Methods of Functional Analysis and Topology
Vol. 20 (2014), no. 4, pp. 328–348

ON GENERALIZED RESOLVENTS AND CHARACTERISTIC MATRICES OF FIRST-ORDER SYMMETRIC SYSTEMS

VADIM MOGILEVSKII

Dedicated with respect to D. Z. Arov and V. M. Adamyan on the occasion of their anniversaries

Abstract. We study general (not necessarily Hamiltonian) first-order symmetric system $Jy' - B(t)y = \Delta(t)f(t)$ on an interval $\mathcal{I} = [a, b)$ with the regular endpoint a and singular endpoint b. It is assumed that the deficiency indices $n_\pm(T_{\min})$ of the corresponding minimal relation $T_{\min}$ in $L^2_\Delta(\mathcal{I})$ satisfy $n_-(T_{\min}) \le n_+(T_{\min})$. We describe all generalized resolvents $y = R(\lambda)f$, $f \in L^2_\Delta(\mathcal{I})$, of $T_{\min}$ in terms of boundary problems with λ-depending boundary conditions imposed on regular and singular boundary values of a function y at the endpoints a and b respectively. We also parametrize all characteristic matrices $\Omega(\lambda)$ of the system immediately in terms of boundary conditions. Such a parametrization is given both by the block representation of $\Omega(\lambda)$ and by the formula similar to the well-known Krein formula for resolvents. These results develop the Štraus' results on generalized resolvents and characteristic matrices of differential operators.

1. Introduction

Let H and $\widehat{H}$ be finite dimensional Hilbert spaces and let $\mathbb{H} := H \oplus \widehat{H} \oplus H$. Denote also by $[\mathbb{H}]$ the set of all linear operators in $\mathbb{H}$. We study first-order symmetric systems of differential equations defined on an interval $\mathcal{I} = [a, b), -\infty < a < b \le \infty$, with the regular endpoint a and regular or singular endpoint b. Such a system is of the form [3, 13]

$$Jy' - B(t)y = \Delta(t)f(t), \quad t \in \mathcal{I}, \tag{1.1}$$

where $B(t) = B^*(t)$ and $\Delta(t) \ge 0$ are $[\mathbb{H}]$-valued functions on $\mathcal{I}$ and

$$J = \begin{pmatrix} 0 & 0 & -I_H \\ 0 & iI_{\widehat{H}} & 0 \\ I_H & 0 & 0 \end{pmatrix} : H \oplus \widehat{H} \oplus H \to H \oplus \widehat{H} \oplus H. \tag{1.2}$$

With (1.1) one associates the homogeneous system

$$Jy' - B(t)y = \lambda\Delta(t)y, \quad \lambda \in \mathbb{C}. \tag{1.3}$$

We assume that system (1.1) is definite (see Definition 3.1). Recall also that system (1.1) is called a Hamiltonian system if $\widehat{H} = \{0\}$ and hence

$$J = \begin{pmatrix} 0 & -I_H \\ I_H & 0 \end{pmatrix} : H \oplus H \to H \oplus H. \tag{1.4}$$

A function-theoretic approach to systems (1.1) can be found in the works of Kogan and Rofe-Beketov [18], Krall [19], Arov and Dym [2]. Another approach is based on the extension theory of symmetric linear relations (see [4, 9, 10, 17, 20, 27] and references therein). According to [17, 20, 27] the system (1.1) generates the minimal linear relation

2010 *Mathematics Subject Classification.* 34B08, 34B40, 47A06, 47B25.
Key words and phrases. First-order symmetric system, boundary problem with a spectral parameter, generalized resolvent, characteristic matrix.

$T_{\min}$ and the maximal linear relation $T_{\max}$ in the Hilbert space $L^2_\Delta(\mathcal{I})$ of all functions $f(\cdot):\mathcal{I}\to\mathbb{H}$ satisfying $||f||^2_\Delta := \int\limits_{\mathcal{I}}(\Delta(t)f(t),f(t))_{\mathbb{H}}\,dt<\infty$. It turns out that $T_{\min}$ is a closed symmetric relation and $T_{\max}=T^*_{\min}$. Moreover, the deficiency indices $n_\pm(T_{\min})$ of $T_{\min}$ satisfy $\dim H\le n_\pm(T_{\min})\le \dim\mathbb{H}$.

According to [6, 9, 29] each generalized resolvent $R(\lambda)$ of $T_{\min}$ admits the representation

$$(R(\lambda)f)(x)=\int_{\mathcal{I}} Y_0(x,\lambda)(\Omega(\lambda)+\tfrac{1}{2}\operatorname{sgn}(t-x)J)Y_0^*(t,\overline{\lambda})\Delta(t)f(t)\,dt,\quad f=f(\cdot)\in L^2_\Delta(\mathcal{I}).$$

Here $Y_0(\cdot,\lambda)$ is an $[\mathbb{H}]$-valued operator solution of Eq. (1.3) satisfying $_0(a,\lambda)=I_{\mathbb{H}}$ and $\Omega(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathbb{H}]$ is a Nevanlinna operator function called a characteristic matrix of the system (1.1) corresponding to $R(\lambda)$. By using the matrix $\Omega(\cdot)$ one constructs a spectral function generating an eigenfunction expansion of the system (1.1) (see e.g. [10]).

A somewhat other approach in the theory of generalized resolvents of $T_{\min}$ is based on an application of boundary problems for the system (1.1). Namely, assume that (1.1) is a Hamiltonian system and that $T_{\min}$ has minimal deficiency indices $n_\pm(T_{\min})=\dim H$. Then for each $\lambda\in\mathbb{C}\setminus\mathbb{R}$ there exists a unique operator solution $v(t,\lambda)(\in[H,H\oplus H])$ of Eq. (1.3) such that $v(\cdot,\lambda)h\in L^2_\Delta(\mathcal{I}),\ h\in H$, and

$$v(a,\lambda)=\begin{pmatrix} m(\lambda)\\ -I_H\end{pmatrix}:H\to H\oplus H,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{1.5}$$

Equality (1.5) defines a Nevanlinna operator function $m(\cdot):\mathbb{C}\setminus\mathbb{R}\to[H]$ called the Titchmarsh-Weyl coefficient (see e.g. [16]). Moreover, the following holds: (1) for each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists a unique holomorphic operator function $C_a(\cdot):\mathbb{C}\setminus\mathbb{R}\to[H\oplus H,H]$ satisfying

$$\operatorname{ran}C_a(\lambda)=H,\quad i\mathrm{Im}\lambda\cdot C_a(\lambda)JC_a^*(\lambda)\ge 0,\quad C_a(\lambda)JC_a^*(\overline{\lambda})=0,\quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{1.6}$$

and such that a function $y(t)=(R(\lambda)f)(t),\ f=f(\cdot)\in L^2_\Delta(\mathcal{I})$, is an L^2_Δ-solution of the following boundary problem with λ-depending boundary condition:

$$Jy'-B(t)y=\lambda\Delta(t)y+\Delta(t)f(t),\quad t\in\mathcal{I}, \tag{1.7}$$

$$C_a(\lambda)y(a)=0,\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{1.8}$$

(2) The characteristic matrix $\Omega(\cdot)$ corresponding to $R(\lambda)$ is of the form

$$\Omega(\lambda)=\begin{pmatrix} m(\lambda)-m(\lambda)(\tau(\lambda)+m(\lambda))^{-1}m(\lambda) & -\frac{1}{2}I+m(\lambda)(\tau\lambda)+m(\lambda))^{-1}\\ -\frac{1}{2}I+(\tau(\lambda)+m(\lambda))^{-1}m(\lambda) & -(\tau(\lambda)+m(\lambda))^{-1}\end{pmatrix}, \tag{1.9}$$

where $\tau(\lambda):=\ker C_a(\lambda),\ \lambda\in\mathbb{C}\setminus\mathbb{R}$, is a Nevanlinna family of linear relations in H.

Statement (1) readily follows from the results of [11, 30], while statement (2) was proved in [9] (for the Sturm-Liouville operator see [28]).

Note that the case $n_+(T_{\min})=n_-(T_{\min})>\dim H$ is more complicated, because in this case only one boundary condition (1.8) at the endpoint a is not sufficient for construction of a spectral function of the system (1.1).

In the present paper we extend the above statements to general (not necessarily Hamiltonian) symmetric systems (1.1) with $n_-(T_{\min})\le n_+(T_{\min})$. Our main result is a description of all generalized resolvents and characteristic matrices of such systems immediately in terms of boundary conditions. We describe all characteristic matrices by analogy with formula (1.9) and also by the formula similar to the well known Krein formula for resolvents.

To simplify the presentation of our results we assume within this section that system (1.1) satisfies $n_+(T_{\min})=n_-(T_{\min})$. We show that in this case there exist a finite-dimensional Hilbert space $\mathcal{H}_b$ and a surjective linear mapping $\Gamma_b:\operatorname{dom}T_{\max}\to\mathbb{H}_b$ such that

$$[y,z]_b (= \lim_{t\uparrow b}(Jy(t),z(t))) = (J_b\Gamma_b y, \Gamma_b z), \quad y,z\in \operatorname{dom} T_{\max}.$$

Here $\mathbb{H}_b = \mathcal{H}_b \oplus \widehat{H} \oplus \mathcal{H}_b$ and J_b is an operator in $\mathbb{H}_b$ given by

$$J_b = \begin{pmatrix} 0 & 0 & -I_{\mathcal{H}_b} \\ 0 & iI_{\widehat{H}} & 0 \\ I_{\mathcal{H}_b} & 0 & 0 \end{pmatrix} : \underbrace{\mathcal{H}_b \oplus \widehat{H} \oplus \mathcal{H}_b}_{\mathbb{H}_b} \to \underbrace{\mathcal{H}_b \oplus \widehat{H} \oplus \mathcal{H}_b}_{\mathbb{H}_b}. \tag{1.10}$$

In fact, $\Gamma_b y$ is a singular boundary value of a function y in the sense of [12, Chapter 13.2] (for more details see Remark 3.5 in [1]).

Assume that $\mathcal{H}_b$ and Γ_b are fixed and let $\mathcal{H} = H \oplus \widehat{H} \oplus \mathcal{H}_b$. With each Nevanlinna family of linear relations (in particular operators) $\tau = \tau(\lambda)$ in $\mathcal{H}$ we associate a pair of holomorphic operator functions $C_a(\lambda) = C_{\tau,a}(\lambda) (\in [\mathbb{H},\mathcal{H}])$ and $C_b(\lambda) = C_{\tau,b}(\lambda)(\in [\mathbb{H}_b,\mathcal{H}])$ satisfying for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$ the relations (cf. (1.6))

$$\operatorname{ran}(C_a(\lambda),\ C_b(\lambda)) = \mathcal{H}, \tag{1.11}$$

$$i\operatorname{Im}\lambda\cdot (C_a(\lambda)JC_a^*(\lambda) - C_b(\lambda)J_bC_b^*(\lambda)) \ge 0, \quad C_a(\lambda)JC_a^*(\overline{\lambda}) = C_b(\lambda)J_bC_b^*(\overline{\lambda}). \tag{1.12}$$

We show that for each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists a unique Nevanlinna family of linear relations $\tau = \tau(\lambda)$ in $\mathcal{H}$ such that a function $y(t) = (R(\lambda)f)(t),\ f = f(\cdot)\in L^2_\Delta(\mathcal{I})$, is an L^2_Δ-solution of the following boundary problem (cf. (1.7), (1.8))

$$Jy' - B(t)y = \lambda\Delta(t)y + \Delta(t)f(t), \quad t\in\mathcal{I}, \tag{1.13}$$

$$C_{\tau,a}(\lambda)y(a) + C_{\tau,b}(\lambda)\Gamma_b y = 0, \quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{1.14}$$

Note, that (1.14) is a boundary condition imposed on boundary values of a function $y\in\operatorname{dom}T_{\max}$ (more precisely, on the regular value $y(a)$ and singular value $\Gamma_b y$). One may consider $\tau = \tau(\lambda)$ as a (Nevanlinna) boundary parameter, since $R(\lambda)$ runs over the set of all generalized resolvents of $T_{\min}$ when τ runs over the set of all Nevanlinna families of linear relations in $\mathcal{H}$. To indicate this fact explicitly we write $R(\lambda) = R_\tau(\lambda)$ and $\Omega(\lambda) = \Omega_\tau(\lambda)$ for the generalized resolvent of $T_{\min}$ and the corresponding characteristic matrix respectively.

The boundary problem (1.13), (1.14) defines a canonical resolvent $R_\tau(\lambda)$ of $T_{\min}$ if and only if $\tau = \tau^*$. In this case $C_{\tau,a}(\lambda)\equiv C_a,\ C_{\tau,b}(\lambda)\equiv C_b$ and the operators C_a and C_b satisfy

$$\operatorname{ran}(C_a,\ C_b) = \mathcal{H} \quad\text{and}\quad C_aJC_a^* = C_bJC_b^*. \tag{1.15}$$

Moreover, $R_\tau(\lambda) = (\widetilde{T}^\tau - \lambda)^{-1}$ with $\widetilde{T}^\tau = (\widetilde{T}^\tau)^*$ given by the boundary conditions

$$\widetilde{T}^\tau = \{\{y,f\}\in T_{\max} : C_a y(a) + C_b\Gamma_b y = 0\}. \tag{1.16}$$

Thus, the equalities (1.16) and (1.15) gives a parametrization of all self-adjoint extensions $\widetilde{T} = \widetilde{T}^\tau$ of $T_{\min}$ in terms of boundary conditions. Note that in the case of the regular endpoint b (1.15) and (1.16) take the form of self-adjoint boundary conditions from [3, 13]. Moreover, for Hamiltonian systems with singular endpoint b the description of all extensions $\widetilde{T} = \widetilde{T}^*$ of $T_{\min}$ in the form (1.15), (1.16) was obtained in [19].

It turns out that for each boundary parameter τ there exists a unique $[\mathbb{H}]$-valued operator solution $Z_\tau(\cdot,\lambda),\ \lambda\in\mathbb{C}\setminus\mathbb{R}$, of Eq. (1.3) such that $Z_\tau(\cdot,\lambda)h\in L^2_\Delta(\mathcal{I}),\ h\in\mathbb{H}$, and the following boundary condition is satisfied:

$$C_{\tau,a}(\lambda)(Z_\tau(a,\lambda) + J)h + C_{\tau,b}(\lambda)\Gamma_b(Z_\tau(\cdot,\lambda)h) = 0, \quad h\in\mathbb{H},\ \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

Moreover, the characteristic matrix $\Omega_\tau(\cdot)$ is

$$\Omega_\tau(\lambda) = Z_\tau(a,\lambda) + \tfrac{1}{2}J, \quad \lambda\in\mathbb{C}\setminus\mathbb{R} \tag{1.17}$$

and the following inequality holds:

$$(\text{Im}\lambda)^{-1}\cdot \text{Im}\Omega_\tau(\lambda)\geq \int_{\mathcal I} Z_\tau^*(t,\lambda)\Delta(t)Z_\tau(t,\lambda)\,dt,\quad \lambda\in\mathbb C\setminus\mathbb R. \tag{1.18}$$

Note that definition of the characteristic matrix $\Omega_\tau(\cdot)$ by means of (1.17) is similar to that of the Titchmarsh-Weyl coefficient $m(\cdot)$ by means of (1.5). Observe also that formula (1.18) is similar to well-known formulas for various classes of boundary problems (see e.g. [14, 5]).

