Methods of Functional Analysis and Topology
Vol. 21 (2015), no. 1, pp. 56–68

ON REGULARITY OF LINEAR SUMMATION METHODS OF TAYLOR SERIES

M. V. GAEVSKIJ AND P. V. ZADEREY

Abstract. The paper specifies necessary and sufficient conditions for regularity of an infinite matrix of real numbers, which determines some summation method for a class of functions that are analytic on the unit disk and continuous on the closed circle.

Let L be the space of integrable 2π-periodic functions with the norm $\|f\|_L = \int\limits_{-\pi}^{\pi} |f(t)|dt$, and let C be a subspace of L that consists of continuous functions with the norm $\|f\|_C = \max\limits_{t\in[-\pi,\pi]} |f|$. Let also $\Lambda = \{\lambda_k^{(n)}\}$, $n, k = 0, 1, \dots,$ be an infinite matrix of real numbers, which determines some summation method. With every 2π-periodic continuous function $f \in C$ with the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=0}^{\infty}(a_k \cos kx + b_k \sin kx)$$

we associate a series

$$U_n(f, \Lambda, x) = \lambda_0^{(n)} \frac{a_0}{2} + \sum_{k=0}^{\infty} \lambda_k^{(n)}(a_k \cos kx + b_k \sin kx).$$

We say that the Fourier series of a 2π-periodic continuous function $f \in C$ is summable by the Λ-method in a point x to the value $f(x)$, if all the series on the right-hand side of the above relation are convergent at this point, and

$$\lim_{n\to\infty} U_n(f, \Lambda, x) = f(x).$$

The summation Λ-method is regular in the space of 2π-periodic continuous functions C, if, for any 2π-periodic function $f \in C$ and any x, its Fourier series is summed by this method to the $f(x)$.

Karamata and Tomić [1] proved that the summation Λ-method defined by an infinite matrix is regular in the space C if and only if the following conditions holds:

(A) for any $k = 0, 1, 2, \dots$

$$\lim_{n\to\infty} \lambda_k^{(n)} = 1;$$

(B) for any n, there exists a number M_n (possibly depending on n) such that, for all m,

$$\int_0^{\pi} \Big|\frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{m} \lambda_k^{(n)} \cos kx\Big|dx \le M_n;$$

2000 *Mathematics Subject Classification.* Primary 46E15; Secondary 30A82, 30E10, 40C05.
Key words and phrases. Summation method, infinite matrix, regularity, Taylor series.

(C) the total variation of the function

$$\bar{K}_n(x) = \lim_{m\to\infty} \int_0^x \Big\{\frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{m} \lambda_k^{(n)} \cos kx\Big\} dx = \frac{\lambda_0^{(n)}}{2} x + \sum_{k=1}^{\infty} \frac{\lambda_k^{(n)}}{k} \sin kx$$

is uniformly bounded, i.e.,

$$\int_0^\pi |d\bar{K}_n(x)| \le M.$$

(Here and below, M means some absolute constants which are, perhaps, different in different formulas.)

For triangular matrices Λ, i.e., matrices, for which $\lambda_k^{(n)} = 0$ for $n < k$, these conditions become simpler [2], [3]. In this case, for the regularity of the summation method Λ, it is necessary and sufficient that condition *(A)* and the following condition be valid:

$$(B*) \qquad \int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{n-1} \lambda_k^{(n)} \cos kx \Big| dx \le M.$$

Let $D = \{w \in \mathbb{C} : |w| < 1\}$. By $A(\overline{D})$, we denote the space of functions $f(\cdot)$ analytic in D and continuous in $\overline{D} = \{w \in \mathbb{C} : |w| \le 1\}$ with the norm

$$\|f\|_{A(\overline{D})} = \max_{z\in\overline{D}} |f(z)|.$$

If the infinite matrix $\Lambda = \{\lambda_k^{(n)}\}, n, k = 0, 1, \dots,$ of real numbers $\lambda_k^{(n)}$ is given, then any function $f \in A(\overline{D})$ with the Taylor series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z \in D \tag{1}$$

can be associated with a sequence of series,

$$U_n(f, \Lambda, z) = \sum_{k=0}^{\infty} \lambda_k^{(n)} c_k z^k, \quad z \in \overline{D}. \tag{2}$$

We say that the series (1) is summable by the Λ-method at the point $z \in \overline{D}$ to the value $f(z)$, if all series (2) are convergent for any $n = 0, 1, \dots$, and the following relation holds:

$$\lim_{n\to\infty} U_n(f, \Lambda, z) = f(z), \quad z \in \overline{D}. \tag{3}$$

The summation Λ-method is called regular in space $A(\overline{D})$, if, for each $f \in A(\overline{D})$ and any $z \in \overline{D}$, series (1) is summed by this method to $f(z)$.

If the summation method is regular in the space $A(\overline{D})$, then, for each $f \in A(\overline{D})$, series (2) converges uniformly, and the convergence in (3) is uniform, i.e., there is the convergence in the norm of the space $A(\overline{D})$.

For triangular matrices in the space $A(\overline{D})$, L.V. Taikov [4] obtained the following result:

The triangular matrix of real numbers Λ is regular in the space $A(\overline{D})$ if and only if it satisfies the condition (A) and the following condition:

(D) there exists $M > 0$ and such a decomposition $\lambda_k^{(n)} = \alpha_k^{(n)} + \beta_k^{(n)}$ into real numbers $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ $(k = 1, 2, \dots, n)$ that

$$\int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{n} (\alpha_k^{(n)} \cos kx + \beta_k^{(n)} \sin kx) \Big| dx \le M \quad (n = 1, 2, \dots).$$

He also pointed out that the conditions for the regularity of the Λ-method in the space $A(\overline{D})$ are weaker than the conditions for the regularity in the space C.

