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SMOOTH FUNCTIONS ON 2-TORUS WHOSE KRONROD-REEB GRAPH CONTAINS A CYCLE

SERGIY MAKSYMENKO AND BOHDAN FESHCHENKO

Dedicated to the memory of our teacher Sharko Volodymyr Vasylyovych

Abstract. Let $f : M \to \mathbb{R}$ be a Morse function on a connected compact surface M, and $\mathcal{S}(f)$ and $\mathcal{O}(f)$ be respectively the stabilizer and the orbit of f with respect to the right action of the group of diffeomorphisms $\mathcal{D}(M)$. In a series of papers the first author described the homotopy types of connected components of $\mathcal{S}(f)$ and $\mathcal{O}(f)$ for the cases when M is either a 2-disk or a cylinder or $\chi(M) < 0$. Moreover, in two recent papers the authors considered special classes of smooth functions on 2-torus T^2 and shown that the computations of $\pi_1\mathcal{O}(f)$ for those functions reduces to the cases of 2-disk and cylinder.

In the present paper we consider another class of Morse functions $f : T^2 \to \mathbb{R}$ whose KR-graphs have exactly one cycle and prove that for every such function there exists a subsurface $Q \subset T^2$, diffeomorphic with a cylinder, such that $\pi_1\mathcal{O}(f)$ is expressed via the fundamental group $\pi_1\mathcal{O}(f|_Q)$ of the restriction of f to Q.

This result holds for a larger class of smooth functions $f : T^2 \to \mathbb{R}$ having the following property: for every critical point z of f the germ of f at z is smoothly equivalent to a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

1. Introduction

Let M be a smooth compact surface, $X \subset M$ be a closed (possibly empty) subset, and $\mathcal{D}(M, X)$ be the group of diffeomorphisms of M fixed on some neighborhood of X. Then $\mathcal{D}(M, X)$ acts from the right on $C^\infty(M)$ by following rule: if $h \in \mathcal{D}(M, X)$ and $f \in C^\infty(M)$ then the result of the action of h on f is the composition map

$$f \circ h : M \xrightarrow{h} M \xrightarrow{f} \mathbb{R}. \tag{1}$$

Given $f \in C^\infty(M)$ let

$$\mathcal{S}(f, X) = \{f \in \mathcal{D}(M, X) \mid f \circ h = f\}, \quad \mathcal{O}(f, X) = \{f \circ h \mid h \in \mathcal{D}(M, X)\}$$

be respectively the *stabilizer* and the *orbit* of f under the action (1). Let also

$$\mathcal{S}'(f, X) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M, X).$$

If X is empty, then we omit it from notation and write $\mathcal{D}(M) = \mathcal{D}(M, \varnothing)$, $\mathcal{S}(f) = \mathcal{S}(f, \varnothing)$, $\mathcal{O}(f) = \mathcal{O}(f, \varnothing)$, and so on. We will also endow the spaces $\mathcal{D}(M, X)$, $C^\infty(M)$, $\mathcal{S}(f, X)$, and $\mathcal{O}(f, X)$ with the corresponding Whitney C^∞-topologies.

Denote by $\mathcal{S}_{\mathrm{id}}(f, X)$ and $\mathcal{D}_{\mathrm{id}}(M, X)$ the identity path components $\mathcal{S}(f, X)$ and $\mathcal{D}(M, X)$ respectively, and $\mathcal{O}_f(f, X)$ be the path component of f in $\mathcal{O}(f, X)$.

Let $\mathcal{F}(M)$ be a subset in $C^\infty(M)$ consisting of functions f having the following two properties:

(B) f takes a constant value at each connected components of ∂M, and all critical points of f are contained in the interior of M;

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(L) for every critical point z of f the germ of f at z is smoothly equivalent to a certain **homogeneous polynomial** $f_z : \mathbb{R}^2 \to \mathbb{R}$ **without multiple factors**.

Let $\mathrm{Morse}(M) \subset C^\infty(M)$ be an open and everywhere dense subset consisting of all Morse functions having the above property (B), that is functions having only *non-degenerate* critical points. By the Morse lemma every non-degenerate singularity is smoothly equivalent to a homogeneous polynomial $\pm x^2 \pm y^2$ having no multiple factors. Therefore $\mathrm{Morse}(M) \subset \mathcal{F}(M)$. This shows that the class $\mathcal{F}(M)$ is large.

Let $f \in \mathcal{F}(M)$ and $c \in \mathbb{R}$. A connected component C of the level set $f^{-1}(c)$ is called *critical* if C contains at least one critical point of f; otherwise C is *regular*. Consider a partition Δ of M into connected component of level sets of f. It is well known that the corresponding quotient M/Δ has a structure of a finite one-dimensional CW-complex and is called *Kronrod-Reeb graph* or simply KR-graph of the function f. We will denote it by $\Gamma(f)$. The vertices of $\Gamma(f)$ are critical components of level sets of f.

This graph was introduced by A. S. Kronrod in [4] for studying functions on surfaces and also used by by G. Reeb in [20]. Applications of $\Gamma(f)$ to study Morse functions on surfaces are given e.g. in [1, 8, 5, 22, 23, 19].