The main result of the paper is a parametrization of all characteristic matrices $\Omega(\cdot)$ of the system (1.1) immediately in terms of the boundary parameter τ. This result can be formulated in the form of the following theorem.

Theorem 1.1. *There exist operator functions $\Omega_0(\lambda)(\in[\mathbb H])$, $S(\lambda)(\in[\mathcal H,\mathbb H])$ and a Nevanlinna operator function $M(\lambda)(\in[\mathcal H])$, $\lambda\in\mathbb C\setminus\mathbb R$, such that the equality*

$$\Omega(\lambda)=\Omega_\tau(\lambda)=\Omega_0(\lambda)-S(\lambda)(\tau(\lambda)+M(\lambda))^{-1}S^*(\overline{\lambda}),\quad \lambda\in\mathbb C\setminus\mathbb R \tag{1.19}$$

establishes a bijective correspondence between all (Nevanlinna) boundary parameters $\tau=\tau(\lambda)$ and all characteristic matrices $\Omega(\cdot)$ of the system (1.1)*. Moreover, for each boundary parameter τ the corresponding characteristic matrix $\Omega(\lambda)=\Omega_\tau(\lambda)$ admits the representation*

$$\Omega(\lambda)=X\widetilde\Omega_\tau(\lambda)X^*,\quad \lambda\in\mathbb C\setminus\mathbb R, \tag{1.20}$$

where $X\in[\mathcal H\oplus\mathcal H,\mathbb H]$ is a certain operator (see (4.84)*) and $\widetilde\Omega_\tau(\cdot):\mathbb C\setminus\mathbb R\to[\mathcal H\oplus\mathcal H]$ is a Nevanlinna operator function given by the block matrix representation (cf.* (1.9)*)*

$$\widetilde\Omega_\tau(\lambda)=\begin{pmatrix} M(\lambda)-M(\lambda)(\tau(\lambda)+M(\lambda))^{-1}M(\lambda) & -\frac12 I_{\mathcal H}+M(\lambda)(\tau\lambda)+M(\lambda))^{-1}\\ -\frac12 I_{\mathcal H}+(\tau(\lambda)+M(\lambda))^{-1}M(\lambda) & -(\tau(\lambda)+M(\lambda))^{-1}\end{pmatrix} \tag{1.21}$$

Note that the operator functions $\Omega_0(\cdot)$, $S(\cdot)$ and $M(\cdot)$ in (1.19) are defined in terms of the boundary values of respective L^2_Δ-operator solutions of Eq. (1.3) at the endpoints a and b. Observe also that in the case of the Hamiltonian system (1.1) with $n_\pm(T_{\min})=\dim H$ one has $\mathcal H=H$, $X=I_{\mathbb H}$, $M(\lambda)=m(\lambda)$ and hence $\Omega(\lambda)(=\Omega_\tau(\lambda))=\widetilde\Omega_\tau(\lambda)$. This implies that equality (1.9) is a particular case of (1.20), (1.21).

In conclusion note that an analog of Theorem 1.1 for differential operators of an even order was proved in our paper [25].

2. Preliminaries

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak H$, $\mathcal H$ denote Hilbert spaces; $[\mathcal H_1,\mathcal H_2]$ is the set of all bounded linear operators defined on the Hilbert space $\mathcal H_1$ with values in the Hilbert space $\mathcal H_2$; $[\mathcal H]:=[\mathcal H,\mathcal H]$; $\mathbb C_+\,(\mathbb C_-)$ is the upper (lower) half-plane of the complex plane.

Let $\widetilde{\mathcal H}$ be a Hilbert space and let $\mathcal H$ be a subspace in $\widetilde{\mathcal H}$. We denote by $P_{\widetilde{\mathcal H},\mathcal H}(\in[\widetilde{\mathcal H},\mathcal H])$ the orthoprojection in $\widetilde{\mathcal H}$ onto $\mathcal H$. Moreover, we denote by $I_{\mathcal H,\widetilde{\mathcal H}}(\in[\mathcal H,\widetilde{\mathcal H}])$ the embedding operator of the subspace $\mathcal H$ into $\widetilde{\mathcal H}$. It is clear that $P^*_{\widetilde{\mathcal H},\mathcal H}=I_{\mathcal H,\widetilde{\mathcal H}}$.

Recall that a closed linear relation from $\mathcal H_0$ to $\mathcal H_1$ is a closed linear subspace in $\mathcal H_0\oplus\mathcal H_1$. The set of all closed linear relations from $\mathcal H_0$ to $\mathcal H_1$ (in $\mathcal H$) will be denoted by $\widetilde{\mathcal C}(\mathcal H_0,\mathcal H_1)$ ($\widetilde{\mathcal C}(\mathcal H)$). A closed linear operator T from $\mathcal H_0$ to $\mathcal H_1$ is identified with its graph $\text{gr}\,T\in\widetilde{\mathcal C}(\mathcal H_0,\mathcal H_1)$.

For a linear relation $T\in\widetilde{\mathcal C}(\mathcal H_0,\mathcal H_1)$ we denote by $\text{dom}\,T$, $\text{ran}\,T$, $\ker T$ and $\text{mul}\,T$ the domain, range, kernel and the multi-valued part of T respectively. Recall that $\text{mul}\,T$ is a linear manifold in $\mathcal H_1$ defined by

$$\text{mul}\,T:=\{h_1\in\mathcal H_1:\{0,h_1\}\in T\}.$$

Recall also that the inverse and adjoint linear relations of T are the relations $T^{-1} \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ and $T^* \in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0)$ defined by

$$
\begin{aligned}
T^{-1} &= \{\{h_1, h_0\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : \{h_0, h_1\} \in T\},\\
T^* &= \{\{k_1, k_0\} \in \mathcal{H}_1 \oplus \mathcal{H}_0 : \ (k_0, h_0) - (k_1, h_1) = 0, \ \{h_0, h_1\} \in T\}.
\end{aligned} \tag{2.1}
$$

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H})$ we denote by $\rho(T) := \{\lambda \in \mathbb{C} : (T - \lambda)^{-1} \in [\mathcal{H}]\}$ the resolvent set of T.

Recall also that an operator function $\Phi(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [\mathcal{H}]$ is called a Nevanlinna function if it is holomorphic and satisfies $\operatorname{Im}\lambda \cdot \operatorname{Im}\Phi(\lambda) \ge 0$ and $\Phi^*(\lambda) = \Phi(\overline{\lambda}), \ \lambda \in \mathbb{C} \setminus \mathbb{R}$.

2.2. **The classes $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ and $\widetilde{R}(\mathcal{H})$.** Let $\mathcal{H}_0$ be a Hilbert space, let $\mathcal{H}_1$ be a subspace in $\mathcal{H}_0$ and let $\tau = \{\tau_+, \tau_-\}$ be a collection of holomorphic functions $\tau_\pm(\cdot) : \mathbb{C}_\pm \to \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$. In the paper we systematically deal with collections $\tau = \{\tau_+, \tau_-\}$ of the special class $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$. Definition and detailed characterization of this class can be found in our paper [26] (see also [23, 25, 1], where the notation $\widetilde{R}(\mathcal{H}_0, \mathcal{H}_1)$ were used instead of $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$). If $\dim \mathcal{H}_1 < \infty$, then according to [26] the collection $\tau = \{\tau_+, \tau_-\} \in \widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ admits the representation

$$
\tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}_0\}, \ \ \lambda \in \mathbb{C}_+; \quad \tau_-(\lambda) = \{(D_0(\lambda), D_1(\lambda)); \mathcal{H}_1\}, \ \ \lambda \in \mathbb{C}_- \tag{2.2}
$$

by means of two pairs of holomorphic operator functions

$$
(C_0(\lambda), C_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0, \ \ \lambda \in \mathbb{C}_+, \ \text{ and } \ (D_0(\lambda), D_1(\lambda)) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_1, \ \ \lambda \in \mathbb{C}_-
$$

(more precisely, by equivalence classes of such pairs). The equalities (2.2) mean that

$$
\begin{aligned}
\tau_+(\lambda) &= \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : C_0(\lambda)h_0 + C_1(\lambda)h_1 = 0\}, \quad \lambda \in \mathbb{C}_+,\\
\tau_-(\lambda) &= \{\{h_0, h_1\} \in \mathcal{H}_0 \oplus \mathcal{H}_1 : D_0(\lambda)h_0 + D_1(\lambda)h_1 = 0\}, \quad \lambda \in \mathbb{C}_-.
\end{aligned} \tag{2.3}
$$

In [26] the class $\widetilde{R}_+(\mathcal{H}_0, \mathcal{H}_1)$ is characterized both in terms of $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$-valued functions $\tau_\pm(\cdot)$ and in terms of operator functions $C_j(\cdot)$ and $D_j(\cdot)$, $j \in \{0, 1\}$, from (2.2).

If $\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H}$, then the class $\widetilde{R}(\mathcal{H}) := \widetilde{R}_+(\mathcal{H}, \mathcal{H})$ coincides with the well-known class of Nevanlinna $\widetilde{\mathcal{C}}(\mathcal{H})$-valued functions $\tau(\cdot)$ (see, for instance, [7]). In this case the collection (2.2) turns into the Nevanlinna pair

$$
\tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathcal{H}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.4}
$$

with $C_0(\lambda), C_1(\lambda) \in [\mathcal{H}]$. Recall also that the subclass $\widetilde{R}^0(\mathcal{H}) \subset \widetilde{R}(\mathcal{H})$ is defined as the set of all $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$ such that $\tau(\lambda) \equiv \theta (= \theta^*), \ \lambda \in \mathbb{C} \setminus \mathbb{R}$. This implies that $\tau(\cdot) \in \widetilde{R}^0(\mathcal{H})$ if and only if

$$
\tau(\lambda) \equiv \{(C_0, C_1); \mathcal{H}\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.5}
$$

with some operators $C_0, C_1 \in [\mathcal{H}]$ satisfying $\operatorname{Im}(C_1 C_0^*) = 0$ and $0 \in \rho(C_0 \pm iC_1)$ (for more details see e.g. [1, Remark 2.5]).

2.3. **Boundary triplets and Weyl functions for symmetric relations.** Recall that a linear relation $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is called symmetric (self-adjoint) if $A \subset A^*$ (resp. $A = A^*$).

Let A be a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$, let $\mathfrak{N}_\lambda(A) = \ker(A^* - \lambda)$ $(\lambda \in \mathbb{C})$ be a defect subspace of A, let $\widehat{\mathfrak{N}}_\lambda(A) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(A)\}$ and let $n_\pm(A) := \dim \mathfrak{N}_\lambda(A) \le \infty, \ \lambda \in \mathbb{C}_\pm$, be deficiency indices of A.

The following definitions are well known.

Definition 2.1. The operator function $R(\cdot):\mathbb{C}\backslash\mathbb{R}\to[\mathfrak{H}]$ is called a generalized resolvent of A if there exist a Hilbert space $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ and a self-adjoint relation $\widetilde{A}\in\widetilde{\mathcal{C}}(\widetilde{\mathfrak{H}})$ such that $A\subset\widetilde{A}$ and the following equality holds:

$$R(\lambda)=P_{\mathfrak{H}}(\widetilde{A}-\lambda)^{-1}\upharpoonright\mathfrak{H},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}. \tag{2.6}$$

The relation $\widetilde{A}$ in (2.6) is called an exit space extension of A.

Definition 2.2. The generalized resolvent (2.6) is called canonical if $\widetilde{\mathfrak{H}}=\mathfrak{H}$, i.e., if $R(\lambda)=(\widetilde{A}-\lambda)^{-1}$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, is the resolvent of $\widetilde{A}=\widetilde{A}^*\in\widetilde{\mathcal{C}}(\mathfrak{H})$, $\widetilde{A}\supset A$.

Clearly, canonical resolvents exist if and only if $n_+(A)=n_-(A)$.

Next we recall definitions of boundary triplets, the corresponding Weyl functions, and γ-fields following [8, 21, 24, 26].

Assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$ and $\mathcal{H}_2:=\mathcal{H}_0\ominus\mathcal{H}_1$, so that $\mathcal{H}_0=\mathcal{H}_1\oplus\mathcal{H}_2$. Denote by P_j the orthoprojection in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j\in\{1,2\}$.

Definition 2.3. A collection $\Pi_+=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$, where $\Gamma_j:A^*\to\mathcal{H}_j$, $j\in\{0,1\}$, are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma:\widehat{f}\to\{\Gamma_0\widehat{f},\Gamma_1\widehat{f}\}$, $\widehat{f}\in A^*$, from A^* into $\mathcal{H}_0\oplus\mathcal{H}_1$ is surjective and the following Green's identity

$$(f',g)-(f,g')=(\Gamma_1\widehat{f},\Gamma_0\widehat{g})_{\mathcal{H}_0}-(\Gamma_0\widehat{f},\Gamma_1\widehat{g})_{\mathcal{H}_0}+i(P_2\Gamma_0\widehat{f},P_2\Gamma_0\widehat{g})_{\mathcal{H}_2}$$

holds for all $\widehat{f}=\{f,f'\}$, $\widehat{g}=\{g,g'\}\in A^*$.

Proposition 2.4. ([24]). *Let $\Pi_+=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*. Then*

(1) $\dim\mathcal{H}_1=n_-(A)\le n_+(A)=\dim\mathcal{H}_0$.
(2) $\ker\Gamma_0\cap\ker\Gamma_1=A$ *and Γ_j is a bounded operator from A^* onto $\mathcal{H}_j$.*
(3) *The equality $A_0:=\ker\Gamma_0=\{\widehat{f}\in A^*:\Gamma_0\widehat{f}=0\}$ defines a maximal symmetric extension A_0 of A such that $\mathbb{C}_+\subset\rho(A_0)$.*

Proposition 2.5. ([24]). *Let $\Pi_+=\{\mathcal{H}_0\oplus\mathcal{H}_1,\Gamma_0,\Gamma_1\}$ be a boundary triplet for A^*. Denote also by π_1 the orthoprojection in $\mathfrak{H}\oplus\mathfrak{H}$ onto $\mathfrak{H}\oplus\{0\}$. Then the operators $\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_\lambda(A)$, $\lambda\in\mathbb{C}_+$, and $P_1\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_z(A)$, $z\in\mathbb{C}_-$, isomorphically map $\widehat{\mathfrak{N}}_\lambda(A)$ onto $\mathcal{H}_0$ and $\widehat{\mathfrak{N}}_z(A)$ onto $\mathcal{H}_1$ respectively. Therefore the equalities*

$$\begin{aligned}\gamma_+(\lambda)&=\pi_1(\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_\lambda(A))^{-1},\quad \lambda\in\mathbb{C}_+,\\ \gamma_-(z)&=\pi_1(P_1\Gamma_0\upharpoonright\widehat{\mathfrak{N}}_z(A))^{-1},\quad z\in\mathbb{C}_-,\end{aligned} \tag{2.7}$$

$$M_+(\lambda)h_0=\Gamma_1\{\gamma_+(\lambda)h_0,\lambda\gamma_+(\lambda)h_0\},\quad h_0\in\mathcal{H}_0,\quad \lambda\in\mathbb{C}_+ \tag{2.8}$$

correctly define the operator functions $\gamma_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathfrak{H}]$, $\gamma_-(\cdot):\mathbb{C}_-\to[\mathcal{H}_1,\mathfrak{H}]$ and $M_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathcal{H}_1]$, which are holomorphic on their domains. Moreover,

$$M_+(\mu)-M_+^*(\lambda)P_1+iP_2=(\mu-\overline{\lambda})\gamma_+^*(\lambda)\gamma_+(\mu),\quad \mu,\lambda\in\mathbb{C}_+, \tag{2.9}$$

where $M_+(\mu)$ is considered as an operator in $\mathcal{H}_0$ (it is possible in view of the inclusion $\mathcal{H}_1\subset\mathcal{H}_0$).

