

STABILITY OF N-EXTREMAL MEASURES

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ABSTRACT. A positive Borel measure μ on \mathbb{R} , which possesses all power moments, is N-extremal if the space of all polynomials is dense in $L^2(\mu)$. If, in addition, μ generates an indeterminate Hamburger moment problem, then it is discrete. It is known that the class of N-extremal measures that generate an indeterminate moment problem is preserved when a finite number of mass points are moved (not “removed”!). We show that this class is preserved even under change of infinitely many mass points if the perturbations are asymptotically small. Thereby “asymptotically small” is understood relative to the distribution of $\text{supp } \mu$; for example, if $\text{supp } \mu = \{n^\sigma \log n : n \in \mathbb{N}\}$ with some $\sigma > 2$, then shifts of mass points behaving asymptotically like, e.g. $n^{\sigma-2}[\log \log n]^{-2}$ are permitted.

A sequence $\vec{s} = (s_n)_{n=0}^\infty$ of real numbers is called a *Hamburger moment sequence* if there exists a positive Borel measure μ on \mathbb{R} which has \vec{s} as its sequence of power moments

$$(1) \quad s_n = \int_{\mathbb{R}} x^n d\mu(x), \quad n = 0, 1, 2, \dots$$

If \vec{s} is a Hamburger moment sequence, we denote by $V_{\vec{s}}$ the set of all positive Borel measures μ on \mathbb{R} such that (1) holds.

Hamburger moment sequences can be characterized by a determinant criterion (see, e.g. [1, Theorem 2.1.1]). If \vec{s} is a Hamburger moment sequence and $V_{\vec{s}}$ contains only one element, the sequence \vec{s} is called *determinate*; otherwise, it is called *indeterminate*. Which of these alternatives takes place can also be characterized by a determinant criterion (see, e.g. [1, Addendum 9 in Chapter 2]). A measure μ that leads to an indeterminate moment sequence \vec{s} via (1) is also called *indeterminate*; similarly, one speaks of a *determinate* measure.

For an indeterminate sequence \vec{s} , the set $V_{\vec{s}}$ is infinite and can be described as follows.

The Nevanlinna parameterization. Let \vec{s} be an indeterminate Hamburger moment sequence. Then there exist four entire functions A, B, C, D of minimal exponential type (here and in the following we understand by this a function of order one and type zero or a function of order less than one) such that the formula

$$(2) \quad \int_{\mathbb{R}} \frac{d\mu(x)}{x-z} = \frac{A(z)\tau(z) + B(z)}{C(z)\tau(z) + D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

establishes a bijective correspondence $\mu \leftrightarrow \tau$ between $V_{\vec{s}}$ and the set

$$\mathcal{N} := \{\tau : \tau \text{ analytic in } \mathbb{C} \setminus \mathbb{R}, \tau(\bar{z}) = \overline{\tau(z)}, \text{Im } z \cdot \text{Im } \tau(z) \geq 0\} \cup \{\infty\}.$$

For $\tau = \infty$, the right-hand side of (2) is interpreted as $A(z)/C(z)$. ◇

A measure μ , which possesses all moments, is called *N-extremal* if the space of all polynomials is dense in $L^2(\mu)$. By a theorem of M. Riesz, a measure, which possesses all

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moments, is N-extremal if and only if it is either determinate or it is indeterminate and corresponds to a constant parameter $\tau \in \mathbb{R} \cup \{\infty\}$ in (2) (see, e.g. [1, Theorem 2.3.2]). The support of an indeterminate N-extremal measure is always discrete and infinite (see, e.g. [1, §3.4.1]). Due to an operator-theoretic interpretation, sometimes the term *canonical* is used for an indeterminate N-extremal measure (see, e.g. [1, Definition 3.4.1]).

All what we have mentioned so far are classical notions, and results go back to H. Hamburger, M. Riesz and R. Nevanlinna in the 1920s. More information about the classical theory of power moment problems can be found in the monographs [1] and [12].

In the paper [3], C. Berg and J. P. Christensen showed that the property of being indeterminate and N-extremal is stable with respect to moving finitely many points of its support. More precisely, they proved the following theorem.

