FACTORIZATION FORMULAS FOR SOME CLASSES OF GENERALIZED J-INNER MATRIX VALUED FUNCTIONS

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Dedicated to Professor D. Z. Arov in the occasion of his 80-th birthday with great respect

ABSTRACT. The class $\mathcal{U}_\kappa(j_{pq})$ of generalized j_{pq}-inner matrix valued functions (mvf's) was introduced in [2]. For a mvf W from a subclass $\mathcal{U}_\kappa^r(j_{pq})$ of $\mathcal{U}_\kappa(j_{pq})$ the notion of the right associated pair was introduced in [13] and some factorization formulas were found. In the present paper we introduce a dual subclass $\mathcal{U}_\kappa^\ell(j_{pq})$ and for every mvf $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ a left associated pair $\{\beta_1, \beta_2\}$ is defined and factorization formulas for W in terms of β_1, β_2 are found. The notion of a singular generalized j_{pq}-inner mvf W is introduced and a characterization of singularity of W is given in terms of associated pair.

1. INTRODUCTION

Let J be an $m \times m$ signature matrix, i.e., $J = J^* = J^{-1}$. Recall that an $m \times m$ meromorphic in $\mathbb{D} = \{\lambda : |\lambda| < 1\}$ matrix valued function (mvf) $W(\lambda)$ is said to belong to the Potapov class $\mathcal{P}(J)$ of J-contractive mvf's, if

$$(1.1) \qquad W(\lambda)^* J W(\lambda) \leq J$$

for all $\lambda \in \mathfrak{h}_W^+$ that is the domain of holomorphy of W in $\mathbb{D}$. If $W \in \mathcal{P}(J)$, then the nontangential limits $W(\mu)$ exists a.e. on $\mathbb{T}$, the boundary of $\mathbb{D}$. A J-contractive mvf $W(\lambda)$ is called J-inner, and is written as $W \in \mathcal{U}(J)$ if

$$(1.2) \qquad W(\mu)^* J W(\mu) = J \quad \text{for} \quad \text{a.e.} \quad \mu \in \mathbb{T} = \partial\mathbb{D}.$$

J-inner mvf's play an important role in the theory of classical problems of analysis. As is known (see [1], [6]–[10], [15]) the set of solutions of many classical interpolation problems coincides with the range of a linear fractional transformation generated by a J-inner mvf.

In the case where $J = j_{pq} := \text{diag}(I_p, -I_q)$ $(p, q \in \mathbb{N})$ the Potapov-Ginzburg transform $S = PG(W)$ of a j_{pq}-contractive mvf $W \in \mathcal{P}(j_{pq})$,

$$(1.3) \qquad S(\lambda) = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} := \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1},$$

belongs to the Schur class $\mathcal{S}^{m \times m}$ of $m \times m$ mvf's holomorphic and contractive in $\mathbb{D}$ and, moreover, for every $W \in \mathcal{U}(j_{pq})$ the mvf $S = PG(W)$ belongs to the class $\mathcal{S}_{\text{in}}^{m \times m}$ of inner mvf's, i.e., $S \in \mathcal{S}^{m \times m}$ and $S(\mu)^* S(\mu) = I_m$ a.e. on $\mathbb{T}$. For every $W \in \mathcal{U}(j_{pq})$ the mvf's $s_{11}(\lambda)$ and $s_{22}(\lambda)$ admit left and right inner-outer factorization,

$$s_{11}(\lambda) = b_1(\lambda)\varphi_1(\lambda), \quad s_{22}(\lambda) = \varphi_2(\lambda)b_2(\lambda),$$

where $b_1 \in \mathcal{S}_{\text{in}}^{p \times p}$, $b_2 \in \mathcal{S}_{\text{in}}^{q \times q}$ and φ_1, φ_2 are outer mvf's (see definition in Subsection 2.1). The pair $\{b_1, b_2\}$ is called an associated pair of $W \in \mathcal{U}(j_{pq})$ and is designated by $\{b_1, b_2\} \in \text{ap}(W)$ (see [6]).

A subclass $\mathcal{U}_S(j_{pq})$ of singular j_{pq}-inner mvf's was introduced by D. Arov in [5] by the equivalence

$$W \in \mathcal{U}_S(j_{pq}) \Longleftrightarrow W \in \mathcal{U}(j_{pq}) \quad \text{and} \quad W \quad \text{is outer.}$$

The class $\mathcal{U}_S(j_{pq})$ was completely characterized in terms of associated pairs. As was shown in [6] a mvf $W \in \mathcal{U}(j_{pq})$ belongs to $\mathcal{U}_S(j_{pq})$ if and only if its associated pair is trivial, i.e., $b_1(\lambda) \equiv I_p$ and $b_2(\lambda) \equiv I_q$.

A class $\mathcal{U}_\kappa(j_{pq})$ of generalized j_{pq}-inner mvf's was introduced in [2] in connection with some indefinite interpolation problems. Recall that an $m \times m$ mvf $W(\lambda)$ meromorphic in $\mathbb{D}$ is said to belong to the class $\mathcal{U}_\kappa(j_{pq})$, if

1) the kernel

$$\mathsf{K}_\omega^W(\lambda) = \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{1 - \lambda\omega^*}$$

has κ negative squares in $\mathfrak{h}_W^+$;

2) $W(\mu)$ is j_{pq}-unitary a.e. on $\mathbb{T}$.

The class of mvf's $\mathcal{S}_\kappa^{m \times m}$, which satisfy the first condition with $j_{pq} = I_m$, is called a generalized Schur class and is denoted by $\mathcal{S}_\kappa^{m \times m}$ (see Subsection 2.2 for details). As is known [2], the Potapov-Ginzburg transform $S = PG(W)$ of a mvf $W \in \mathcal{U}_\kappa(j_{pq})$ belongs to the class $\mathcal{S}_\kappa^{m \times m}$.

In [13] a subclass

$$\mathcal{U}_\kappa^r(j_{pq}) = \{W \in \mathcal{U}_\kappa(j_{pq}) : s_{21} := -w_{22}^{-1}w_{21} \in \mathcal{S}_\kappa^{q \times p}\}$$

was introduced and for every mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ the notion of the (right) associated pair $\{b_1, b_2\}$ was introduced. The mvf's b_1, b_2 were used in order to get a factorization for the mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$.

In the present paper we introduce another subclass

$$\mathcal{U}_\kappa^\ell(j_{pq}) = \{W \in \mathcal{U}_\kappa(j_{pq}) : s_{12} := w_{12}w_{22}^{-1} \in \mathcal{S}_\kappa^{q \times p}\}$$

and define the notion of the left associated pair $\{\beta_1, \beta_2\}$ for $W \in \mathcal{U}_\kappa^\ell(j_{pq})$. We find new factorization formulas for $W \in \mathcal{U}_\kappa^\ell(j_{pq})$.

The classes $\mathcal{U}_\kappa^r(j_{pq})$ and $\mathcal{U}_\kappa^\ell(j_{pq})$ do not coincide as is shown in Example 1. Moreover, if $W \in \mathcal{U}_\kappa^r(j_{pq}) \bigcap \mathcal{U}_\kappa^\ell(j_{pq})$, then the corresponding right associated pair $\{b_1, b_2\}$ not necessarily coincides with the left associated pair $\{\beta_1, \beta_2\}$ (see Example 3). However, Theorem 3.14 says that in this case

$$\det b_1 = \det \beta_1 \quad \text{and} \quad \det b_2 = \det \beta_2,$$

and b_j, and β_j have the same degrees and the same zero sets for every $j \in \{1, 2\}$.

In the present paper we introduce also the notion of singular mvf in classes $\mathcal{U}_\kappa^r(j_{pq})$, $\mathcal{U}_\kappa^\ell(j_{pq})$ and obtain a characterization of singular mvf's in terms of associated pairs. The proof of this result is essentially based on factorization theorems for the class $\mathcal{U}_\kappa^r(j_{pq})$ from [13].

