\mathbb{N})$. Then due to (8) we have $\gamma^{-1}\left(h_{+}^{-1}(\mathbb{N})\right) = H^{-1}(\mathbb{T})$, and therefore $\gamma^{-1}\left(\{\gamma(n)\}_{n\in\mathbb{N}}\right) \subset H^{-1}(\mathbb{T})$. Taking into account (7) we conclude that $\gamma^{-1}\left(\{\gamma(n)\}_{n\in\mathbb{N}}\right) \subset H^{\omega}(\mathbb{T})$, that is,

$$\gamma^{-1}\left(h_{+}^{\omega}(\mathbb{N})\right) \subset H^{\omega}(\mathbb{T}). \tag{23}$$

Inclusions (22) and (23) give the equality (ii) in Theorem 2.

The proof of Theorem 2 is complete. □

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