[3, Theorem 8]. *Let μ be an indeterminate N-extremal measure. Write*

$$\mu = \sum_{x \in \text{supp } \mu} \mu_x \delta_x,$$

where $\mu_x := \mu(\{x\})$ and δ_x denotes the Dirac measure concentrated at x .

Let M be a finite subset of $\text{supp } \mu$, let $\tilde{M} \subseteq \mathbb{R}$ be such that

$$|\tilde{M}| = |M|, \quad \tilde{M} \cap (\text{supp } \mu \setminus M) = \emptyset,$$

and let $\tilde{\mu}_x > 0$, $x \in \tilde{M}$. Then the measure

$$\tilde{\mu} := \sum_{x \in \text{supp } \mu \setminus M} \mu_x \delta_x + \sum_{x \in \tilde{M}} \tilde{\mu}_x \delta_x$$

is indeterminate and N-extremal.

In the present note we show that this result can be improved significantly. Based on the characterization of indeterminate N-extremal measures given by A. Borichev and M. Sodin in [5, Corollary 1.1] and the recent stability result [10, Theorem 3.3] we prove the theorem below. It states that the class of indeterminate N-extremal measures is preserved under asymptotically small shifts of support points. What ‘‘asymptotically small’’ quantitatively means, depends on the distribution of the support of the measure. Intuitively speaking, shifts must be smaller than the separation of neighbouring points of $\text{supp } \mu$ and must not produce (or remove) lumps of support points close to each other. As a conclusion: if the support of the measure is sparse and regularly distributed, large shifts are allowed.

In order to formulate our theorem, we recall the following. A function $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a *growth function* if it satisfies the following axioms:

(gf1) the limit $\rho_\lambda := \lim_{r \rightarrow \infty} \frac{\log \lambda(r)}{\log r}$ exists and is finite and non-negative;

(gf2) for all sufficiently large values of r , the function λ is differentiable and

$$\lim_{r \rightarrow \infty} \left(r \frac{\lambda'(r)}{\lambda(r)} \Big/ \frac{\log \lambda(r)}{\log r} \right) = 1;$$

moreover, $\lim_{r \rightarrow \infty} \lambda(r) = \infty$.

Sometimes also the term *proximate order* for the logarithm of a growth function is used. For details see [8, I.12] or [9, I.6]. Classical examples of growth functions are functions of the form $\lambda(r) := r^\sigma (\log r)^a (\log \log r)^b$ (for large r) or inverses of such functions.

Further, we write $\phi_n \asymp \psi_n$ if there exist positive constants c, C such that $c\phi_n \leq \psi_n \leq C\phi_n$ for all $n \in \mathbb{N}$.

Theorem. Let μ be an indeterminate N-extremal measure. Write $\text{supp } \mu = \{x_n : n \in \mathbb{N}\}$ where $(x_n)_{n=1}^\infty$ is a sequence of pairwise distinct real numbers, set $\mu_n := \mu(\{x_n\})$ and

$$s(n) := \min\{|x_k - x_n| : k \neq n\}, \quad r_\rho(n) := \#\left\{k : \frac{1}{\rho}x_n < x_k < \rho x_n\right\}$$

for $\rho > 1$. Moreover, choose a growth function λ such that $r^{-1}\lambda(r)$ is non-increasing or non-decreasing for large r and that

$$(3) \quad \sum_{\substack{n \in \mathbb{N} \\ x_n \neq 0}} \frac{1}{\lambda(|x_n|)} < \infty.$$

Let $(\tilde{x}_n)_{n=1}^\infty$ be a sequence of pairwise distinct real points, and assume that

$$(4) \quad |\tilde{x}_n - x_n| = O\left(\frac{|x_n|}{\lambda(|x_n|)}\right), \quad n \rightarrow \infty,$$

$$(5) \quad \left(\frac{\tilde{x}_n - x_n}{s(n)}\right)_{n=1}^\infty \in \ell^1,$$

$$(6) \quad \exists \rho > 1 : \quad \frac{\tilde{x}_n - x_n}{s(n)} = O\left(\frac{1}{r_\rho(n)}\right), \quad n \rightarrow \infty.$$

Moreover, let $\tilde{\mu}_n > 0$ be such that $\mu_n \asymp \tilde{\mu}_n$. Then the measure

$$\tilde{\mu} := \sum_{n \in \mathbb{N}} \tilde{\mu}_n \delta_{\tilde{x}_n}$$

is indeterminate and N-extremal.

