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SCALE-INVARIANT SELF-ADJOINT EXTENSIONS OF SCALE-INVARIANT SYMMETRIC OPERATORS: CONTINUOUS VERSUS DISCRETE

MIRON B. BEKKER, MARTIN J. BOHNER, MARK A. NUDEL′MAN, AND HRISTO VOULOV

Abstract. We continue our study of a q-difference version of a second-order differential operator which depends on a real parameter. This version was introduced in our previous three articles on the subject. First we study general symmetric and scale-invariant operators on a Hilbert space. We show that if the index of defect of the operator under consideration is $(1, 1)$, then the operator either does not admit any scale-invariant self-adjoint extension, or it admits exactly one scale-invariant self-adjoint extension, or it admits exactly two scale-invariant self-adjoint extensions, or all self-adjoint extensions are scale invariant. We then apply these results to the differential operator and the corresponding difference operator under consideration. For the continuous case, we show that the interval of the parameter, for which the differential operator is not semi-bounded, contains an infinite sequence of values for which all self-adjoint extensions are scale-invariant, while for the remaining values of the parameter from that interval, there are no scale-invariant self-adjoint extensions. For the corresponding difference operator, we show that if it is not semi-bounded, then it does not admit any scale-invariant self-adjoint extension. We also show that both differential and difference operators, at value(s) of the parameter that correspond to the endpoint(s) of the interval(s) of semi-boundedness, have exactly one scale-invariant self-adjoint extension.

1. Introduction

In this article, we continue our investigation of a scale-invariant difference operator that was initiated in [6, 7, 8]. At first, we briefly recall some notation and previously obtained results.

For a fixed number $q > 1$, we consider the Hilbert space $\mathcal{T} = l^2(\mathbb{Z}; q)$ of bi-infinite sequences $\{x_n\}_{n=-\infty}^{\infty}$ with complex entries that satisfy the condition

$$\sum_{n=-\infty}^{\infty} q^n |x_n|^2 < \infty.$$

The inner product $\langle \cdot, \cdot \rangle$ in the space $\mathcal{T}$ is defined by

$$\langle x, y \rangle = \frac{q-1}{q} \sum_{n=-\infty}^{\infty} q^n x(n)\overline{y(n)}, \quad x, y \in \mathcal{T}. \tag{1}$$

We study the difference operator which is defined by means of the difference expression

$$\begin{aligned}(\mathcal{L}x)(n) &= -\frac{q}{(q-1)^2} \frac{x(n+1) - (1+q)x(n) + qx(n-1)}{q^{2n}} + \frac{\alpha}{q^{2n-1}} x(n) \\ &= -\frac{q}{(q-1)^2} \frac{x(n+1) - \beta x(n) + qx(n-1)}{q^{2n}},\end{aligned} \tag{2}$$

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where $\alpha \in \mathbb{R}$ and $\beta = 1+q+(q-1)^2\alpha$. Denote by L_0 the operator on $\mathcal{T}$ with the domain $\mathcal{D}(L_0)$ consisting of elements $x \in \mathcal{T}$ with finite support and put $L_0 x = \mathcal{L}x$, $x \in \mathcal{D}(L_0)$. The operator L under consideration is the closure of L_0. We consider the operator L as a discrete analogue of the differential operator $\mathcal{H}$ which is generated by the differential expression

$$(Tf)(t) = -\frac{d^2 f}{\mathrm{d}t^2} + \frac{\alpha}{t^2} f(t) \tag{3}$$

in the space $L^2(\mathbb{R}_+)$. Namely, for a given number $q > 1$, we consider points $t_n = q^n$, $n \in \mathbb{Z}$, as points of discretization. The first and the second derivative of the function x defined on $(0, \infty)$ is replaced by the expressions

$$(D_q x)(n) = \frac{x(n+1) - x(n)}{q^{n+1} - q^n}$$

and

$$\begin{aligned} (D_q^2 x)(n-1) &= \frac{(D_q x)(n) - (D_q x)(n-1)}{q^n - q^{n-1}} \\ &= \frac{x(n+1) - (1+q)x(n) + qx(n-1)}{q^{2n-1}(q-1)^2}, \end{aligned}$$

respectively, where $x(n) := x(q^n)$. The term

$$\frac{\alpha}{t^2} x(t) \quad \text{is replaced by} \quad \frac{\alpha}{q^{2n-1}} x(n).$$

Denote by $\mathcal{H}_0$ the differential operator in the space $L^2(\mathbb{R}_+)$ with the domain $\mathcal{D}(\mathcal{H}_0)$ consisting of all smooth functions with compact support within $\mathbb{R}_+$, i.e.,

$$\mathcal{H}_0 f = Tf, \quad f \in \mathcal{D}(\mathcal{H}_0). \tag{4}$$

The operator $\mathcal{H}$ is the closure of $\mathcal{H}_0$. The following facts about the differential operator $\mathcal{H}$ generated by (3) are well known (see, for example [15, 17]):

1. For $\alpha \geq 3/4$, the operator $\mathcal{H}$ is self-adjoint with simple absolutely continuous spectrum.
2. For $-1/4 \leq \alpha < 3/4$, the operator $\mathcal{H}$ is positive and symmetric with index of defect $(1, 1)$.
3. For $\alpha < -1/4$, the operator $\mathcal{H}$ is symmetric with index of defect $(1, 1)$ and not semibounded.

In [6], the following results about the difference operator L generated by the difference expression (2) in the space $\mathcal{T}$ were proved:

1. For
$$\alpha \geq \alpha_{++} := \frac{\sqrt{q} + 1 + 1/\sqrt{q}}{(\sqrt{q}+1)^2}, \quad \text{i.e.,} \quad \beta \geq \beta_{++} := \frac{q^2+1}{\sqrt{q}}$$
and for
$$\alpha \leq \alpha_{--} := -\frac{\sqrt{q} - 1 + 1/\sqrt{q}}{(\sqrt{q}-1)^2}, \quad \text{i.e.,} \quad \beta \leq \beta_{--} := -\frac{q^2+1}{\sqrt{q}},$$
the operator L is self-adjoint.
2. For
$$-\frac{1}{(\sqrt{q}+1)^2} =: \alpha_+ \leq \alpha < \alpha_{++}, \quad \text{i.e.,} \quad 2\sqrt{q} =: \beta_+ \leq \beta < \beta_{++}$$
and for
$$\alpha_{--} < \alpha \leq \alpha_- := -\frac{1}{(\sqrt{q}-1)^2}, \quad \text{i.e.,} \quad \beta_{--} < \beta \leq \beta_- := -2\sqrt{q},$$

the operator L is symmetric with index of defect $(1,1)$, positive for $\alpha \geq \alpha_+$, and negative for $\alpha \leq \alpha_-$.

3. For

$$\alpha_- < \alpha < \alpha_+, \quad \text{i.e.,} \quad \beta_- < \beta < \beta_+,$$

the operator L is symmetric with index of defect $(1,1)$, but not semibounded.

Observe that for $q \downarrow 1$, we have $\alpha_+ \to -1/4$ and $\alpha_{++} \to +3/4$, while $\alpha_- \to -\infty$ and $\alpha_{--} \to -\infty$, that is, we obtain the classical results for the corresponding differential operator.