In a series of papers, [10], [12], [13], [14], [16], [15], the first author calculated the homotopy types of spaces $\mathcal{S}(f)$ and $\mathcal{O}(f)$ for all $f \in \mathcal{F}(M)$, see §2 for some details. In particular, it was proved, [10, Theorem 1.5(3)], that if f is a *generic* Morse function, i.e. it takes distinct values at distinct critical point, then $\mathcal{O}_f(f)$ is homotopy equivalent to a finite-dimensional torus.

This result was improved by E. Kudryavtseva [6, Theorem 2.5(B)], [7, Theorem 2.6(C)]: using another approach she shown that if M is orientable, $\chi(M) < 0$, and f is Morse, then $\mathcal{O}_f(f)$ is homotopy equivalent to a quotient T^k/G of a finite-dimensional torus T^k by the free action of some finite group G.

Recently, [15], the first author established such a statement for all $f \in \mathcal{F}(M)$ provided M is distinct from 2-torus, 2-sphere, projective plane, and Klein bottle. It was also shown in [11, Theorem 1.8] that under the same restrictions on M, the computation of the homotopy type of $\mathcal{O}(f)$, reduces to the case when M is either 2-disk, or a cylinder, or a Möbius band.

In two recent papers, [17], [18], the authors studied smooth functions on 2-torus and shown that under some conditions on $f \in \mathcal{F}(T^2)$ the computation of the homotopy type of $\mathcal{O}(f)$ also reduces to the cases when M is a 2-disk or a cylinder.

In the present paper we study functions $f \in \mathcal{F}(T^2)$ whose Kronrod-Reeb graph has one cycle. The main result, see Theorem 1.6, reduces the computation of $\mathcal{O}_f(f)$ to the restriction of f onto some subsurface $Q \subset T^2$ diffeomorphic to a cylinder. We also give exact formula expressing $\pi_1\mathcal{O}_f(f)$ via $\pi_1\mathcal{O}(f|_Q)$. This extends the result of [18].

Remark 1.1. In [18] the group $\mathcal{D}(M, X)$ means the group of diffeomorphisms *fixed on* X, while in the present paper we denote by $\mathcal{D}(M, X)$ the group of diffeomorphisms *fixed on some neighborhood* of X. In fact, if X is a finite collection of regular components of some level-sets of $f \in \mathcal{F}(M)$, such a restriction does not impact on the homotopy types of $\mathcal{D}(M, X)$, $\mathcal{S}(f, X)$ and $\mathcal{O}(f)$, see [13].

1.2. **Wreath products** $G \underset{\mathbb{Z}_n}{\wr} \mathbb{Z}$**.** Let G be a group with unit e, and $n \geq 1$. Denote by $\mathrm{Map}(\mathbb{Z}_n, G)$ the group of all *maps*, not necessarily homomorphisms, from cyclic group $\mathbb{Z}_n$ into G, with respect to point wise multiplication. That is if $\alpha, \beta : \mathbb{Z}_n \to G$ two elements from $\mathrm{Map}(\mathbb{Z}_n, G)$, then their product is given by the formula $(\alpha\beta)(i) = \alpha(i) \cdot \beta(i)$ for $i \in \mathbb{Z}_n$, where the multiplication $\cdot$ is taken in the group G.

Notice that the group $\mathbb{Z}$ acts from the right on $\mathrm{Map}(\mathbb{Z}_n, G)$ by the following rule: if $\alpha \in \mathrm{Map}(\mathbb{Z}_n, G)$ and $a \in \mathbb{Z}$, then the result $\alpha^k : \mathbb{Z}_n \to G$ of the action of k on α is given by the formula:

$$\alpha^k(i) = \alpha(i + k \bmod n), \quad i \in \mathbb{Z}_n. \tag{2}$$

The semidirect product $\mathrm{Map}(\mathbb{Z}_n, G) \rtimes \mathbb{Z}$ corresponding to this action is called a *wreath product of* G *and* $\mathbb{Z}$ *over* $\mathbb{Z}_n$ and denoted by

$$G \underset{\mathbb{Z}_n}{\wr} \mathbb{Z} := \mathrm{Map}(\mathbb{Z}_n, G) \rtimes \mathbb{Z}.$$

More precisely, $G \underset{\mathbb{Z}_n}{\wr} \mathbb{Z}$ is the *set* $\mathrm{Map}(\mathbb{Z}_n, G) \times \mathbb{Z}$ with the following operation

$$(\alpha, k)\,(\beta, l) = (\alpha\beta^k, k + l) \tag{3}$$

for all $(\alpha, k), (\beta, l) \in \mathrm{Map}(\mathbb{Z}_n, G) \times \mathbb{Z}$.

In particular, we have the following short exact sequence:

$$1 \longrightarrow \mathrm{Map}(\mathbb{Z}_n, G) \xrightarrow{\zeta} G \underset{\mathbb{Z}_n}{\wr} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow 1, \tag{4}$$

where $\zeta(\alpha) = (\alpha, 0)$ is a *canonical inclusion* and $p(\alpha, k) = k$ is a *canonical projection*.

Notice also that for $n = 1$, there is a natural isomorphism $G \underset{\mathbb{Z}_n}{\wr} \mathbb{Z} \cong G \times \mathbb{Z}$.

1.3. **Parallel curves on T^2.** A finite non-empty family of $C_0, \dots, C_{n-1} \subset T^2$ of simple closed curves will be called *parallel* if these curves are mutually disjoint and non-separating.