Here we consider the question of the conditions for the regularity of the method Λ in the space $A(\overline{D})$ in the case of infinite matrices. It will also be shown that the conditions for the regularity of the method Λ in the space $A(\overline{D})$ are weaker than the conditions for the regularity in the space C.

The main results were presented at the International Conference "Approximation theory and its applications" (May 28–June 3, 2012, Kamianets–Podilsky, Ukraine) [5].

Theorem 1. *For the regularity of the summation Λ-method in the space $A(\overline{D})$, it is necessary and sufficient that condition (A) and the following conditions be satisfied:*

(E) there exist numbers $M_n > 0$, perhaps, dependent on n and the decomposition

$$\lambda_k^{(n)} = \alpha_k^{(n)} + \beta_k^{(n)}, \quad k = 1, 2, \dots$$

into real numbers $\alpha_k^{(n)}$ and $\beta_k^{(n)}$ such that each of the functions

$$t_m^{(n)}(x) = \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{m} (\alpha_k^{(n)} \cos kx + \beta_k^{(n)} \sin kx), \quad n = 0, 1, \dots, \quad m = 0, 1, \dots,$$

satisfies

$$\int_0^{2\pi} |t_m^{(n)}(x)| dx \le M_n, \quad n = 0, 1, \dots,$$

where M_n is independent of m;

(F) the total variation of the functions

$$\begin{aligned} K_n(x) &= \lim_{m\to\infty} \int_0^x \Big(\frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{m} (\alpha_k^{(n)} \cos kt + \beta_k^{(n)} \sin kt) \Big) dt \\ &= \frac{\lambda_0^{(n)}}{2} x + \sum_{k=1}^{\infty} \frac{1}{k} (\alpha_k^{(n)} \sin kx - \beta_k^{(n)} \cos kx) \end{aligned}$$

is uniformly bounded on $[0, 2\pi]$,

$$\int_0^{2\pi} |dK_n(x)| = \int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{\infty} (\alpha_k^{(n)} \cos kx + \beta_k^{(n)} \sin kx) \Big| dx \le M.$$

Proof. Necessity. Let Λ-method be regular in $A(\overline{D})$, i.e., for each $f \in A(\overline{D})$ and for any $z \in \overline{D}$, series (2) are convergent, and relation (3) holds. Let us show that relations *(A)*, *(E)*, and *(F)* hold.

Let, for each $f \in A(\overline{D})$,

$$\begin{aligned} U_{m,n}(f) = U_{m,n}(f, \Lambda, z) &= \sum_{k=0}^{m} \lambda_k^{(n)} c_k z^k \\ &= \frac{1}{2\pi i} \int_{|t|=1} f(t) \Big(\sum_{k=0}^{m} \lambda_k^{(n)} \frac{z^k}{t^{k+1}} + \sum_{k=1}^{m} \mu_k^{(n)} \frac{t^{k-1}}{z^k} \Big) dt, \\ U_n(f) &= \lim_{m\to\infty} U_{m,n}(f). \end{aligned}$$

Here $\mu_k^{(n)}, k = 1, 2, \dots, m; n = 0, 1, \dots$, are arbitrary complex numbers.

For fixed $z = e^{i\tau}$, the last expressions are linear functionals on $A(\overline{D})$ and can be represented as

$$U_{m,n}(f, \Lambda, e^{i\tau}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i(\tau+\theta)}) b_m^{(n)}(\theta) d\theta, \tag{4}$$

where

$$b_m^{(n)}(\theta) = \lambda_0^{(n)} + \sum_{k=1}^{m} \Big(\lambda_k^{(n)} e^{-ik\theta} + \mu_k^{(n)} e^{ik\theta}\Big);$$

$$U_n(f,\Lambda,e^{i\tau}) = \lim_{m\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(\tau+\theta)}) b_m^{(n)}(\theta)d\theta. \tag{5}$$

F. Riesz [6] showed that there exist complex numbers $\bar{\mu}_k^{(n)}, k = 1,2,\dots,m;\ n = 0,1,\dots,$ such that

$$\begin{aligned}\|U_{m,n}\| &= \sup_{\|f\|_{A(\overline{D})}=1} \|U_{m,n}(f,\Lambda,e^{i\tau})\|_{A(\overline{D})} \\ &= \min_{\mu_k^{(n)}} \frac{1}{2\pi}\int_0^{2\pi} |b_m^{(n)}(\theta)|d\theta = \frac{1}{2\pi}\int_0^{2\pi} |\bar{b}_m^{(n)}(\theta)|d\theta,\end{aligned} \tag{6}$$

where

$$\bar{b}_m^{(n)}(\theta) = \lambda_0^{(n)} + \sum_{k=1}^{m} \Big(\lambda_k^{(n)} e^{-ik\theta} + \bar{\mu}_k^{(n)} e^{ik\theta}\Big).$$

Then the functionals $U_{m,n}(f)$ and $U_n(f)$ can be written as

$$U_{m,n}(f,\Lambda,e^{i\tau}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(\tau+\theta)})\bar{b}_m^{(n)}(\theta)d\theta, \tag{7}$$

$$U_n(f,\Lambda,e^{i\tau}) = \lim_{m\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(\tau+\theta)})\bar{b}_m^{(n)}(\theta)d\theta. \tag{8}$$