It follows from (2.7) that

$$\Gamma_0\{\gamma_+(\lambda)h_0,\lambda\gamma_+(\lambda)h_0\}=h_0,\quad h_0\in\mathcal{H}_0. \tag{2.10}$$

Definition 2.6. ([24]). The operator functions $\gamma_\pm(\cdot)$ and $M_+(\cdot)$ defined in Proposition 2.5 are called the γ-fields and the Weyl function, respectively, corresponding to the boundary triplet Π_+.

Remark 2.7. (1) If $\mathcal{H}_0 = \mathcal{H}_1 := \mathcal{H}$, then the boundary triplet in the sense of Definition 2.3 turns into the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* in the sense of [15, 21]. In this case $n_+(A) = n_-(A)(= \dim \mathcal{H})$, $A_0(= \ker \Gamma_0)$ is a self-adjoint extension of A and the functions $\gamma_\pm(\cdot)$ and $M_+(\cdot)$ turn into the γ-field $\gamma(\cdot) : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and Weyl function $M(\cdot) : \rho(A_0) \to [\mathcal{H}]$ respectively introduced in [8, 21]. Moreover, in this case $M(\cdot)$ is a Nevanlinna operator function.

To avoid misleading with using other definitions, a boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in the sense of [15, 21] will be called an *ordinary boundary triplet* for A^*.

(2) Along with Π_+ we define in [26] a boundary triplet $\Pi_- = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^*. Such a triplet is applicable to symmetric relations A with $n_+(A) \leq n_-(A)$.

3. Decomposing boundary triplets for symmetric systems

3.1. **Notations.** Let $\mathcal{I} = [a, b\rangle$ $(-\infty < a < b \leq \infty)$ be an interval of the real line (the symbol $\rangle$ means that the endpoint $b < \infty$ might be either included to $\mathcal{I}$ or not). For a given finite-dimensional Hilbert space $\mathbb{H}$ denote by $AC(\mathcal{I}; \mathbb{H})$ the set of functions $f(\cdot) : \mathcal{I} \to \mathbb{H}$ which are absolutely continuous on each segment $[a, \beta] \subset \mathcal{I}$ and let $AC(\mathcal{I}) := AC(\mathcal{I}; \mathbb{C})$.

Next assume that $\Delta(\cdot)$ is an $[\mathbb{H}]$-valued Borel measurable functions on $\mathcal{I}$ integrable on each compact interval $[a, \beta] \subset \mathcal{I}$ and such that $\Delta(t) \geq 0$. Denote by $\mathcal{L}^2_\Delta(\mathcal{I})$ the semi-Hilbert space of Borel measurable functions $f(\cdot) : \mathcal{I} \to \mathbb{H}$ satisfying $||f||^2_\Delta := \int_{\mathcal{I}} (\Delta(t)f(t), f(t))_{\mathbb{H}}\, dt < \infty$ (see e.g. [12, Chapter 13.5]). The semi-definite inner product $(\cdot, \cdot)_\Delta$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ is defined by $(f, g)_\Delta = \int_{\mathcal{I}} (\Delta(t)f(t), g(t))_{\mathbb{H}}\, dt$, $f, g \in \mathcal{L}^2_\Delta(\mathcal{I})$. Moreover, let $L^2_\Delta(\mathcal{I})$ be the Hilbert space of the equivalence classes in $\mathcal{L}^2_\Delta(\mathcal{I})$ with respect to the semi-norm $|| \cdot ||_\Delta$ and let π be the quotient map from $\mathcal{L}^2_\Delta(\mathcal{I})$ onto $L^2_\Delta(\mathcal{I})$.

For a given finite-dimensional Hilbert space $\mathcal{K}$ denote by $\mathcal{L}^2_\Delta[\mathcal{K}, \mathbb{H}]$ the set of all Borel measurable operator-functions $F(\cdot) : \mathcal{I} \to [\mathcal{K}, \mathbb{H}]$ such that $F(t)h \in \mathcal{L}^2_\Delta(\mathcal{I})$ for each $h \in \mathcal{K}$ (this condition is equivalent to $\int_{\mathcal{I}} ||\Delta^{\frac{1}{2}}(t)F(t)||^2\, dt < \infty$). Moreover, we let $\mathcal{L}^2_\Delta[\mathbb{H}] := \mathcal{L}^2_\Delta[\mathbb{H}, \mathbb{H}]$.

3.2. **Symmetric systems.** In this subsection we provide some known results on symmetric systems of differential equations following [13, 17, 20, 27].

Let H and $\widehat{H}$ be finite-dimensional Hilbert spaces and let

$$H_0 = H \oplus \widehat{H}, \quad \mathbb{H} = H_0 \oplus H = H \oplus \widehat{H} \oplus H. \tag{3.1}$$

In the following we denote by $\mathcal{P}_0$, $\widehat{\mathcal{P}}$ and $\mathcal{P}_1$ the orthoprojections in $\mathbb{H}$ onto the first, second and third component in the decomposition $\mathbb{H} = H \oplus \widehat{H} \oplus H$ respectively.

Let as above $\mathcal{I} = [a, b\rangle$ $(-\infty < a < b \leq \infty)$ be an interval in $\mathbb{R}$. Moreover, let $B(\cdot)$ and $\Delta(\cdot)$ be $[\mathbb{H}]$-valued Borel measurable functions on $\mathcal{I}$ integrable on each compact interval $[a, \beta] \subset \mathcal{I}$ and satisfying $B(t) = B^*(t)$ and $\Delta(t) \geq 0$ a.e. on $\mathcal{I}$ and let $J \in [\mathbb{H}]$ be operator (1.2).

A first-order symmetric system on an interval $\mathcal{I}$ (with the regular endpoint a) is a system of differential equations of the form

$$Jy' - B(t)y = \Delta(t)f(t), \quad t \in \mathcal{I}, \tag{3.2}$$

where $f(\cdot) \in \mathcal{L}^2_\Delta(\mathcal{I})$. Together with (3.2) we consider also the homogeneous system

$$Jy'(t) - B(t)y(t) = \lambda\Delta(t)y(t), \quad t \in \mathcal{I}, \quad \lambda \in \mathbb{C}. \tag{3.3}$$

A function $y \in AC(\mathcal{I}; \mathbb{H})$ is a solution of (3.2) (resp. (3.3)) if equality (3.2) (resp. (3.3) holds a.e. on $\mathcal{I}$. A function $Y(\cdot, \lambda) : \mathcal{I} \to [\mathcal{K}, \mathbb{H}]$ is an operator solution of equation (3.3) if $y(t) = Y(t, \lambda)h$ is a (vector) solution of this equation for every $h \in \mathcal{K}$ (here $\mathcal{K}$ is a Hilbert space with $\dim \mathcal{K} < \infty$).

As is known there exists a unique $[\mathbb{H}]$-valued operator solution $Y_0(\cdot,\lambda)$ of Eq. (3.3) satisfying $Y_0(a,\lambda) = I_{\mathbb{H}}$. Moreover, each operator solution $Y(\cdot,\lambda)$ of Eq. (3.3) admits the representation

$$Y(t,\lambda) = Y_0(t,\lambda)Y(a,\lambda), \quad t \in \mathcal{I}. \tag{3.4}$$

In what follows we always assume that system (3.2) is definite in the sense of the following definition.

Definition 3.1. ([13]). Symmetric system (3.2) is called definite if for each $\lambda \in \mathbb{C}$ and each solution y of (3.3) the equality $\Delta(t)y(t) = 0$ (a.e. on $\mathcal{I}$) implies $y(t) = 0,\ t \in \mathcal{I}$.

As it is known [27, 17, 20] definite system (3.2) gives rise to the *maximal linear relations* $\mathcal{T}_{\max}$ and $T_{\max}$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ and $L^2_\Delta(\mathcal{I})$, respectively. They are given by

$$\begin{aligned}\mathcal{T}_{\max} = \{\{y,f\} \in (\mathcal{L}^2_\Delta(\mathcal{I}))^2 : y \in AC(\mathcal{I};\mathbb{H}) \ \text{and} \\ Jy'(t) - B(t)y(t) = \Delta(t)f(t) \ \text{a.e. on } \mathcal{I}\}\end{aligned} \tag{3.5}$$

and $T_{\max} = \{\{\pi y, \pi f\} : \{y,f\} \in \mathcal{T}_{\max}\}$. Moreover the Lagrange's identity

$$(f,z)_\Delta - (y,g)_\Delta = [y,z]_b - (Jy(a), z(a)), \quad \{y,f\},\ \{z,g\} \in \mathcal{T}_{\max} \tag{3.6}$$

holds with

$$[y,z]_b := \lim_{t\uparrow b}(Jy(t), z(t)), \quad y,z \in \operatorname{dom}\mathcal{T}_{\max}. \tag{3.7}$$

Formula (3.7) defines the skew-Hermitian bilinear form $[\cdot,\cdot]_b$ on $\operatorname{dom}\mathcal{T}_{\max}$. By using this form one defines the *minimal relations* $\mathcal{T}_{\min}$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ and $T_{\min}$ in $L^2_\Delta(\mathcal{I})$ via

$$\mathcal{T}_{\min} = \{\{y,f\} \in \mathcal{T}_{\max} : y(a) = 0 \ \text{and} \ [y,z]_b = 0 \ \text{for each} \ z \in \operatorname{dom}\mathcal{T}_{\max}\}$$

and $T_{\min} = \{\{\pi y, \pi f\} : \{y,f\} \in \mathcal{T}_{\min}\}$. According to [27, 17, 20] $T_{\min}$ is a closed symmetric linear relation in $L^2_\Delta(\mathcal{I})$ and $T^*_{\min} = T_{\max}$.

Remark 3.2. It is known (see e.g. [20]) that the maximal relation $T_{\max}$ induced by the definite symmetric system (3.2) possesses the following property: for any $\{\widetilde{y}, \widetilde{f}\} \in T_{\max}$ there exists a unique function $y \in AC(\mathcal{I};\mathbb{H}) \cap \mathcal{L}^2_\Delta(\mathcal{I})$ such that $y \in \widetilde{y}$ and $\{y,f\} \in \mathcal{T}_{\max}$ for any $f \in \widetilde{f}$. Below we associate such a function $y \in AC(\mathcal{I};\mathbb{H}) \cap \mathcal{L}^2_\Delta(\mathcal{I})$ with each pair $\{\widetilde{y}, \widetilde{f}\} \in T_{\max}$.

Denote by $\mathcal{N}_\lambda,\ \lambda \in \mathbb{C}$, the linear space of solutions of the homogeneous system (3.3) belonging to $\mathcal{L}^2_\Delta(\mathcal{I})$. Definition (3.5) of $\mathcal{T}_{\max}$ implies

$$\mathcal{N}_\lambda = \ker(\mathcal{T}_{\max} - \lambda) = \{y \in \mathcal{L}^2_\Delta(\mathcal{I}) : \ \{y, \lambda y\} \in \mathcal{T}_{\max}\}, \quad \lambda \in \mathbb{C},$$

and hence $\mathcal{N}_\lambda \subset \operatorname{dom}\mathcal{T}_{\max}$. As usual, denote by $n_\pm(T_{\min}) := \dim \mathfrak{N}_\lambda(T_{\min}), \quad \lambda \in \mathbb{C}_\pm$, the deficiency indices of $T_{\min}$. Since the system (3.2) is definite, $\pi\mathcal{N}_\lambda = \mathfrak{N}_\lambda(T_{\min})$ and $\ker(\pi \upharpoonright \mathcal{N}_\lambda) = \{0\},\ \ \lambda \in \mathbb{C}$. This implies that $n_\pm(T_{\min}) = \dim \mathcal{N}_\lambda \le \dim \mathbb{H},\ \lambda \in \mathbb{C}_\pm$.

The following lemma will be useful in the sequel.

Lemma 3.3. ([1]). *For each operator solution $Y(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathcal{K},\mathbb{H}]$ of Eq.* (3.3) *the relation*

$$\mathcal{K} \ni h \to (Y(\lambda)h)(t) = Y(t,\lambda)h \in \mathcal{N}_\lambda \tag{3.8}$$

defines the linear mapping $Y(\lambda) : \mathcal{K} \to \mathcal{N}_\lambda$. Moreover, if $F(\lambda) := \pi Y(\lambda) (\in [\mathcal{K}, L^2_\Delta(\mathcal{I})])$, then

$$F^*(\lambda)\widetilde{f} = \int_{\mathcal{I}} Y^*(t,\lambda)\Delta(t)f(t)\,dt, \quad \widetilde{f} \in L^2_\Delta(\mathcal{I}), \quad f \in \widetilde{f}. \tag{3.9}$$

3.3. **Decomposing boundary triplets.** We start this subsection with the following lemma.

Lemma 3.4. ([1]). *If $n_-(T_{\min}) \le n_+(T_{\min})$, then there exist a finite dimensional Hilbert space $\widetilde{\mathcal{H}}_b$, a subspace $\mathcal{H}_b \subset \widetilde{\mathcal{H}}_b$ and a surjective linear mapping*

$$\Gamma_b = \begin{pmatrix} \Gamma_{0b} \\ \widehat{\Gamma}_b \\ \Gamma_{1b} \end{pmatrix} : \operatorname{dom} \mathcal{T}_{\max} \to \widetilde{\mathcal{H}}_b \oplus \widehat{H} \oplus \mathcal{H}_b \tag{3.10}$$

such that for all $y, z \in \operatorname{dom} \mathcal{T}_{\max}$ the following identity is valid:

$$[y,z]_b = (\Gamma_{0b}y, \Gamma_{1b}z)_{\widetilde{\mathcal{H}}_b} - (\Gamma_{1b}y, \Gamma_{0b}z)_{\widetilde{\mathcal{H}}_b} + i(P_{\mathcal{H}_b^\perp}\Gamma_{0b}y, P_{\mathcal{H}_b^\perp}\Gamma_{0b}z)_{\widetilde{\mathcal{H}}_b} + i(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z)_{\widehat{H}} \tag{3.11}$$

(here $\mathcal{H}_b^\perp = \widetilde{\mathcal{H}}_b \ominus \mathcal{H}_b$). Moreover, in the case $n_+(T_{\min}) = n_-(T_{\min})$ (and only in this case) one has $\widetilde{\mathcal{H}}_b = \mathcal{H}_b$ and the identity (3.11) *takes the form*

$$[y,z]_b = (\Gamma_{0b}y, \Gamma_{1b}z)_{\mathcal{H}_b} - (\Gamma_{1b}y, \Gamma_{0b}z)_{\mathcal{H}_b} + i(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z)_{\widehat{H}}.$$

The following proposition is immediate from [1, Proposition 3.6].