If $n = 1$, then $T^2 \setminus C$ is an open cylinder, we will regard T^2 as a cylinder Q_0 with identified boundary components, see Figure 1a).

Suppose $n \geq 2$. Then all curves in a parallel family must be isotopic each other. In this case we will always assume that they are *cyclically enumerated along* T^2, that is C_i and C_{i+1} bound a cylinder Q_i containing no other curves C_j, where all indices are taken modulo n, see Figure 1b). We will also use the following notation:

$$\mathcal{C} = \bigcup_{i=0}^{n-1} C_i, \quad C_i := C_{i \bmod n}, \quad Q_i := Q_{i \bmod n}$$

for all integers $i \in \mathbb{Z}$.

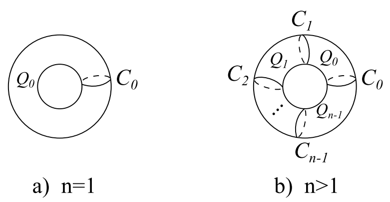


FIGURE 1

1.4. **Cyclic index of f.** Let $f \in \mathcal{F}(T^2)$ be such that its KR-graph $\Gamma(f)$ is not a tree. It is easy to show, [18], that then $\Gamma(f)$ has a unique simple cycle, which we will denote by Λ, see Figure 2.

Let also $C \subset T^2$ be a regular component of some level set $f^{-1}(c)$, $c \in \mathbb{R}$, and z be the corresponding point on $\Gamma(f)$. It is easy to check, see [18], that $z \in \Lambda$ if and only if C does not separate T^2. Notice that $f^{-1}(c)$ consists of finitely many connected components and is invariant with respect to each $h \in \mathcal{S}(f)$. Let

$$\mathcal{C} = \{h(C) \mid h \in \mathcal{S}'(f)\}$$

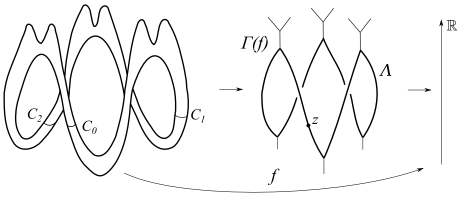


FIGURE 2

be the set of images of C under the action of $\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2)$. Then $\mathcal{C}$ consists of finitely many connected components of $f^{-1}(c)$:

$$\mathcal{C} \ = \ \{ C_0 = C, \ C_1, \ \ldots, \ C_{n-1} \}$$

for some $n \geq 1$. Emphasize that we only consider the images of C for all diffeomorphisms h that preserve f and are *isotopic to* $\mathcal{C}$. However, there may exist $h \in \mathcal{S}(f)$ that is not isotopic to id_{T^2} and such that $h(C) \subset f^{-1}(c) \setminus \mathcal{C}$.

It follows that the curves in $\mathcal{C}$ are mutually disjoint, and neither of them separates T^2, since C does not do this. Thus they are *parallel* in the sense of §1.3, and therefore we will assume that they are cyclically ordered along T^2, and that C_i and C_{i+1} bound a cylinder Q_i whose interior does not intersect $\mathcal{C}$.

Definition 1.5. *The number n of curves in $\mathcal{C}$ will be called the* ***cyclic index*** *of f.*

It is easy to see that the cyclic index of f does not depend on a particular choice of a regular component C of some level-set of f that does not separate T^2.

Let $f|_{Q_0}$ be the restriction of f onto Q_0 and $\mathcal{O}(f|_{Q_0}, \partial Q_0)$ be the orbit of $f|_{Q_0}$ with respect to the action of the group $\mathcal{D}(Q_0, \partial Q_0)$ of diffeomorphisms of Q_0 fixed on some neighborhood of ∂Q_0. Now we can formulate the main result of the present paper.

Theorem 1.6. cf. [18]. *Let $f \in \mathcal{F}(T^2)$ be such that $\Gamma(f)$ has a cycle, C be a regular connected component of certain level set $f^{-1}(c)$ of f that does not separate T^2, $\mathcal{C} = \{h(C) \mid h \in \mathcal{S}'(f)\}$, and n be the cyclic index of f, i.e. the number of curves in $\mathcal{C}$.*

If $n = 1$, then there is an isomorphism

$$\xi : \pi_1 \mathcal{O}(f) \ \cong \ \pi_1 \mathcal{O}(f, C) \ \times \ \mathbb{Z}.$$

Suppose $n \geq 2$ and let Q_0 be the cylinder bounded by C_0 and C_1. Then we have an isomorphism

$$\xi : \pi_1 \mathcal{O}(f) \ \cong \ \pi_1 \mathcal{O}(f|_{Q_0}, \partial Q_0) \underset{\mathbb{Z}_n}{\wr} \mathbb{Z}.$$

For $n = 1$ this theorem is proved in [18], therefore we will assume that $n \geq 2$.

1.7. **Structure of the paper.** In §2 we recall some results about the homotopy types of stabilizers and orbits of $f \in \mathcal{F}(M)$, and in §3 present some formulas for the multiplication in the relative homotopy group $\pi_1(D, S)$, where D is a topological group and S is its subgroup.