The convergence of series (2) for each $f \in A(\overline{D})$ at a fixed point $z \in \overline{D}$ is the convergence of functionals (4) to the continuous functional (5). According to the Banach–Steinhaus theorem [7, p. 266], the sequence $\{U_{m,n}(f)\}$ converges on $A(\overline{D})$ as $m \to \infty$ to the continuous functional $U_n(f)$, if and only if

a) norms $U_{m,n}(f)$ are bounded in aggregate,

$$\|U_{m,n}\| = \frac{1}{2\pi}\int_0^{2\pi} |\bar{b}_m^{(n)}(\theta)|d\theta \le M_n,$$

where M_n depends, perhaps, on n, but not on m;

b) the sequence $\{U_{m,n}(e^{i\nu\tau})\}$ converges to $U_n(e^{i\nu\tau})$ for any $\nu = 0,1,\dots$ (linear combinations of the functions $e^{i\nu\tau}, \nu = 0,1,\dots,$ are dense in $A(\overline{D})$),

$$\lim_{m\to\infty} U_{m,n}(e^{i\nu\tau}) = \lim_{m\to\infty} \lambda_\nu^{(n)} e^{i\nu\tau} = \lambda_\nu^{(n)} e^{i\nu\tau} = U_n(e^{i\nu\tau}). \tag{9}$$

If $\bar{\mu}_k^{(n)} = \bar{\eta}_k^{(n)} + i\bar{\bar{\eta}}_k^{(n)}$, then

$$\begin{aligned}\bar{b}_m^{(n)}(x) &= \mathrm{Re}\bar{b}_m^{(n)}(x) + i\mathrm{Im}\bar{b}_m^{(n)}(x) \\ &= \lambda_0^{(n)} + \sum_{k=1}^{m}(\lambda_k^{(n)} + \bar{\eta}_k^{(n)})\cos kx - \sum_{k=1}^{m}\bar{\bar{\eta}}_k^{(n)}\sin kx \\ &\quad - i\Big(\sum_{k=1}^{m}(\lambda_k^{(n)} - \bar{\eta}_k^{(n)})\sin kx - \sum_{k=1}^{m}\bar{\bar{\eta}}_k^{(n)}\cos kx\Big).\end{aligned} \tag{10}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} |\bar{b}_m^{(n)}(x)|dx \ge \frac{1}{2}\Big(\frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Re}\bar{b}_m^{(n)}(x)|dx + \frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Im}\bar{b}_m^{(n)}(x)|dx\Big) \tag{11}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Re}\bar{b}_m^{(n)}(x)|dx = \frac{1}{2\pi}\int_0^{2\pi} |\mathrm{Re}\bar{b}_m^{(n)}(-x)|dx,$$

$$\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Im}\bar{b}_m^{(n)}(x)|dx=\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Im}\bar{b}_m^{(n)}(-x)|dx,$$

we have

$$\begin{aligned}\frac{1}{4\pi}\int_0^{2\pi}|\mathrm{Re}\bar{b}_m^{(n)}(x)|dx&=\frac{1}{8\pi}\int_0^{2\pi}|\mathrm{Re}\bar{b}_m^{(n)}(x)|dx+\frac{1}{8\pi}\int_0^{2\pi}|\mathrm{Re}\bar{b}_m^{(n)}(x)|dx\\&=\frac{1}{8\pi}\int_0^{2\pi}|\mathrm{Re}\bar{b}_m^{(n)}(x)|dx+\frac{1}{8\pi}\int_0^{2\pi}|\mathrm{Re}\bar{b}_m^{(n)}(-x)|dx\\&\geq\frac{1}{8\pi}\int_0^{2\pi}|\mathrm{Re}\bar{b}_m^{(n)}(x)+\mathrm{Re}\bar{b}_m^{(n)}(-x)|dx.\end{aligned}\tag{12}$$

Similarly, we obtain

$$\frac{1}{4\pi}\int_0^{2\pi}|\mathrm{Im}\bar{b}_m^{(n)}(x)|dx\geq\frac{1}{8\pi}\int_0^{2\pi}|\mathrm{Im}\bar{b}_m^{(n)}(x)-\mathrm{Im}\bar{b}_m^{(n)}(-x)|dx.\tag{13}$$

We note that

$$\begin{aligned}\frac{1}{4}\Big(\mathrm{Re}\bar{b}_m^{(n)}(x)+\mathrm{Re}\bar{b}_m^{(n)}(-x)\Big)&=\frac{\lambda_0^{(n)}}{2}+\sum_{k=1}^m\frac{1}{2}(\lambda_k^{(n)}+\bar{\eta}_k^{(n)})\cos kx,\\\frac{1}{4}\Big(\mathrm{Im}\bar{b}_m^{(n)}(x)-\mathrm{Im}\bar{b}_m^{(n)}(-x)\Big)&=-\sum_{k=1}^m\frac{1}{2}(\lambda_k^{(n)}-\bar{\eta}_k^{(n)})\sin kx,\end{aligned}\tag{14}$$

then, assuming

$$\frac{1}{2}(\lambda_k^{(n)}+\bar{\eta}_k^{(n)})=\alpha_k^{(n)}\quad\text{and}\quad\frac{1}{2}(\lambda_k^{(n)}-\bar{\eta}_k^{(n)})=\beta_k^{(n)},\tag{15}$$

$$t_m^{(n)}(x)=\frac{\lambda_0^{(n)}}{2}+\sum_{k=1}^m\Big(\alpha_k^{(n)}\cos kx+\beta_k^{(n)}\sin kx\Big),\tag{16}$$

relations (6) and (11)–(14) yield

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}|t_m^{(n)}(x)|dx\\&\leq\frac{1}{2\pi}\int_0^{2\pi}\Big|\frac{\lambda_0^{(n)}}{2}+\sum_{k=1}^m\alpha_k^{(n)}\cos kx\Big|dx\\&+\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{k=1}^m\beta_k^{(n)}\sin kx\Big|dx\leq\|U_{m,n}\|.\end{aligned}\tag{17}$$

So, the convergence of series (2) implies that norms (6) are bounded, and, hence, condition *(E)* is fulfilled by (17). Due to (9), equality (3) ensures that condition *(A)* holds.