2. Preliminaries

2.1. **Notations.** Let Ω_+ be equal to either $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, or $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \Im\lambda > 0\}$;

$$\rho_\omega(\lambda) = \begin{cases} 1 - \lambda\omega^*, & \text{if} \quad \Omega_+ = \mathbb{D}, \\ -2\pi i(\lambda - \omega^*), & \text{if} \quad \Omega_+ = \mathbb{C}_+. \end{cases}$$

Thus, $\Omega_+ = \{\omega \in \mathbb{C} : \rho_\omega(\omega) > 0\}$ and $\Omega_0 = \{\omega \in \mathbb{C} : \rho_\omega(\omega) = 0\}$ is the boundary of Ω_+. Correspondingly, we set $\Omega_- = \{\omega \in \mathbb{C} : \rho_\omega(\omega) < 0\}$.

For a mvf $f(\lambda)$ let us set

$$f^{\#}(\lambda) = f(\lambda^{\circ})^{*}, \quad \text{where} \quad \lambda^{\circ} = \begin{cases} 1/\lambda^{*} & : \text{ if } \Omega_{+} = \mathbb{D}, \lambda \neq 0, \\ \lambda^{*} & : \text{ if } \Omega_{+} = \mathbb{C}_{+}. \end{cases}$$

Denote by $\mathfrak{h}_{f}$ the domain of holomorphy of the mvf f and let $\mathfrak{h}_{f}^{\pm} = \mathfrak{h}_{f} \cap \Omega_{\pm}$.

The following basic classes of mvf's will be used in this paper:

$H_{r}(\Omega_{\pm})$ $(0 < r \leq \infty)$ is the class of holomorphic functions in $\Omega_{\pm}$ such that $\|u\|_{r} < \infty$,

$$\|u\|_{r} = \begin{cases} \sup\limits_{0 < \rho < 1} [\frac{1}{2\pi} \int_{0}^{2\pi} |u(\rho e^{it})| dt]^{\frac{1}{r}}, & (0 < r < \infty), \\ \sup\limits_{z \in \Omega_{\pm}} |u(z)|, & (p = \infty). \end{cases}$$

$H_{r}^{p \times q}(\Omega_{\pm})$ is the class of $p \times q$ -mvf's with entries in $H_{r}(\Omega_{\pm})$,

$H_{r} := H_{r}^{1 \times 1}(\Omega_{+}), \quad H_{r}^{p} := H_{r}^{p \times 1}(\Omega_{+}) \quad (1 \leq r \leq \infty),$

$\mathcal{S}_{\text{in}}^{p \times q} = \{s \in \mathcal{S}^{p \times q} : s(\mu)^{*} s(\mu) = I_{p} \text{ a.e. on } \Omega_{0}\},$

$\mathcal{S}_{\text{out}}^{p \times q} = \{s \in \mathcal{S}^{p \times q} : \overline{sH_{2}^{q}} = H_{2}^{p}\}, \quad \mathcal{S}_{\text{out}} = \mathcal{S}_{\text{out}}^{1 \times 1},$

$\mathcal{N}_{\pm}^{p \times q} = \{f = h^{-1}g : g \in H_{\infty}^{p \times q}(\Omega_{\pm}), h \in \mathcal{S}_{\text{out}}(\Omega_{\pm})\},$

$\mathcal{N}_{\text{out}}^{p \times q} = \{f = h^{-1}g : g \in \mathcal{S}_{\text{out}}^{p \times q}, h \in \mathcal{S}_{\text{out}}\}, \quad \mathcal{N}_{\text{out}} = \mathcal{N}_{\text{out}}^{1 \times 1}.$

In particular, $f \in \mathcal{N}_{-}^{p \times q}$ if and only if $f^{\#} \in \mathcal{N}_{+}^{q \times p}$. As is known [8], a $p \times p$ mvf f belongs to the class $\mathcal{N}_{\text{out}}^{p \times p}$ if and only if $\det f \in \mathcal{N}_{\text{out}}$. This implies, in particular, that such a mvf should be invertible in Ω_{+}. Another criterion for $f \in \mathcal{N}_{\text{out}}^{p \times p}$ is formulated in terms of the Smirnov class

$$f \in \mathcal{N}_{\text{out}}^{p \times p} \Longleftrightarrow f, f^{-1} \in \mathcal{N}_{+}^{p \times p}.$$

An important connection between these classes is given by the following

Theorem 2.1. ([8], Th. 3.59). (THE SMIRNOV MAXIMUM PRINCIPLE).

$$\mathcal{N}_{+}^{p \times q} \cap L_{r}^{p \times q} = H_{r}^{p \times q} \quad (1 \leq r \leq \infty).$$

2.2. **The generalized Schur class.** Let $\kappa \in \mathbb{Z}_{+}$. Recall [9], [16] that a Hermitian kernel $\mathsf{K}_{\omega}(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m \times m}$ is said to have κ negative squares, if for every positive integer n and every choice of $\omega_{j} \in \Omega$ and $u_{j} \in \mathbb{C}^{m}$ $(j = 1, \dots, n)$ the matrix

$$(\langle \mathsf{K}_{\omega_{j}}(\omega_{k}) u_{j}, u_{k} \rangle)_{j,k=1}^{n}$$

has at most κ negative eigenvalues, and for some choice of $\omega_{1}, \dots, \omega_{n} \in \Omega$ and $u_{1}, \dots, u_{n} \in \mathbb{C}^{m}$ exactly κ negative eigenvalues.

Let $\mathcal{S}_{\kappa}^{q \times p}$ denote *the generalized Schur class* of $q \times p$ mvf's s that are meromorphic in Ω_{+} and for which the kernel

$$(2.1) \qquad \qquad \Lambda_{\omega}^{s}(\lambda) = \frac{I_{p} - s(\lambda)s(\omega)^{*}}{\rho_{\omega}(\lambda)}$$

has κ negative squares on $\mathfrak{h}_{s}^{+} \times \mathfrak{h}_{s}^{+}$ (see [16]).

In the case where $\kappa = 0$ the class $\mathcal{S}_{0}^{q \times p}$ coincides with the Schur class $\mathcal{S}^{q \times p}$ of contractive mvf's holomorphic in Ω_{+}. Every mvf $s \in \mathcal{S}^{p \times p}$ with $\det s(\lambda) \not\equiv 0$ admits an inner-outer factorization

$$s = b_{\ell} a_{\ell} = a_{r} b_{r},$$

where $b_{\ell}, b_{r} \in \mathcal{S}_{\text{in}}^{p \times p}$, $a_{\ell}, a_{r} \in \mathcal{S}_{\text{out}}^{p \times p}$.

Let $b_\omega(\lambda)$, be an elementary Blaschke factor ($b_\omega(\lambda) = \frac{\lambda-\omega}{1-\lambda\omega^*}$ in the case $\Omega_+ = \mathbb{D}$, $b_\omega(\lambda) = \frac{\lambda-\omega}{\lambda-\omega^*}$ in the case $\Omega_+ = \mathbb{C}_+$), and let P be an orthogonal projection in $\mathbb{C}^p$. Then the mvf

$$B_\alpha(\lambda) = I_p - P + b_\alpha(\lambda)P, \quad \omega \in \Omega_+,$$

belongs to the Schur class $\mathcal{S}^{p\times p}$ and is called *an elementary Blaschke–Potapov (BP) factor* and $B(\lambda)$ is called *primary* if rank $P = 1$. The product

$$B(\lambda) = \prod_{j=1}^{\overset{\curvearrowright}{\kappa}} B_{\alpha_j}(\lambda),$$

where $B_{\alpha_j}(\lambda)$ are primary Blaschke–Potapov factors is called *a Blaschke–Potapov product* of degree κ.

As shown in [16] every mvf $s \in \mathcal{S}_\kappa^{q\times p}$ admits a factorization of the form

$$(2.2) \qquad\qquad s(\lambda) = b_\ell(\lambda)^{-1}s_\ell(\lambda), \quad \lambda \in \mathfrak{h}_s^+,$$

where $b_\ell \in \mathcal{S}^{q\times q}$ is a $q \times q$ Blaschke–Potapov product of degree κ, s_ℓ is in the Schur class $\mathcal{S}^{q\times p}$ and

$$(2.3) \qquad\qquad \text{rank} \begin{bmatrix} b_\ell(\lambda) & s_\ell(\lambda) \end{bmatrix} = q \quad (\lambda \in \Omega_+).$$

The representation (2.2) is called a *left Kreĭn–Langer factorization*.