A critical rôle in our investigation is played by the fact that the operator L is (q^2, U)-scale invariant (see Definition 1 in Section 2). Note that the operator $\mathcal{H}$ is also (q^2, U)-scale invariant for any $q > 0$. Using this fact, in [7], it was proved that for $\alpha \geq \alpha_{++}$ and for $a \leq \alpha_{--}$, that is, in the case when the operator L is self-adjoint, its spectrum is simple, discrete and located along a geometric sequence with the ratio q^2. In [5], it was proved that a semibounded symmetric (q^2, U)-scale-invariant operator always admits positive (q^2, U)-scale-invariant and positive self-adjoint extensions. In particular, the extreme extensions, the so-called Friedrichs extensions and Kreĭn extensions ('hard' and 'soft' in the terminology of Mark G. Kreĭn) are (q^2, U)-scale invariant. In addition, it was proved that if a symmetric positive (q^2, U)-scale-invariant operator has index of defect $(1,1)$, then the only self-adjoint extensions that are (q^2, U)-scale invariant are the extreme extensions.

In [5], the above mentioned results were obtained in an indirect way, using extension theory of nondensely defined Hermitian contractions. Reformulation of those results in terms of scale-invariant semibounded densely defined symmetric operators was also pointed out in [5, Theorem 5]. In [14], invariance of the Friedrichs and Kreĭn extensions for scale-invariant semibounded operators (*μ-scale-invariant* in the terminology of [14]) was proved directly.

Now, using invariance of the extreme extensions, in [8], it was proved that for $\alpha_+ < \alpha < \alpha_{++}$ and $\alpha_{--} < \alpha < \alpha_-$, the operator L has distinct extreme extensions, and the resolvent operators of those extensions were described. It is known that for $-1/4 < \alpha < 3/4$, extreme self-adjoint extensions of the differential operator $\mathcal{H}$ are also distinct.

In this article, we investigate the case $\alpha_- \leq \alpha \leq \alpha_+$ for the operator L and the case $\alpha \leq -1/4$ for the operator $\mathcal{H}$. The article is organized as follows. In Section 2, we consider the problem of existence of (q^2, U)-scale-invariant self-adjoint extensions of a given (q^2, U)-scale-invariant symmetric operator with index of defect $(1,1)$. With each such symmetric operator, we associate a linear fractional transformation that maps the unit circle onto itself and the interior of the unit circle onto itself. We show that the operator possesses (q^2, U)-scale-invariant self-adjoint extensions if and only if the corresponding linear fractional transformation has fixed points on the unit circle (Theorem 1). In Section 3, we apply Theorem 1 to the case of the differential operator $\mathcal{H}$ defined by the differential expression (3) for $\alpha = -1/4$ ($\mathcal{H}$ is a positive operator) and $\alpha < -1/4$ ($\mathcal{H}$ is not semibounded operator) and q fixed. We show that for $\alpha = -1/4$, the operator $\mathcal{H}$ has exactly one (q^2, U)-scale-invariant self-adjoint extension. According to the above mentioned result from [5], this means that for $\alpha = -1/4$, the operator $\mathcal{H}$ has exactly one positive self-adjoint extension (Theorem 3). This statement is, perhaps, known, but the authors could not find any reference. We also show that for $\alpha < -1/4$, there exists an infinite sequence of values of α_k, $\alpha_k \downarrow -\infty$, depending on q, such that for $\alpha = \alpha_k$, all self-adjoint extensions of the operator $\mathcal{H}$ are (q^2, U)-scale invariant, and for $\alpha \neq \alpha_k$, the operator $\mathcal{H}$ does not possess (q^2, U)-scale-invariant self-adjoint extensions. To the best of the authors' knowledge, this result is new (Theorem 2). In Section 4, we apply Theorem 1 to the difference operator L defined by (2) in the space $l^2(\mathbb{Z}; q)$ for the values of $\alpha = \alpha_+$ and $\alpha = \alpha_-$ and for $\alpha_- < \alpha < \alpha_+$. We show that for $\alpha = \alpha_+$ and for

$\alpha = \alpha_-$, when the operator L is semibounded, it has exactly one (q^2, U)-scale-invariant (i.e. positive, respectively, negative) self-adjoint extension (Theorem 7). At the same time, for $\alpha_- < \alpha < \alpha_+$, the operator L does not have any (q^2, U)-scale-invariant self-adjoint extension (Theorem 5)! This demonstrates an essential difference in the behavior of the differential operator $\mathcal{H}$ and the difference operator L.

2. Self-adjoint extensions

Definition 1. Let $\mathcal{H}$ be a closed operator in a Hilbert space $\mathcal{T}$ with the domain $\mathcal{D}(\mathcal{H})$. Let $q \neq 1$ be a positive real number. We say that the operator $\mathcal{H}$ is (q^2, U)-scale invariant if there exists a unitary operator U on the Hilbert space $\mathcal{T}$ such that

$$U\mathcal{D}(\mathcal{H}) = \mathcal{D}(\mathcal{H}) \tag{5}$$

and

$$U\mathcal{H}f = q^2\mathcal{H}Uf, \quad f \in \mathcal{D}(\mathcal{H}). \tag{6}$$

The notion of (q^2, U)-scale invariance is a particular case of a more general notion of *$p(t)$-homogeneity* of a symmetric operator with respect to a $*$-closed family of unitary operators, introduced in [10, Definition 1.1].

From Definition 1, it follows that if the densely defined operator $\mathcal{H}$ is (q^2, U)-scale invariant, then the operator $\mathcal{H}^*$ is also (q^2, U)-scale invariant. The proof of this statement is given in [4].

Recall that a point $z \in \mathbb{C}$ belongs to the field of regularity of an operator $\mathcal{H}$ if there exists $k > 0$ (depending on z) such that

$$\|(\mathcal{H} - zI)f\| \geq k\,\|f\|, \quad f \in \mathcal{D}(\mathcal{H}).$$

The field of regularity of the operator $\mathcal{H}$ is an open set and, therefore, consists of a finite or countable number of components. Put

$$\mathcal{M}_z = (\mathcal{H} - zI)\mathcal{D}(\mathcal{H}) \quad \text{and} \quad \mathcal{N}_{\bar{z}} = \mathcal{M}_z^{\perp}.$$

Every nonzero vector from $\mathcal{N}_z$ is an eigenvector of the operator $\mathcal{H}^*$ that corresponds to the eigenvalue z. If z belongs to the field of regularity of the closed operator $\mathcal{H}$, then $\mathcal{M}_z$ is a closed linear manifold, that is, a subspace. For z and ζ from the one and same component of the field of regularity, $\dim \mathcal{N}_{\bar{z}} = \dim \mathcal{N}_{\bar{\zeta}}$, and this dimension is called the defect number of the operator $\mathcal{H}$ in the corresponding component of the field of regularity.

Lemma 1. *Let $\mathcal{H}$ be a closed operator in a Hilbert space $\mathcal{T}$ with the domain $\mathcal{D}(\mathcal{H})$. Suppose that there exist a unitary operator U on the space $\mathcal{T}$ and $q \in (0, \infty) \setminus \{1\}$ such that*

1. *$U\mathcal{D}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H})$;*
2. *$U\mathcal{H}f = q^2\mathcal{H}Uf$, $f \in \mathcal{D}(\mathcal{H})$.*

If for some $z \in \mathbb{C}$, both points z and z/q^2 belong to the one and same component of the field of regularity and the defect number of the operator $\mathcal{H}$ in that component is finite, then $U\mathcal{D}(\mathcal{H}) = \mathcal{D}(\mathcal{H})$, and the operator $\mathcal{H}$ is (q^2, U)-scale invariant.