With regard to relations (10), (15) and (16), it is easy to obtain that

$$\bar{b}_m^{(n)}(x)=2\bar{\bar{b}}_m^{(n)}(x)+i\sum_{k=1}^m\bar{\bar{\eta}}_k^{(n)}e^{ikx},\tag{18}$$

where

$$\begin{aligned}\bar{\bar{b}}_m^{(n)}(x)&=\frac{\lambda_0^{(n)}}{2}+\sum_{k=1}^m\alpha_k^{(n)}\cos kx-i\sum_{k=1}^m\beta_k^{(n)}\sin kx\\&=\frac{1}{2}\Big(t_m^{(n)}(x)+t_m^{(n)}(-x)\Big)-\frac{i}{2}\Big(t_m^{(n)}(x)-t_m^{(n)}(-x)\Big).\end{aligned}\tag{19}$$

Since $f \in A(\overline{D})$, we have $\int_0^{2\pi} f(e^{i(\tau+\theta)}) \sum_{k=1}^{m} \bar{\bar{\eta}}_k^{(n)} e^{ik\theta} d\theta = 0$. Then functionals (7) and (8) can be written as

$$U_{m,n}(f, \Lambda, e^{i\tau}) = \frac{1}{\pi} \int_0^{2\pi} f(e^{i(\tau+\theta)}) \bar{\bar{b}}_m^{(n)}(\theta) d\theta, \tag{20}$$

$$U_n(f, \Lambda, e^{i\tau}) = \lim_{m\to\infty} \frac{1}{\pi} \int_0^{2\pi} f(e^{i(\tau+\theta)}) \bar{\bar{b}}_m^{(n)}(\theta) d\theta. \tag{21}$$

Let us represent $\bar{\bar{b}}_m^{(n)}(x)$ in the form

$$\bar{\bar{b}}_m^{(n)}(x) = \frac{1-i}{2} t_m^{(n)}(x) + \frac{1+i}{2} t_m^{(n)}(-x).$$

Then the functionals $U_{m,n}(f)$ and $U_n(f)$ take the form

$$\begin{aligned} U_{m,n}(f) &= \frac{1}{\pi} \int_0^{2\pi} f(e^{i(\tau+\theta)}) \Big(\frac{1-i}{2} t_m^{(n)}(\theta) + \frac{1+i}{2} t_m^{(n)}(-\theta) \Big) d\theta \\ &= \frac{1}{\pi} \int_0^{2\pi} f(e^{i(\tau+\theta)}) \frac{1-i}{2} t_m^{(n)}(\theta) d\theta + \frac{1}{\pi} \int_0^{2\pi} f(e^{i(\tau-\theta)}) \frac{1+i}{2} t_m^{(n)}(\theta) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} g_\tau(\theta) t_m^{(n)}(\theta) d\theta, \end{aligned}$$

where

$$g_\tau(\theta) = (1-i) f(e^{i(\tau+\theta)}) + (1+i) f(e^{i(\tau-\theta)}),$$

$$U_n(f) = \lim_{m\to\infty} \frac{1}{2\pi} \int_0^{2\pi} g_\tau(\theta) t_m^{(n)}(\theta) d\theta.$$

Obviously, if $f \in A(\overline{D})$, then $g_\tau \in A(\overline{D})$.

It is known [8, p. 125] that the linear functional on the space of continuous functions can be represented as a Stieltjes integral, so functionals (20) and (21) can be written as

$$U_{m,n}(f) = \frac{1}{2\pi} \int_0^{2\pi} g_\tau(\theta) dB_m^{(n)}(\theta),$$

$$U_n(f) = \lim_{m\to\infty} \frac{1}{2\pi} \int_0^{2\pi} g_\tau(\theta) dB_m^{(n)}(\theta),$$

where

$$\Big(B_m^{(n)}(\theta) \Big)' = \Big(\frac{\lambda_0^{(n)}}{2} \theta + \sum_{k=1}^{m} \frac{1}{k} \Big(\alpha_k^{(n)} \sin k\theta - \beta_k^{(n)} \cos k\theta \Big) \Big)' = t_m^{(n)}(\theta).$$

In view of convergence of series (2) and, hence, the uniform boundedness of the norms $\|U_{m,n}\|$, the functions $B_m^{(n)}(\theta)$ have uniformly bounded variation, according to (17).

According to the Helly theorem on the limit transition under the sign of a Stieltjes integral [9, p. 366], we obtain

$$U_n(f) = \frac{1}{2\pi} \int_0^{2\pi} g_\tau(\theta) dK_n(\theta), \tag{22}$$

where the functions

$$K_n(\theta) = \lim_{m\to\infty} B_m^{(n)}(\theta) = \frac{\lambda_0^{(n)}}{2} \theta + \sum_{k=1}^{\infty} \frac{1}{k} \Big(\alpha_k^{(n)} \sin k\theta - \beta_k^{(n)} \cos k\theta \Big)$$

have uniformly bounded variation, i.e.,

$$\int_0^{2\pi} |dK_n(\theta)| \le M.$$

The necessity is proved.