Similarly, every generalized Schur function $s \in \mathcal{S}_\kappa^{q\times p}$ admits a *right Kreĭn-Langer factorization*

$$(2.4) \qquad\qquad s(\lambda) = s_r(\lambda)b_r(\lambda)^{-1} \quad \text{for} \quad \lambda \in \mathfrak{h}_s^+,$$

where $b_r \in \mathcal{S}^{p\times p}$ is a Blaschke–Potapov product of degree κ, $s_r \in \mathcal{S}^{q\times p}$ and

$$(2.5) \qquad\qquad \text{rank} \begin{bmatrix} b_r(\lambda)^* & s_r(\lambda)^* \end{bmatrix} = p \quad (\lambda \in \Omega_+).$$

As is known (see [8]) the factors b_ℓ and s_ℓ in (2.2) meet the rank condition (2.3) if and only if the factorization (2.2) is left coprime, i.e., there exists a pair of mvf's $c_\ell \in H_\infty^{q\times q}$ and $d_\ell \in H_\infty^{q\times p}$ such that

$$(2.6) \qquad\qquad b_\ell(\lambda)c_\ell(\lambda) + s_\ell(\lambda)d_\ell(\lambda) = I_q \quad \text{for} \quad \lambda \in \Omega_+.$$

Therefore mvf's c_ℓ and d_ℓ do not have a common right inner divisor.

Similarly, the factors b_r and s_r in (2.4) meet the rank condition (2.3) if and only if the factorization (2.4) is right coprime, i.e., there exists a pair of mvf's $c_r \in H_\infty^{p\times p}$ and $d_r \in H_\infty^{p\times q}$ such that

$$(2.7) \qquad\qquad c_r(\lambda)b_r(\lambda) + d_r(\lambda)s_r(\lambda) = I_p \quad \text{for} \quad \lambda \in \Omega_+.$$

Therefore mvf's c_r and d_r don't have a common left inner divisor.

Theorem 2.2. ([13]). *Let* $s \in \mathcal{S}_\kappa^{q\times p}$ *have Kreĭn-Langer factorizations*

$$(2.8) \qquad\qquad s = b_\ell^{-1}s_\ell = s_r b_r^{-1}.$$

Then there exists a set of mvf's $c_\ell = c_\ell(s) \in H_\infty^{q\times q}$, $d_\ell = d_\ell(s) \in H_\infty^{p\times q}$, $c_r = c_r(s) \in H_\infty^{p\times p}$ *and* $d_r = d_r(s) \in H_\infty^{p\times q}$, *such that*

$$(2.9) \qquad\qquad \begin{bmatrix} c_r & d_r \\ -s_\ell & b_\ell \end{bmatrix} \begin{bmatrix} b_r & -d_\ell \\ s_r & c_\ell \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}.$$

3. Generalized j_{pq}-inner mvf's

Let j_{pq} be an $m \times m$ signature matrix

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad \text{where} \quad p + q = m,$$

Definition 3.1. ([13]). An $m \times m$ mvf $W(\lambda) = [w_{ij}(\lambda)]_{i,j=1}^2$ that is meromorphic in Ω_+ is said to belong to the class $\mathcal{U}_\kappa(j_{pq})$ of *generalized j_{pq}-inner* mvf's, if

(i) the kernel

(3.1) $$\mathsf{K}_\omega^W(\lambda) = \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)}$$

has κ negative squares in $\mathfrak{h}_W^+ \times \mathfrak{h}_W^+$;

(ii) $j_{pq} - W(\mu)j_{pq}W(\mu)^* = 0$ a.e. on Ω_0.

As is known [2, Th. 6.8], for every $W \in \mathcal{U}_\kappa(j_{pq})$, $w_{22}(\lambda)$ is invertible for all $\lambda \in \mathfrak{h}_W^+$ except for at most κ points in Ω_+. Thus the Potapov-Ginzburg transform of W

(3.2) $$S(\lambda) = PG(W) := \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix}^{-1}$$

is well defined for those $\lambda \in \mathfrak{h}_W^+$ for which $w_{22}(\lambda)$ is invertible. As is easily seen, $S(\lambda)$ belongs to the class $\mathcal{S}_\kappa^{m \times m}$ and $S(\mu)$ is unitary for a.e. $\mu \in \Omega_0$ (see [2], [13]).

3.1. The class $\mathcal{U}_\kappa^r(j_{pq})$.

Definition 3.2. ([13]). An $m \times m$ mvf $W \in \mathcal{U}_\kappa(j_{pq})$ is said to be in the class $\mathcal{U}_\kappa^r(j_{pq})$, if

(3.3) $$s_{21} := -w_{22}^{-1}w_{21} \in \mathcal{S}_\kappa^{q \times p}.$$

Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and let the Kreĭn-Langer factorization of s_{21} be written as

(3.4) $$s_{21}(\lambda) = b_\ell(\lambda)^{-1}s_\ell(\lambda) = s_r(\lambda)b_r(\lambda)^{-1} \quad (\lambda \in \mathfrak{h}_{s_{21}}^+),$$

where $b_\ell \in \mathcal{S}_{\text{in}}^{q \times q}$, $b_r \in \mathcal{S}_{\text{in}}^{p \times p}$, $s_\ell, s_r \in \mathcal{S}^{q \times p}$. Then, as was shown in [13], the mvf's $b_\ell s_{22}$ and $s_{11}b_r$ are holomorphic in Ω_+, and

(3.5) $$b_\ell s_{22} \in \mathcal{S}^{q \times q} \quad \text{and} \quad s_{11}b_r \in \mathcal{S}^{p \times p}.$$

Definition 3.3. ([13]). Consider the inner-outer factorizations of $s_{11}b_r$ and $b_\ell s_{22}$

(3.6) $$s_{11}b_r = b_1 a_1, \quad b_\ell s_{22} = a_2 b_2,$$

where $b_1 \in \mathcal{S}_{\text{in}}^{p \times p}$, $b_2 \in \mathcal{S}_{\text{in}}^{q \times q}$, $a_1 \in \mathcal{S}_{\text{out}}^{p \times p}$, $a_2 \in \mathcal{S}_{\text{out}}^{q \times q}$. The pair b_1, b_2 of inner factors in the factorizations (3.6) is called the associated pair of the mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$ and is written as $\{b_1, b_2\} \in \text{ap}^r(W)$, for short.

From now onwards this pair $\{b_1, b_2\}$ will be called also a right associated pair since it is related to the right linear fractional transformation

$$T_W[\varepsilon] = (w_{11}\varepsilon + w_{12})(w_{21}\varepsilon + w_{22})^{-1},$$

see [6], [8]. Such transformations play an important role in describing solutions of different interpolation problems, see [1], [6]–[10], [4], [12], [14]. In case $\kappa = 0$ a definition of associated pair was introduced in [6].

As was shown in [13] for every $W \in \mathcal{U}_\kappa^r(j_{pq})$ and c_r, d_r, c_ℓ and d_ℓ as in (2.9) the mvf

(3.7) $$K_r = K_r^W := (-w_{11}d_\ell + w_{12}c_\ell)(-w_{21}d_\ell + w_{22}c_\ell)^{-1},$$

belongs to $H_\infty^{p \times q}$ and admits the representations

(3.8) $$K_r = (-w_{11}d_\ell + w_{12}c_\ell)a_2 b_2 = b_1 a_1 (c_r w_{21}^\# - d_r w_{22}^\#),$$

where $\{b_1, b_2\} \in \mathrm{ap}^r(W)$. It is clear that $K_r^\# \in H_\infty^{q \times p}(\Omega_-)$.

In the future we shall need the following factorizations of the mvf $W \in \mathcal{U}_\kappa^r(j_{pq})$, which were obtained in [13].