In §4 we consider families of parallel curves on 2-torus and relations between Dehn twists and slides along these curves. Given $f \in \mathcal{F}(T^2)$ such that its KR-graph has one cycle, we introduce in §5 some special coordinates and flows adopted with f. In §6 we define two epimorphisms $\varphi : \pi_1(\mathcal{D}(T^2), \mathcal{S}'(f)) \to \mathbb{Z}$ and $\kappa : \pi_0 \mathcal{S}'(f) \to \mathbb{Z}_n$ and study their properties, see Theorem 6.1.

As an interpretation of (c) Theorem 6.1 we show in §7 that there exists a f-invariant $\mathbb{Z}_n$-action on T^2, see Theorem 7.1. This interpretation is not used in the paper, but it gives a new view point of such functions f. Finally, in §8 we complete Theorem 1.6.

2. Homotopy types of $\mathcal{S}(f)$ and $\mathcal{O}(f)$

Let $f \in \mathcal{F}(M)$ and X be a finite (possibly empty) union of regular components of some level sets of f. We will briefly recall description of the homotopy types of $\mathcal{S}(f, X)$ and $\mathcal{O}(f, X)$.

Theorem 2.1. [21, 10, 13]. *The following map*

$$p : \mathcal{D}(M, X) \longrightarrow \mathcal{O}(f, X), \quad p(h) = f \circ h.$$

is a Serre fibration with fiber $\mathcal{S}(f, X)$, that is it has a homotopy lifting property for CW-complexes.

Hence $p(\mathcal{D}_{\mathrm{id}}(M, X)) = \mathcal{O}_f(f, X)$ and the restriction map

$$p|_{\mathcal{D}_{\mathrm{id}}(M,X)} : \mathcal{D}_{\mathrm{id}}(M, X) \longrightarrow \mathcal{O}_f(f, X) \tag{5}$$

is also a Serre fibration with fiber $\mathcal{S}'(f, X) = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(M, X)$.

Moreover, for each $k \geq 0$ there is an isomorphism

$$\lambda_k : \pi_k\big(\mathcal{D}(M, X), \mathcal{S}(f, X)\big) \to \pi_k\mathcal{O}(f, X)$$

defined by $\lambda_k[\omega] = [f \circ \omega]$ for a continuous map $\omega : (I^k, \partial I^k, 0) \to \big(\mathcal{D}(M), \mathcal{S}(f), \mathrm{id}_M\big)$, and making commutative the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k\mathcal{D}(M,X) \arrow[r, "q"] \arrow[dr, "p"'] & \pi_k\big(\mathcal{D}(M,X),\mathcal{S}(f,X)\big) \arrow[r, "\partial"] \arrow[d, "\lambda_k", "\cong"'] & \pi_{k-1}\mathcal{S}(f,X) \arrow[r] & \cdots \\
& & \pi_k\mathcal{O}(f,X) \arrow[ur, "\partial\circ\lambda_k^{-1}"'] & &
\end{tikzcd}$$

see for example [3, §4.1, Theorem 4.1].

Theorem 2.2. [10, 12, 13]. *$\mathcal{O}_f(f, X) = \mathcal{O}_f(f, X \cup \partial M)$, and so*

$$\pi_k\mathcal{O}(f, X) \;\cong\; \pi_k\mathcal{O}(f, X \cup \partial M), \quad k \geq 1.$$

*Suppose either f has a critical point which is not a **nondegenerate local extremum** or M is a non-orientable surface. Then $\mathcal{S}_{\mathrm{id}}(f)$ is contractible, $\pi_n\mathcal{O}(f) = \pi_n M$ for $n \geq 3$, $\pi_2\mathcal{O}(f) = 0$, and for $\pi_1\mathcal{O}(f)$ we have the following short exact sequence of fibration p:*

$$1 \longrightarrow \pi_1\mathcal{D}(M) \xrightarrow{\;p\;} \pi_1\mathcal{O}(f) \xrightarrow{\;\partial\circ\lambda_1^{-1}\;} \pi_0\mathcal{S}'(f) \longrightarrow 1. \tag{6}$$

Moreover, $p\big(\pi_1\mathcal{D}(M)\big)$ is contained in the center of $\pi_1\mathcal{O}(f)$.

If either $\chi(M) < 0$ or $X \neq \varnothing$. Then $\mathcal{D}_{\mathrm{id}}(M, X)$ and $\mathcal{S}_{\mathrm{id}}(f, X)$ are contractible, whence from the exact sequence of homotopy groups of the fibration (5) *we get $\pi_k\mathcal{O}(f, X) = 0$ for $k \geq 2$, and that the boundary map*

$$\partial \circ \lambda_1^{-1} : \pi_1\mathcal{O}(f, X) \;\longrightarrow\; \pi_0\mathcal{S}'(f, X)$$

is an isomorphism.

Suppose M is differs from 2-sphere S^2, 2-torus, projective plane, and Klein bottle, and let $X = \partial M$. Then M and X satisfy assumptions of Theorem 2.2, and we get the following isomorphisms

$$\pi_1\mathcal{O}(f) \;\cong\; \pi_1\mathcal{O}(f, \partial M) \;\cong\; \pi_0\mathcal{S}'(f, \partial M).$$

A possible structure of $\pi_0\mathcal{S}'(f, \partial M)$ for this case is completely described in [16].

However in the remained four cases of M we have that $\pi_1\mathcal{D}(M) \neq 0$, and all terms in the short exact sequence (6) can be non-trivial.