Sufficiency. Let conditions *(A)*, *(E)* and *(F)* be fulfilled. We will show that series (2) converge at each point $z \in \overline{D}$, and equality (3) holds.

Indeed, relations (6), (16) and (18)–(19) yield

$$
\begin{aligned}
\|U_{m,n}\| &= \frac{1}{2\pi}\int_0^{2\pi} |\bar{b}_m^{(n)}(x)|dx \le \frac{1}{2\pi}\int_0^{2\pi} |\bar{\bar{b}}_m^{(n)}(x)|dx \\
&= \frac{1}{4\pi}\int_0^{2\pi} |t_m^{(n)}(x) + t_m^{(n)}(-x) - i(t_m^{(n)}(x) - t_m^{(n)}(-x))|dx \\
&\le \frac{1}{\pi}\int_0^{2\pi} |t_m^{(n)}(x)|dx.
\end{aligned} \tag{23}
$$

According to (23), condition *(E)* implies that the norms $\|U_{m,n}\|$ are bounded. The sequence $\{U_{m,n}(e^{i\nu\tau})\}$ converges to $U_n(e^{i\nu\tau})$ for any $\nu = 0, 1, \dots$ by condition *(A)*, i.e., it converges on a dense set of polynomials in the space $A(\overline{D})$. Therefore, according to the Banach–Steinhaus theorem, series (2) converge at each point $z \in \overline{D}$. Moreover, the function $K_n(x)$ has bounded variation. So, by (22) the functionals $U_n(f)$ defined on $A(\overline{D})$ have the norms

$$\|U_n\| \le M \int_0^{2\pi} |dK_n(\theta)|.$$

In view of condition *(F)*, the sequence of norms of the functionals $U_n(f)$ is uniformly bounded, and, according to *(A)*, equality (3) is valid for any polynomial. Therefore, by the Banach-Steinhaus theorem, equality (3) holds for any $f \in A(\overline{D})$.

The theorem is proved. □

Let us show that the conditions for the regularity of the Λ-method in the space $A(\overline{D})$ is weaker than regularity conditions in the space of C.

It is known [10] that the sum of the trigonometric series

$$\sum_{\nu=2}^{\infty} \frac{\cos \nu x}{\ln \nu}$$

is a function $f \in L$, and

$$\int_0^{2\pi} |S_n(f,x)|dx = \int_0^{2\pi} \Big|\sum_{\nu=2}^{n} \frac{\cos \nu x}{\ln \nu}\Big|dx = O(1).$$

In addition, the function $\varphi(x) = \frac{1}{2}\Big(f\big(x+\frac{\pi}{2}\big) - f\big(x - \frac{\pi}{2}\big)\Big) \in L$, and

$$\varphi(x) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}\sin(2\nu+1)x}{\ln(2\nu+1)}.$$

Moreover, we have

$$\int_0^{2\pi} |S_n(\varphi,x)|dx = \int_0^{2\pi} \Big|\sum_{\nu=1}^{n} \frac{(-1)^{\nu+1}\sin(2\nu+1)x}{\ln(2\nu+1)}\Big|dx = O(1).$$

At the same time, the partial sums of the series

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}\cos(2\nu+1)x}{\ln(2\nu+1)}$$

are unbounded in L.

Indeed,

$$S_{[\frac{n+1}{2}]}(x)=\sum_{\nu=1}^{[\frac{n+1}{2}]}\frac{(-1)^{\nu+1}\cos(2\nu+1)x}{\ln(2\nu+1)}=\frac{1}{2}\sum_{\nu=2}^{n}\frac{\sin\nu(x-\frac{\pi}{2})}{\ln\nu}-\frac{1}{2}\sum_{\nu=2}^{n}\frac{\sin\nu(x+\frac{\pi}{2})}{\ln\nu}$$

and

$$\begin{aligned}\int_0^{2\pi}|S_{[\frac{n+1}{2}]}(x)|dx&=\frac{1}{2}\int_0^{2\pi}\Big|\sum_{\nu=2}^{n}\frac{\sin\nu(x-\frac{\pi}{2})}{\ln\nu}-\sum_{\nu=2}^{n}\frac{\sin\nu(x+\frac{\pi}{2})}{\ln\nu}\Big|dx\\&=\frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\Big|\sum_{\nu=2}^{n}\frac{\sin\nu x}{\ln\nu}-\sum_{\nu=2}^{n}\frac{\sin\nu(x-\pi)}{\ln\nu}\Big|dx\\&+\frac{1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\Big|\sum_{\nu=2}^{n}\frac{\sin\nu x}{\ln\nu}-\sum_{\nu=2}^{n}\frac{\sin\nu(x+\pi)}{\ln\nu}\Big|dx\\&=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\Big|\sum_{\nu=2}^{n}\frac{\sin\nu x}{\ln\nu}\Big|dx+O\Big(\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\Big|\sum_{\nu=2}^{n}\frac{\sin\nu(x+\pi)}{\ln\nu}\Big|dx\Big).\end{aligned}$$