Proof. Suppose, in contrary, that $U\mathcal{D}(\mathcal{H})$ is properly contained in $\mathcal{D}(\mathcal{H})$. Then

$$U\mathcal{M}_z = U(\mathcal{H} - zI)\mathcal{D}(\mathcal{H}) = (q^2\mathcal{H} - zI)U\mathcal{D}(\mathcal{H}) \subset \mathcal{M}_{z/q^2},$$

where the inclusion is proper. Therefore,

$$U\mathcal{N}_{\bar{z}} \supset \mathcal{N}_{\bar{z}/q^2},$$

and the inclusion is also proper. But

$$\dim U\mathcal{N}_{\bar{z}} = \dim \mathcal{N}_{\bar{z}} = \dim \mathcal{N}_{\bar{z}/q^2} < \infty,$$

a contradiction, which completes the proof. $\square$

Since for a symmetric operator, the upper half plane $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ and the lower half plane $\mathbb{C}_- = \{z : \operatorname{Im} z < 0\}$ belong to the field of regularity, we have the following corollary.

Corollary 1. *Let $\mathcal{H}$ be a closed symmetric operator and suppose that there exist a unitary operator U and $q \in (0,\infty)\setminus\{1\}$ such that conditions 1 and 2 of Lemma 1 are fulfilled. If at least one of the defect numbers of the operator $\mathcal{H}$ is finite, then $\mathcal{H}$ is (q^2, U)-scale invariant.*

Theorem 1. *Let $\mathcal{H}$ be a symmetric (q^2, U)-scale-invariant operator in a Hilbert space $\mathcal{T}$. Suppose that the index of defect of the operator $\mathcal{H}$ is $(1,1)$. Then exactly one of the following four cases holds:*

1. *$\mathcal{H}$ does not admit any (q^2, U)-scale-invariant self-adjoint extension in $\mathcal{T}$.*
2. *$\mathcal{H}$ admits exactly one (q^2, U)-scale-invariant self-adjoint extension in $\mathcal{T}$.*
3. *$\mathcal{H}$ admits exactly two (q^2, U)-scale-invariant self-adjoint extensions in $\mathcal{T}$.*
4. *All self-adjoint extensions of $\mathcal{H}$ in $\mathcal{T}$ are (q^2, U)-scale invariant.*

Proof. Let $\hat{H}$ be a self-adjoint (dissipative) extension of the (q^2, U)-scale-invariant symmetric operator $\mathcal{H}$ with index of defect $(1,1)$. Denote by $\varphi(z)$ a defect vector of $\mathcal{H}$ that corresponds to $z \notin \mathbb{R}$. Note that from our assumption about (q^2, U)-scale invariance of $\mathcal{H}$, it follows that $U\varphi(z) = \varphi(z/q^2)$. According to the von Neumann formulas (see, for example, [2]), the operator $\hat{H}$ is described by

$$\mathcal{D}(\hat{H}) = \{f \in \mathcal{T} :\ f = f_0 + \xi(\varphi(i) + \rho\varphi(-i)),\ f_0 \in \mathcal{D}(\mathcal{H})\} \tag{7}$$

$(\|\varphi(i)\| = \|\varphi(-i)\|)$ and

$$\hat{H}f = \mathcal{H}f_0 + i\xi(\varphi(i) - \rho\varphi(-i)), \tag{8}$$

where $|\rho| = 1$ for a self-adjoint extension and $|\rho| < 1$ for a dissipative extension. Since

$$U\mathcal{D}(\hat{H}) = \left\{f \in \mathcal{T} :\ f = f_0 + \xi(\varphi(i/q^2) + \rho\varphi(-i/q^2)),\ f_0 \in \mathcal{D}(\mathcal{H})\right\}$$

is a domain of another self-adjoint (dissipative) extension, say $\hat{H}'$, of the operator $\mathcal{H}$, it admits the description

$$U\mathcal{D}(\hat{H}) = \{f \in \mathcal{T} :\ f = f_0 + \xi(\varphi(i) + \rho'\varphi(-i)),\ f_0 \in \mathcal{D}(\mathcal{H})\},$$

where $|\rho'| = 1$ if $|\rho| = 1$ and $|\rho'| < 1$ if $|\rho| < 1$. In particular,

$$\varphi(i/q^2) + \rho\varphi(-i/q^2) = f_0 + \xi(\varphi(i) + \rho'\varphi(-i)) \tag{9}$$

for some $f_0 \in \mathcal{D}(\mathcal{H})$ and $\xi \in \mathbb{C}$. In order to find a relation between parameters ρ and ρ', we use the fact that the domain $\mathcal{D}(\mathcal{H}^*)$ of the operator $\mathcal{H}^*$ is a complete Hilbert space with respect to the inner product

$$\langle x, y\rangle^* = \langle x, y\rangle + \langle \mathcal{H}^*x, \mathcal{H}^*y\rangle, \quad x, y \in \mathcal{D}(\mathcal{H}^*)$$

and admits the decomposition (see, e.g., [9, Chapter XII, Sec. 4])

$$\mathcal{D}(\mathcal{H}^*) = \mathcal{D}(\mathcal{H}) \oplus^* \mathcal{N}_i \oplus^* \mathcal{N}_{-i}, \tag{10}$$

where $\oplus^*$ stands for the orthogonal sum with respect to the inner product $\langle\cdot,\cdot\rangle^*$ and $\mathcal{N}_i = \text{l. h.}\{\varphi(i)\}$, $\mathcal{N}_{-i} = \text{l. h.}\{\varphi(-i)\}$ are the defect subspaces (l. h. means linear hull). Using (9) and (10), one obtains

$$\left\langle\varphi(i/q^2), \varphi(i)\right\rangle^* + \rho\left\langle\varphi(-i/q^2), \varphi(i)\right\rangle^* = \xi\left\langle\varphi(i), \varphi(i)\right\rangle^*$$

and

$$\left\langle\varphi(i/q^2), \varphi(-i)\right\rangle^* + \rho\left\langle\varphi(-i/q^2), \varphi(-i)\right\rangle^* = \rho'\xi\left\langle\varphi(-i), \varphi(-i)\right\rangle^*.$$

From the last two equations, it follows that

$$\rho'\frac{\langle\varphi(-i), \varphi(-i)\rangle^*}{\langle\varphi(i), \varphi(i)\rangle^*} = \frac{\left\langle\varphi(i/q^2), \varphi(-i)\right\rangle^* + \rho\left\langle\varphi(-i/q^2), \varphi(-i)\right\rangle^*}{\left\langle\varphi(i/q^2), \varphi(i)\right\rangle^* + \rho\left\langle\varphi(-i/q^2), \varphi(i)\right\rangle^*}.$$

Taking into account that

$$\begin{aligned} \langle \varphi(i), \varphi(i) \rangle^* &= 2 \left\| \varphi(i) \right\|^2, \\ \langle \varphi(-i), \varphi(-i) \rangle^* &= 2 \left\| \varphi(-i) \right\|^2, \\ \left\langle \varphi(i/q^2), \varphi(-i) \right\rangle^* &= \frac{q^2-1}{q^2} \left\langle \varphi(i/q^2), \varphi(-i) \right\rangle, \\ \left\langle \varphi(-i/q^2), \varphi(-i) \right\rangle^* &= \frac{q^2+1}{q^2} \left\langle \varphi(-i/q^2), \varphi(-i) \right\rangle, \\ \left\langle \varphi(i/q^2), \varphi(i) \right\rangle^* &= \frac{q^2+1}{q^2} \left\langle \varphi(i/q^2), \varphi(i) \right\rangle, \end{aligned}$$

and

$$\left\langle \varphi(-i/q^2), \varphi(i) \right\rangle^* = \frac{q^2-1}{q^2} \left\langle \varphi(-i/q^2), \varphi(i) \right\rangle,$$