In particular, suppose $M = T^2$. Then the sequence (6) has the following form:

$$1 \longrightarrow \mathbb{Z}^2 \xrightarrow{\;p\;} \pi_1\mathcal{O}_f(f) \xrightarrow{\;\partial\;} \pi_0\mathcal{S}'(f) \longrightarrow 1. \tag{7}$$

It is shown in [17] that if a KR-graph $\Gamma(f)$ of $f \in \mathcal{F}(T^2)$ is a tree, then under some additional "triviality" assumption on the action $\mathcal{S}'(f)$ on $\Gamma(f)$, the sequence (7) splits.

Moreover, in [18] the authors considered the case when $\Gamma(f)$ of $f \in \mathcal{F}(T^2)$ has one cycle, and f has cyclic index $n = 1$.

3. Multiplication in $\pi_1(D, S, e)$

Let D be a topological space, S be its subset, and $e \in S$ be a point. Then, in general, the relative homotopy set $\pi_1(D, S, e)$, as well as $\pi_0(D, e)$ and $\pi_0(S, e)$ ***have no natural group structure***. However, if D is a topological group, S is a subgroup of D, and e is the unit of D, then such group structures exist. We leave the following lemma for the reader.

Lemma 3.1. cf. [2, Ch. 1, §4]. *Let D be a topological group with multiplication $\circ$, S be a subgroup of D, and e be the unit of D. Then $\pi_0(D, e)$, $\pi_1(D, S, e)$, $\pi_0(S, e)$ have a group structures such that in the corresponding sequence of homotopy groups of the triple (D, S, e)*

$$\cdots \to \pi_1(D, e) \xrightarrow{q} \pi_1(D, S, e) \xrightarrow{\partial} \pi_0(S, e) \xrightarrow{i} \pi_0 D \to \cdots$$

the maps q, ∂, and i are homomorphisms. Moreover $q(\pi_1(D, e))$ is contained in the center of $\pi_1(D, S, e)$.

In what follows we will assume that D, S, and e are the same as in Lemma 3.1. We will recall a formula for the multiplication in $\pi_1(D, S, e)$.

Let $g, h : (I, \partial I, 0) \to (D, S, e)$ be two paths representing some elements of $\pi_1(D, S, e)$. For simplicity we will denote $g(t)$ by g_t and similarly for h. The class of $[g] \in \pi_1(D, S, e)$ will also be denoted by $[g_t]$. Define another path $r : (I, \partial I, 0) \to (D, S, e)$ by

$$r(t) = \begin{cases} g_{2t}, & t \in [0, \frac{1}{2}], \\ g_1 \circ h_{2t-1}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Then $[r_t] = [g_t]\ [h_t]$ in $\pi_1(D, S, e)$.

As an immediate consequence of this formula we get the following lemma:

Lemma 3.2. *Let $g, h : I \to D$ be two paths such that $g(0) = e$, $g(1) = h(0) \in S$ and $h(1) \in S$ as well, and $s : (I, \partial I, 0) \to (D, S, e)$ be a path defined by*

$$s(t) = \begin{cases} g_{2t}, & t \in [0, \frac{1}{2}], \\ h_{2t-1}, & t \in [\frac{1}{2}, 1], \end{cases}$$

so it is obtained by joining g and h, see Figure 3(a). Then

$$[s_t] = [g_t]\ [g_1^{-1} \circ h_t] \tag{8}$$

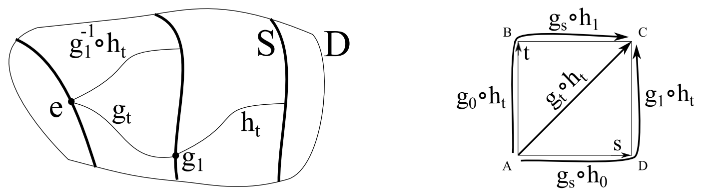
in $\pi_1(D, S, e)$, where $[g_1^{-1} \circ h_t]$ is a class of a path $(I, \partial I, 0) \to (D, S, e)$ defined by $t \mapsto g_1^{-1} \circ h_t$.

Lemma 3.3. *Let $g_t, h_t : (I, \partial I, 0) \to (D, S, e)$ be two paths. Then in $\pi_1(D, S, e)$ we have the following identities:*

$$[g_t \circ h_t] = [g_s]\ [h_t] = [h_t]\ [h_1^{-1} \circ g_s \circ h_1], \tag{9}$$

$$[h_t]\ [g_s]\ [h_t^{-1}] = [h_1^{-1} \circ g_s \circ h^{-1}], \tag{10}$$

where $[g_t \circ h_t]$ means the class of the path $(I, \partial I, 0) \to (D, S, e)$ given by $t \mapsto g_t \circ h_t$, and similarly for all other classes.



(a) (b)

FIGURE 3

Proof. Let $\gamma : I \times I \to D$ be a continuous map defined by

$$\gamma(s,t) = g_s \circ h_t, \quad (s,t) \in I \times I,$$

see Figure 3(b).