We note that the sequence $\{\frac{1}{\ln\nu}\}$ is convex, and, for the conjugate Fejer kernel

$$\widetilde{F}_n(x)=\sum_{\nu=1}^{n}\Big(1-\frac{\nu}{n+1}\Big)\sin\nu x,$$

the estimation

$$|\widetilde{F}_n(x+\pi)|=\Big|\frac{(n+1)\sin(x+\pi)-\sin(n+1)(x+\pi)}{4(n+1)\sin^2\frac{x+\pi}{2}}\Big|\le\frac{n+2}{4(n+1)\sin^2\frac{\pi}{4}}<1$$

holds for $|x|\le\frac{\pi}{2}$. Then, using the Abel transformation, it is easy to obtain the estimation

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\Big|\sum_{\nu=2}^{n}\frac{\sin\nu(x+\pi)}{\ln\nu}\Big|dx\le M.$$

Hence,

$$\int_0^{2\pi}|S_{[\frac{n+1}{2}]}(x)|dx=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\Big|\sum_{\nu=2}^{n}\frac{\sin\nu x}{\ln\nu}\Big|dx+O(1).$$

According to the results of [11], the partial sums of the series $\sum_{\nu=2}^{\infty}\frac{\sin\nu x}{\ln\nu}$ are unbounded in L:

Let the coefficients of the series $\sum_{\nu=1}^{\infty}a_\nu\sin\nu x$ satisfy the condition

$$\sum_{\nu=0}^{\infty}|\triangle a_\nu|+\sum_{\nu=2}^{\infty}\Big|\sum_{k=1}^{[\nu/2]}\frac{\triangle a_{\nu-k}-\triangle a_{\nu+k}}{k}\Big|<\infty,\quad \textit{where}\quad \triangle a_\nu=a_\nu-a_{\nu+1}.$$

Then the partial sums of the series are bounded in L if and only if, for any $m\in\mathbb{N}$,

$$\sum_{k=1}^{m}\frac{|a_{m+k}|}{k}\le M$$

and

$$\sum_{\nu=1}^{\infty}\frac{|a_\nu|}{\nu}<\infty.$$

The last condition for the series $\sum_{\nu=2}^{\infty}\frac{\sin\nu x}{\ln\nu}$ is not fulfilled. Note that this condition is not required for the boundedness of partial sums of $\sum_{\nu=1}^{\infty}a_\nu\cos\nu x$.

We define the matrix $\Lambda = \{\lambda_\nu^{(n)}\}$ as

$$\lambda_\nu^{(n)} = \begin{cases} 1 - \frac{\nu}{n+1}, & \text{for } \nu \le n, \\ 0, & \text{when } \nu > n \text{ and } \nu = 2k, \\ \frac{(-1)^{k+1}}{\ln \nu}, & \text{when } \nu > n \text{ and } \nu = 2k+1. \end{cases}$$

Note that the polynomial $\frac{1}{2} + \sum_{\nu=1}^n \left(1 - \frac{\nu}{n+1}\right) \cos \nu x := F_n(x)$ is Fejer kernel, and the relation $\frac{1}{\pi} \int_0^{2\pi} |F_n(x)| dx = \frac{1}{\pi} \int_0^{2\pi} F_n(x) dx = 1$ holds.

It is obvious that $\lambda_\nu^{(n)} \to 1$ as $n \to \infty$ for any fixed ν. Let us show that condition *(B)* of the Karamata and Tomić theorem is not satisfied. Let $m > n$

$$\begin{aligned} &\int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^m \lambda_k^{(n)} \cos kx \Big| dx \\ &\quad = \int_0^{2\pi} \Big| \frac{1}{2} + \sum_{k=1}^n \Big(1 - \frac{k}{n+1}\Big) \cos kx + \sum_{k=n+1}^m \lambda_k^{(n)} \cos kx \Big| dx \\ &\quad \ge \int_0^{2\pi} \Big| \sum_{k=n+1}^m \lambda_k^{(n)} \cos kx \Big| dx - \int_0^{2\pi} |F_n(x)| dx \\ &\quad = \int_0^{2\pi} \Big| \sum_{2l+1=n+1}^m \lambda_{2l+1}^{(n)} \cos (2l+1)x \Big| dx - M \\ &\quad = \int_0^{2\pi} \Big| \sum_{l=[\frac{n}{2}]}^{[\frac{m-1}{2}]} \frac{(-1)^l}{\ln (2l+1)} \cos (2l+1)x \Big| dx - M \\ &\quad \ge \int_0^{2\pi} \Big| S_{[\frac{m-1}{2}]}(x) \Big| dx - \int_0^{2\pi} \Big| S_{[\frac{n}{2}]}(x) \Big| dx - M \to \infty, \quad \text{as} \quad m \to \infty. \end{aligned}$$

We now set $\lambda_\nu^{(n)} = \alpha_\nu^{(n)} + \beta_\nu^{(n)}$, where

$$\alpha_\nu^{(n)} = \begin{cases} 1 - \frac{\nu}{n+1}, & \text{for } \nu \le n, \\ 0, & \text{when } \nu > n \end{cases}$$

and

$$\beta_\nu^{(n)} = \begin{cases} 0, & \text{for } \nu \le n \text{ or } \nu > n \text{ and } \nu = 2k, \\ \frac{(-1)^{k+1}}{\ln \nu}, & \text{when } \nu > n \text{ and } \nu = 2k+1, \end{cases}$$

and check the conditions *(E)* and *(F)*.