one obtains

$$\rho' = \frac{\rho(q^2+1) \left\langle \varphi(-i/q^2), \varphi(-i) \right\rangle + (q^2-1) \left\langle \varphi(i/q^2), \varphi(-i) \right\rangle}{\rho(q^2-1) \left\langle \varphi(-i/q^2), \varphi(i) \right\rangle + (q^2+1) \left\langle \varphi(i/q^2), \varphi(i) \right\rangle},$$

that is, ρ' is a linear fractional transformation of ρ. Put

$$\begin{aligned} \mathcal{A} &= (q^2+1) \left\langle \varphi(-i/q^2), \varphi(-i) \right\rangle, \\ \mathcal{B} &= (q^2-1) \left\langle \varphi(i/q^2), \varphi(-i) \right\rangle, \\ \mathcal{C} &= (q^2-1) \left\langle \varphi(-i/q^2), \varphi(i) \right\rangle, \\ \mathcal{D} &= (q^2+1) \left\langle \varphi(i/q^2), \varphi(i) \right\rangle \end{aligned}$$

so that

$$\rho' = \frac{\mathcal{A}\rho + \mathcal{B}}{\mathcal{C}\rho + \mathcal{D}} = \Gamma(\rho). \tag{11}$$

The transformation (11) maps the unit circle onto itself and the interior of the unit circle onto itself. From this fact it follows that the matrix of coefficients (we denote it also by Γ) satisfies

$$\Gamma = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \quad \Gamma^* J \Gamma = kJ, \quad J = \operatorname{diag}\{1, -1\}, \quad k \neq 0.$$

The extension $\hat{H}$ is (q^2, U)-scale invariant if and only if $\Gamma(\rho) = \rho$, that is, ρ is a fixed point of Γ. If the fixed point is on the unit circle, then any (q^2, U)-scale-invariant extension is self-adjoint. If the fixed point is inside the unit circle, then the extension is dissipative. Now the result follows immediately. If $\mathcal{C} = \mathcal{B} = 0$ and $\mathcal{A} = \mathcal{D}$, then Γ is the identity transformation, and every self-adjoint or dissipative extension is (q^2, U)-scale invariant. □

Remark 1. (a) In [11], a symmetric operator commuting with a $*$-closed family of unitary operators was considered. Necessary as well as sufficient conditions for the existence of a self-adjoint extension of the symmetric operator that also commutes with each operator of the family were obtained. The corresponding conditions were expressed in terms of the characteristic function of the symmetric operator.

(b) Let U be a unitary operator. It is well known that there exists a symmetric operator A with index of defect $(1, 1)$ which commutes with U but does not admit any self-adjoint extension which also commutes with U. The first example of such an operator was constructed by R. S. Phillips (see [13, Chapter 2] for details). Statement 1. of our Theorem 1 states that the same is true if we replace commutativity by (q^2, U)-invariance. Corresponding examples are, in fact, given by Theorem 2 and 5 below.

(c) Statements 2., 3., and 4. of our Theorem 1 are related to [10, Proposition 4.16] in which, however, only semibounded operators were considered.

Remark 2. If $\mathcal{H}$ is a (q^2, U)-scale-invariant operator with index of defect (n, n), then its self-adjoint (q^2, U)-scale-invariant extensions are defined by unitary operators which are fixed points of some linear fractional transformation with matrix coefficients. That linear fractional transformation maps the interior of the unit ball in the space of $n \times n$-matrices onto itself, and the set of unitary $n \times n$-matrices onto itself. From these properties, it is possible to deduce (see [12, 16]) that the matrix Γ of the coefficients of that linear fractional transformation satisfies

$$\Gamma^* J \Gamma = kJ, \quad J = \operatorname{diag}\{I_n, -I_n\}, \quad k \neq 0.$$

3. Differential operator

In this section, we apply the results obtained in Section 2 to the differential operator describe in Section 1. Recall that the differential operator $\mathcal{H}$ is generated by the differential expression

$$(Tf)(t) = -\frac{d^2 f}{\mathrm{d}t^2} + \frac{\alpha}{t^2} f(t).$$

Denote by $\mathcal{H}_0$ the differential operator in the space $L^2(\mathbb{R}_+)$ with the domain $\mathcal{D}(\mathcal{H}_0)$ consisting of all smooth functions with compact support within $\mathbb{R}_+$ (see (4)). The operator $\mathcal{H}$ is the closure of $\mathcal{H}_0$. It is easily seen that the operator $\mathcal{H}^*$ is described as follows. Its domain $\mathcal{D}(\mathcal{H}^*)$ consists of those and only those functions $f \in L^2(\mathbb{R}_+)$ for which

1. f is absolutely continuous;
2. f' is absolutely continuous;
3. $Tf \in L^2(\mathbb{R}_+)$

and

$$\mathcal{H}^* f = Tf, \quad f \in \mathcal{D}(\mathcal{H}^*).$$

For any $q > 0$, denote by U_q the operator on $\mathcal{T} = L^2(\mathbb{R}_+)$ defined by

$$(U_q f)(t) = \sqrt{q} f(qt).$$

It is easily seen that U_q are unitary operators and $U_q^* = U_{1/q}$. Moreover,

$$U_q \mathcal{H} f = q^2 \mathcal{H} U_q f, \quad f \in \mathcal{D}(\mathcal{H}_0),$$

and thus the operator $\mathcal{H}_0$ is (q^2, U_q)-scale-invariant. Therefore, the same is valid for the operator $\mathcal{H} = \overline{\mathcal{H}_0}$. We fix a value of $q \in (0, \infty) \setminus \{1\}$ and denote the corresponding U_q by U. Since the operator $\mathcal{H}_0$ has real coefficients, its defect vectors $\varphi(z) = \psi_z(t)$ satisfy $\psi_{\bar{z}}(t) = \overline{\psi_z(t)}$. From this fact, it follows that the transformation (11) takes the form

$$\rho' = \frac{\mathcal{A}\rho + \overline{\mathcal{B}}}{\mathcal{B}\rho + \overline{\mathcal{A}}}, \tag{12}$$

that is, it is a usual Möbius transformation.

At first, we consider the case $\alpha < -1/4$. We put $\alpha = -1/4 - \nu^2$, $\nu > 0$. It is well known (see, for example, [17]), that $\mathcal{H}$ is a symmetric but not semibounded operator with index of defect $(1, 1)$.

Theorem 2. *Define*

$$\nu_k := \frac{\pi k}{\ln q}, \quad k \in \mathbb{Z}.$$

For $\nu = \nu_k$, all self-adjoint extensions of the operator $\mathcal{H}$ in $\mathcal{T}$ are (q^2, U)-scale invariant. For $\nu \neq \nu_k$, the operator $\mathcal{H}$ does not have (q^2, U)-scale-invariant extensions in the space $\mathcal{T}$.