Note that the support $\{x_n : n \in \mathbb{N}\}$ of an indeterminate N-extremal measure is the zero set of an entire function of minimal exponential type and therefore $(|x_n|)_{n=1}^\infty$ grows at least linearly if the latter sequence is arranged in a non-increasing way. Hence one can always choose the growth function

$$(7) \quad \lambda(r) := r^{1+\varepsilon} \quad \text{with } \varepsilon > 0.$$

However, the choice of λ can be adapted to $\text{supp } \mu$ and lead to much stronger results than what one obtains with (7).

Example. Consider a growth function ω with $\rho_\omega \in (0, \frac{1}{2}]$ such that

$$(8) \quad \omega(r) = O(r^{\rho_\omega}), \quad r \rightarrow \infty \quad \text{with “o” instead of “O” if } \rho_\omega = \frac{1}{2}.$$

As a concrete instance, one could think of a function ω whose inverse is

$$\omega^{-1}(s) = s^\sigma (\log s)^\alpha (\log \log s)^\beta \quad \text{for large } s,$$

where $\sigma \geq 2$, $\alpha, \beta \geq 0$ and $\alpha > 0$ if $\sigma = 2$.

Set $x_n := \omega^{-1}(n) \log n$, $n \in \mathbb{N}$, let F be the canonical product with zeros at x_n , and set $\mu_n := |F'(x_n)|^{-2}$. By [5, Theorem D] the measure

$$\mu := \sum_{n \in \mathbb{N}} \mu_n \delta_{x_n}$$

is indeterminate and N-extremal.

We show that, for each sequence $(\tilde{x}_n)_{n=1}^\infty$ of pairwise distinct real numbers which satisfies

$$(9) \quad |\tilde{x}_n - x_n| = O\left(\frac{n^{\frac{1}{\rho_\omega}-2}}{[\log \log n]^{1+\varepsilon}}\right), \quad n \rightarrow \infty, \quad \text{for some } \varepsilon > 0,$$

the hypotheses of the Theorem are fulfilled, and hence that the measure $\tilde{\mu} := \sum_{n \in \mathbb{N}} \tilde{\mu}_n \delta_{\tilde{x}_n}$, where $\tilde{\mu}_n \asymp \mu_n$, is indeterminate and N-extremal. Note that in the case $\rho_\omega < \frac{1}{2}$ the shifts are allowed to be unbounded.

Define a growth function λ such that

$$\lambda(r) = \omega(r)[\log \omega(r)]^2 \quad \text{for large enough } r.$$

Then $\rho_\lambda = \rho_\omega < 1$, and hence $r^{-1}\lambda(r)$ is decreasing for large r (see, e.g. [10, Remark 2.2 (iii)]). Moreover, $\omega(x_n) \gtrsim n$, and hence $\lambda(x_n) \gtrsim n(\log n)^2$ (we write $\phi_n \gtrsim \psi_n$ if there exists a positive constant C such that $\phi_n \geq C\psi_n$ for all $n \in \mathbb{N}$). This shows that (3) holds.

Let $\varepsilon' \in (0, 1)$. By (8) and the definition of ρ_ω we have

$$n^{\frac{1}{\rho_\omega}} \lesssim \omega^{-1}(n) \lesssim x_n \lesssim n^{\frac{1}{\rho_\omega} + \varepsilon'} \quad \text{and hence} \quad \omega(x_n) \lesssim n^{1 + \varepsilon' \rho_\omega}.$$

It follows that (this rough estimate is enough) $\lambda(x_n) \lesssim n^2$. Now (9) gives

$$\frac{|\tilde{x}_n - x_n|}{x_n / \lambda(x_n)} \lesssim \frac{n^{\frac{1}{\rho_\omega} - 2}}{[\log \log n]^{1 + \varepsilon}} \cdot \frac{n^2}{n^{\frac{1}{\rho_\omega}}} \leq 1,$$

which is (4).