Let $m > n$, then

$$\begin{aligned} &\int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{\nu=1}^m \left(\alpha_\nu^{(n)} \cos \nu x + \beta_\nu^{(n)} \sin \nu x\right) \Big| dx \\ &\quad \le \int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{\nu=1}^n \alpha_\nu^{(n)} \cos \nu x \Big| dx + \int_0^{2\pi} \Big| \sum_{\nu=n}^m \beta_\nu^{(n)} \sin \nu x \Big| dx \\ &\quad \le \int_0^{2\pi} \Big| \frac{1}{2} + \sum_{\nu=1}^n \Big(1 - \frac{\nu}{n+1}\Big) \cos \nu x \Big| dx \\ &\quad + \int_0^{2\pi} |S_m(\varphi, x)| dx + \int_0^{2\pi} |S_n(\varphi, x)| dx = O(1). \end{aligned}$$

Hence the condition *(E)* of the theorem holds.

For any m, we have

$$\int_0^{2\pi} \Big|\frac{\lambda_0^{(n)}}{2} + \sum_{\nu=1}^{m} \big(\alpha_\nu^{(n)} \cos \nu x + \beta_\nu^{(n)} \sin \nu x\big)\Big| dx = O(1).$$

Then the condition *(F)* of the theorem also hold.

In the following theorem, we indicate simpler, although sufficient conditions for the regularity of the summation method. The verification of conditions *(E)* and *(F)* of Theorem 1, that is, checking the boundedness of integrals of the absolute values of functions, causes certain difficulties. Further, these conditions are replaced by the test for the boundedness of certain sums, which is much simpler.

Theorem 2. *Let the number $\lambda_k^{(n)}$ be represented in the form $\lambda_k^{(n)} = \alpha_k^{(n)} + \beta_k^{(n)}$ where the real numbers $\alpha_k^{(n)}$ and $\beta_k^{(n)}$, $k = 1, 2, \ldots$, satisfy the conditions*

$$S_1 := \sum_{k=0}^{\infty} |\triangle \alpha_k^{(n)}| + \sum_{k=0}^{\infty} |\triangle \beta_k^{(n)}| \le M, \tag{24}$$

$$\sum_{k=1}^{\infty} \frac{|\beta_k^{(n)}|}{k} \le M \tag{25}$$

and

$$\sum_{\nu=2}^{\infty} \Big| \sum_{k=1}^{[\nu/2]} \frac{\triangle \alpha_{\nu-k}^{(n)} - \triangle \alpha_{\nu+k}^{(n)}}{k} \Big| + \sum_{\nu=2}^{\infty} \Big| \sum_{k=1}^{[\nu/2]} \frac{\triangle \beta_{\nu-k}^{(n)} - \triangle \beta_{\nu+k}^{(n)}}{k} \Big| \le M. \tag{26}$$

Then, in order that the summation Λ-method be regular in the space $A(\overline{D})$, it is necessary and sufficient that condition (A) and the following conditions be satisfied for any $m \in \mathbb{N}$:

$$\sum_{k=1}^{m} \frac{|\alpha_{m+k}^{(n)}| + |\beta_{m+k}^{(n)}|}{k} \le M_n, \tag{G}$$

where M_n is independent of m, and

$$\sum_{k=1}^{m} \frac{|\alpha_{m-k}^{(n)} - \alpha_{m+k}^{(n)}|}{k} + \sum_{k=1}^{m} \frac{|\beta_{m-k}^{(n)} - \beta_{m+k}^{(n)}|}{k} \le M, \tag{H}$$

Proof. The proof uses Theorem 1. We note that if relations (24) and (26) hold, then, for any $m \in \mathbb{N}$, the following quantities are finite (see. [14, p. 73]):

$$S_2 := \sum_{i=2}^{m-2} \left(\left| \sum_{k=1}^{q_{i,m}} \frac{\triangle \alpha_{i-k}^{(n)} - \triangle \alpha_{i+k}^{(n)}}{k} \right| + \left| \sum_{k=1}^{q_{i,m}} \frac{\triangle \beta_{i-k}^{(n)} - \triangle \beta_{i+k}^{(n)}}{k} \right| \right),$$

$$S_3 := \sum_{i=2}^{\infty} \left(\left| \sum_{k=1}^{[\frac{i}{2}]} \frac{\triangle \alpha_{m+i-k}^{(n)} - \triangle \alpha_{m+i+k}^{(n)}}{k} \right| + \left| \sum_{k=1}^{[\frac{i}{2}]} \frac{\triangle \beta_{m+i-k}^{(n)} - \triangle \beta_{m+i+k}^{(n)}}{k} \right| \right),$$

where $q_{i,m} = \min([\frac{i}{2}], [\frac{m-i}{2}])$.

Necessity. Let the summation Λ-method be regular in the space $A(\overline{D})$. Then conditions *(A)*, *(E)*, and *(F)* hold and yield conditions *(G)* and *(H)*, if relations (24)–(26) hold.

By [12, p. 453], condition *(E)* yields

$$\lim_{k\to\infty} \alpha_k^{(n)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \beta_k^{(n)} = 0. \tag{27}$$

By virtue of relations (24)–(27), the series

$$\frac{\lambda_0^{(n)}}{2} + \sum_{\nu=1}^{\infty} (\alpha_\nu^{(n)} \cos \nu x + \beta_\nu^{(n)} \sin \nu x)$$

is a Fourier series [13, §1]. On the basis of condition *(E)*, its partial sums are bounded in L. It was shown in [11] that condition *(G)* is necessary and sufficient in this case. This yields the necessity of *(G)* in Theorem 2.