Proposition 3.5. *Assume that $n_-(T_{\min}) \le n_+(T_{\min})$. Moreover, let*

$$y(t) = \{y_0(t),\ \widehat{y}(t),\ y_1(t)\} (\in \underbrace{H \oplus \widehat{H} \oplus H}_{\mathbb{H}}), \quad t \in \mathcal{I}, \tag{3.12}$$

be the representation of a function $y \in \operatorname{dom} \mathcal{T}_{\max}$ in accordance with the decomposition (3.1) *of $\mathbb{H}$ and let Γ_b be the surjective linear mapping* (3.10) *satisfying the identity* (3.11). *Assume also that $\mathcal{H}_0$ and $\mathcal{H}_1 (\subset \mathcal{H}_0)$ are finite dimensional Hilbert spaces defined by*

$$\mathcal{H}_0 = \underbrace{H \oplus \widehat{H}}_{H_0} \oplus \widetilde{\mathcal{H}}_b = H_0 \oplus \widetilde{\mathcal{H}}_b, \quad \mathcal{H}_1 = \underbrace{H \oplus \widehat{H}}_{H_0} \oplus \mathcal{H}_b = H_0 \oplus \mathcal{H}_b \tag{3.13}$$

and $\Gamma_j : T_{\max} \to \mathcal{H}_j$, $j \in \{0,1\}$, are the operators given by

$$\Gamma_0\{\widetilde{y}, \widetilde{f}\} = \{-y_1(a),\ i(\widehat{y}(a) - \widehat{\Gamma}_b y),\ \Gamma_{0b}y\} (\in H \oplus \widehat{H} \oplus \widetilde{\mathcal{H}}_b), \tag{3.14}$$

$$\Gamma_1\{\widetilde{y}, \widetilde{f}\} = \{y_0(a),\ \tfrac{1}{2}(\widehat{y}(a) + \widehat{\Gamma}_b y),\ -\Gamma_{1b}y\} (\in H \oplus \widehat{H} \oplus \mathcal{H}_b), \quad \{\widetilde{y}, \widetilde{f}\} \in T_{\max} \tag{3.15}$$

(here $y_0(a), \widehat{y}(a)$ and $y_1(a)$ are taken from the representation (3.12) *of a function $y \in \operatorname{dom} \mathcal{T}_{\max}$, which corresponds to $\{\widetilde{y}, \widetilde{f}\} \in T_{\max}$ according to Remark 3.2). Then the collection $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $T_{\max}$ and the (maximal symmetric) relation $A_0 (= \ker \Gamma_0)$ is*

$$A_0 = \{\{\widetilde{y}, \widetilde{f}\} \in T_{\max} : y_1(a) = 0,\ \widehat{y}(a) = \widehat{\Gamma}_b y,\ \Gamma_{0b}y = 0\}. \tag{3.16}$$

If in addition $n_+(T_{\min}) = n_-(T_{\min})$, then Π_+ turns into an ordinary boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $T_{\max}$, with $\mathcal{H} = H_0 \oplus \mathcal{H}_b$ and the mappings $\Gamma_0, \Gamma_1 : T_{\max} \to \mathcal{H}$, given by (3.14) *and* (3.15) *with $\widetilde{\mathcal{H}}_b = \mathcal{H}_b$. Moreover, in this case $A_0 = A_0^*$.*

Definition 3.6. The boundary triplet $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ constructed in Proposition 3.5 is called a decomposing boundary triplet for $T_{\max}$.

In the sequel we suppose (unless otherwise stated) that the following assumptions are fulfilled:

- (A1) The system (3.2) satisfies $n_-(T_{\min}) \le n_+(T_{\min})$;
- (A2) $\widetilde{\mathcal{H}}_b$ and $\mathcal{H}_b (\subset \widetilde{\mathcal{H}}_b)$ are finite dimensional Hilbert spaces and Γ_b is a surjective linear mapping (3.10) such that (3.11) holds.

The following two propositions directly follows from [1, Propositions 4.4 and 4.5].

Proposition 3.7. (1) *For every* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *there exists a unique operator solution* $v_0(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H_0,\mathbb{H}]$ *of Eq.* (3.3) *such that*

$$\begin{gathered}\mathcal{P}_1 v_0(a,\lambda) = -P_{H_0,H}, \quad i(\widehat{\mathcal{P}} v_0(a,\lambda) - \widehat{\Gamma}_b v_0(\lambda)) = P_{H_0,\widehat{H}}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R};\\ \Gamma_{0b} v_0(\lambda) = 0, \quad \lambda \in \mathbb{C}_+; \quad P_{\mathcal{H}_b}\Gamma_{0b} v_0(\lambda) = 0, \quad \lambda \in \mathbb{C}_-.\end{gathered} \tag{3.17}$$

(2) *For every* $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$) *there exists a unique operator solution* $u_+(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\widetilde{\mathcal{H}}_b,\mathbb{H}]$ *(resp.* $u_-(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_b,\mathbb{H}]$*) of Eq.* (3.3) *such that*

$$\begin{gathered}\mathcal{P}_1 u_\pm(a,\lambda) = 0, \quad i(\widehat{\mathcal{P}} u_\pm(a,\lambda) - \widehat{\Gamma}_b u_\pm(\lambda)) = 0, \quad \lambda \in \mathbb{C}_\pm;\\ \Gamma_{0b} u_+(\lambda) = I_{\widetilde{\mathcal{H}}_b}, \quad \lambda \in \mathbb{C}_+; \quad P_{\mathcal{H}_b}\Gamma_{0b} u_-(\lambda) = I_{\mathcal{H}_b}, \quad \lambda \in \mathbb{C}_-.\end{gathered} \tag{3.18}$$

Here $v_0(\lambda)$ *and* $u_\pm(\lambda)$ *denote linear mappings from Lemma 3.3 for the solutions* $v_0(\cdot,\lambda)$ *and* $u_\pm(\cdot,\lambda)$*, respectively.*

Proposition 3.8. *Let* $v_0(\cdot,\lambda)$ *and* $u_\pm(\cdot,\lambda)$ *be the operator solutions from Proposition 3.7, let* $Z_+(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_0,\mathbb{H}]$ *and* $Z_-(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_1,\mathbb{H}]$ *be the operator solutions of Eq.* (3.3) *given by*

$$Z_+(t,\lambda) = (v_0(t,\lambda),\, u_+(t,\lambda)) : H_0 \oplus \widetilde{\mathcal{H}}_b \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+; \tag{3.19}$$

$$Z_-(t,\lambda) = (v_0(t,\lambda),\, u_-(t,\lambda)) : H_0 \oplus \mathcal{H}_b \to \mathbb{H}, \quad \lambda \in \mathbb{C}_- \tag{3.20}$$

and let $Z_\pm(\lambda)$ *be the linear mappings from Lemma 3.3 for the solutions* $Z_\pm(\cdot,\lambda)$*. Moreover, let* $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a decomposing boundary triplet* (3.13)–(3.15) *for* $T_{\max}$*. Then the* γ*-fields* $\gamma_\pm(\cdot)$ *of the triplet* Π_+ *are*

$$\gamma_+(\lambda) = \pi Z_+(\lambda), \quad \lambda \in \mathbb{C}_+; \quad \gamma_-(\lambda) = \pi Z_-(\lambda), \quad \lambda \in \mathbb{C}_- \tag{3.21}$$

and the corresponding Weyl function $M_+(\cdot)$ *admits the block-matrix representation*

$$M_+(\lambda) = \begin{pmatrix} m_0(\lambda) & M_{2+}(\lambda)\\ M_{3+}(\lambda) & M_{4+}(\lambda)\end{pmatrix} : \underbrace{H_0 \oplus \widetilde{\mathcal{H}}_b}_{\mathcal{H}_0} \to \underbrace{H_0 \oplus \mathcal{H}_b}_{\mathcal{H}_1}, \quad \lambda \in \mathbb{C}_+ \tag{3.22}$$

with the entries defined by

$$m_0(\lambda) = (\mathcal{P}_0 + \widehat{\mathcal{P}}) v_0(a,\lambda) + \tfrac{i}{2} P_{\widehat{H}}, \quad M_{2+}(\lambda) = (\mathcal{P}_0 + \widehat{\mathcal{P}}) u_+(a,\lambda), \tag{3.23}$$

$$M_{3+}(\lambda) = -\Gamma_{1b} v_0(\lambda), \quad M_{4+}(\lambda) = -\Gamma_{1b} u_+(\lambda). \tag{3.24}$$

Moreover, for each $\lambda \in \mathbb{C}_-$ *the following equalities hold:*

$$m_0^*(\overline{\lambda}) = (\mathcal{P}_0 + \widehat{\mathcal{P}}) v_0(a,\lambda) + \tfrac{i}{2} P_{\widehat{H}}, \quad M_{3+}^*(\overline{\lambda}) = (\mathcal{P}_0 + \widehat{\mathcal{P}}) u_-(a,\lambda). \tag{3.25}$$

In (3.23) *and* (3.25) $P_{\widehat{H}} \in [H_0]$ *is the orthoprojection in* H_0 *onto* $\widehat{H}$*.*

Corollary 3.9. ([1]). *Let* $n_+(T_{\min}) = n_-(T_{\min})$ *and let* $A_0 = A_0^*$ *be the extension* (3.16) *of* $T_{\min}$*. Then for every* $\lambda \in \rho(A_0)$ *there exists a unique pair of operator-valued solutions* $v_0(\cdot,\lambda) \in \mathcal{L}^2_\Delta[H_0,\mathbb{H}]$ *and* $u(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_b,\mathbb{H}]$ *of Eq.* (3.3) *satisfying*

$$\mathcal{P}_1 v_0(a,\lambda) = -P_{H_0,H}, \quad i(\widehat{\mathcal{P}} v_0(a,\lambda) - \widehat{\Gamma}_b v_0(\lambda)) = P_{H_0,\widehat{H}}, \quad \Gamma_{0b} v_0(\lambda) = 0, \ \ \lambda \in \rho(A_0);$$

$$\mathcal{P}_1 u(a,\lambda) = 0, \quad i(\widehat{\mathcal{P}} u(a,\lambda) - \widehat{\Gamma}_b u(\lambda)) = 0, \quad \Gamma_{0b} u(\lambda) = I_{\mathcal{H}_b}, \ \ \lambda \in \rho(A_0).$$

Assume, in addition, that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is an ordinary decomposing boundary triplet* (3.13)–(3.15) *for* $T_{\max}$*. Then the corresponding Weyl function* $M(\cdot)$ *is*

$$M(\lambda) = \begin{pmatrix} m_0(\lambda) & M_2(\lambda)\\ M_3(\lambda) & M_4(\lambda)\end{pmatrix} : H_0 \oplus \mathcal{H}_b \to H_0 \oplus \mathcal{H}_b, \quad \lambda \in \rho(A_0); \tag{3.26}$$

$$m_0(\lambda) = (\mathcal{P}_0 + \widehat{\mathcal{P}}) v_0(a,\lambda) + \tfrac{i}{2} P_{\widehat{H}}, \quad M_2(\lambda) = (\mathcal{P}_0 + \widehat{\mathcal{P}}) u(a,\lambda), \tag{3.27}$$

$$M_3(\lambda) = -\Gamma_{1b} v_0(\lambda), \quad M_4(\lambda) = -\Gamma_{1b} u(\lambda), \quad \lambda \in \rho(A_0). \tag{3.28}$$

4. Generalized resolvents and characteristic matrices of symmetric systems

4.1. Generalized resolvents.

Definition 4.1. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be finite dimensional Hilbert spaces (3.13). Then a boundary parameter τ is a collection $\tau=\{\tau_+,\tau_-\}\in\widetilde{R}_+(\mathcal{H}_0,\mathcal{H}_1)$ of the form (2.2).

In the case of equal deficiency indices $n_+(T_{\min})=n_-(T_{\min})$ one has $\widetilde{\mathcal{H}}_b=\mathcal{H}_b$, $\mathcal{H}_0=\mathcal{H}_1=:\mathcal{H}$ and a boundary parameter is an operator pair $\tau\in\widetilde{R}(\mathcal{H})$ defined by (2.4). If in addition $\tau\in\widetilde{R}^0(\mathcal{H})$, then a boundary parameter will be called self-adjoint. Such a boundary parameter τ admits the representation as a self-adjoint operator pair (2.5).

For each boundary parameter $\tau=\{\tau_+,\tau_-\}$ of the form (2.2) we assume that

$$C_0(\lambda)=(C_{0a}(\lambda),\ \widehat{C}_0(\lambda),\ C_{0b}(\lambda)):H\oplus\widehat{H}\oplus\widetilde{\mathcal{H}}_b\to\mathcal{H}_0,\quad\lambda\in\mathbb{C}_+,\tag{4.1}$$

$$C_1(\lambda)=(C_{1a}(\lambda),\ \widehat{C}_1(\lambda),\ C_{1b}(\lambda)):H\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H}_0,\quad\lambda\in\mathbb{C}_+,\tag{4.2}$$

$$D_0(\lambda)=(D_{0a}(\lambda),\ \widehat{D}_0(\lambda),\ D_{0b}(\lambda)):H\oplus\widehat{H}\oplus\widetilde{\mathcal{H}}_b\to\mathcal{H}_1,\quad\lambda\in\mathbb{C}_-,\tag{4.3}$$

$$D_1(\lambda)=(D_{1a}(\lambda),\ \widehat{D}_1(\lambda),\ D_{1b}(\lambda)):H\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H}_1,\quad\lambda\in\mathbb{C}_-\tag{4.4}$$

are the block matrix representations of $C_j(\lambda)$ and $D_j(\lambda)$, $j\in\{0,1\}$.

If $n_+(T_{\min})=n_-(T_{\min})$, then for each boundary parameter (2.4) we assume that

$$C_0(\lambda)=(C_{0a}(\lambda),\ \widehat{C}_0(\lambda),\ C_{0b}(\lambda)):H\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H},\quad\lambda\in\mathbb{C}\setminus\mathbb{R},\tag{4.5}$$

$$C_1(\lambda)=(C_{1a}(\lambda),\ \widehat{C}_1(\lambda),\ C_{1b}(\lambda)):H\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H},\quad\lambda\in\mathbb{C}\setminus\mathbb{R}\tag{4.6}$$

are the block matrix representations of $C_0(\lambda)$ and $C_1(\lambda)$.