Then the path $[g_t \circ h_t]$ corresponds to the restriction of γ to the diagonal $AC = \{s = t \mid (s,t) \in I \times I\}$. Evidently, this path is homotopic relatively to its ends to the composition of paths along sides AB and BC as well as to the composition of paths along sides AD and DC. Hence by (8) we get the following relations in $\pi_1(D, S, e)$:

$$[g_t \circ h_t] = [g_s \circ h_0]\ [(g_1 \circ h_0)^{-1} \circ g_1 \circ h_t] = [g_s]\ [h_t],$$
$$[g_t \circ h_t] = [g_0 \circ h_t]\ [(g_0 \circ h_1)^{-1} \circ g_s \circ h_1] = [h_t]\ [h_1^{-1} \circ g_s \circ h_1],$$
$$[h_t]\ [g_s]\ [h_t^{-1}] = [h_t]\ [h_t^{-1}]\ [h_1 \circ g_s \circ h_1^{-1}] = [h_t \circ h_t^{-1}]\ [g_s \circ h_1^{-1}] = [g_s \circ h_1^{-1}],$$

where we take to account that $g_0 = h_0 = e$. □

4. PARALLEL CURVES ON T^2

4.1. **Twists and slides along curves.** Let $\alpha, \beta : [-1,1] \to [0,1]$ be two C^∞-functions such that $\alpha = 0$ on $[-1, -\frac{1}{2}]$ and $\alpha = 1$ on $[\frac{1}{2}, 1]$, while $\beta = 0$ on $[-1, -\frac{2}{3}] \cup [\frac{2}{3}, 1]$ and $\beta = 1$ on $[-\frac{1}{3}, \frac{1}{3}]$, see Figure 4.

Let also $Q = S^1 \times [-1,1]$ be a cylinder and $C = S^1 \times 0$. Define the following two diffeomorphisms $\tau, \theta : Q \to Q$ by

$$\tau(z,t) = (ze^{\alpha(t)}, t), \quad \theta(z,t) = (ze^{\beta(t)}, t)$$

for $(z,t) \in Q$, see Figure 4. Then τ is called a *Dehn twist* and θ is called a *slide* along the curve C. Notice that τ is fixed on some neighborhood of ∂Q, while θ is fixed on some neighborhood of $C \cup \partial Q$.

(a) Dehn twist (b) Slide

FIGURE 4

Lemma 4.2. *Let $\mathcal{D}(Q, \partial Q)$ be the group of diffeomorphisms fixed on some neighborhood of $\partial Q = S^1 \times \{0,1\}$, and $\tau \in \mathcal{D}(Q, \partial Q)$ be a Dehn twist along the curve C. Then*

$$\pi_0 \mathcal{D}(Q, \partial Q) = \langle [\tau] \rangle \cong \mathbb{Z},$$

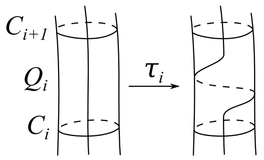
i.e. it is an infinite cyclic group generated by the isotopy class of the Dehn twist τ.

Now let $C \subset M$ be a simple closed curve. Suppose C *preserves orientation*, that is it has a closed neighborhood W diffeomorphic to a cylinder Q. Fix any $\phi : Q \to W$ such that $\phi(S^1 \times 0) = C$.

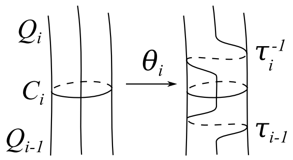
Since τ is fixed on some neighborhood of ∂Q, we see that $\phi\circ\tau\circ\phi^{-1} : W \to W$ extends by the identity to some diffeomorphism $\bar{\tau}$ and $\bar{\theta}$ of M respectively. Any diffeomorphism $h : M \to M$ isotopic to $\bar{\tau}$ or $\bar{\tau}^{-1}$ will be called a *Dehn twist* along C.

Also notice that θ is fixed on some neighborhood of $(S^1 \times 0) \cup \partial Q$, whence the diffeomorphism $\phi \circ \theta \circ \phi^{-1} : W \to W$ extends by the identity to some diffeomorphisms $\bar{\theta}$ of M. Any diffeomorphism $h : M \to M$ *fixed on some neighborhood of* C, supported in some cylindrical neighborhood W of C, and isotopic to $\bar{\theta}$ or $\bar{\theta}^{-1}$ relatively to some neighborhood of $C \cup \overline{M \setminus Q}$ will be called a *slide* along C.

4.3. **Diffeomorphisms of T^2 fixed on parallel family of curves.** Let $C_0, \dots, C_{n-1} \subset T^2$ be a parallel family of curves cyclically ordered along T^2, see §1.3 and Figure 1. For each $i = 0, \dots, n-1$ let $\tau_i \in \mathcal{D}(T^2)$ be a Dehn twist such that $\mathrm{supp}\,(\tau_i) \subset \mathrm{Int}Q_i$ and its restriction $\tau_i|_{Q_i}$ generates $\pi_0\mathcal{D}(Q_i, \partial Q_i) \cong \mathbb{Z}$, see Figure 5(a). Replacing, if necessary, τ_i with τ_i^{-1} we can assume that *all τ_i are isotopic each other as diffeomorphisms of T^2.*



(a) Dehn twist



(b) Slide

FIGURE 5

Let

$$\mathcal{G} = \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, \mathcal{C}) \tag{11}$$

be the group of diffeomorphisms fixed on some neighborhood of each C_i and isotopic to the identity via an isotopy that is not necessarily fixed near $\mathcal{C}$. Evidently, $\mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$ is the path component of $\mathcal{G}$ containing id_{T^2}, whence

$$\pi_0\mathcal{G} \;\cong\; \mathcal{G}/\mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C}).$$