Theorem 3.4. ([13]). *Let* $W \in \mathcal{U}_\kappa^r(j_{pq})$, $\{b_1, b_2\} \in ap^r(W)$ *and let* b_ℓ, s_ℓ, b_r, s_r *be defined by the Kreĭn-Langer factorization (3.4). Then* W *admits the factorization*

$$(3.9) \qquad W = \begin{bmatrix} b_1 & 0 \\ 0 & b_2^{-1} \end{bmatrix} \begin{bmatrix} a_1^{-*} & 0 \\ 0 & a_2^{-1} \end{bmatrix} \begin{bmatrix} b_r^* & -s_r^* \\ -s_\ell & b_\ell \end{bmatrix} \quad a.e. \quad in \quad \Omega_0.$$

Theorem 3.5. ([13]). *Let* $W \in \mathcal{U}_\kappa^r(j_{pq})$, $\{b_1, b_2\} \in ap^r(W)$, *let* K_r *be defined by (3.7), c_r, d_r, c_ℓ and d_ℓ be as in Theorem 2.2. Then* W *admits the factorization*

$$(3.10) \qquad W = \Theta\, \Phi \quad in \quad \Omega_+ \quad and \quad W = \Theta^- \Phi^- \quad in \quad \Omega_-,$$

where

$$(3.11) \qquad \Theta = \begin{bmatrix} b_1 & K_r b_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix} \quad in \quad \Omega_+, \quad \Theta^- = \begin{bmatrix} b_1 & 0 \\ K_r^\# b_1 & b_2^{-1} \end{bmatrix} \quad in \quad \Omega_-,$$

$$(3.12) \qquad \Phi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2^{-1} \end{bmatrix} \begin{bmatrix} c_r & d_r \\ -s_\ell & b_\ell \end{bmatrix} \quad in \quad \Omega_+,$$

$$(3.13) \qquad \Phi^- = \begin{bmatrix} \varphi_{11}^- & \varphi_{12}^- \\ \varphi_{21}^- & \varphi_{22}^- \end{bmatrix} = \begin{bmatrix} a_1^{-\#} & 0 \\ 0 & a_2^\# \end{bmatrix} \begin{bmatrix} b_r^\# & -s_r^\# \\ d_\ell^\# & c_\ell^\# \end{bmatrix} \quad in \quad \Omega_-.$$

Moreover, Φ *and* Φ^- *are invertible in* Ω_+ *and* Ω_-, *respectively, and*

$$(3.14) \qquad \begin{aligned} \Phi^{-1} &= \begin{bmatrix} b_r & -d_\ell \\ s_r & c_\ell \end{bmatrix} \begin{bmatrix} a_1^{-1} & 0 \\ 0 & a_2 \end{bmatrix} \quad in \quad \Omega_+, \\ (\Phi^-)^{-1} &= \begin{bmatrix} c_r^\# & s_\ell^\# \\ -d_r^\# & b_\ell^\# \end{bmatrix} \begin{bmatrix} a_1^\# & 0 \\ 0 & a_2^{-\#} \end{bmatrix} \quad in \quad \Omega_-. \end{aligned}$$

It follows from (3.12)–(3.14) that,

$$(3.15) \qquad \Phi \in \mathcal{N}_{\mathrm{out}}^{m \times m}(\Omega_+), \quad \Phi^- \in \mathcal{N}_{\mathrm{out}}^{m \times m}(\Omega_-) \quad (m = p + q).$$

3.2. **The class** $\mathcal{U}_\kappa^\ell(j_{pq})$.

Definition 3.6. *An* $m \times m$ *mvf* $W \in \mathcal{U}_\kappa(j_{pq})$ *is said to be in the class* $\mathcal{U}_\kappa^\ell(j_{pq})$, *if*

$$(3.16) \qquad s_{12} := w_{12} w_{22}^{-1} \in \mathcal{S}_\kappa^{q \times p}.$$

The classes $\mathcal{U}_\kappa^\ell(j_{pq})$ and $\mathcal{U}_\kappa^r(j_{pq})$ do not coincide.

Example 1. Let $W = \begin{bmatrix} \sqrt{2} & -\lambda \\ -1 & \sqrt{2}\lambda \end{bmatrix}$, then $S = PG(W) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ \frac{1}{\lambda} & \frac{1}{\lambda} \end{bmatrix}$, i.e., $s_{21} \in \mathcal{S}_1$ and $s_{12} \in \mathcal{S}$, therefore $W \in \mathcal{U}_1^r(j_{11})$, but $W \notin \mathcal{U}_1^\ell(j_{11})$.

Introduce the notation

$$(3.17) \qquad \widetilde{W}(\lambda) = \begin{cases} W(\bar{\lambda})^*, & \text{if} \quad \Omega_+ = \mathbb{D}, \\ W(-\bar{\lambda})^*, & \text{if} \quad \Omega_+ = \mathbb{C}_+. \end{cases}$$

Proposition 3.7. *Let* $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ *and let* $S(\lambda) = [s_{ij}(\lambda)]_{i,j=1}^2$ *be its PG-transform. Then* $\widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq})$ *and* $\widehat{S}(\lambda) = PG(\widetilde{W})$ *has the form*

$$(3.18) \qquad \widehat{S}(\lambda) = \begin{bmatrix} \widetilde{s}_{11}(\lambda) & -\widetilde{s}_{21}(\lambda) \\ -\widetilde{s}_{12}(\lambda) & \widetilde{s}_{22}(\lambda) \end{bmatrix}.$$

Proof. If $\widehat{S}(\lambda) = PG(\widetilde{W}) = \begin{bmatrix} \widehat{s}_{11}(\lambda) & \widehat{s}_{21}(\lambda) \\ \widehat{s}_{12}(\lambda) & \widehat{s}_{22}(\lambda) \end{bmatrix}$ then

$$\widehat{s}_{21}(\lambda) = -\widetilde{w}_{22}(\lambda)^{-1}\widetilde{w}_{12}(\lambda) = -(w_{12}w_{22}^{-1})^{\sim} = -\widetilde{s}_{12}(\lambda) \in \mathcal{S}_\kappa.$$

Therefore $\widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq})$.

Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and let the Kreĭn-Langer factorization of s_{12} be written as

(3.19) $$s_{12}(\lambda) = \beta_\ell(\lambda)^{-1}\sigma_\ell(\lambda) = \sigma_r(\lambda)\beta_r(\lambda)^{-1}, \quad (\lambda \in \mathfrak{h}_{s_{12}}^+),$$

where $\beta_\ell \in \mathcal{S}_{\text{in}}^{q \times q}$, $\beta_r \in \mathcal{S}_{\text{in}}^{p \times p}$, $\sigma_\ell, \sigma_r \in \mathcal{S}^{q \times p}$. Then

(3.20) $$\widehat{s}_{21}(\lambda) = -\widetilde{s}_{12}(\lambda) = -\widetilde{\sigma}_\ell(\lambda)\widetilde{\beta}_\ell(\lambda)^{-1} = -\widetilde{\beta}_r(\lambda)^{-1}\widetilde{\sigma}_r(\lambda),$$

(3.21) $$\widehat{s}_{22}(\lambda) = \widetilde{w}_{22}(\lambda)^{-1} = \widetilde{s}_{22}(\lambda),$$

(3.22) $$\begin{aligned} \widehat{s}_{11}(\lambda) &= \widetilde{w}_{11}(\lambda) - \widetilde{w}_{21}(\lambda)\widetilde{w}_{22}(\lambda)^{-1}\widetilde{w}_{12}(\lambda) \\ &= \widetilde{w}_{11}(\lambda) - (w_{12}(\lambda)w_{21}(\lambda)^{-1}w_{21}(\lambda))^{\sim} = \widetilde{s}_{11}(\lambda). \end{aligned}$$

$\square$

Theorem 3.8. *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and let the Blaschke-Potapov factors β_ℓ and β_r be defined by the Kreĭn-Langer factorizations (3.19) of s_{21}. Then*

(3.23) $$s_{22}\beta_r \in \mathcal{S}^{q \times q} \quad \text{and} \quad \beta_\ell s_{11} \in \mathcal{S}^{p \times p}.$$

Proof. It follows from (3.5), (3.21) and (3.22) that

$$\widetilde{\beta}_r\widetilde{s}_{22} = (s_{22}\beta_r)^{\sim} \in \mathcal{S}^{q \times q}, \quad \widetilde{s}_{11}\widetilde{\beta}_\ell = (\beta_\ell s_{11})^{\sim} \in \mathcal{S}^{p \times p},$$

Therefore, $s_{22}\beta_r \in \mathcal{S}^{q \times q}$ and $\beta_\ell s_{11} \in \mathcal{S}^{p \times p}$.