In the case of a self-adjoint boundary parameter (2.5) the equalities (4.5) and (4.6) take the form

$$C_0=(C_{0a},\ \widehat{C}_0,\ C_{0b}):H\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H},\quad C_1=(C_{1a},\ \widehat{C}_1,\ C_{1b}):H\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H}.\tag{4.7}$$

Lemma 4.2. *Let $\widetilde{\mathcal{H}}_b$ be decomposed as $\widetilde{\mathcal{H}}_b=\mathcal{H}_b\oplus\mathcal{H}_b^\perp$ and let*

$$\mathbb{H}_b:=\widetilde{\mathcal{H}}_b\oplus\widehat{H}\oplus\mathcal{H}_b=\mathcal{H}_b\oplus(\mathcal{H}_b^\perp\oplus\widehat{H})\oplus\mathcal{H}_b,$$

$$J_b=\begin{pmatrix}0&0&-I_{\mathcal{H}_b}\\0&iI_{\mathcal{H}_b^\perp\oplus\widehat{H}}&0\\I_{\mathcal{H}_b}&0&0\end{pmatrix}:\underbrace{\mathcal{H}_b\oplus(\mathcal{H}_b^\perp\oplus\widehat{H})\oplus\mathcal{H}_b}_{\mathbb{H}_b}\to\underbrace{\mathcal{H}_b\oplus(\mathcal{H}_b^\perp\oplus\widehat{H})\oplus\mathcal{H}_b}_{\mathbb{H}_b}.\tag{4.8}$$

Then the equalities

$$C_a(\lambda)=(-C_{1a}(\lambda),\ i\widehat{C}_0(\lambda)-\tfrac12\widehat{C}_1(\lambda),\ -C_{0a}(\lambda)):H\oplus\widehat{H}\oplus H\to\mathcal{H}_0,\quad\lambda\in\mathbb{C}_+,\tag{4.9}$$

$$C_b(\lambda)=(C_{0b}(\lambda),\ -i\widehat{C}_0(\lambda)-\tfrac12\widehat{C}_1(\lambda),\ C_{1b}(\lambda)):\widetilde{\mathcal{H}}_b\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H}_0,\quad\lambda\in\mathbb{C}_+,\tag{4.10}$$

$$D_a(\lambda)=(-D_{1a}(\lambda),\ i\widehat{D}_0(\lambda)-\tfrac12\widehat{D}_1(\lambda),\ -D_{0a}(\lambda)):H\oplus\widehat{H}\oplus H\to\mathcal{H}_1,\quad\lambda\in\mathbb{C}_-,\tag{4.11}$$

$$D_b(\lambda)=(D_{0b}(\lambda),\ -i\widehat{D}_0(\lambda)-\tfrac12\widehat{D}_1(\lambda),\ D_{1b}(\lambda)):\widetilde{\mathcal{H}}_b\oplus\widehat{H}\oplus\mathcal{H}_b\to\mathcal{H}_1,\quad\lambda\in\mathbb{C}_-\tag{4.12}$$

establish a bijective correspondence between all boundary parameters $\tau=\{\tau_+,\tau_-\}$ of the form (2.2) *and* (4.1)–(4.4) *and all collections of holomorphic operator functions*

$$(C_a(\lambda),\ C_b(\lambda)):\mathbb{H}\oplus\mathbb{H}_b\to\mathcal{H}_0,\ \lambda\in\mathbb{C}_+;\quad(D_a(\lambda),\ D_b(\lambda)):\mathbb{H}\oplus\mathbb{H}_b\to\mathcal{H}_1,\ \lambda\in\mathbb{C}_-\tag{4.13}$$

satisfying

$$\operatorname{ran}(C_a(\lambda),\ C_b(\lambda))=\mathcal{H}_0,\ \lambda\in\mathbb{C}_+;\quad\operatorname{ran}(D_a(\lambda),\ D_b(\lambda))=\mathcal{H}_1,\ \lambda\in\mathbb{C}_-;\tag{4.14}$$

$$i(C_a(\lambda)JC_a^*(\lambda)-C_b(\lambda)J_bC_b^*(\lambda))\ge0,\quad i(D_a(\lambda)JD_a^*(\lambda)-D_b(\lambda)J_bD_b^*(\lambda))\le0,\tag{4.15}$$

$$C_a(\lambda)JD_a^*(\overline{\lambda})=C_b(\lambda)J_bD_b^*(\overline{\lambda}),\ \lambda\in\mathbb{C}_+.\tag{4.16}$$

If in addition $n_+(T_{\min}) = n_-(T_{\min})$, *then* $\mathcal{H}_b^\perp = \{0\}$, $\mathbb{H}_b = \mathcal{H}_b \oplus \widehat{H} \oplus \mathcal{H}_b$, J_b *takes the form* (1.10) *and the equalities*

$$C_a(\lambda)=(-C_{1a}(\lambda),\ i\widehat{C}_0(\lambda)-\tfrac{1}{2}\widehat{C}_1(\lambda),\ -C_{0a}(\lambda)) : H \oplus \widehat{H} \oplus H \to \mathcal{H}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{4.17}$$

$$C_b(\lambda)=(C_{0b}(\lambda),\ -i\widehat{C}_0(\lambda)-\tfrac{1}{2}\widehat{C}_1(\lambda),\ C_{1b}(\lambda)) : \mathcal{H}_b \oplus \widehat{H} \oplus \mathcal{H}_b \to \mathcal{H}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R} \tag{4.18}$$

establish a bijective correspondence between all boundary parameters τ *of the form* (2.4) *and* (4.5), (4.6) *and all holomorphic operator functions*

$$(C_a(\lambda),\ C_b(\lambda)) : \mathbb{H} \oplus \mathbb{H}_b \to \mathcal{H},\ \ \lambda \in \mathbb{C}\setminus\mathbb{R} \tag{4.19}$$

satisfying for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ *the relations* (1.11) *and* (1.12). *Moreover, in the case* $n_+(T_{\min}) = n_-(T_{\min})$ *the equalities*

$$C_a = (-C_{1a},\ i\widehat{C}_0 - \tfrac{1}{2}\widehat{C}_1,\ -C_{0a}) : H \oplus \widehat{H} \oplus H \to \mathcal{H}, \tag{4.20}$$

$$C_b = (C_{0b},\ -i\widehat{C}_0 - \tfrac{1}{2}\widehat{C}_1,\ C_{1b}) : \mathcal{H}_b \oplus \widehat{H} \oplus \mathcal{H}_b \to \mathcal{H} \tag{4.21}$$

give a bijective correspondence between all self-adjoint boundary parameters τ *of the form* (2.5) *and* (4.7) *and all operators* $(C_a, C_b) : \mathbb{H} \oplus \mathbb{H}_b \to \mathcal{H}$ *satisfying* (1.15).

Proof. It follow from (3.13) that $\mathcal{H}_2(= \mathcal{H}_0 \ominus \mathcal{H}_1) = \mathcal{H}_b^\perp$. Therefore by (4.1) and (4.3)

$$\begin{aligned}
C_{01}(\lambda) &:= C_0(\lambda) \restriction \mathcal{H}_1 = (C_{0a}(\lambda),\ \widehat{C}_0(\lambda),\ C_{0b}(\lambda) \restriction \mathcal{H}_b),\\
C_{02}(\lambda) &:= C_0(\lambda) \restriction \mathcal{H}_2 = C_{0b}(\lambda) \restriction \mathcal{H}_b^\perp,\\
D_{01}(\lambda) &:= D_0(\lambda) \restriction \mathcal{H}_1 = (D_{0a}(\lambda),\ \widehat{D}_0(\lambda),\ D_{0b}(\lambda) \restriction \mathcal{H}_b),\\
D_{02}(\lambda) &:= D_0(\lambda) \restriction \mathcal{H}_2 = D_{0b}(\lambda) \restriction \mathcal{H}_b^\perp
\end{aligned}$$

and the immediate calculations give

$$\begin{aligned}
2\mathrm{Im}(C_1(\lambda)C_{01}^*(\lambda)) + C_{02}(\lambda)C_{02}^*(\lambda) &= i(C_a(\lambda)JC_a^*(\lambda) - C_b(\lambda)J_bC_b^*(\lambda)),\ \ \lambda \in \mathbb{C}_+,\\
2\mathrm{Im}(D_1(\lambda)D_{01}^*(\lambda)) + D_{02}(\lambda)D_{02}^*(\lambda) &= i(D_a(\lambda)JD_a^*(\lambda) - D_b(\lambda)J_bD_b^*(\lambda)),\ \ \lambda \in \mathbb{C}_-,
\end{aligned}$$

$$C_1(\lambda)D_{01}^*(\overline{\lambda}) - C_{01}(\lambda)D_1^*(\overline{\lambda}) + iC_{02}(\lambda)D_{02}^*(\overline{\lambda}) = C_b(\lambda)J_bD_b^*(\overline{\lambda}) - C_a(\lambda)JD_a^*(\overline{\lambda}),\ \ \lambda \in \mathbb{C}_+.$$

Moreover, the following equivalences are obvious:

$$\begin{aligned}
\mathrm{ran}\,(C_0(\lambda),\ C_1(\lambda)) = \mathcal{H}_0 &\iff \mathrm{ran}\,(C_a(\lambda),\ C_b(\lambda)) = \mathcal{H}_0,\\
\mathrm{ran}\,(D_0(\lambda),\ D_1(\lambda)) = \mathcal{H}_1 &\iff \mathrm{ran}\,(D_a(\lambda),\ D_b(\lambda)) = \mathcal{H}_1.
\end{aligned}$$

This and [1, Proposition 2.5] yield the desired statements. □

Let $\tau = \{\tau_+, \tau_-\}$ be a boundary parameter defined by (2.2) and (4.1)–(4.4) and let $C_a(\lambda)$, $C_b(\lambda)$ and $D_a(\lambda)$, $D_b(\lambda)$ be the operator-functions (4.9)–(4.12) (hence the relations (4.14)–(4.16) hold). For a given function $f \in \mathcal{L}^2_\Delta(\mathcal{I})$ consider the boundary problem

$$Jy' - B(t)y = \lambda\Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I}; \tag{4.22}$$

$$C_a(\lambda)y(a) + C_b(\lambda)\Gamma_b y = 0,\ \ \lambda \in \mathbb{C}_+; \quad D_a(\lambda)y(a) + D_b(\lambda)\Gamma_b y = 0,\ \ \lambda \in \mathbb{C}_-. \tag{4.23}$$

A function $y(\cdot,\cdot) : \mathcal{I} \times (\mathbb{C}\setminus\mathbb{R}) \to \mathbb{H}$ is called a solution of this problem if for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the function $y(\cdot,\lambda)$ belongs to $AC(\mathcal{I};\mathbb{H}) \cap \mathcal{L}^2_\Delta(\mathcal{I})$ and satisfies the equation (4.22) a.e. on $\mathcal{I}$ (so that $y \in \mathrm{dom}\,\mathcal{T}_{\max}$) and the boundary conditions (4.23).

If $n_+(T_{\min}) = n_-(T_{\min})$ and τ is a boundary parameter defined by (2.4) and (4.5), (4.6), then the boundary conditions (4.23) take the form

$$C_a(\lambda)y(a) + C_b(\lambda)\Gamma_b y = 0, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{4.24}$$

with $C_a(\lambda)$ and $C_b(\lambda)$ given by (4.17) and (4.18). Moreover, if τ is a self-adjoint boundary parameter (2.5), (4.7), then (4.24) becomes a self-adjoint boundary condition

$$C_a y(a) + C_b \Gamma_b y = 0, \tag{4.25}$$

where C_a and C_b are the operators (4.20) and (4.21) (hence they satisfy (1.15)).

In the following theorem we describe all generalized resolvents (and, consequently, all exit space self-adjoint extensions) of $T_{\min}$ in terms of λ-depending boundary conditions.

Theorem 4.3. *Let $\tau = \{\tau_+, \tau_-\}$ be a boundary parameter defined by* (2.2) *and* (4.1)–(4.4) *and let $C_a(\lambda)$, $C_b(\lambda)$ and $D_a(\lambda)$, $D_b(\lambda)$ be given by* (4.9)–(4.12). *Then for every $f \in \mathcal{L}^2_\Delta(\mathcal{I})$ the boundary problem* (4.22), (4.23) *has a unique solution $y(t,\lambda) = y_f(t,\lambda)$ and the equality*

$$R(\lambda)\widetilde{f} = \pi(y_f(\cdot,\lambda)), \quad \widetilde{f} \in L^2_\Delta(\mathcal{I}), \quad f \in \widetilde{f}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}$$

defines a generalized resolvent $R(\lambda) =: R_\tau(\lambda)$ of $T_{\min}$. Conversely, for each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists a unique boundary parameter τ such that $R(\lambda) = R_\tau(\lambda)$.

If in addition $n_+(T_{\min}) = n_-(T_{\min})$, then the above statements hold with the boundary parameter τ of the form (2.4), (4.5), (4.6) *and the boundary condition* (4.24) *in place of* (4.23). *Moreover, $R_\tau(\lambda)$ is a canonical resolvent of $T_{\min}$ if and only if τ is a self-adjoint boundary parameter* (2.5), (4.7). *In this case $R_\tau(\lambda) = (\widetilde{T}^\tau - \lambda)^{-1}$, where $\widetilde{T}^\tau$ is given by* (1.16) *with the operators C_a and C_b of the form* (4.20), (4.21).

Proof. Let $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be the decomposing boundary triplet (3.13)–(3.15) for $T_{\max}$. Then the immediate checking shows that

$$C_0(\lambda)\Gamma_0\{\widetilde{y},\widetilde{f}\} - C_1(\lambda)\Gamma_1\{\widetilde{y},\widetilde{f}\} = C_a(\lambda)y(a) + C_b(\lambda)\Gamma_b(y), \quad \{\widetilde{y},\widetilde{f}\} \in T_{\max}. \tag{4.26}$$

Hence the boundary problem (4.22), (4.23) is equivalent to the following one:

$$\{\widetilde{y}, \lambda\widetilde{y} + \widetilde{f}\} \in T_{\max}, \tag{4.27}$$

$$C_0(\lambda)\Gamma_0\{\widetilde{y}, \lambda\widetilde{y} + \widetilde{f}\} - C_1(\lambda)\Gamma_1\{\widetilde{y}, \lambda\widetilde{y} + \widetilde{f}\} = 0, \quad \lambda \in \mathbb{C}_+, \tag{4.28}$$

$$D_0(\lambda)\Gamma_0\{\widetilde{y}, \lambda\widetilde{y} + \widetilde{f}\} - D_1(\lambda)\Gamma_1\{\widetilde{y}, \lambda\widetilde{y} + \widetilde{f}\} = 0, \quad \lambda \in \mathbb{C}_-. \tag{4.29}$$

Now application of [26, Theorem 3.11] gives the required statement. □

4.2. **Characteristic matrices.** The following theorem is well known (see e.g. [6, 10, 29]).

Theorem 4.4. *Let $Y_0(\cdot,\lambda)$ be the $[\mathbb{H}]$-valued operator solution of Eq.* (3.3) *satisfying $Y_0(a,\lambda) = I_{\mathbb{H}}$. Then for each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists a unique operator function $\Omega(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [\mathbb{H}]$ such that for each $\widetilde{f} \in L^2_\Delta(\mathcal{I})$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$*

$$R(\lambda)\widetilde{f} = \pi\left(\int_{\mathcal{I}} Y_0(\cdot,\lambda)(\Omega(\lambda) + \tfrac{1}{2}\operatorname{sgn}(t-x)J)Y_0^*(t,\overline{\lambda})\Delta(t)f(t)\,dt\right), \quad f \in \widetilde{f}.$$

Moreover, $\Omega(\cdot)$ is a Nevanlinna operator function.