Proof. We follow the construction from Section 2. The defect vector $\varphi(z) = \psi_z(t)$, $\operatorname{Im} z \neq 0$, that is, a solution of the differential equation

$$-\frac{d^2\psi_z(t)}{\mathrm{d}t^2} - \frac{v^2 + 1/4}{t^2}\psi_z(t) = z\psi_z(t)$$

is given by

$$\psi_z(t) = \begin{cases} \sqrt{t}H^{(1)}_{i\nu}(\sqrt{z}t), & \operatorname{Im} z > 0, \\ \sqrt{t}H^{(2)}_{-i\nu}(\sqrt{z}t), & \operatorname{Im} z < 0, \end{cases}$$

where $H^{(1)}_\mu$ and $H^{(2)}_\mu$ are the Hankel functions of order μ of first and second kind, respectively (we assume that $-\pi < \arg z < \pi$). For the properties of Hankel functions, see [1, 3]. In the formula (12), the coefficients $\mathcal{A}$ and $\mathcal{B}$ are calculated according to

$$\begin{aligned} \mathcal{A} &= (q^2+1)\int_0^\infty tH^{(2)}_{-i\nu}\left(\frac{e^{-i\pi/4}}{q}t\right)H^{(1)}_{i\nu}(e^{i\pi/4}t)\,\mathrm{d}t, \\ \mathcal{B} &= (q^2-1)\int_0^\infty tH^{(1)}_{i\nu}\left(\frac{e^{i\pi/4}}{q}t\right)H^{(1)}_{i\nu}(e^{i\pi/4}t)\,\mathrm{d}t. \end{aligned}$$

Evaluating the integrals using known formulas for integrals of products of Bessel functions (see, e.g., [3, Sec. 7.14.1, formula (9)]) and asymptotic behavior of such functions, one obtains

$$\begin{aligned} \mathcal{A} &= \frac{2q^2e^{\pi\nu}}{\pi\sinh\pi\nu}\left(e^{\pi\nu/2}e^{i\nu\ln q} - e^{-\pi\nu/2}e^{-i\nu\ln q}\right), \\ \mathcal{B} &= -\frac{4iq^2e^{\pi\nu}\sin(\nu\ln q)}{\pi\sinh\pi\nu}. \end{aligned}$$

In particular,

$$\begin{aligned} \operatorname{Im}\mathcal{A} &= \frac{4q^2e^{\pi\nu}\sin(\nu\ln q)\cosh(\pi\nu/2)}{\pi\sinh\pi\nu}, \\ |\mathcal{B}| &= \frac{4q^2e^{\pi\nu}|\sin(\nu\ln q)|}{\pi\sinh\pi\nu}. \end{aligned}$$

From the last formulas, the result follows immediately. Indeed, for $\nu = \nu_k = \frac{\pi k}{\ln q}$, $k \in \mathbb{Z}$, we have $\mathcal{B} = 0$ while $\mathcal{A} \in \mathbb{R}$, so the transformation (12) becomes the identity transformation. Thus, in such case, every self-adjoint or dissipative extension of the operator $\mathcal{H}$ is (q^2, U)-scale invariant. If $\nu \neq \nu_k$, then $|\mathcal{B}| < |\operatorname{Im}\mathcal{A}|$. This means that transformation (12) is of elliptic type and has one fixed point inside the unit circle and one outside. In other words, the operator $\mathcal{H}$ does not admit any (q^2, U)-scale-invariant self-adjoint extension. This completes the proof. □

Remark 3. It is possible to show that for $\nu = \nu_k$, the spectrum of any self-adjoint extension consists of an absolutely continuous part, filling the positive semiaxis, and simple eigenvalues are located on the negative semiaxis. Those eigenvalues are located along a geometric sequence with ratio q^2.

Now we consider the case $\alpha = -1/4$ in (3). It is known that for such a value of α (in fact, for $-1/4 \leq \alpha < 3/4$), the operator $\mathcal{H}$, the closure of $\mathcal{H}_0$, is a symmetric positive operator with index of defect $(1, 1)$.

Theorem 3. *For $\alpha = -1/4$, the operator $\mathcal{H}$ has exactly one (q^2, U)-scale-invariant self-adjoint extension.*

Proof. We follow the line of proof of Theorem 2. The defect vectors $\varphi(z) = \psi_z(t)$ are given by

$$\psi_z(t) = \begin{cases} tH_0^{(1)}\sqrt{z}t, & \operatorname{Im} z > 0, \\ tH_0^{(2)}\sqrt{z}t, & \operatorname{Im} z < 0. \end{cases}$$

The coefficients $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned} \mathcal{A} &= (q^2+1)\int_0^\infty tH_0^{(2)}\left(\frac{e^{-i\pi/4}}{q}t\right)H_0^{(1)}(e^{i\pi/4}t)\mathrm{d}t = \frac{2q^2}{\pi^2}(-\pi + 2i\ln q), \\ \mathcal{B} &= (q^2-1)\int_0^\infty tH^{(1)}\left(\frac{e^{i\pi/4}}{q}t\right)H^{(1)}(e^{i\pi/4}t)\mathrm{d}t = \frac{4iq^2\ln q}{\pi^2}. \end{aligned}$$

Since $|\operatorname{Im}\mathcal{A}| = |\mathcal{B}|$, one concludes that the transformation (12) has one fixed point on the unit circle. This completes the proof. □

In [5], it was proved that for a symmetric semibounded (q^2, U)-scale-invariant operator $\mathcal{H}$ with index of defect $(1,1)$, the only semibounded and (q^2, U)-scale-invariant extensions are the extreme extensions, the so-called the Friedrichs extension H_F and the Kreĭn extension H_K. Since a semibounded symmetric operator has only one semibounded self-adjoint extension if and only if the extreme extensions coincide, one obtains the following statement.

Corollary 2. *For $\alpha = -1/4$, the positive operator $\mathcal{H}$, the closure of the operator $\mathcal{H}_0$ defined by (4), has only one positive self-adjoint extension, say H_+. Its domain $\mathcal{D}(H_+)$ is described by*

$$\mathcal{D}(H_+) = \Big\{ f \in L^2(\mathbb{R}_+) : \ f \in \mathcal{D}(\mathcal{H}^*), \ \lim_{t\to 0+} W(f, \overline{h})(t) = 0 \Big\}, \tag{13}$$

where

$$h(t) = \sqrt{t}\Big[H_0^{(1)}(e^{i\pi/4}t) - H_0^{(2)}(e^{-i\pi/4}t)\Big]$$

and $W(f,g) = f'g - fg'$ is the Wronskian.

Remark 4. The operator H_+ has a homogeneous Lebesgue spectrum filling the positive semiaxis $\mathbb{R}_+$.

4. Difference operator

In this section, we study the difference operator that was introduced in our previous articles [6, 7, 8] and described in Section 1. Recall that the difference operator acts in the Hilbert space $\mathcal{T} = l^2(\mathbb{Z}; q)$ of bi-infinite sequences $\{x(n)\}_{n=-\infty}^{\infty}$ with complex entries that satisfy the condition

$$\sum_{n=-\infty}^{\infty} q^n |x(n)|^2 < \infty,$$

where $q > 1$ is fixed. For $x, y \in \mathcal{T}$, the inner product $\langle x, y\rangle$ is defined by (1). The difference operator is defined by means of the difference expression

$$\begin{aligned} (\mathcal{L}x)(n) &= -\frac{q}{(q-1)^2}\frac{x(n+1) - (1+q)x(n) + qx(n-1)}{q^{2n}} + \frac{\alpha}{q^{2n-1}}x(n) \\ &= -\frac{q}{(q-1)^2}\frac{x(n+1) - \beta x(n) + qx(n-1)}{q^{2n}}, \end{aligned}$$

where $\alpha \in \mathbb{R}$ and $\beta = 1 + q + (q-1)^2\alpha$. The operator L generated by the difference expression (2) in the space $\mathcal{T}$ is (q^2, U)-scale invariant, where the unitary operator U on the space $\mathcal{T}$ is defined by

$$(Ux)(n) = \frac{1}{\sqrt{q}}x(n-1). \tag{14}$$

The domain of the operator L^* consists of all those $x = \{x(n)\}_{n=-\infty}^{\infty}$, $x \in \mathcal{T}$, for which $\mathcal{L}x \in \mathcal{T}$ and $L^*x = \mathcal{L}x$.