Consider now condition *(F)*. Let relations (24)–(27) be satisfied. For any $m \in \mathbb{N}$, we have [14, Theorem 1]

$$\Big| \int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{\nu=1}^{\infty} (\alpha_\nu^{(n)} \cos \nu x + \beta_\nu^{(n)} \sin \nu x) \Big| dx - \frac{4}{\pi} \sum_{k=1}^{m} \frac{\xi_k}{k} - 2 \sum_{k=2m+1}^{\infty} \frac{|\beta_k^{(n)}|}{k} \Big|$$
$$\leq M(S_1 + S_2 + S_3),$$

where

$$\xi_k = \xi\Big(\beta_k^{(n)}, \sqrt{(\alpha_{m-k}^{(n)} - \alpha_{m+k}^{(n)})^2 + (\beta_{m-k}^{(n)} - \beta_{m+k}^{(n)})^2}\,\Big),$$

and the function $\xi(t,u)$ is defined as

$$\xi(t,u) = \begin{cases} \frac{\pi}{2}|t|, & \text{for} |u| \leq |t|, \\ |t| \arcsin \left|\frac{t}{u}\right| + \sqrt{u^2 - t^2}, & \text{for } |t| < |u|. \end{cases}$$

Therefore, a necessary condition is

$$\sum_{k=1}^{m} \frac{\xi_k}{k} \leq M.$$

Then, using the estimations [14, Lemma 5], we obtain

$$|u| \leq \xi(t,u) \leq \frac{\pi}{2}|t| + |u| \leq M(|t| + |u|). \tag{28}$$

Considering this inequality and the relation

$$\frac{1}{2}|t| + \frac{1}{2}|u| \leq \sqrt{t^2 + u^2},$$

we have

$$\sum_{k=1}^{m} \frac{\xi_k}{k} \geq \sum_{k=1}^{m} \frac{\sqrt{(\alpha_{m-k}^{(n)} - \alpha_{m+k}^{(n)})^2 + (\beta_{m-k}^{(n)} - \beta_{m+k}^{(n)})^2}}{k}$$
$$\geq \frac{1}{2} \sum_{k=1}^{m} \frac{|\alpha_{m-k}^{(n)} - \alpha_{m+k}^{(n)}| + |\beta_{m-k}^{(n)} - \beta_{m+k}^{(n)}|}{k}.$$

The necessity of *(H)* is proved.

Sufficiency. Suppose that conditions (24)–(26), *(G)* and *(H)* are satisfied. Let us show that the conditions of Theorem 1 are satisfied.

Indeed, condition *(G)* yields (27), and relations (24)–(27) imply that the series (see [13, §1])

$$\frac{\lambda_0^{(n)}}{2} + \sum_{\nu=1}^{\infty} (\alpha_\nu^{(n)} \cos \nu x + \beta_\nu^{(n)} \sin \nu x)$$

is the Fourier series of an integrable function. Condition *(G)* provides the boundedness of its partial sums in the metric of L (see [11]), i.e.,

$$\int_0^{2\pi} \Big| \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{m} (\alpha_k^{(n)} \cos kx + \beta_k^{(n)} \sin kx) \Big| dx \leq M_n,$$

and this condition is *(E)*.

Since the series

$$\frac{\lambda_0^{(n)}}{2}+\sum_{\nu=1}^{\infty}(\alpha_\nu^{(n)}\cos\nu x+\beta_\nu^{(n)}\sin\nu x)$$

is a Fourier series, it can be integrated over any interval. Thus, there exists a function with bounded variation

$$\begin{aligned}K_n(x)&=\int_0^x\Big(\frac{\lambda_0^{(n)}}{2}+\sum_{\nu=1}^{\infty}(\alpha_\nu^{(n)}\cos\nu t+\beta_\nu^{(n)}\sin\nu t)\Big)dt\\&=\frac{\lambda_0^{(n)}}{2}x+\sum_{\nu=1}^{\infty}\frac{1}{\nu}(\alpha_\nu^{(n)}\sin\nu x-\beta_\nu^{(n)}\cos\nu x).\end{aligned}$$

Then, for any $m\in\mathbb{N}$ in view of [14, Theorem 1] and inequality (28), we have

$$\begin{aligned}\int_0^{2\pi}|dK_n(x)|&=\int_0^{2\pi}\Big|\frac{\lambda_0^{(n)}}{2}+\sum_{\nu=1}^{\infty}(\alpha_\nu^{(n)}\cos\nu x+\beta_\nu^{(n)}\sin\nu x)\Big|dx\\&\le\frac{4}{\pi}\sum_{\nu=1}^{m}\frac{\xi_\nu}{\nu}+2\sum_{\nu=2m+1}^{\infty}\frac{|\beta_\nu^{(n)}|}{\nu}+M(S_1+S_2+S_3)\\&\le M_1\sum_{k=1}^{m}\frac{|\alpha_{m-k}^{(n)}-\alpha_{m+k}^{(n)}|+|\beta_{m-k}^{(n)}-\beta_{m+k}^{(n)}|}{k}+M_2+M(S_1+S_2+S_3).\end{aligned}$$

As is seen, condition *(F)* is valid, if conditions (24)–(26) and *(H)* are satisfied. This proves the theorem. □

Acknowledgments. The authors would like to thank an anonymous referee for valuable suggestions that improved the presentation of this material.

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Kirovohrad Volodymyr Vynnychenko State Pedagogical University, Kirovohrad, Ukraine
E-mail address: mgaevskij@gmail.com

Kyiv National University of Technology and Design, Kyiv, Ukraine
E-mail address: zadereypv@ukr.net

Received 15/06/2013; Revised 10/11/2014