Definition 4.5. ([6, 29]). The operator function $\Omega(\cdot)$ is called the characteristic matrix of the symmetric system (3.2) corresponding to the generalized resolvent $R(\lambda)$.

In the following the characteristic matrix $\Omega(\cdot)$ will be called canonical if it corresponds to the canonical resolvent $R(\lambda)$ of $T_{\min}$.

Since $\Omega^*(\lambda) = \Omega(\overline{\lambda})$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, it follows that the characteristic matrix $\Omega(\cdot)$ is uniquely defined, in fact, by its restriction onto $\mathbb{C}_+$.

Let the assumptions (A1) and (A2) from Subsection 3.3 be satisfied, let $\mathcal{H}_0$ and $\mathcal{H}_1$ be finite dimensional Hilbert spaces (3.13), let τ be a boundary parameter and let $R_\tau(\lambda)$

be the corresponding generalized resolvent of $T_{\min}$ (see Theorem 4.3). In the following we denote by $\Omega_\tau(\cdot)$ the characteristic matrix corresponding to $R_\tau(\cdot)$.

It follows from Theorem 4.3 that the equality $\Omega(\lambda) = \Omega_\tau(\lambda)$ gives a parametrization of all characteristic matrices of the system (3.2) in terms of the boundary parameter τ. In the following theorem we represent this parametrization in the explicit form.

Theorem 4.6. *Let A_0 be the maximal symmetric extension* (3.16) *of $T_{\max}$ and let $M_+(\cdot)$ be the operator function* (3.22)–(3.24). *Moreover, let $P_{\widehat{H}} \in [H_0]$ be the orthoprojection in H_0 onto $\widehat{H}$ (see* (3.1)*) and let*

$$\Omega_0(\lambda) = \begin{pmatrix} m_0(\lambda) & -\frac{1}{2} I_{H,H_0} \\ -\frac{1}{2} P_{H_0,H} & 0 \end{pmatrix} : \underbrace{H_0 \oplus H}_{\mathbb{H}} \to \underbrace{H_0 \oplus H}_{\mathbb{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.30}$$

$$S_1(\lambda) = \begin{pmatrix} m_0(\lambda) - \frac{i}{2} P_{\widehat{H}} & M_{2+}(\lambda) \\ -P_{H_0,H} & 0 \end{pmatrix} : \underbrace{H_0 \oplus \widetilde{\mathcal{H}}_b}_{\mathcal{H}_0} \to \underbrace{H_0 \oplus H}_{\mathbb{H}}, \quad \lambda \in \mathbb{C}_+, \tag{4.31}$$

$$S_2(\lambda) = \begin{pmatrix} m_0(\lambda) + \frac{i}{2} P_{\widehat{H}} & -I_{H,H_0} \\ M_{3+}(\lambda) & 0 \end{pmatrix} : \underbrace{H_0 \oplus H}_{\mathbb{H}} \to \underbrace{H_0 \oplus \mathcal{H}_b}_{\mathcal{H}_1}, \quad \lambda \in \mathbb{C}_+. \tag{4.32}$$

Then: (1) *$\Omega_0(\cdot)$ is the characteristic matrix corresponding to the generalized resolvent $R(\lambda) = (A_0 - \lambda)^{-1}$, $\lambda \in \mathbb{C}_+$, of $T_{\min}$.*

(2) *For each boundary parameter $\tau = \{\tau_+, \tau_-\}$ of the form* (2.2) *the operator $C_0(\lambda) - C_1(\lambda) M_+(\lambda)$, $\lambda \in \mathbb{C}_+$, is boundedly invertible.*

(3) *The equality*

$$\Omega(\lambda) = \Omega_\tau(\lambda) = \Omega_0(\lambda) + S_1(\lambda)(C_0(\lambda) - C_1(\lambda) M_+(\lambda))^{-1} C_1(\lambda) S_2(\lambda), \quad \lambda \in \mathbb{C}_+ \tag{4.33}$$

establishes a bijective correspondence between all boundary parameters $\tau = \{\tau_+, \tau_-\}$ defined by (2.2) *and all characteristic matrices $\Omega(\cdot)$ of the system* (3.2).

Proof. Let $\tau = \{\tau_+, \tau_-\}$ be a boundary parameter (2.2). Since by Proposition 3.8 $M_+(\cdot)$ is the Weyl function of the decomposing boundary triplet $\Pi = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for $T_{\max}$, it follows from [26, Theorem 3.11] that $(\tau_+(\lambda) + M_+(\lambda))^{-1} \in [\mathcal{H}_1, \mathcal{H}_0]$, $\lambda \in \mathbb{C}_+$. Hence by [22, Lemma 2.1] the operator $C_0(\lambda) - C_1(\lambda) M_+(\lambda)$ is boundedly invertible and

$$T_\tau(\lambda) := -(\tau_+(\lambda) + M_+(\lambda))^{-1} = (C_0(\lambda) - C_1(\lambda) M_+(\lambda))^{-1} C_1(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.34}$$

Next assume that $A_0 (= \ker \Gamma_0)$ is the extension (3.16) of $T_{\min}$ and that $\gamma_\pm(\cdot)$ are the γ-fields of the triplet Π_+. As it was mentioned in the proof of Theorem 4.3 the generalized resolvent $R_\tau(\lambda)$ is generated in fact by the boundary problem (4.27)–(4.29). Therefore according to [26, Theorem 3.11] the following Krein formula for generalized resolvents holds:

$$R_\tau(\lambda) = (A_0 - \lambda)^{-1} + \gamma_+(\lambda) T_\tau(\lambda) \gamma_-^*(\overline{\lambda}), \quad \lambda \in \mathbb{C}_+. \tag{4.35}$$

Let us show that for each $\widetilde{f} \in L^2_\Delta(\mathcal{I})$ and $\lambda \in \mathbb{C}_+$

$$(A_0 - \lambda)^{-1} \widetilde{f} = \pi \left(\int_{\mathcal{I}} Y_0(\cdot, \lambda)(\Omega_0(\lambda) + \tfrac{1}{2} \operatorname{sgn}(t - x) J) Y_0^*(t, \overline{\lambda}) \Delta(t) f(t)\, dt \right), \quad f \in \widetilde{f}. \tag{4.36}$$

It follows from [1, Theorem 6.2] that the equality

$$(A_0 - \lambda)^{-1} \widetilde{f} = \pi \left(\int_{\mathcal{I}} G_0(\cdot, t, \lambda) \Delta(t) f(t)\, dt \right), \quad \widetilde{f} \in L^2_\Delta(\mathcal{I}), \quad f \in \widetilde{f}, \quad \lambda \in \mathbb{C}_+ \tag{4.37}$$

holds with the Green function $G_0(\cdot, \cdot, \lambda)$ of the form

$$G_0(x, t, \lambda) = \begin{cases} v_0(x, \lambda)\, \varphi^*(t, \overline{\lambda}), & x > t \\ \varphi(x, \lambda)\, v_0^*(t, \overline{\lambda}), & x < t \end{cases}, \quad \lambda \in \mathbb{C}_+. \tag{4.38}$$

Here $\varphi(\cdot,\lambda)$ is the $[H_0,\mathbb{H}]$-valued operator solution of Eq. (3.3) satisfying

$$\varphi(a,\lambda)=\begin{pmatrix} I_{H_0}\\ 0\end{pmatrix}: H_0\to H_0\oplus H,\quad \lambda\in\mathbb{C}$$

and $v_0(\cdot,\lambda)\in\mathcal{L}_\Delta^2[H_0,\mathbb{H}]$ is the operator solution from Proposition 3.7. Let

$$Y_+(t,\lambda):=(\varphi(t,\lambda),\,0): H_0\oplus\mathcal{H}_b\to\mathbb{H},\quad \lambda\in\mathbb{C}_+, \tag{4.39}$$

$$Y_-(t,\lambda):=(\varphi(t,\lambda),\,0): H_0\oplus\widetilde{\mathcal{H}}_b\to\mathbb{H},\quad \lambda\in\mathbb{C}_- \tag{4.40}$$

and let $Z_\pm(t,\lambda)$ be given by (3.19) and (3.20). Then (4.38) can be represented as

$$G_0(x,t,\lambda)=\begin{cases} Z_+(x,\lambda)\,Y_-^*(t,\overline{\lambda}), & x>t\\ Y_+(x,\lambda)\,Z_-^*(t,\overline{\lambda}), & x<t\end{cases},\quad \lambda\in\mathbb{C}_+. \tag{4.41}$$

Since

$$Z_\pm(a,\lambda)=\begin{pmatrix} (\mathcal{P}_0+\widehat{\mathcal{P}})v_0(a,\lambda) & (\mathcal{P}_0+\widehat{\mathcal{P}})u_\pm(a,\lambda)\\ \mathcal{P}_1 v_0(a,\lambda) & \mathcal{P}_1 u_\pm(a,\lambda)\end{pmatrix},\quad \lambda\in\mathbb{C}_\pm,$$

it follows from (3.23), (3.25) and the first equalities in (3.17) and (3.18) that

$$\begin{aligned} Z_+(a,\lambda)&=\begin{pmatrix} m_0(\lambda)-\frac{i}{2}P_{\widehat{H}} & M_{2+}(\lambda)\\ -P_{H_0,H} & 0\end{pmatrix}: H_0\oplus\widetilde{\mathcal{H}}_b\to H_0\oplus H,\quad \lambda\in\mathbb{C}_+,\\ Z_-(a,\lambda)&=\begin{pmatrix} m_0^*(\overline{\lambda})-\frac{i}{2}P_{\widehat{H}} & M_{3+}^*(\overline{\lambda})\\ -P_{H_0,H} & 0\end{pmatrix}: H_0\oplus\mathcal{H}_b\to H_0\oplus H,\quad \lambda\in\mathbb{C}_-.\end{aligned} \tag{4.42}$$

Therefore

$$Z_+(a,\lambda)=S_1(\lambda),\quad Z_-(a,\lambda)=S_2^*(\overline{\lambda}) \tag{4.43}$$

and (3.4) yields

$$Z_+(t,\lambda)=Y_0(t,\lambda)S_1(\lambda),\quad \lambda\in\mathbb{C}_+;\quad Z_-(t,\lambda)=Y_0(t,\lambda)S_2^*(\overline{\lambda}),\quad \lambda\in\mathbb{C}_-. \tag{4.44}$$

Moreover, by (4.39) and (4.40)

$$Y_+(a,\lambda)=\begin{pmatrix} I_{H_0} & 0\\ 0 & 0\end{pmatrix}: H_0\oplus\mathcal{H}_b\to H_0\oplus H,\quad Y_-(a,\lambda)=\begin{pmatrix} I_{H_0} & 0\\ 0 & 0\end{pmatrix}: H_0\oplus\widetilde{\mathcal{H}}_b\to H_0\oplus H$$

and (3.4) gives $Y_+(t,\lambda)=Y_0(t,\lambda)Y_+(a,\lambda)$ and $Y_-(t,\lambda)=Y_0(t,\lambda)Y_-(a,\lambda)$. This and (4.41) imply that

$$G_0(x,t,\lambda)=\begin{cases} Y_0(x,\lambda)(S_1(\lambda)Y_-^*(a,\overline{\lambda}))Y_0^*(t,\overline{\lambda}), & x>t\\ Y_0(x,\lambda)(Y_+(a,\lambda)S_2(\lambda))Y_0^*(t,\overline{\lambda}), & x<t\end{cases},\quad \lambda\in\mathbb{C}_+. \tag{4.45}$$

Observe also that the operator J can be represented as

$$J=\begin{pmatrix} iP_{\widehat{H}} & -I_{H,H_0}\\ P_{H_0,H} & 0\end{pmatrix}: H_0\oplus H\to H_0\oplus H \tag{4.46}$$

and the direct calculations with taking (4.30) into account give

$$S_1(\lambda)Y_-^*(a,\overline{\lambda})=\Omega_0(\lambda)-\tfrac12 J,\quad Y_+(a,\lambda)S_2(\lambda)=\Omega_0(\lambda)+\tfrac12 J,\quad \lambda\in\mathbb{C}_+.$$

Combining these equalities with (4.45) and (4.37) one gets (4.36). Hence statement (1) holds.

Next in view of (3.21) and (4.44) $\gamma_-(\overline{\lambda})=\pi Z_-(\overline{\lambda})$. Therefore, by Lemma 3.3 and the second equality in (4.44), for each $\widetilde{f}\in L_\Delta^2(\mathcal{I})$ and $\lambda\in\mathbb{C}_+$ one has

$$\gamma_-^*(\overline{\lambda})\widetilde{f}=\int_{\mathcal{I}} Z_-^*(t,\overline{\lambda})\Delta(t)f(t)\,dt=\int_{\mathcal{I}} S_2(\lambda)Y_0^*(t,\overline{\lambda})\Delta(t)f(t)\,dt,\quad f\in\widetilde{f}.$$

This and the first equalities in (3.21) and (4.44) imply that for each $\widetilde{f} \in L^2_\Delta(\mathcal{I})$ and $\lambda \in \mathbb{C}_+$

$$\gamma_+(\lambda)T_\tau(\lambda)\gamma_-^*(\overline{\lambda})\widetilde{f} = \pi\left(\int_{\mathcal{I}} Y_0(\cdot,\lambda)S_1(\lambda)T_\tau(\lambda)S_2(\lambda)Y_0^*(t,\overline{\lambda})\Delta(t)f(t)\,dt\right),\ f\in\widetilde{f}. \tag{4.47}$$

Now combining (4.35) with (4.36) and (4.47) we obtain the equality

$$R_\tau(\lambda)\widetilde{f} = \pi\left(\int_{\mathcal{I}} Y_0(\cdot,\lambda)(\Omega_\tau(\lambda)+\tfrac{1}{2}\operatorname{sgn}(t-x)J)Y_0^*(t,\overline{\lambda})\Delta(t)f(t)\,dt\right),\ \widetilde{f}\in L^2_\Delta(\mathcal{I}),\ \lambda\in\mathbb{C}_+,$$

where $\Omega_\tau(\cdot)$ is the operator function (4.33). Thus $\Omega_\tau(\cdot)$ is the characteristic matrix of the generalized resolvent $R_\tau(\lambda)$, which in view of Theorem 4.3 yields statement (3) of the theorem. □

Let as before $M_+(\lambda)$, $\lambda\in\mathbb{C}_+$, be given by (3.22)–(3.24). With each boundary parameter $\tau = \{\tau_+,\tau_-\}$ of the form (2.2) we associate a holomorphic operator function $\widetilde{\Omega}_\tau(\cdot):\mathbb{C}_+\to[\mathcal{H}_0\oplus\mathcal{H}_1,\mathcal{H}_1\oplus\mathcal{H}_0]$ given by

$$\widetilde{\Omega}_\tau(\lambda) = \begin{pmatrix}\widetilde{\omega}_1(\lambda) & \widetilde{\omega}_2(\lambda)\\ \widetilde{\omega}_3(\lambda) & \widetilde{\omega}_4(\lambda)\end{pmatrix}:\mathcal{H}_0\oplus\mathcal{H}_1\to\mathcal{H}_1\oplus\mathcal{H}_0,\quad \lambda\in\mathbb{C}_+, \tag{4.48}$$

$$\widetilde{\omega}_1(\lambda) = M_+(\lambda) - M_+(\lambda)(\tau_+(\lambda)+M_+(\lambda))^{-1}M_+(\lambda), \tag{4.49}$$

$$\widetilde{\omega}_2(\lambda) = -\tfrac{1}{2}I_{\mathcal{H}_1} + M_+(\lambda)(\tau_+(\lambda)+M_+(\lambda))^{-1}, \tag{4.50}$$

$$\widetilde{\omega}_3(\lambda) = -\tfrac{1}{2}I_{\mathcal{H}_0} + (\tau_+(\lambda)+M_+(\lambda))^{-1}M_+(\lambda),\quad \widetilde{\omega}_4(\lambda) = -(\tau_+(\lambda)+M_+(\lambda))^{-1}. \tag{4.51}$$

It follows from (4.34) that the equalities (4.49)–(4.51) can be represented as

$$\widetilde{\omega}_1(\lambda) = M_+(\lambda)(C_0(\lambda)-C_1(\lambda)M_+(\lambda))^{-1}C_0(\lambda), \tag{4.52}$$

$$\widetilde{\omega}_2(\lambda) = -\tfrac{1}{2}I_{\mathcal{H}_1} - M_+(\lambda)(C_0(\lambda)-C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda), \tag{4.53}$$

$$\widetilde{\omega}_3(\lambda) = \tfrac{1}{2}I_{\mathcal{H}_0} - (C_0(\lambda)-C_1(\lambda)M_+(\lambda))^{-1}C_0(\lambda), \tag{4.54}$$

$$\widetilde{\omega}_4(\lambda) = (C_0(\lambda)-C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda). \tag{4.55}$$

In the following proposition we give a somewhat other parametrization of all characteristic matrices $\Omega(\lambda)$ (cf. (4.33)).