$\square$

Definition 3.9. *Consider inner-outer factorizations of $\beta_\ell s_{11}$ and $s_{22}\beta_r$,*

(3.24) $$\beta_\ell s_{11} = \alpha_1\beta_1, \quad s_{22}\beta_r = \beta_2\alpha_2,$$

where $\beta_1 \in \mathcal{S}_{\text{in}}^{p \times p}$, $\beta_2 \in \mathcal{S}_{\text{in}}^{q \times q}$, $\alpha_1 \in \mathcal{S}_{\text{out}}^{p \times p}$, $\alpha_2 \in \mathcal{S}_{\text{out}}^{q \times q}$. The pair β_1, β_2 of inner factors in the factorizations (3.24) is called the left associated pair of the mvf $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ and is written as $\{\beta_1, \beta_2\} \in \text{ap}^\ell(W)$, for short.

If $\{\beta_1, \beta_2\} \in \text{ap}^\ell(W)$, then

(3.25) $$\widetilde{s}_{11}\widetilde{\beta}_\ell = \widetilde{\beta}_1\widetilde{\alpha}_1, \quad \widetilde{\beta}_r\widetilde{s}_{22} = \widetilde{\alpha}_2\widetilde{\beta}_2$$

and, therefore, $\{\widetilde{\beta}_1, \widetilde{\beta}_2\} \in \text{ap}^r(\widetilde{W})$.

Applying equation (2.9) to mvf $\widehat{s}_{21}$ one gets mvf's $\widehat{\gamma}_\ell, \widehat{\delta}_\ell, \widehat{\gamma}_r, \widehat{\delta}_r$ such that

(3.26) $$\begin{bmatrix} \widehat{\gamma}_\ell & \widehat{\delta}_\ell \\ \widehat{\sigma}_r & \widehat{\beta}_r \end{bmatrix} \begin{bmatrix} \widehat{\beta}_\ell & -\widehat{\delta}_r \\ -\widehat{\sigma}_r & \widehat{\gamma}_r \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}.$$

Then by Theorem 3.4

$$\widetilde{W} = \begin{bmatrix} \widetilde{w}_{11} & \widetilde{w}_{21} \\ \widetilde{w}_{12} & \widetilde{w}_{22} \end{bmatrix} = \begin{bmatrix} \widetilde{\beta}_1 & 0 \\ 0 & \widetilde{\beta}_2^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{\alpha}_1^{-*} & 0 \\ 0 & \widetilde{\alpha}_2^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{\beta}_\ell^* & \widetilde{\sigma}_\ell^* \\ \widetilde{\sigma}_r & \widetilde{\sigma}_r \end{bmatrix}.$$

Passing to the W one obtains

$$W = \begin{bmatrix} \beta_\ell^* & \sigma_r \\ \sigma_\ell^* & \beta_r \end{bmatrix} \begin{bmatrix} \alpha_1^{-*} & 0 \\ 0 & \alpha_2^{-1} \end{bmatrix} \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2^{-1} \end{bmatrix}.$$

Note that equation (3.26) is equivalent to (3.9).

Since $\widetilde{W} \in \mathcal{U}_\kappa^r(j_{pq})$, $(\widetilde{\beta}_1, \widetilde{\beta}_2) \in \text{ap}^r(\widetilde{W})$ and

$$c_\ell(\widehat{s}_{21}) = \widetilde{\gamma}_r, \quad d_\ell(\widehat{s}_{21}) = \widetilde{\delta}_r,$$

$$c_r(\widehat{s}_{21}) = \widetilde{\gamma}_\ell, \quad d_r(\widehat{s}_{21}) = \widetilde{\delta}_\ell,$$

then the mvf

$$\widetilde{K}_\ell := K_r^{\widetilde{W}} = (-\widetilde{w}_{11}\widetilde{\delta}_r + \widetilde{w}_{21}\widetilde{\gamma}_r)(-\widetilde{w}_{12}\widetilde{\delta}_r + \widetilde{w}_{22}\widetilde{\gamma}_r)^{-1}$$

belongs to $H_\infty^{p \times q}$ and in view of (3.8) it admits the representations

$$\widetilde{K}_\ell = (-\widetilde{w}_{11}\widetilde{\delta}_r + \widetilde{w}_{21}\widetilde{\gamma}_r)\widetilde{\alpha}_2\widetilde{\beta}_2 = \widetilde{\beta}_1\widetilde{\alpha}_1(\widetilde{\gamma}_\ell\widetilde{w}_{12}^{\#} - \widetilde{\delta}_\ell\widetilde{w}_{22}^{\#}).$$

Hence the mvf $K_\ell = (K_r^{\widetilde{W}})\widetilde{}$ can be represented as

$$(3.27) \qquad K_\ell = \beta_2\alpha_2(-\delta_r w_{11} + \gamma_r w_{21}) = (w_{12}^{\#}\gamma_\ell - w_{22}^{\#}\delta_\ell)\alpha_1\beta_1 \in H_\infty^{q \times p}.$$

By Theorem 3.5 $\widetilde{W}$ admits the factorization

$$(3.28) \qquad \widetilde{W} = \widetilde{\Theta}_1\widetilde{\Phi}_1, \quad \text{in} \quad \Omega_+,$$

where

$$(3.29) \qquad \widetilde{\Theta}_1 = \begin{bmatrix} \widetilde{\beta}_1 & \widetilde{K}_\ell\widetilde{\beta}_2^{-1} \\ 0 & \widetilde{\beta}_2^{-1} \end{bmatrix}, \quad \widetilde{\Phi}_1 = \begin{bmatrix} \widetilde{\alpha}_1 & 0 \\ 0 & \widetilde{\alpha}_2^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{\gamma}_\ell & \widetilde{\delta}_\ell \\ \widetilde{\sigma}_r & \widetilde{\beta}_r \end{bmatrix}.$$

Hence

$$(3.30) \qquad W = \Phi_1\Theta_1 = \begin{bmatrix} \gamma_\ell & \sigma_r \\ \delta_\ell & \beta_r \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2^{-1} \end{bmatrix} \begin{bmatrix} \beta_1 & 0 \\ \beta_2^{-1}K_\ell & \beta_2^{-1} \end{bmatrix}.$$

Similarly, the second identity in (3.9) has the form

$$\widetilde{W} = \widetilde{\Theta}_1^-\widetilde{\Phi}_1^-,$$

where

$$\widetilde{\Theta}_1^- = \begin{bmatrix} \widetilde{\beta}_1 & 0 \\ \widetilde{K}_\ell^{\#}\widetilde{\beta}_1 & \widetilde{\beta}_2^{-1} \end{bmatrix}, \quad \widetilde{\Phi}_1^- = \begin{bmatrix} \widetilde{\alpha}_1^{-\#} & 0 \\ 0 & \widetilde{\alpha}_2^{\#} \end{bmatrix} \begin{bmatrix} \widetilde{\beta}_\ell^{\#} & \widetilde{\sigma}_\ell^{\#} \\ \widetilde{\delta}_r^{\#} & \widetilde{\gamma}_r^{\#} \end{bmatrix}.$$

Therefore

$$(3.31) \qquad W = \Phi_1^-\Theta_1^- = \begin{bmatrix} \beta_\ell^{\#} & \delta_r^{\#} \\ \sigma_\ell^{\#} & \gamma_r^{\#} \end{bmatrix} \begin{bmatrix} \alpha_1^{-\#} & 0 \\ 0 & \alpha_2^{\#} \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_1 K_\ell^{\#} \\ 0 & \beta_2^{-1} \end{bmatrix}.$$

Thus one obtains the following.