At first, we assume that α satisfies

$$-\frac{1}{(\sqrt{q}-1)^2} = a_- < \alpha < \alpha_+ = -\frac{1}{(\sqrt{q}+1)^2},$$

i.e.,

$$-2\sqrt{q} < \beta < 2\sqrt{q}. \tag{15}$$

For such values of α, and β, the operator L, generated by the difference expression (2), is symmetric, but not semibounded, with index of defect $(1,1)$ (see [6]).

The key rôle in our consideration of the operator L is played by the following theorem proved in [8].

Theorem 4. *Suppose that λ_1 and λ_2 are distinct roots of the quadratic equation $\lambda^2 - (\beta/q)\lambda + (1/q)\lambda = 0$. Then:*

1. *For $i \in \{1,2\}$, there exist solutions f_i, $f_i(0) = 1$, of the functional equation*

$$\frac{1}{\lambda_i} f_i(zq^2) - \beta f_i(z) + q\lambda_i f_i(z/q^2) = -z\frac{(q-1)^2}{q} f_i(z) \tag{16}$$

that are entire functions of $z \in \mathbb{C}$.

2. *If we put*

$$w_i(n,z) = \lambda_i^{-n} f_i(zq^{2n}), \quad n \in \mathbb{Z}, \tag{17}$$

then $w_i = \{w_i(n,\cdot)\}_{n=-\infty}^{\infty}$ are linearly independent solutions of the equation $\mathcal{L}x = zx$. In particular, $w_i(n,\cdot)$ are entire functions for each $n \in \mathbb{Z}$.

The solutions w_i, $i \in \{1,2\}$, clearly satisfy the condition

$$\lim_{n\to-\infty} \frac{w_i(n-1,z)}{w_i(n,z)} = \lambda_i, \quad i \in \{1,2\},$$

and we call them Poincaré–Perron solutions (PP-solutions). More precisely, since $f_i = 1 + zg_i$, where g_i is an entire function, one has

$$w_i(n,z) = \lambda_i^{|n|}(1 + \delta_i(n,z)), \tag{18}$$

where $\delta_i(n,z) = q^{-2|n|} z g_i(zq^{-2|n|})$. Therefore $\delta_i(n,z) \to 0$ as $n \to -\infty$. For the interval of α, and β, under consideration, the roots

$$\lambda_1 = (\beta + i\sqrt{4q-\beta^2})/(2q) \quad \text{and} \quad \lambda_2 = (\beta - i\sqrt{4q-\beta^2})/(2q)$$

are complex conjugates, from which it follows that

$$f_2(z) = \overline{f_1(\bar{z})}$$

and, therefore

$$\overline{w_2(z)} = w_1(\bar{z}). \tag{19}$$

In [6], it was pointed out that the domain of the adjoint operator L^* consists of all vectors $x \in \mathcal{T}$ for which $\mathcal{L}x \in \mathcal{T}$ and $(L^*x)(n) = (\mathcal{L}x)(n)$. For the operator L, we have the limit point case as $n \to \infty$ and the limit circle case as $n \to -\infty$ (see [8]). Using this observation, one can easily deduce that the operator L admits the description

$$\mathcal{D}(L) = \left\{ x \in \mathcal{T} : \ \mathcal{L}x \in \mathcal{T}, \ \lim_{n\to-\infty} q^{-n} W_n(x,\bar{y}) = 0 \, \forall y \in \mathcal{D}(L^*) \right\},$$
$$Lx = \mathcal{L}x.$$

Here $W_n(x_1,x_2) = x_1(n)x_2(n+1) - x_1(n+1)x_2(n)$.

The defect vector $\varphi(z) = \{\varphi(n,z)\}_{n=-\infty}^{\infty}$, that is, the solution of the equation $\mathcal{L}x = zx$, $\operatorname{Im} z \neq 0$, that belongs to the space $\mathcal{T}$, can be written in terms of PP-solutions w_1 and w_2 as

$$\varphi(n,z) = a(z)w_1(n,z) + b(z)w_2(n,z). \tag{20}$$

Since the coefficients of the difference expression (2) are real, one has $\varphi(\bar{z}) = \overline{\varphi(z)}$. By virtue of (19), this means that $\overline{a(\bar{z})} = b(z)$. Therefore, in the representation (20), $a(z) \neq 0$ and $b(z) \neq 0$ for $\operatorname{Im} z \neq 0$.

Theorem 5. *Let β satisfy condition* (15). *Then the operator L does not admit (q^2, U)-scale-invariant self-adjoint extensions.*

In order to prove Theorem 5, we first provide the following inequality.

Lemma 2. *Let the functions a and b be defined by* (20). *Then*

$$\frac{|b(z)|^2 - |a(z)|^2}{\operatorname{Im} z} > 0.$$

Proof. We show

$$\|\varphi(z)\|^2 = \frac{q}{q-1} \frac{\sqrt{4q-\beta^2}}{2q} \frac{|b(z)|^2 - |a(z)|^2}{\operatorname{Im} z}, \tag{21}$$

from which the statement follows. In [8, p. 876], it was shown that

$$\begin{aligned} \|\varphi(z)\|^2 &= \frac{q-1}{q} \sum_{k=-\infty}^{\infty} q^k |\varphi(k,z)|^2 \\ &= \frac{i}{2(q-1)\operatorname{Im} z} \lim_{m\to-\infty} \Big\{ q^{-(m-1)} \Big[\varphi(m-1,z)\overline{\varphi(m,z)} \\ &\qquad - \varphi(m,z)\overline{\varphi(m-1,z)} \Big] \Big\}. \end{aligned}$$

Using (20), one obtains

$$\begin{aligned} &\lim_{m\to-\infty} q^{-(m-1)} \Big[\varphi(m-1,z)\overline{\varphi(m,z)} - \varphi(m,z)\overline{\varphi(m-1,z)} \Big] \\ &= |a(z)|^2 \lim_{m\to-\infty} q^{|m|+1} \Big[w_1(m-1,z)\overline{w_1(m,z)} - w_1(m,z)\overline{w_1(m-1,z)} \Big] \\ &+ |b(z)|^2 \lim_{m\to-\infty} q^{|m|+1} \Big[w_2(m-1,z)\overline{w_2(m,z)} - w_2(m,z)\overline{w_2(m-1,z)} \Big] \\ &+ a(z)\overline{b(z)} \lim_{m\to-\infty} q^{|m|+1} \Big[w_1(m-1,z)\overline{w_2(m,z)} - w_1(m,z)\overline{w_2(m-1,z)} \Big] \\ &+ \overline{a(z)}b(z) \lim_{m\to-\infty} q^{|m|+1} \Big[\overline{w_1(m,z)}w_2(m-1,z) - \overline{w_1(m-1,z)}w_2(m,z) \Big]. \end{aligned}$$

Now we evaluate the limits using (18) and the fact that $\lambda_2 = \overline{\lambda_1}$:

$$\begin{aligned} &|a(z)|^2 \lim_{m\to-\infty} q^{|m|+1} \Big[w_1(m-1,z)\overline{w_1(m,z)} - w_1(m,z)\overline{w_1(m-1,z)} \Big] \\ &= |a(z)|^2 \lim_{m\to-\infty} q^{|m|+1} |\lambda_1|^{2|m|} \\ &\times \Big(\lambda_1 [1+\delta_1(m-1,z)] \Big[1+\overline{\delta_1(m,z)}\Big] - \overline{\lambda_1} [1+\delta_1(m,z)] \Big[1+\overline{\delta_1(m-1,z)}\Big] \Big) \\ &= |a(z)|^2 q(\lambda_1 - \lambda_2). \end{aligned}$$