Denote by $s_\omega(n)$ the separation of the sequence $(\omega^{-1}(n))_{n=1}^\infty$, and let $s(n)$ be the separation of $(x_n)_{n=1}^\infty$ as in the Theorem. Then $s(n) \geq s_\omega(n) \log n$. Using [10, Lemma 2.12], we can deduce that

$$(10) \quad \frac{|\tilde{x}_n - x_n|}{s(n)} \lesssim \frac{n^{\frac{1}{\rho_\omega} - 2}}{[\log n]^\varepsilon} \cdot \frac{n}{\omega^{-1}(n)} \cdot \frac{1}{\log n} \lesssim \frac{1}{n(\log n)[\log \log n]^{1 + \varepsilon}},$$

which gives (5).

Denote by $r_{\omega, \rho}(n)$ the expression defined analogously to $r_\rho(n)$ for the sequence $(\omega^{-1}(n))_{n=1}^\infty$ instead of $(x_n)_{n=1}^\infty$. Then $r_{\omega, \rho}(n) \lesssim r_\rho(n)$ and hence (see the proof of [10, Lemma 2.12]) $r_{\omega, \rho}(n) \lesssim n$. The above estimate (10) thus also shows that (6) holds.

For more (and explicit) examples of indeterminate N-extremal measures see, e.g. [5, Appendix 2]. Examples with very sparse support (corresponding to order 0) can be found in [11].

For the proof of the Theorem, we need to recall the notion of entire functions of the Hamburger class.

The Hamburger class of entire functions. In what follows we use the letter \mathcal{H} to denote the set of all transcendental entire functions H of minimal exponential type which have only real and simple zeros, say $(y_n)_{n=1}^\infty$, and satisfy

$$(11) \quad \lim_{n \rightarrow \infty} \frac{|y_n|^l}{|H'(y_n)|} = 0, \quad l = 0, 1, 2, \dots$$

This class \mathcal{H} of functions is known as the *Hamburger class*¹.

Let $H \in \mathcal{H}$, denote by $(y_n)_{n=1}^\infty$ the zeros of H in any order, let (y_n^+) be the (finite or infinite) sequence of positive zeros arranged in an increasing order and let (y_n^-) be

¹We use the definition given in [5]. In many other sources, e.g. in [1], the definition of the Hamburger class reads differently. However, these definitions are equivalent.

the negative zeros arranged in a decreasing order, both with indices $n = 1, 2, \dots$. By Lindelöf's theorem (see, e.g. [2, Theorem 2.10.3]) we have

$$\lim_{n \rightarrow \infty} \frac{y_n^+}{n} = \lim_{n \rightarrow \infty} \frac{y_n^-}{n} = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} \sum_{\substack{|y_n| \leq r \\ y_n \neq 0}} \frac{1}{y_n} \quad \text{exists in } \mathbb{R},$$

where we tacitly understand the limit of a finite sequence as being equal to 0.

If $H(0) = 1$, then the function H has the product representation

$$H(z) = \lim_{r \rightarrow \infty} \prod_{|y_n| \leq r} \left(1 - \frac{z}{y_n}\right).$$

◇

The following lemma, which is used in the proof of the Theorem, is a perturbation result for the Hamburger class.

Lemma. *Let $H \in \mathcal{H}$ with $H(0) = 1$, and write the set of zeros of H as a sequence $(y_n)_{n=1}^\infty$. Let $(\tilde{y}_n)_{n=1}^\infty$ be a sequence of pairwise distinct real non-zero points such that the conditions (3)–(6) are satisfied for the pair $(y_n)_{n=1}^\infty, (\tilde{y}_n)_{n=1}^\infty$ of sequences and some growth function as in the Theorem. Then the following statements hold.*