Theorem 3.10. Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$, let $\{\beta_1, \beta_2\} \in ap^\ell(W)$ and let K_ℓ be defined as in (3.27). Then W admit the factorizations

$$(3.32) \qquad W = \Phi_1\Theta_1 \quad \text{in} \quad \Omega_+ \quad \text{and} \quad W = \Phi_1^-\Theta_1^- \quad \text{in} \quad \Omega_+,$$

where

$$(3.33) \qquad \Theta_1 = \begin{bmatrix} \beta_1 & 0 \\ \beta_2^{-1}K_\ell & \beta_2^{-1} \end{bmatrix}, \quad \Theta_1^- = \begin{bmatrix} \beta_1 & \beta_1 K_\ell^{\#} \\ 0 & \beta_2^{-1} \end{bmatrix},$$

$$(3.34) \qquad \Phi_1 = \begin{bmatrix} \gamma_\ell & \sigma_r \\ \delta_\ell & \beta_r \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2^{-1} \end{bmatrix}, \quad \Phi_1^- = \begin{bmatrix} \beta_\ell^{\#} & \delta_r^{\#} \\ \sigma_\ell^{\#} & \gamma_r^{\#} \end{bmatrix} \begin{bmatrix} \alpha_1^{-\#} & 0 \\ 0 & \alpha_2^{\#} \end{bmatrix}.$$

Similarly, Theorem 3.4 and formulas (3.25), (3.26) yield

Theorem 3.11. Let $W \in \mathcal{U}_\kappa^\ell(j_{pq})$, and let $\{\beta_1, \beta_2\} \in ap^\ell(W)$. Then W can be expressed in terms of the factors in (3.24) as follows:

$$(3.35) \qquad W = \begin{bmatrix} \beta_\ell^* & \sigma_r \\ \sigma_\ell^* & \beta_r \end{bmatrix} \begin{bmatrix} \alpha_1^{-*} & 0 \\ 0 & \alpha_2^{-1} \end{bmatrix} \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2^{-1} \end{bmatrix} \quad a.e. \quad \text{in} \quad \Omega_0.$$

3.3. **The class $\mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{U}_\kappa^r(j_{pq})$.** Let $G(\lambda)$ be a $p \times q$ mvf that is meromorphic on Ω_+ with a Laurent expansion

(3.36) $$G(\lambda) = (\lambda - \lambda_0)^{-k}G_{-k} + \cdots + (\lambda - \lambda_0)^{-1}G_{-1} + G_0 + o(\lambda - \lambda_0)$$

in a neighborhood of a pole $\lambda_0 \in \Omega_+$, $G_{-j} \in \mathbb{C}^{p \times q}$ $(j = 0, 1, \ldots, k)$. The pole multiplicity $M_\pi(G, \lambda_0)$ is defined by (see [16])

(3.37) $$M_\pi(G, \lambda_0) = \mathrm{rank}L(G, \lambda_0), \quad L(G, \lambda_0) = \begin{bmatrix} G_{-k} & & \mathbf{0} \\ \vdots & \ddots & \\ G_{-1} & \ldots & G_{-k} \end{bmatrix}.$$

The pole multiplicity of G over Ω_+ is given by

(3.38) $$M_\pi(G, \Omega_+) = \sum_{\lambda \in \Omega_+} M_\pi(G, \lambda).$$

This definition of pole multiplicity coincides with the definition based on the Smith-McMillan representation of G (see [10]).

Proposition 3.12. ([13]). *Let $H_\ell, H_r \in H_\infty^{p \times q}$ and let $G_\ell \in H_\infty^{p \times p}$ and $G_r \in H_\infty^{q \times q}$ be a pair of mvf's such that $G_\ell^{-1} \in H_{\kappa,\infty}^{p \times p}$ and $G_r^{-1} \in H_{\kappa,\infty}$ for some $\kappa \in \mathbb{N} \cup \{0\}$. Then*

  (i) *The pair G_ℓ, H_ℓ is left coprime over $\Omega_+ \iff M_\pi(G_\ell^{-1}H_\ell, \Omega_+) = M_\pi(G_\ell^{-1}, \Omega_+)$.*

  (ii) *The pair G_r, H_r is right coprime over $\Omega_+ \iff M_\pi(H_rG_r^{-1}, \Omega_+) = M_\pi(G_r^{-1}, \Omega_+)$.*

Lemma 3.13. ([13]). *If $S = [s_{ij}]_1^2 \in \mathcal{S}_\kappa^{m \times m}$ and $s_{21} \in S_\kappa^{q \times p}$ and if $\begin{bmatrix} 0 & I_q \end{bmatrix} Sh \in H_2^q$ for some $h \in H_2^m$, then $Sh \in H_2^m$.*

Theorem 3.14. *Let $W \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{U}_\kappa^r(j_{pq})$, let $S(\lambda)$ be its Potapov-Ginzburg transform and let (b_1, b_2) and (β_1, β_2) be its right and left associated pairs defined by (3.6) and (3.24), respectively, and let the Blaschke-Potapov factors $b_\ell, b_r, \beta_\ell, \beta_r$ be defined by the Kreĭn-Langer factorizations (3.4) and (3.19). Then*

  (i) *the factorization $s_{22} = b_\ell^{-1}(a_2b_2)$ is left coprime over Ω_+,*
  (ii) *the factorization $s_{22} = (\beta_2\alpha_2)\beta_r^{-1}$ is right coprime over Ω_+,*
  (iii) *the factorization $s_{11} = \beta_\ell^{-1}(\alpha_1\beta_1)$ is left coprime over Ω_+,*
  (iv) *the factorization $s_{11} = (b_1a_1)b_r^{-1}$ is right coprime over Ω_+,*
  (v) *$\det b_1 = \theta_1 \det \beta_1$ and $\det b_2 = \theta_2 \det \beta_2$ for some $\theta_1, \theta_2 \in \mathbb{T}$. In particular,*

$$\mathfrak{h}_{b_1} = \mathfrak{h}_{\beta_1} \quad and \quad \mathfrak{h}_{b_2} = \mathfrak{h}_{\beta_2}.$$

Proof. (i) Let us denote by κ' the pole multiplicity of s_{22} over Ω_+,

(3.39) $$\kappa' := M_\pi(s_{22}, \Omega_+)(\leq \kappa),$$

and denote by θ the Blaschke-Potapov factor of degree κ' such that $s_{22}\theta \in S^{q \times q}$. Then $s_{22}\theta u \in H_2^q$ for every $u \in \mathbb{C}^q$. By Lemma 3.13 one gets the inclusion

(3.40) $$\begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} \theta u \in \mathcal{H}_2^m, \quad \text{for all} \quad u \in \mathbb{C}^q,$$

and hence

(3.41) $$M_\pi(s_{12}, \Omega_+) \leq M_\pi\left(\begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix}, \Omega_+ \right) = \kappa' \leq \kappa.$$

On the other hand $M_\pi(s_{12}, \Omega_+) = \kappa$, since $W \in \mathcal{U}_\kappa^\ell(j_{pq})$. This proves the equality

(3.42) $$M_\pi(s_{22}, \Omega_+) = \kappa = M_\pi(b_\ell^{-1}, \Omega_+).$$

By Proposition 3.12 this means that the factorization $s_{22} = b_\ell^{-1}(b_\ell s_{22})$ is left coprime, so by (2.8) the factorization $s_{22} = b_\ell^{-1}(a_2b_2)$ is left coprime, too.

(ii) Let the mvf $\widetilde{W}(\lambda)$ be given by (3.17) and let $\widehat{S}$ be the Potapov-Ginzburg transform of $\widetilde{W}(\lambda)$. Then by Proposition 3.7 $\widehat{S}$ takes the form (3.18) and, hence, $\widetilde{W} \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{U}_\kappa^r(j_{pq})$. Application of the statement proved in (i) shows that the factorization

$$\widetilde{s}_{22} = \widetilde{\beta}_r^{-1}(\widetilde{\alpha}_2 \widetilde{\beta}_2)$$

is left coprime over Ω_+. This implies (ii).