In the same way, one obtains

$$\begin{aligned} &|b(z)|^2 \lim_{m\to-\infty} q^{|m|+1} \Big[w_2(m-1,z)\overline{w_2(m,z)} - w_2(m,z)\overline{w_2(m-1,z)} \Big] \\ &\quad = |b(z)|^2 (\lambda_2 - \lambda_1). \end{aligned}$$

Now

$$\begin{aligned}
&a(z)\overline{b(z)} \lim_{m\to-\infty} q^{|m|+1}\left[w_1(m-1,z)\overline{w_2(m,z)} - w_1(m,z)\overline{w_2(m-1,z)}\right] \\
&= a(z)\overline{b(z)} \lim_{m\to-\infty} q^{|m|+1} q^{|m|+1} \lambda_1^{|m|}\overline{\lambda_2^{|m|}} \\
&\times \left(\lambda_1\left[1+\delta_1(m-1,z)\right]\left[1+\overline{\delta_2(m,z)}\right] - \overline{\lambda_2}\left[1+\delta_1(m,z)\right]\left[1+\overline{\delta_2(m-1,z)}\right]\right) = 0
\end{aligned}$$

because $\lambda_2 = \overline{\lambda_1}$ and $|\lambda_1| = q^{-1/2}$. Similarly,

$$\overline{a(z)}b(z) \lim_{m\to-\infty} q^{|m|+1}\left[\overline{w_1(m,z)}w_2(m-1,z) - \overline{w_1(m-1,z)}w_2(m,z)\right] = 0.$$

Combining the previous equalities and taking into account that

$$\operatorname{Im}\lambda_1 = \sqrt{4q-\beta^2}/2q,$$

one obtains (21). This completes the proof. □

Proof of Theorem 5. Since $\varphi(\bar z) = \overline{\varphi(z)}$, the transformation (11) is again Möbius transformation, that is, of the form (12). The coefficients $\mathcal{A}$ and $\mathcal{B}$ are calculated according to the formulas

$$\begin{aligned}
\mathcal{A} &= (q^2+1)\sum_{k=-\infty}^{\infty} q^k \overline{\varphi(k-1,i)}\varphi(k,i) \\
&= \frac{iq^3}{(q-1)^2}\lim_{m\to-\infty} q^{-(m-1)}\left[|\varphi(m-1,i)|^2 - \overline{\varphi(m-2,i)}\varphi(m,i)\right]
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\mathcal{B} &= (q^2-1)\sum_{k=-\infty}^{\infty} q^k \overline{\varphi(k-1,i)\varphi(k,i)} \\
&= \frac{iq^3}{(q-1)^2}\lim_{m\to-\infty} q^{-(m-1)}\left[\overline{\varphi(m-2,i)\varphi(m,i)} - \left(\overline{\varphi(m-1,i)}\right)^2\right].
\end{aligned} \tag{23}$$

Evaluating the limits in (22) and (23) in the same way as above, one obtains the expressions

$$\begin{aligned}
\mathcal{A} &= \frac{iq^3}{(q-1)^2}\left[|a(i)|^2(1-e^{-2i\psi}) + |b(i)|^2(1-e^{2i\psi})\right], \\
\mathcal{B} &= \frac{2iq^3}{(q-1)^2}\overline{a(i)b(i)}(\cos 2\psi - 1),
\end{aligned} \tag{24}$$

where $\psi = \arg\lambda_1$, $0<\psi<\pi$. As was pointed out above $a(i)\neq 0$ and $b(i)\neq 0$, i.e., $\mathcal{B}\neq 0$. This means that the transformation (12) is not the identical transformation. From (24), it follows that

$$|\mathcal{B}| - |\operatorname{Im}\mathcal{A}| = -\frac{2q^3\sin^2\psi}{(q-1)^2}(|b(i)|-|a(i)|)^2 < 0$$

(we also used (21)). The last inequality means that the transformation (12) is of elliptic type and does not have fixed points on the unit circle. This completes the proof. □

Now we consider the cases $\alpha = \alpha_+ = -\frac{1}{(\sqrt{q}+1)^2}$ and $\alpha = \alpha_- = -\frac{1}{(\sqrt{q}-1)^2}$. In both cases, $\beta^2 = 4q$. Since the arguments for $\alpha=\alpha_+$ and for $\alpha=\alpha_-$ are the same, we provide them only for $\alpha=\alpha_+$.

For the value of $\beta = 2\sqrt{q}$, instead of Theorem 4, we have the following result.

Theorem 6. *Let $\mathcal{L}$ be the difference expression (2). Then for $\beta = 2\sqrt{q}$ and any $z \in \mathbb{C}$, the equation*

$$\mathcal{L}x = zx$$

has two linearly independent solutions, $w_1 = \{w_1(n,\cdot)\}_{n=-\infty}^{\infty}$ and $w_2 = \{w_2(n,\cdot)\}_{n=-\infty}^{\infty}$, such that

$$\lim_{n\to-\infty} \frac{w_1(n,z)}{\lambda^{-n}} = 1, \tag{25}$$

$$\lim_{n\to-\infty} \frac{w_2(n,z)}{n\lambda^{-n}} = 1, \tag{26}$$

where $\lambda = q^{-1/2}$ is the double root of the equation $\lambda^2 - (\beta/q)\lambda + 1/q = 0$.

Proof. The solution w_1 is constructed in the same way as in Theorem 4, that is, $w_1(n,z) = \lambda^{-n} f(zq^{2n})$, where $f(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is an entire function, which is a solution of the functional equation

$$\frac{1}{\lambda} f(zq^2) - \beta f(z) + q\lambda f(z/q^2) = -z\frac{(q-1)^2}{q} f(z).$$

Since $f(0) = 1$ and $q > 1$, the condition (25) is fulfilled. Note that from the proof of Theorem 4 (see [8]), it follows that $c_1 \neq 0$. For the solution w_2, we are looking at the form $w_2(n,z) = u(n,z)w_1(n,z)$. For the bi-infinite sequence $u(z) = \{u(n,z)\}_{n=-\infty}^{\infty}$, one obtains the equation

$$-f(zq^{2n+2})\,[u(n+1,z) - u(n,z)] + f(zq^{2n-2})\,[u(n,z) - u(n-1,z)] = 0 \tag{27}$$

(we used the fact that $\lambda q^2 = 1$). From (27), it follows that

$$\lim_{n\to-\infty} \frac{u(n,z) - u(n-1,z)}{u(n+1,z) - u(n,z)} = \lim_{n\to-\infty} \frac{f(zq^{2n+2})}{f(zq^{2n-2})} = 1$$

(as $f(0) = 1$, one has $f(zq^{2n}) \neq 0$ for $n < 0$ and $|n|$ large enough). Now we show that the limit

$$\lim_{n\to-\infty} [u(n,z) - u(n-1,z)] =: K(z) \neq 0 \tag{28}$$

exists. Consider the infinite product

$$\prod_{n=-1}^{-\infty} \frac{u(n,z) - u(n-1,z)}{u(n+1,z) - u(n,z)}. \tag{29}$$

From (27), it follows that

$$\left| \frac{u(n,z) - u(n-1,z)}{u(n+1,z) - u(n,z)} - 1 \right| = \left| \frac{f(zq^{2n+2}) - f(zq^{2n-2})}{f(zq^{2n-2})} \right|.$$

The numerator of the last fraction behaves as $|c_1 z|(1-q^{-4})q^{2n+2}[1+\mathrm{o}(1/n)]$ as $n \to -\infty$, while the denominator approaches 1. This means that

$$\left| \frac{u(n,z) - u(n-1,z)}{u(n+1,z) - u(n,z)} - 1 \right| \sim q^{2n+2} \quad \text{as} \quad n \to -\infty.$$