Proposition 4.7. *Let $P_{\widehat{H}}\in[H_0]$ be the orthoprojection in H_0 onto $\widehat{H}$ and let*

$$X_1 = \begin{pmatrix}P_{\mathcal{H}_1,H_0} & \tfrac{i}{2}P_{\widehat{H}}P_{\mathcal{H}_0,H_0}\\ 0 & P_{\mathcal{H}_0,H}\end{pmatrix}:\mathcal{H}_1\oplus\mathcal{H}_0\to\underbrace{H_0\oplus H}_{\mathbb{H}}, \tag{4.56}$$

$$X_2 = \begin{pmatrix}P_{\mathcal{H}_0,H_0} & \tfrac{i}{2}P_{\widehat{H}}P_{\mathcal{H}_1,H_0}\\ 0 & P_{\mathcal{H}_1,H}\end{pmatrix}:\mathcal{H}_0\oplus\mathcal{H}_1\to\underbrace{H_0\oplus H}_{\mathbb{H}} \tag{4.57}$$

(clearly the operators $P_{\mathcal{H}_j,H_0}$ and $P_{\mathcal{H}_j,H}$ make sense, because in view of (3.13) *$H\subset H_0\subset\mathcal{H}_j$, $j\in\{0,1\}$). Then for each boundary parameter $\tau=\{\tau_+,\tau_-\}$ the corresponding characteristic matrix $\Omega(\lambda)=\Omega_\tau(\lambda)$ of the system* (3.2) *admits the representation*

$$\Omega_\tau(\lambda) = X_1\widetilde{\Omega}_\tau(\lambda)X_2^*,\quad \lambda\in\mathbb{C}_+. \tag{4.58}$$

Proof. Let $T_\tau(\lambda)$ be given by (4.34). Since

$$X_2^* = \begin{pmatrix} I_{H_0,\mathcal{H}_0} & 0\\ -\tfrac{i}{2}I_{H_0,\mathcal{H}_1}P_{\widehat{H}} & I_{H,\mathcal{H}_1}\end{pmatrix}:H_0\oplus H\to\mathcal{H}_0\oplus\mathcal{H}_1, \tag{4.59}$$

it follows from (4.56) and (4.48)–(4.51) that

$$X_1\widetilde{\Omega}_\tau(\lambda)X_2^* = \begin{pmatrix}\omega_1(\lambda) & \omega_2(\lambda)\\ \omega_3(\lambda) & \omega_4(\lambda)\end{pmatrix}:H_0\oplus H\to H_0\oplus H,\quad \lambda\in\mathbb{C}_+, \tag{4.60}$$

where

$$\omega_1(\lambda) = m_0(\lambda) + P_{\mathcal{H}_1,H_0} M_+(\lambda) T_\tau(\lambda) M_+(\lambda) \upharpoonright H_0 + \tfrac{i}{2} P_{\mathcal{H}_1,H_0} M_+(\lambda) T_\tau(\lambda) \upharpoonright H_0 \cdot P_{\widehat{H}}$$
$$- \tfrac{i}{2} P_{\widehat{H}} P_{\mathcal{H}_0,H_0} T_\tau(\lambda) M_+(\lambda) \upharpoonright H_0 + \tfrac{1}{4} P_{\widehat{H}} P_{\mathcal{H}_0,H_0} T_\tau(\lambda) \upharpoonright H_0 \cdot P_{\widehat{H}},$$
$$\omega_2(\lambda) = -\tfrac{1}{2} I_{H,H_0} - P_{\mathcal{H}_1,H_0} M_+(\lambda) T_\tau(\lambda) \upharpoonright H + \tfrac{i}{2} P_{\widehat{H}} P_{\mathcal{H}_0,H_0} T_\tau(\lambda) \upharpoonright H,$$
$$\omega_3(\lambda) = -\tfrac{1}{2} P_{H_0,H} - P_{\mathcal{H}_0,H} T_\tau(\lambda) M_+(\lambda) \upharpoonright H_0 - \tfrac{i}{2} P_{\mathcal{H}_0,H} T_\tau(\lambda) \upharpoonright H_0 \cdot P_{\widehat{H}},$$
$$\omega_4(\lambda) = P_{\mathcal{H}_0,H} T_\tau(\lambda) \upharpoonright H$$

(in the equality for $\omega_1(\lambda)$ we made use of the relation $m_0(\lambda) = P_{\mathcal{H}_1,H_0} M_+(\lambda) \upharpoonright H_0$ implied by (3.22)). Next, in view of (3.22) the equalities (4.31) and (4.32) can be written as

$$S_1(\lambda) = \begin{pmatrix} P_{\mathcal{H}_1,H_0} M_+(\lambda) - \frac{i}{2} P_{\widehat{H}} P_{\mathcal{H}_0,H_0} \\ -P_{\mathcal{H}_0,H} \end{pmatrix} : \mathcal{H}_0 \to H_0 \oplus H, \tag{4.61}$$

$$S_2(\lambda) = (M_+(\lambda) \upharpoonright H_0 + \tfrac{i}{2} I_{H_0,\mathcal{H}_1} P_{\widehat{H}},\ -I_{H,\mathcal{H}_1}) : H_0 \oplus H \to \mathcal{H}_1. \tag{4.62}$$

This and (4.33) yield

$$\Omega_\tau(\lambda) = \begin{pmatrix} m_0(\lambda) & -\frac{1}{2} I_{H,H_0} \\ -\frac{1}{2} P_{H_0,H} & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} P_{\mathcal{H}_1,H_0} M_+(\lambda) - \frac{i}{2} P_{\widehat{H}} P_{\mathcal{H}_0,H_0} \\ -P_{\mathcal{H}_0,H} \end{pmatrix} \cdot T_\tau(\lambda) \cdot (M_+(\lambda) \upharpoonright H_0 + \tfrac{i}{2} I_{H_0,\mathcal{H}_1} P_{\widehat{H}},\ -I_{H,\mathcal{H}_1})$$

and the immediate calculations show that

$$\Omega_\tau(\lambda) = \begin{pmatrix} \omega_1(\lambda) & \omega_2(\lambda) \\ \omega_3(\lambda) & \omega_4(\lambda) \end{pmatrix} : H_0 \oplus H \to H_0 \oplus H, \quad \lambda \in \mathbb{C}_+. \tag{4.63}$$

Now comparing (4.60) and (4.63) we arrive at the equality (4.58). □

Theorem 4.8. *Assume the hypotheses of Lemma 4.2. Moreover, let $\tau = \{\tau_+, \tau_-\}$ be a boundary parameter defined by* (2.2) *and* (4.1)–(4.4) *and let $C_a(\lambda)$ and $C_b(\lambda)$ be given by* (4.9) *and* (4.10)*. Then*

(1) *For each $\lambda \in \mathbb{C}_+$ there exists a unique operator solution $Z_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathbb{H}]$ of Eq.* (3.3) *satisfying the boundary condition*

$$C_a(\lambda)(Z_\tau(a, \lambda) + J) + C_b(\lambda) \Gamma_b Z_\tau(\lambda) = 0, \quad \lambda \in \mathbb{C}_+ \tag{4.64}$$

(here $Z_\tau(\lambda)$ is the mapping (3.8) *for $Z_\tau(\cdot, \lambda)$).*

(2) *The corresponding characteristic matrix $\Omega_\tau(\cdot)$ satisfies*

$$\Omega_\tau(\lambda) = Z_\tau(a, \lambda) + \tfrac{1}{2} J, \quad \lambda \in \mathbb{C}_+. \tag{4.65}$$

(3) *The following inequality holds*

$$(\mathrm{Im}\lambda)^{-1} \cdot \mathrm{Im}\Omega_\tau(\lambda) \geq \int_{\mathcal{I}} Z^*_\tau(t, \lambda) \Delta(t) Z_\tau(t, \lambda)\, dt, \quad \lambda \in \mathbb{C}_+. \tag{4.66}$$

Proof. (1) Let $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be the decomposing boundary triplet (3.13)–(3.15) for $T_{\max}$ and let $M_+(\cdot)$ and $\gamma_+(\cdot)$ be the Weyl function and the γ-field of Π_+ respectively. Moreover, let $Z_+(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathcal{H}_0, \mathbb{H}]$ be the operator solution of Eq. (3.3) defined in Proposition 3.8 and let

$$Z_\tau(t, \lambda) := -Z_+(t, \lambda)(C_0(\lambda) - C_1(\lambda) M_+(\lambda))^{-1} C_a(\lambda) J, \quad \lambda \in \mathbb{C}_+. \tag{4.67}$$

Clearly, $Z_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathbb{H}]$ and $Z_\tau(\cdot, \lambda)$ is an operator solution of Eq. (3.3). Let us show that $Z_\tau(\cdot, \lambda)$ satisfies (4.64).

Assume that $h \in \mathbb{H}$, $h_0 := -(C_0(\lambda) - C_1(\lambda) M_+(\lambda))^{-1} C_a(\lambda) J h$ and

$$\widetilde{y} = \pi(Z_\tau(\cdot, \lambda) h). \tag{4.68}$$

Then by (4.67) $\widetilde{y} = \pi Z_+(\lambda)h_0$ and the equality (3.21) yields

$$\widetilde{y} = \gamma_+(\lambda)h_0. \tag{4.69}$$

Combining (4.68) with (4.26) one gets

$$C_0(\lambda)\Gamma_0\{\widetilde{y}, \lambda\widetilde{y}\} - C_1(\lambda)\Gamma_1\{\widetilde{y}, \lambda\widetilde{y}\} = (C_a(\lambda)Z_\tau(a,\lambda) + C_b(\lambda)\Gamma_b Z_\tau(\lambda))h. \tag{4.70}$$

On the other hand, combining of (4.69) with (2.10) and (2.8) yields $\Gamma_0\{\widetilde{y}, \lambda\widetilde{y}\} = h_0$ and $\Gamma_1\{\widetilde{y}, \lambda\widetilde{y}\} = M_+(\lambda)h_0$. Therefore

$$C_0(\lambda)\Gamma_0\{\widetilde{y}, \lambda\widetilde{y}\} - C_1(\lambda)\Gamma_1\{\widetilde{y}, \lambda\widetilde{y}\} = (C_0(\lambda) - C_1(\lambda)M_+(\lambda))h_0 = -C_a(\lambda)Jh.$$

Comparing this equality with (4.70) one obtains

$$C_a(\lambda)Z_\tau(a,\lambda) + C_b(\lambda)\Gamma_b Z_\tau(\lambda) = -C_a(\lambda)J, \quad \lambda \in \mathbb{C}_+.$$

This implies (4.64).

To prove uniqueness of $Z_\tau(\cdot,\lambda)$ assume that $\widetilde{Z}_\tau(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathbb{H}]$ is another solution of Eq. (3.3) satisfying (4.64). Then for each $h \in \mathbb{H}$ the function $y = (Z_\tau(t,\lambda) - \widetilde{Z}_\tau(t,\lambda))h$ is a solution of the homogeneous boundary problem (4.22), (4.23) (with $f = 0$). Since by Theorem 4.3 such a problem has a unique solution $y = 0$, it follows that $Z_\tau(t,\lambda) = \widetilde{Z}_\tau(t,\lambda)$.

(2) Assume that $S_2(\lambda)$ is given by (4.32) and that

$$Z_0(t,\lambda) := (Z_+(t,\lambda) \upharpoonright H_0,\ 0) : H_0 \oplus H \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+. \tag{4.71}$$

Then by (4.42)

$$Z_0(a,\lambda) = \begin{pmatrix} m_0(\lambda) - \frac{i}{2}P_{\widehat{H}} & 0 \\ -P_{H_0,H} & 0 \end{pmatrix} : H_0 \oplus H \to H_0 \oplus H$$

and the equalities (4.30) and (4.46) yield

$$Z_0(a,\lambda) = \Omega_0(\lambda) - \tfrac{1}{2}J, \quad \lambda \in \mathbb{C}_+. \tag{4.72}$$

Next we show that

$$Z_\tau(t,\lambda) = Z_0(t,\lambda) + Z_+(t,\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda)S_2(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.73}$$

Since by (4.9)

$$C_a(\lambda)J = (-C_{0a}(\lambda),\ -\widehat{C}_0(\lambda) - \tfrac{i}{2}\widehat{C}_1(\lambda),\ C_{1a}(\lambda)) : H \oplus \widehat{H} \oplus H \to \mathcal{H}_0,$$

it follows from (4.1) and (4.2) that

$$C_a(\lambda)Jh = -C_0(\lambda)(\mathcal{P}_0 h + \widehat{\mathcal{P}}h) - \tfrac{i}{2}C_1(\lambda)\widehat{\mathcal{P}}h + C_1(\lambda)\mathcal{P}_1 h, \quad h \in \mathbb{H}. \tag{4.74}$$

Let $P_{\mathbb{H},H_0} (\in [\mathbb{H}, H_0])$ be the orthoprojection in $\mathbb{H}$ onto H_0 (see (3.1)) and let as before $P_{\widehat{H}} (\in [H_0])$ be the orthoprojection in H_0 onto $\widehat{H}$. Then $\mathcal{P}_0 h + \widehat{\mathcal{P}}h = P_{\mathbb{H},H_0}h$, $\widehat{\mathcal{P}}h = P_{\widehat{H}}P_{\mathbb{H},H_0}h$ and the equality (4.74) can be written as

$$C_a(\lambda)Jh = -C_0(\lambda)P_{\mathbb{H},H_0}h - \tfrac{i}{2}C_1(\lambda)P_{\widehat{H}}P_{\mathbb{H},H_0}h + C_1(\lambda)\mathcal{P}_1 h, \quad h \in \mathbb{H}.$$

Moreover, by (4.62)

$$S_2(\lambda)h = M_+(\lambda)P_{\mathbb{H},H_0}h + \tfrac{i}{2}P_{\widehat{H}}P_{\mathbb{H},H_0}h - \mathcal{P}_1 h, \quad h \in \mathbb{H},$$

and combining of the last two equalities yields

$$C_a(\lambda)Jh = -(C_0(\lambda) - C_1(\lambda)M_+(\lambda))P_{\mathbb{H},H_0}h - C_1(\lambda)S_2(\lambda)h, \quad h \in \mathbb{H}.$$

This and (4.67) imply that for each $h \in \mathbb{H}$

$$Z_\tau(t,\lambda)h = Z_+(t,\lambda)P_{\mathbb{H},H_0}h + Z_+(t,\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda)S_2(\lambda)h. \tag{4.75}$$

Since by (4.71) $Z_+(t,\lambda)P_{\mathbb{H},H_0}h = Z_0(t,\lambda)h$, it follows from (4.75) that $Z_\tau(t,\lambda)$ admits the representation (4.73).