(iii) & (iv) Consider the mvf

$$U(\lambda) = \begin{bmatrix} 0 & I_q \\ I_p & 0 \end{bmatrix} W^\#(\lambda) \begin{bmatrix} 0 & I_p \\ I_q & 0 \end{bmatrix} = \begin{bmatrix} w_{22}^\#(\lambda) & w_{12}^\#(\lambda) \\ w_{21}^\#(\lambda) & w_{11}^\#(\lambda) \end{bmatrix}.$$

The Potapov-Ginzburg transform $S' = PG(U) = \begin{bmatrix} s'_{11} & s'_{12} \\ s'_{21} & s'_{22} \end{bmatrix}$ of this matrix takes the form

$$
\begin{aligned}
S' = PG(U) &= \begin{bmatrix} w_{22}^\#(\lambda) & w_{12}^\#(\lambda) \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ w_{21}^\#(\lambda) & w_{11}^\#(\lambda) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} w_{22}^\#(\lambda) - w_{12}^\#(\lambda)w_{11}^\#(\lambda)^{-1}w_{21}^\#(\lambda) & w_{12}^\#(\lambda)w_{11}^\#(\lambda)^{-1} \\ -w_{11}^\#(\lambda)^{-1}w_{21}^\#(\lambda) & w_{11}^\#(\lambda)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} s_{22}(\lambda) & -s_{21}(\lambda) \\ -s_{12}(\lambda) & s_{11}(\lambda) \end{bmatrix}.
\end{aligned}
$$

(3.43)

(see [8, Lemma 4.24]). Therefore, $U \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{U}_\kappa^r(j_{pq})$. Applying the statements (i) and (ii) to the mvf U one proves that the factorizations

(3.44) $s_{11} = \beta_\ell^{-1}(\alpha_1\beta_1) \quad \text{and} \quad s_{11} = (b_1 a_1)b_r^{-1}$

are left coprime and right coprime, respectively, over Ω_+.

(v) By (3.6), (3.24),

(3.45) $\det s_{11} \det b_r = \det b_1 \det a_1, \quad \det s_{11} \det \beta_\ell = \det \alpha_1 \det \beta_1.$

Since β_ℓ and b_r are left and right inner factors in the Kreĭn-Langer factorization (3.44), we see that $\det b_r = \theta \det \beta_\ell$ for some $\theta \in \mathbb{T}$ (see [3]). Therefore, the formulas in (3.45) represent two inner-outer factorizations of the same function. The uniqueness of inner-outer factorization implies that

(3.46) $\det b_1 = \theta_1 \det \beta_1$

for some $\theta_1 \in \mathbb{T}$. Similarly as $s_{22}\beta_r = \beta_2\alpha_2$ and $b_\ell s_{22} = a_2 b_2$,

$$\det s_{22} \det b_\ell = \det b_2 \det a_2, \quad \det s_{22} \det \beta_r = \det \beta_2 \det \alpha_2$$

and hence, using the equality $\det b_\ell = \vartheta \beta_r$ $(\vartheta \in \mathbb{T})$ one obtains for some $\theta_2 \in \mathbb{T}$

(3.47) $\det b_2 = \theta_2 \det \beta_2.$

Equalities (3.46) and (3.47) imply that $\mathfrak{h}_{b_1} = \mathfrak{h}_{\beta_1}$, $\mathfrak{h}_{b_2} = \mathfrak{h}_{\beta_2}$.   □

Example 2. Let a 4×4 mvf $W(\lambda)$ be given by

$$W(\lambda) = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix} = \begin{bmatrix} \frac{4-\lambda}{3\lambda} & 0 & 0 & \frac{2\lambda-2}{3\lambda} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2-2\lambda}{3\lambda} & 0 & 0 & \frac{4\lambda-1}{3\lambda} \end{bmatrix}.$$

Then the Potapov-Ginzburg transformation $S = PG(W)$ of W takes the form

$$S(\lambda) = \begin{bmatrix} \frac{4-\lambda}{3\lambda} & 0 & 0 & \frac{2\lambda-2}{3\lambda} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2-2\lambda}{3\lambda} & 0 & 0 & \frac{4\lambda-1}{3\lambda} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{4\lambda-1} & 0 & 0 & \frac{2\lambda-2}{4\lambda-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2\lambda-2}{4\lambda-1} & 0 & 0 & \frac{3\lambda}{4\lambda-1} \end{bmatrix}.$$

The mvf $s_{21}(\lambda)$ admits the left Kreĭn–Langer factorization

$$s_{21} = \begin{bmatrix} 0 & 0 \\ \frac{2\lambda-2}{4\lambda-1} & 0 \end{bmatrix} = b_\ell^{-1} s_\ell = \begin{bmatrix} 1 & 0 \\ 0 & \frac{4\lambda-1}{4-\lambda} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ \frac{2\lambda-2}{4-\lambda} & 0 \end{bmatrix}$$

with a Blaschke–Potapov factor b_ℓ of degree 1 and hence $s_{21} \in \mathcal{S}_1^{2\times2}$.
   Thus, considering the outer-inner factorization of $b_\ell s_{22}$

$$b_\ell s_{22} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{4\lambda-1}{4-\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{3\lambda}{4\lambda-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3\lambda}{4-\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4-\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} = a_2 b_2,$$

one obtains $b_2 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$.
   On the other hand the mvf s_{12} admits the right Kreĭn–Langer factorization

$$s_{12} = \begin{bmatrix} 0 & \frac{2\lambda-2}{4\lambda-1} \\ 0 & 0 \end{bmatrix} = \sigma_r \beta_r^{-1} = \begin{bmatrix} 0 & \frac{2\lambda-2}{4-\lambda} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{4\lambda-1}{4-\lambda} \end{bmatrix}^{-1},$$

and (3.24) takes the form

$$s_{22}\beta_r = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3\lambda}{4\lambda-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{4\lambda-1}{4-\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3\lambda}{4-\lambda} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4-\lambda} \end{bmatrix} = \beta_2 \alpha_2.$$

Therefore $\beta_2 = b_2 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$.
   Similarly, one obtains $b_1 = \beta_1 = I_2$.

   Modifying this example one can get a mvf $W \in \mathcal{U}_\kappa^\ell(j_{pq}) \cap \mathcal{U}_\kappa^r(j_{pq})$ such that the left and the right associated pairs do not coincide.

Example 3. Let U be a unitary 2×2 matrix and let

$$\widetilde{W}(\lambda) = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda)U \\ w_{21}(\lambda) & w_{22}(\lambda)U \end{bmatrix},$$

where w_{ij} are as in Example 2. Then the corresponding Potapov-Ginzburg transformation $\widetilde{S} = PG(\widetilde{W})$ of $\widetilde{W}$ takes the form

$$\widetilde{S}(\lambda) = \begin{bmatrix} \widetilde{s}_{11}(\lambda) & \widetilde{s}_{12}(\lambda) \\ \widetilde{s}_{21}(\lambda) & \widetilde{s}_{22}(\lambda) \end{bmatrix} = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ U^{-1}s_{21}(\lambda) & U^{-1}s_{22}(\lambda) \end{bmatrix}$$

and hence

$$\widetilde{s}_{21} = (b_\ell U)^{-1} s_\ell, \quad \widetilde{b}_\ell = b_\ell U, \quad \widetilde{s}_{12} = \sigma_r \beta_r^{-1}, \quad \widetilde{\beta}_r = \beta_r.$$

Since

$$\widetilde{b}_\ell \widetilde{s}_{22} = b_\ell s_{22} = a_2 b_2 \quad \text{and} \quad \widetilde{s}_{22}\widetilde{\beta}_r = U^{-1}s_{22}\beta_r = U^{-1}\beta_2\alpha_2,$$

we see that

$$\widetilde{b}_2 = b_2 = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \quad \text{and} \quad \widetilde{\beta}_2 = U^{-1}\beta_2 = U^{-1}\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}.$$

4. SINGULAR GENERALIZED J-INNER MVF'S

4.1. Definition of singular generalized J-inner mvf.

Definition 4.1. *A mvf $U \in \mathcal{U}_\kappa(J)$ is said to be singular, if $U, U^{-1} \in \mathcal{N}_+^{m \times m}$. The class of singular generalized J-inner mvf's will be denoted by $\mathcal{U}_{\kappa,S}(J)$.*

In the case $\kappa = 0$ this definition was introduced by D. Arov in [7]. The simplest examples of singular J-inner mvf's are the elementary BP factors of the third and fourth kind (see [7]). We will present below an example of a singular generalized J-inner mvf in the case $\kappa = 1$.