Theorem 4.4. *Let $\theta_i \in \mathcal{G}$, $i = 0, \dots, n-1$, be a slide along C_i such that*

(i) $\mathrm{supp}\,(\theta_i) \subset \mathrm{Int}Q_{i-1} \cup \mathrm{Int}Q_i$*, and, in particular, θ_i is fixed near Q_i;*
(ii) $\mathrm{supp}\,(\theta_i) \cap \mathrm{supp}\,(\theta_j) = \varnothing$ *for $i \neq j$;*
(iii) $\theta|_{Q_i}$ *is isotopic to $\tau_{i-1}\circ\tau_i^{-1}$ relatively to some neighborhood of $C_i \cup M \setminus (Q_{i-1} \cup Q_i)$, see Figure 5(b).*

Denote $\theta = \theta_0 \circ \theta_1 \circ \cdots \circ \theta_{n-1}$. Then $\theta \in \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$, i.e. it is isotopic to id_{T^2} relatively to some neighborhood of $\mathcal{C}$. Moreover,

$$\pi_0\mathcal{G} \;\cong\; \langle[\theta_1], \dots, [\theta_{n-1}]\rangle \;\cong\; \mathbb{Z}^{n-1}, \tag{12}$$

i.e. this group is freely generated by isotopy classes of slides $\theta_1, \dots, \theta_{n-1}$ in $\mathcal{G}$.

In particular, if $n = 1$, $\pi_0\mathcal{G} = \{1\}$, and so $\mathcal{G} = \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, \mathcal{C}) = \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$.

Proof. For $n = 1$ this statement is established in [18], therefore we will assume that $n \geq 2$.

It follows from (iii) that θ is isotopic relatively to some neighborhood of $\mathcal{C}$ to

$$\tau_0 \circ \tau_1^{-1} \circ \tau_1 \circ \tau_2^{-1} \circ \cdots \circ \tau_{n-1} \circ \tau_0^{-1} = \mathrm{id}_{T^2},$$

that is $\theta \in \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$.

It remains to prove (12). Evidently, if $h \in \mathcal{G}$, then $h(Q_i) = Q_i$ and h is fixed on some neighborhood of $\partial Q_i = C_i \cup C_{i+1}$. In other words, the restriction $h|_{Q_i} \in \mathcal{D}(Q_i, \partial Q_i)$. Hence, by Lemma 4.2, $h|_{Q_i}$ is isotopic relatively to some neighborhood ∂Q_i to $\tau_i^{a_i}|_{Q_i}$ for a unique $a_i \in \mathbb{Z}$. Therefore h itself is isotopic relatively to some neighborhood of $\mathcal{C}$ to the product

$$\tau_0^{a_0} \circ \tau_1^{a_1} \circ \cdots \circ \tau_{n-1}^{a_{n-1}} \tag{13}$$

for unique integers $a_0, \dots, a_{n-1} \in \mathbb{Z}^n$.

It easily follows that the correspondence $h \longmapsto (a_0, \dots, a_{n-1})$ is a well-defined homomorphism

$$q : \mathcal{G} \longrightarrow \mathbb{Z}^n.$$

Consider the following subgroup of $\mathbb{Z}^n$:

$$\Delta = \{(a_0, \dots, a_{n-1}) \in \mathbb{Z}^n \mid a_0 + \cdots + a_{n-1} = 0\}.$$

Lemma 4.5. $\ker(q) = \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$ *and* $q(\mathcal{G}) = \Delta$*, so we have the following exact sequence:*

$$1 \longrightarrow \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C}) \xrightarrow{\subset} \mathcal{G} \xrightarrow{q} \Delta \longrightarrow 1.$$

Hence $\pi_0 \mathcal{G} \cong \mathcal{G}/\mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C}) \cong \Delta \cong \mathbb{Z}^{n-1}$.

Proof. The identity $\ker(q) = \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$ easily follows from Lemma 4.2.

Let us prove that $q(\mathcal{G}) = \Delta$. Suppose $q(h) = (a_0, \dots, a_{n-1})$, so h is isotopic relatively to some neighborhood of $\mathcal{C}$ to the product $\tau_0^{a_0} \circ \tau_1^{a_1} \circ \cdots \circ \tau_{n-1}^{a_{n-1}}$. But by construction all τ_i are mutually isotopic as diffeomorphisms of T^2. Hence h is isotopic to $\tau_0^{a_0 + \cdots + a_{n-1}}$. On the other hand, by assumption h is isotopic to id_{T^2}, while τ_0 is not isotopic to the identity and its isotopy class in $\pi_0 \mathcal{D}(T^2)$ has infinite order. Therefore $a_0 + \cdots + a_{n-1} = 0$, i.e. $q(h) \in \Delta$. □

Now we can complete the proof of Theorem 4.4. By (ii) θ_i is isotopic relatively $\mathcal{C}$ to the product $\tau_{i-1} \circ \tau_i^{-1}$, see Figure 5(b). This means that

$$q(\theta_i) = (\underbrace{0, \dots, 0, 1}_{i}, -1, 0, \dots, 0), \quad i = 1, \dots, n-1.$$

It remains to note that the elements $q(\theta_i)$, $i = 1, \dots, n-1$, constitute a basis for Δ, whence their isotopy classes in $\mathcal{G}$ constitute a basis for $\pi_0 \mathcal{G}$. □

Corollary 4.6. *For each* $h \in \mathcal{G}$ *there exist unique* $b_1, \dots, b_{n-1} \in \mathbb{Z}$ *and* $g \in \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$ *such that* $h = \theta_1^{b_1} \circ \cdots \circ \theta_{n-1}^{b_{n-1}} \circ g$.