(i) *The sequences $(y_n)_{n=1}^\infty$ and $(\tilde{y}_n)_{n=1}^\infty$ satisfy*

$$(12) \quad \lim_{n \rightarrow \infty} \frac{\tilde{y}_n}{y_n} = 1.$$

(ii) *For the sequence $(\tilde{y}_n)_{n=1}^\infty$ one has*

$$(13) \quad \lim_{n \rightarrow \infty} \frac{\tilde{y}_n^+}{n} = \lim_{n \rightarrow \infty} \frac{\tilde{y}_n^-}{n} = 0, \quad \lim_{r \rightarrow \infty} \sum_{|\tilde{y}_n| \leq r} \frac{1}{\tilde{y}_n} \quad \text{exists in } \mathbb{R}.$$

(iii) *The function $\tilde{H}(z) := \lim_{r \rightarrow \infty} \prod_{|\tilde{y}_n| \leq r} \left(1 - \frac{z}{\tilde{y}_n}\right)$ satisfies*

$$(14) \quad |\tilde{H}'(\tilde{y}_n)| \asymp |H'(y_n)|.$$

(iv) *\tilde{H} belongs to the Hamburger class.*

Proof. Conditions (3) and (4) imply that

$$(15) \quad \sum_{n \in \mathbb{N}} \left| \frac{\tilde{y}_n}{y_n} - 1 \right| < \infty.$$

In particular, we obtain (12). We conclude that the sequence $(\tilde{y}_n)_{n=1}^\infty$ satisfies the first relation in (13). Since we have $\left| \frac{1}{y_n} - \frac{1}{\tilde{y}_n} \right| = \frac{1}{|\tilde{y}_n|} \cdot \left| \frac{\tilde{y}_n}{y_n} - 1 \right|$, the convergence of (15) also implies that

$$\sum_{n \in \mathbb{N}} \left| \frac{1}{y_n} - \frac{1}{\tilde{y}_n} \right| < \infty.$$

Now it follows from the relation

$$\sum_{|\tilde{y}_n| \leq r} \frac{1}{\tilde{y}_n} = \sum_{|\tilde{y}_n| \leq r} \frac{1}{y_n} + \sum_{|\tilde{y}_n| \leq r} \left(\frac{1}{\tilde{y}_n} - \frac{1}{y_n} \right)$$

that the second limit relation in (13) holds; here we also use the argument in [10, Remark 3.16] and (5) to control the effect of situations where $|y_n| \leq r$ and $|\tilde{y}_n| > r$ or vice versa. This readily implies that the function \tilde{H} is well defined and, by Lindelöf's theorem, of minimal exponential type.

The assumptions (3)–(6) are exactly the hypotheses of [10, Theorem 3.3] for the sequences $(y_n)_{n=1}^\infty$ and $(\tilde{y}_n)_{n=1}^\infty$. An application of this theorem gives $|\tilde{H}'(\tilde{y}_n)| \asymp |H'(y_n)|$. Together with (12) it follows that \tilde{H} satisfies (11) and therefore $\tilde{H} \in \mathcal{H}$. \square

The Hamburger class plays an important role in the characterization of indeterminate N-extremal measures: Corollary 1.1 in [5] says that a measure $\mu = \sum_{n=1}^\infty \mu_n \delta_{x_n}$ is indeterminate and N-extremal if and only if

- (a) $F(z) := \lim_{r \rightarrow \infty} \prod_{|x_n| \leq r} \left(1 - \frac{z}{x_n}\right)$ converges and belongs to \mathcal{H} ;
- (b) $\sum_{n=1}^\infty |x_n|^l \mu_n < \infty$ for all $l = 0, 1, \dots$;
- (c) $\sum_{n=1}^\infty \frac{1}{\mu_n |F'(x_n)|^2 (1 + x_n^2)} < \infty$;
- (d) for every function $G \in \mathcal{H}$ such that every zero of G is also a zero of F ,

$$\sum_{k=1}^\infty \frac{1}{\mu_{n(k)} |G'(x_{n(k)})|^2} = \infty,$$

where $(x_{n(k)})_{k=1}^\infty$ is the sequence of zeros of G .