Hence

$$\sum_{n=-1}^{-\infty} \left| \frac{u(n,z) - u(n-1,z)}{u(n+1,z) - u(n,z)} - 1 \right| < \infty,$$

which means that the infinite product (29) converges. Since

$$\prod_{n=-1}^{-\infty} \frac{u(n,z)-u(n-1,z)}{u(n+1,z)-u(n,z)} = \lim_{N\to\infty} \prod_{n=-1}^{-N} \frac{u(n,z)-u(n-1,z)}{u(n+1,z)-u(n,z)} = \lim_{N\to\infty} \frac{u(-N,z)-u(-N-1,z)}{u(-1,z)-u(0,z)},$$

one deduces that $\lim_{n\to\infty}[u(-n,z)-u(-n-1,z)]$ exists and is not equal to zero. This shows (28). Now

$$\lim_{n\to\infty}[u(-n-1,z)-u(-n,z)] = K(z). \tag{30}$$

By (28), $K(z)\neq 0$. From the existence of the limit in (30), it follows that

$$\lim_{n\to-\infty} \frac{u(n,z)}{n} = -K(z).$$

As $w_2(n,z)=w_1(n,z)u(n,z)$, one concludes that

$$\lim_{n\to-\infty} \frac{w_2(n,z)}{n\lambda^{-n}} = -K(z).$$

Since solutions of the equation $\mathcal{L}x = zx$ are defined up to a multiplicative constant, the proof is complete. □

As above, we represent the defect vector $\varphi(z)$ in the form

$$\varphi(z) = a(z)w_1(z) + b(z)w_2(z)$$

and have the following lemma (compare with [8, Lemma 3.1]).

Lemma 3. *For $\beta = 2\sqrt{q}$, one has*

$$\|\varphi(z)\|^2 = \frac{\sqrt{q}}{q-1}\frac{\operatorname{Im}[a(z)\overline{b(z)}]}{\operatorname{Im} z}.$$

In particular,

$$\frac{\operatorname{Im}[a(z)\overline{b(z)}]}{\operatorname{Im} z} > 0, \quad \operatorname{Im} z \neq 0. \tag{31}$$

The proof of Lemma 3 is based on calculations similar to those performed in the proof of Lemma 2, and we omit them. Note that (31) implies that $|b(i)|>0$.

Theorem 7. *For $\beta = 2\sqrt{q}$, the operator L has only one (q^2,U)-scale-invariant extension. Hence the operator L has only one positive self-adjoint extension.*

Proof. As in the proof of Theorem 5, the transformation (11) is a Möbius transformation of the form (12). The coefficients $\mathcal{A}$ and $\mathcal{B}$ of the transformation are given by the formulas (22) and (23), respectively. Evaluating those coefficients in the same way as above, one obtains

$$\mathcal{A} = \frac{iq^3}{(q-1)^2}\left\{|b(i)|^2 - 2i\operatorname{Im}[a(i)\overline{b(i)}]\right\}, \quad \mathcal{B} = -\frac{iq^3}{(q-1)^2}\overline{[b^2(i)]}.$$

In particular, $\mathcal{B}\neq 0$. Since $|\mathcal{B}| = |\operatorname{Im}\mathcal{A}|$, one concludes that the transformation (12) in the case under consideration is of parabolic type, that is, has only one fixed point. This completes the proof. □

Remark 5. Theorem 7 proves [8, Conjecture 4.6].

References

1. Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards Applied Mathematics Series, vol. 55. For sale by the Superintendent of Documents. U.S. Government Printing Office, Washington, D.C., 1964.
2. N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space.* Translated from the Russian and with a preface by Merlynd Nestell. Reprint of the 1961 and 1963 translations. Two volumes bound as one. Dover Publications Inc., New York, 1993.
3. G. Beĭtmen and A. Èrdeĭi, *Vysshie transtsendentnye funktsii.* T. II*: Funktsii Besselya, funktsii parabolicheskogo tsilindra, ortogonalnye mnogochleny* [Higher Transcendental Functions. Vol. 2: Bessel Functions, Parabolic Cylinder Functions, Orthogonal Polynomials]. Translated from the English by N. Ja. Vilenkin. Second edition, unrevised. Spravochnaya Matematicheskaya Biblioteka [Mathematical Reference Library]. Nauka, Moscow, 1974. (Russian)
4. Borislava Bekker and Miron B. Bekker, *On self-adjoint homogeneous operators*, Complex Anal. Oper. Theory **7** (2013), no. 1, 9–31.
5. Miron B. Bekker, *On a class of nondensely defined Hermitian contractions*, Adv. Dyn. Syst. Appl. **2** (2007), no. 2, 141–165.
6. Miron B. Bekker, Martin J. Bohner, Alexander N. Herega, and Hristo Voulov, *Spectral analysis of a q-difference operator*, J. Phys. A: Math. Theor. **43** (2010), no. 14, 145–207.
7. Miron B. Bekker, Martin J. Bohner, and Hristo Voulov, *A q-difference operator with discrete and simple spectrum*, Methods Funct. Anal. Topology **17** (2011), no. 4, 281–294.
8. Miron B. Bekker, Martin J. Bohner, and Hristo Voulov, *Extreme self-adjoint extensions of a semibounded q-difference operator*, Math. Nachr. **287** (2014), no. 8, 869–884.
9. N. Dunford and J. T. Schwartz, *Linear Operators.* Part II*: Spectral Theory. Selfadjoint Operators in Hilbert Space.* With the assistance of William G. Bade and Robert G. Bartle. Reprint of the 1963 original. A Wiley-Interscience Publication. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988.
10. Seppo Hassi and Sergii Kuzhel, *On symmetries in the theory of finite rank singular perturbations*, J. Funct. Anal. **256** (2009), no. 3, 777–809.
11. A. N. Kočubeĭ, *Symmetric operators commuting with a family of unitary operators*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 4, 77–78. (Russian)
12. M. G. Kreĭn and Ju. L. Šmul′jan, *Fractional linear transformations with operator coefficients*, Mat. Issled. (Kishinev) **2** (1967), no. 3, 64–96. (Russian); English transl. Amer. Math. Soc. Transl., Ser. 2 **103** (1974), 125–152.
13. A. V. Kuzhel and S. A. Kuzhel, *Regular Extensions of Hermitian Operators.* Translated from the Russian by P. Malyshev and D. Malyshev. VSP, Utrecht, 1998.
14. K. A. Makarov and E. Tsekanovskii, *On μ-scale invariant operators*, Methods Funct. Anal. Topology **13** (2007), no. 2, 181–186.
15. M. A. Naĭmark, *Linear Differential Operators*, Second edition, revised and augmented. With an appendix by V. È. Ljance. Nauka, Moscow, 1969. (Russian)
16. Ju. L. Šmul′jan, *Operator balls*, Teor. Funktsii Funktsional. Anal. i Prilozhen. **6** (1968), 68–81. (Russian); English transl. Integral Equations Operator Theory **13** (1990), no. 6, 864–882.
17. Joachim Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, vol. 1258, Springer-Verlag, Berlin, 1987.

Department of Mathematics, University of Pittsburgh at Johnstown, Johnstown, PA, USA
E-mail address: `bekker@pitt.edu`

Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO, USA
E-mail address: `bohner@mst.edu`

Integrated Banking Information Systems, Odessa, 65125, Ukraine
E-mail address: `mark@te.net.ua`

Department of Mathematics and Statistics, University of Missouri–Kansas City, Kansas City, MO, USA
E-mail address: `voulovh@umkc.edu`

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