Example 4. Let $W(\lambda) = I_m - \frac{1+\lambda}{1-\lambda} \frac{\delta}{2} u u^* J$, $\Omega_+ = \mathbb{D}$, where

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \delta = -1.$$

Then

$$W(\lambda) = \frac{1}{2(1-\lambda)} \begin{bmatrix} 3-\lambda & -1-\lambda \\ 1+\lambda & 1-3\lambda \end{bmatrix},$$

$$W(\lambda) J W(\mu)^* = \frac{1}{(1-\lambda)(1-\mu^*)} \begin{bmatrix} 2-\lambda-\mu^* & 1-\lambda\mu^* \\ 1-\lambda\mu^* & \lambda+\mu^*-2\lambda\mu^* \end{bmatrix},$$

and

$$(4.1) \qquad K_\mu^W(\lambda) = \frac{1}{(1-\lambda)(1-\mu^*)} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

The kernel $K_\mu^W(\lambda)$ has 1 negative square in $\mathbb{D}$ since

$$(4.2) \qquad K_\mu^W(\lambda) = -f(\lambda) f(\mu)^*, \quad \text{with} \quad f(\lambda) = \frac{1}{1-\lambda} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Moreover, $J - W(\mu) J W(\mu)^* = O$ for $\mu \in \mathbb{T} \setminus \{1\}$, and, therefore, $W \in \mathcal{U}_1^r(J)$.
Next, since $\det W(\lambda) \equiv 1$, W and W^{-1} are outer, and thus, $W \in \mathcal{U}_{1,S}(J)$.

Other examples of singular generalized J-inner mvf's for $J = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ can be found in [11].

Proposition 4.2. *Let $U = U_1 U_2$, where $U_i \in \mathcal{U}_{\kappa_i,S}(J)$, $i = 1, 2$, are singular generalized J-inner mvf's. Then $U \in \mathcal{U}_{\kappa,S}(J)$ for some $\kappa \in \mathbb{Z}_+$ such that $\kappa \leq \kappa_1 + \kappa_2$.*

Proof. Let U_1, U_2 be singular generalized J-inner mvf's, then $U_1, U_1^{-1} \in \mathcal{N}_+^{m \times m}$ and $U_2, U_2^{-1} \in \mathcal{N}_+^{m \times m}$, hence $U = U_1 U_2$ belongs to $\mathcal{N}_+^{m \times m}$, moreover, $U^{-1} = U_2^{-1} U_1^{-1}$ belongs to $\mathcal{N}_+^{m \times m}$. Therefore $U \in \mathcal{N}_{\text{out}}$, i.e., the mvf U is singular.
The inclusion $U \in \mathcal{U}_\kappa(J)$ with $\kappa \leq \kappa_1 + \kappa_2$ is a general fact that is implied by the identity

$$K_\omega^U(\lambda) = K_\omega^{U_1}(\lambda) + U_1(\lambda) K_\omega^{U_2}(\lambda) U_1(\omega)^*.$$

$\square$

4.2. Characterization of singular mvf's in terms of associated pairs.
In what follows we suppose that $J = j_{pq}$.

Lemma 4.3. *Let $W \in \mathcal{U}_\kappa^r(j_{pq})$ and $\{b_1, b_2\} \in \mathrm{ap}^r(W)$. Then $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_+^{m \times m}$ if and only if $b_2 \equiv \text{const}$.*

Proof. 1) Without loss of generality we may assume that $b_2 = I_q$. Then it follows from (3.10) and (3.11) that

$$W = \begin{bmatrix} b_1 & K \\ 0 & I \end{bmatrix} \Phi,$$

where $\Phi \in \mathcal{N}_{\text{out}}^{m \times m}(\Omega_+) \subset \mathcal{N}_+^{m \times m}$. Therefore, $W \in \mathcal{N}_+^{m \times m} \cap \mathcal{U}_\kappa^r(j_{pq})$.

2) Assume now that $W \in \mathcal{N}_+^{m \times m}$. Then also $W\Phi^{-1} \in \mathcal{N}_+^{m \times m}$ and by the formula (3.10)

$$W\Phi^{-1} = \Theta = \begin{bmatrix} b_1 & Kb_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix} \in \mathcal{N}_+^{m \times m} \cap L_\infty.$$

Therefore, by the Smirnov maximum principle

$$\begin{bmatrix} b_1 & Kb_2^{-1} \\ 0 & b_2^{-1} \end{bmatrix} \in H_\infty^{m \times m},$$

and hence $b_2^{-1} \in H_\infty^{m \times m}$. Thus $b_2 \equiv \text{const}$. $\qquad \square$

Lemma 4.4. *Let* $W \in \mathcal{U}_\kappa^r(j_{pq})$ *and* $\{b_1, b_2\} \in ap^r(W)$. *Then* $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_-^{m \times m}$ *if and only if* $b_1 = \text{const}$.

Proof. 1) Without loss of generality we may assume that $b_1 = I_p$. Then it follows from (3.10), (3.11) and (3.12) that

$$W = \begin{bmatrix} I & 0 \\ K^\# & b_2^{-1}I \end{bmatrix} \Phi^-,$$

where $\Phi^- \in \mathcal{N}_{\text{out}}^{m \times m}(\Omega_-) \subset \mathcal{N}_-^{m \times m}$. Hence $W \in \mathcal{N}_-^{m \times m} \cap \mathcal{U}_\kappa^r(j_{pq})$.

2) Conversely, let $W \in \mathcal{N}_-^{m \times m}$. Then $W(\Phi^-)^{-1} \in \mathcal{N}_-^{m \times m}$ and by the formula (3.10)

$$W(\Phi^-)^{-1} = \Theta^- = \begin{bmatrix} b_1 & 0 \\ K^\# b_1 & b_2^{-1} \end{bmatrix} \in H_\infty^{m \times m}(\Omega_-).$$

Thus, by the Smirnov maximum principle

$$\begin{bmatrix} b_1 & 0 \\ K^\# b_1 & b_2^{-1} \end{bmatrix} \in \mathcal{N}_-^{m \times m} \cap L_\infty,$$

and hence $b_1 \in H_\infty^{m \times m}(\Omega_-)$. This proves that $b_1 \equiv \text{const}$. $\qquad \square$

Theorem 4.5. *Let* $W \in \mathcal{U}_\kappa^r(j_{pq})$ *and* $\{b_1, b_2\} \in ap^r(W)$. *Then* W *is singular if and only if* $b_1 \equiv \text{const}$ *and* $b_2 \equiv \text{const}$.

Proof. If $b_2 \equiv \text{const}$, then by Lemma 4.3

$$(4.3) \qquad\qquad W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_+^{m \times m}.$$

If $b_1 \equiv \text{const}$, then by Lemma 4.4 $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_-^{m \times m}$. It follows from the identity

$$W(\lambda) j_{pq} W^\#(\lambda) = j_{pq} \quad (\lambda \in \mathfrak{h}_W \cap \mathfrak{h}_{W^\#})$$

that

$$(4.4) \qquad\qquad W(\lambda)^{-1} = j_{pq} W^\#(\lambda) j_{pq}.$$

Therefore, since $W \in \mathcal{N}_-^{m \times m}$, then $W^\# \in \mathcal{N}_+^{m \times m}$. Consequently,

$$(4.5) \qquad\qquad W^{-1} \in \mathcal{N}_+^{m \times m}.$$

With regard to these two conditions (4.3), (4.5) we obtain that $W \in \mathcal{U}_{\kappa,S}(j_{pq})$ by Definition 4.1.

Conversely, let $W \in \mathcal{U}_{\kappa,S}^r(j_{pq})$. Then

$$W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_{\text{out}}^{m \times m}(\Omega_+),$$

and hence $W \in \mathcal{U}_\kappa^r(j_{pq}) \cap \mathcal{N}_+^{m \times m}$. By Lemma 4.3 this condition is equivalent to $b_2 \equiv$ const.

Next, it follows from (4.4) and (4.5) that $W^\# \in \mathcal{N}_{out}^{m \times m}$. Hence $W \in \mathcal{N}_-^{m \times m}$ and by Lemma 4.4 this condition is equivalent to $b_1 \equiv$ const. $\qquad\square$

Corollary 4.6. *Let* $W \in \mathcal{U}_\kappa^\ell(j_{pq})$ *and* $\{\beta_1, \beta_2\} \in ap^\ell(W)$. *Then* W *is singular if and only if* $\beta_1 \equiv$ const *and* $\beta_2 \equiv$ const.

Proof. By Lemma 3.14 $|\deg \beta_1| = |\deg b_1|$, $|\deg \beta_2| = |\deg b_2|$. Therefore, the statements concerning W are implied by Theorem 4.5. $\qquad\square$

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