Remark. This statement has a remarkable history. In [6, § 8, Theorem 4], H. Hamburger stated the *false* result that μ is indeterminate and N-extremal if and only if the conditions (a)–(c) hold and

$$(d') \sum_{k=1}^\infty \frac{1}{\mu_{n(k)} |F'(x_{n(k)})|^2} = \infty.$$

This mistake remained unnoticed for a long time; it was even reproduced in [1, Ch. 4, Addenda and Problems, Corollary 2]. The first person who noticed that Hamburger's proof is incomplete was H. Pedersen in 1989. Counterexamples were constructed, e.g. by P. Koosis in [7]. The correction of Hamburger's original statement was posed as an open problem in [4]. Finally, the correct statement was established in [5] in 1998 as a consequence of a more general and stronger theorem.

It should be added that, by considering the canonical system corresponding to the given measure, it is easy to see that Hamburger's original statement is false, and one can construct a variety of counterexamples. However, this approach does not (or is not known to) lead to a proof of the correct result.

Proof of the Theorem. Without loss of generality we can assume that neither of the sequences $(x_n)_{n=1}^\infty$, $(\tilde{x}_n)_{n=1}^\infty$ contains the point 0 because otherwise, we could shift zeros from 0 to a non-zero point using [3, Theorem 8]; the conditions in the Theorem are unaffected by this. We check the conditions (a)–(d) stated above from [5, Corollary 1.1] for $\tilde{\mu}$.

1. The Hamburger conditions (a)–(c). Since μ is an indeterminate N-extremal measure, the set $\{x_n : n \in \mathbb{N}\}$ is the zero set of some Hamburger class function F satisfying (a)–(c). It follows from the Lemma applied to the sequences $(x_n)_{n=1}^\infty$ and $(\tilde{x}_n)_{n=1}^\infty$ that $\{\tilde{x}_n : n \in \mathbb{N}\}$ is the zero set of a Hamburger class function, namely $\tilde{F}(z) := \lim_{r \rightarrow \infty} \prod_{|\tilde{x}_n| \leq r} \left(1 - \frac{z}{\tilde{x}_n}\right)$. Moreover, (14), (12) and our hypothesis " $\tilde{\mu}_n \asymp \mu_n$ " imply that (b) and (c) are satisfied with x_n , μ_n and F replaced by \tilde{x}_n , $\tilde{\mu}_n$ and \tilde{F} .

2. The Borichev–Sodin condition (d). Let $\tilde{G} \in \mathcal{H}$ be such that each zero of \tilde{G} is also a zero of \tilde{F} , and write the set of zeros of \tilde{G} as a subsequence $(\tilde{x}_{n(k)})_{k \in \mathbb{N}}$ of $(\tilde{x}_n)_{n=1}^\infty$.

By [10, Proposition 3.23], the conditions (3)–(6) also hold with the roles of $(x_n)_{n=1}^\infty$ and $(\tilde{x}_n)_{n=1}^\infty$ exchanged. The quantity $s(n)$, measuring the separation of a sequence, does not decrease when one passes to a subsequence. The quantity $r_\rho(n)$, measuring the size of lumps of points in a sequence, does not increase when one passes to a subsequence. Hence, the sequences $(\tilde{x}_{n(k)})_{k \in \mathbb{N}}$ and $(x_{n(k)})_{k \in \mathbb{N}}$ satisfy the conditions corresponding to (3)–(6) (with the same growth function λ).

The Lemma implies that

$$G(z) := \lim_{r \rightarrow \infty} \prod_{|x_{n(k)}| \leq r} \left(1 - \frac{z}{x_{n(k)}}\right)$$

belongs to the Hamburger class. Each zero of G is also a zero of F and therefore condition (d) is satisfied. We obtain from the Lemma that $|G'(x_{n(k)})| \asymp |\tilde{G}'(\tilde{x}_{n(k)})|$ and hence (d) is also satisfied for \tilde{x}_n , $\tilde{\mu}_n$ and G . This shows that all conditions (a)–(d) are satisfied for $\tilde{\mu}$ and therefore $\tilde{\mu}$ is indeterminate and N-extremal. \square

Remark. A result analogous to the above theorem can be formulated and proved, which asserts stability of density of polynomials in weighted ℓ^p -spaces, and is based on [5, Theorem A] rather than [5, Corollary 1.1]. The proof is just the same application of [10, Theorem 3.3].

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