Now combining (4.73) with (4.72) and the first equality in (4.43) and then taking (4.33) into account one obtains the equality (4.65).

(3) Let us show that

$$\Omega_0(\mu) - \Omega_0^*(\lambda) = (\mu - \overline{\lambda})\gamma_1^*(\lambda)\gamma_1(\mu), \quad S_1(\mu) - S_2^*(\lambda)P_1 = (\mu - \overline{\lambda})\gamma_1^*(\lambda)\gamma_+(\mu), \tag{4.76}$$

where $\mu, \lambda \in \mathbb{C}_+$, $P_1 = P_{\mathcal{H}_0, \mathcal{H}_1}$ is the orthoprojection in $\mathcal{H}_0$ onto $\mathcal{H}_1$ and

$$\gamma_1(\lambda) = (\gamma_+(\lambda) \upharpoonright H_0, \ 0) : H_0 \oplus H \to L_\Delta^2(\mathcal{I}). \tag{4.77}$$

The first equality in (4.76) is immediate from (2.9). Next, by (4.62) one has

$$S_2^*(\lambda)P_1 = \begin{pmatrix} P_{\mathcal{H}_0,H_0} M_+^*(\lambda) - \frac{i}{2} P_{\widehat{H}} P_{\mathcal{H}_1,H_0} \\ -P_{\mathcal{H}_1,H} \end{pmatrix} P_1 = \begin{pmatrix} P_{\mathcal{H}_0,H_0} M_+^*(\lambda) P_1 - \frac{i}{2} P_{\widehat{H}} P_{\mathcal{H}_0,H_0} \\ -P_{\mathcal{H}_0,H} \end{pmatrix},$$

which in view of (4.61) yields

$$S_1(\mu) - S_2^*(\lambda)P_1 = \begin{pmatrix} P_{\mathcal{H}_1,H_0} M_+(\mu) - P_{\mathcal{H}_0,H_0} M_+^*(\lambda) P_1 \\ 0 \end{pmatrix} : \mathcal{H}_0 \to H_0 \oplus H, \quad \mu, \lambda \in \mathbb{C}_+. \tag{4.78}$$

Since $H_0 \subset \mathcal{H}_1$, it follows that $H_0 \perp \mathcal{H}_2$ and hence $P_{\mathcal{H}_0,H_0} P_2 = 0$. Therefore application of the operator $P_{\mathcal{H}_0,H_0}$ to the identity (2.9) yields

$$P_{\mathcal{H}_1,H_0} M_+(\mu) - P_{\mathcal{H}_0,H_0} M_+^*(\lambda) P_1 = (\mu - \overline{\lambda}) P_{\mathcal{H}_0,H_0} \gamma_+^*(\lambda)\gamma_+(\mu), \quad \mu, \lambda \in \mathbb{C}_+.$$

Combining this equality with (4.78) one gets the second equality in (4.76).

Now application of [25, Lemma 21] to the representation (4.33) of $\Omega_\tau(\cdot)$ with taking (4.76) and (2.9) into account yields

$$\mathrm{Im}\Omega_\tau(\lambda) \geq \mathrm{Im}\lambda \cdot \gamma_\tau^*(\lambda)\gamma_\tau(\lambda), \quad \lambda \in \mathbb{C}_+, \tag{4.79}$$

where

$$\gamma_\tau(\lambda) = \gamma_1(\lambda) + \gamma_+(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1} C_1(\lambda) S_2(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.80}$$

Since by (3.21) $\gamma_+(\lambda) = \pi Z_+(\lambda)$, it follows from (4.77) and (4.71) that $\gamma_1(\lambda) = \pi Z_0(\lambda)$. Combining these equalities with (4.80) and (4.73) we obtain

$$\gamma_\tau(\lambda) = \pi Z_\tau(\lambda), \tag{4.81}$$

where $Z_\tau(\lambda)$ is the mapping (3.8) for $Z_\tau(\cdot, \lambda)$. Therefore by Lemma 3.3

$$\gamma_\tau^*(\lambda)\widetilde{f} = \int_{\mathcal{I}} Z_\tau^*(t, \lambda)\Delta(t) f(t)\, dt, \quad \widetilde{f} \in L_\Delta^2(\mathcal{I}), \quad f \in \widetilde{f}, \tag{4.82}$$

and combining of (4.79) with (4.81) and (4.82) gives (4.66). □

4.3. **The case of equal deficiency indices.** In the case of equal deficiency indices $n_+(T_{\min}) = n_-(T_{\min})$ the above results can be rather simplified. Namely, the following theorems are immediate from Theorems 4.6, 4.8 and Proposition 4.7.

Theorem 4.9. *Let* $n_+(T_{\min}) = n_-(T_{\min})$ *(so that* $\widetilde{\mathcal{H}}_b = \mathcal{H}_b$*), let* $\mathcal{H} = H_0 \oplus \mathcal{H}_b$*, let* $A_0 = A_0^*$ *be given by* (3.16) *and let* $M(\cdot)$ *be the (Nevanlinna) operator function defined by* (3.26)–(3.28)*. Moreover, let* $\Omega_0(\cdot)$ *be the operator function* (4.30) *and let*

$$S(\lambda) = \begin{pmatrix} m_0(\lambda) - \frac{i}{2} P_{\widehat{H}} & M_2(\lambda) \\ -P_{H_0,H} & 0 \end{pmatrix} : \underbrace{H_0 \oplus \mathcal{H}_b}_{\mathcal{H}} \to \underbrace{H_0 \oplus H}_{\mathbb{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.83}$$

Then: (1) $\Omega_0(\cdot)$ *is the characteristic matrix corresponding to the canonical resolvent* $(A_0 - \lambda)^{-1}$ *and the equality*

$$\Omega(\lambda) = \Omega_\tau(\lambda) = \Omega_0(\lambda) + S(\lambda)(C_0(\lambda) - C_1 M(\lambda))^{-1} C_1(\lambda) S^*(\overline{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}$$

establishes a bijective correspondence between all boundary parameters τ *of the form* (2.4) *and all characteristic matrices* $\Omega(\cdot)$ *of the system* (3.2). *Moreover, the characteristic matrix* $\Omega(\cdot) = \Omega_\tau(\cdot)$ *is canonical if and only if the boundary parameter* τ *is self-adjoint.*

(2) *For each boundary parameter* $\tau = \tau(\lambda)$ *the corresponding characteristic matrix* $\Omega(\lambda) = \Omega_\tau(\lambda)$ *of the system* (3.2) *admits the representation*

$$\Omega_\tau(\lambda) = X\widetilde{\Omega}_\tau(\lambda)X^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

with the Nevanlinna operator function $\widetilde{\Omega}_\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \to [\mathcal{H} \oplus \mathcal{H}])$ *of the form* (1.21) *and the operator* $X \in [\mathcal{H} \oplus \mathcal{H}, \mathbb{H}]$ *given by*

$$X = \begin{pmatrix} P_{\mathcal{H},H_0} & \frac{i}{2} P_{\widehat{H}} P_{\mathcal{H},H_0} \\ 0 & P_{\mathcal{H},H} \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \to \underbrace{H_0 \oplus H}_{\mathbb{H}}. \tag{4.84}$$

Theorem 4.10. *Let* $n_+(T_{\min}) = n_-(T_{\min})$, *let* $\mathbb{H}_b = \mathcal{H}_b \oplus \widehat{H} \oplus \mathcal{H}_b$ *and let* J_b *be the operator* (1.10). *Moreover, let* τ *be a boundary parameter defined by* (2.4) *and* (4.5), (4.6) *and let* $C_a(\lambda)$ *and* $C_b(\lambda)$ *be the operator functions* (4.17) *and* (4.18). *Then for each* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *there exists a unique operator solution* $Z_\tau(\cdot, \lambda) \in \mathcal{L}^2_\Delta[\mathbb{H}]$ *of Eq.* (3.3) *satisfying* (4.64) *(with* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*). Moreover,* $\Omega_\tau(\lambda) = Z_\tau(a, \lambda) + \frac{1}{2}J$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *and the inequality* (4.66) *is valid for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

If in addition τ *is a self-adjoint boundary parameter, then the following identity holds:*

$$\Omega_\tau(\mu) - \Omega^*_\tau(\lambda) = (\mu - \overline{\lambda}) \int_{\mathcal{I}} Z^*_\tau(t, \lambda)\Delta(t) Z_\tau(t, \mu)\, dt, \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

This implies that for the canonical characteristic matrix $\Omega_\tau(\cdot)$ *the inequality* (4.66) *turns into the equality, which holds for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

References

1. S. Albeverio, M. M. Malamud, V. I. Mogilevskii, *On Titchmarsh-Weyl functions and eigenfunction expansions of first-order symmetric systems*, Integr. Equ. Oper. Theory **77** (2013), 303–354.
2. D. Z. Arov, H. Dym, *J-inner matrix functions, interpolation and inverse problems for canonical systems,* I*: Foundations*, Integr. Equ. Oper. Theory **29** (1997), 373–454.
3. F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1963.
4. J. Behrndt, S. Hassi, H. de Snoo, R. Wiestma, *Square-integrable solutions and Weyl functions for singular canonical systems*, Math. Nachr. **284** (2011), no. 11–12, 1334–1383.
5. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
6. V. M. Bruk, *Linear relations in a space of vector functions*, Math. Notes **24** (1978), no. 4, 767–773.
7. V. A. Derkach, S. Hassi, M. M. Malamud, H. S. V. de Snoo, *Generalized resolvents of symmetric operators and admissibility*, Methods Funct. Anal. Topology **6** (2000), no. 3, 24–55.
8. V. A. Derkach, M. M. Malamud, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, J. Funct. Anal. **95** (1991), 1–95.
9. A. Dijksma, H. Langer, H. S. V. de Snoo, *Hamiltonian systems with eigenvalue depending boundary conditions*, Oper. Theory Adv. Appl. **35** (1988), 37–83.
10. A. Dijksma, H. Langer, H. S. V. de Snoo, *Eigenvalues and pole functions of Hamiltonian systems with eigenvalue depending boundary conditions*, Math. Nachr. **161** (1993), 107–153.
11. A. Dijksma, H. S. V. de Snoo, *Self-adjoint extensions of symmetric subspaces*, Pacific J. Math. **54** (1974), no. 1, 71–100.
12. N. Dunford and J. T. Schwartz, *Linear Operators, Part* 2*: Spectral Theory*, Interscience Publishers, New York—London, 1963.
13. I. Gohberg, M. G. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Transl. Math. Monographs, Vol. 24, Amer. Math. Soc., Providence, RI, 1970.
14. M. L. Gorbachuk, *On spectral functios of a differential equation of the second order with operator-valued coefficients*, Ukrain. Mat. Zh. **18** (1966), no. 2, 3–21. (Russian)

15. V. I. Gorbachuk, M. L. Gorbachuk, *Boundary Value Problems for Operator Differential Equations*, Kluwer Academic Publishers, Dordrecht—Boston—London, 1991. (Russian edition: Naukova Dumka, Kiev, 1984)
16. D. B. Hinton, A. Schneider, *On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems* I, Math. Nachr. **163** (1993), 323–342.
17. I. S. Kats, *Linear relations generated by the canonical differential equation of phase dimension 2, and eigenfunction expansions*, St. Petersburg Math. J. **14** (2003), no. 3, 429–452.
18. V. I. Kogan, F. S. Rofe-Beketov, *On square-integrable solutions of symmetric systems of differential equations of arbitrary order*, Proc. Roy. Soc. Edinburgh Sect. A **74** (1974/75), 5–40.
19. A. M. Krall, *$M(\lambda)$-theory for singular Hamiltonian systems with one singular endpoint*, SIAM J. Math. Anal. **20** (1989), 664–700.
20. M. Lesch, M. M. Malamud, *On the deficiency indices and self-adjointness of symmetric Hamiltonian systems*, J. Differential Equations **189** (2003), 556–615.
21. M. M. Malamud, *On a formula for the generalized resolvents of a non-densely defined Hermitian operator*, Ukrain. Mat. Zh. **44** (1992), no. 12, 1658–1688. (Russian); English transl. Ukrainian Math. J. **44** (1992), no. 12, 1522-1547.
22. M. M. Malamud, V. I. Mogilevskii, *Krein type formula for canonical resolvents of dual pairs of linear relations*, Methods Funct. Anal. Topology **8** (2002), no. 4, 72–100.
23. V. I. Mogilevskii, *Nevanlinna type families of linear relations and the dilation theorem*, Methods Funct. Anal. Topology **12** (2006), no. 1, 38–56.
24. V. I. Mogilevskii, *Boundary triplets and Krein type resolvent formula for symmetric operators with unequal defect numbers*, Methods Funct. Anal. Topology **12** (2006), no. 3, 258–280.
25. V. I. Mogilevskii, *Description of generalized resolvents and characteristic matrices of differential operators in terms of the boundary parameter*, Math. Notes **90** (2011), no. 4, 548–570.
26. V. I. Mogilevskii, *On exit space extensions of symmetric operators with applications to first order symmetric systems*, Methods Funct. Anal. Topology **19** (2013), no. 3, 268–292.
27. B. C. Orcutt, *Canonical Differential Equations*, Dissertation, University of Virginia, 1969.
28. A. V. Štraus, *On spectral functions of differential operators*, Izv. Akad. Nauk SSSR Ser. Mat. **19** (1955), no. 4, 201–220.
29. A. V. Štraus, *On generalized resolvents and spectral functions of differential operators of even order*, Izv. Akad. Nauk. SSSR Ser. Mat. **21** (1957), no. 6, 785–808.
30. A. V. Štraus, *Extensions and generalized resolvents of a symmetric operator which is not densely defined*, Izv. Akad. Nauk. SSSR Ser. Mat. **34** (1970), no. 1, 175–202.

Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, 74 R. Luxemburg, Donets'k, 83050, Ukraine

Department of Mathematics, Lugans'k National University, 2 Oboronna, Lugans'k, 91011, Ukraine

E-mail address: vim@mail.dsip.net

Received 13/01/2014