4.7. **Smooth shifts along trajectories of a flow.** Let $\mathbf{F} : M \times \mathbb{R} \to M$ be a smooth flow on a manifold M. Then for every smooth function $\alpha : M \to \mathbb{R}$ one can define the following map $\mathbf{F}_\alpha : T^2 \to \mathbb{R}$ by the formula:

$$\mathbf{F}_\alpha(z) = \mathbf{F}(z, \alpha(z)), \quad z \in M. \tag{14}$$

Lemma 4.8. [10, Claim 4.14.1]. *Suppose* $\mathbf{F}_\alpha$ *is a* ***diffeomorphism****. Then for each* $t \in [0,1]$ *the map*

$$\mathbf{F}_{t\alpha} : M \to M, \quad \mathbf{F}_{t\alpha}(z) = \mathbf{F}(z, t\alpha(z))$$

is a diffeomorphism as well.

In particular, $\{\mathbf{F}_{t\alpha}\}_{t \in I}$ *is an isotopy between* $\mathrm{id}_M = \mathbf{F}_0$ *and* $\mathbf{F}_\alpha$.

5. Some constructions associated with f

In the sequel we will regard the circle S^1 and the torus T^2 as the corresponding factor-groups $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}^2/\mathbb{Z}^2$. For $s \in S^1$ and $\varepsilon \in (0, 0.5)$ let

$$J_\varepsilon(s) = (s-\varepsilon, s+\varepsilon) \subset S^1$$

be an open ε-neighborhood of $s \in S^1$.

Let $f \in \mathcal{F}(T^2)$ be a function such that its KR-graph $\Gamma(f)$ has only one cycle, C be a regular connected component of certain level set of f not separating T^2, and

$$\mathcal{C} = \{h(C) \mid h \in \mathcal{S}'(f)\} = \{C_0 = C,\ C_1,\ \dots,\ C_{n-1}\},$$

see Figure 2. We will now define several constructions "adopted" with f.

Special coordinates. As the curves $\{C_i \mid i = 0, \dots, n-1\}$ are "parallel", one can assume (by a proper choice of coordinates on T^2) that the following two conditions hold:

(a) $C_i = \frac{i}{n} \times S^1 \ \subset\ \mathbb{R}^2/\mathbb{Z}^2 \equiv T^2$;

(b) there exists $\varepsilon > 0$ such that for all $t \in J_\varepsilon(\frac{i}{n}) = (\frac{i}{n}-\varepsilon, \frac{i}{n}+\varepsilon)$ the curve $t \times S^1$ is a regular connected component of some level set of f.

It is convenient to regard each C_k as a *meridian* of T^2. Let $C' = S^1 \times 0$ be the corresponding *parallel*. Then $C' \cap C_i = \frac{i}{n}$.

Isotopies L and M. Let $\mathbf{L}, \mathbf{M} : T^2 \times [0,1] \to T^2$ be two isotopies defined by

$$\mathbf{L}(x,y,t) = (x+t \bmod 1,\ y), \quad \mathbf{M}(x,y,t) = (x,\ y+t \bmod 1) \tag{15}$$

for $x \in C'$, $y \in C$, and $t \in [0,1]$. Geometrically, $\mathbf{L}$ is a "*rotation*" of the torus along its parallels and $\mathbf{M}$ is a rotation along its meridians. We can regard them as loops in $\pi_1\mathcal{D}(T^2)$. Denote by $\mathcal{L}$ and $\mathcal{M}$ the subgroups of $\pi_1\mathcal{D}^{\mathrm{id}}$ generated by loops $\mathbf{L}$ and $\mathbf{M}$ respectively. It is well known that that $\mathcal{L}$ and $\mathcal{M}$ are commuting free cyclic groups, and so we get an isomorphism

$$\pi_1\mathcal{D}^{\mathrm{id}} \cong \mathcal{L} \times \mathcal{M}.$$

Also notice that $\mathbf{L}$ and $\mathbf{M}$ can be also regarded as *flows* $\mathbf{L}, \mathbf{M} : T^2 \times \mathbb{R} \to T^2$ defined by the same formulas Eq. (15) for $(x,y,t) \in T^2 \times \mathbb{R}$. All orbits of the *flows* $\mathbf{L}$ and $\mathbf{M}$ are periodic of period 1.

A flow F. Since T^2 is an orientable surface, one can construct a "Hamiltonian like" flow $\mathbf{F} : T^2 \times \mathbb{R} \to T^2$ having the following properties, see e.g. [10, Lemma 5.1]:

1) a point $z \in T^2$ is fixed for $\mathbf{F}$ if and only if z is a critical point of f;
2) f is constant along orbits of $\mathbf{F}$, that is $f(z) = f(\mathbf{F}(z,t))$ for all $z \in T^2$ and $t \in \mathbb{R}$.

It follows that every critical point of f and every regular components of every level set of f is an orbit of $\mathbf{F}$.

In particular, each curve $t \times S^1$ for $t \in J_\varepsilon(\frac{i}{n})$, $i = 0, \dots, n-1$, is an orbit of $\mathbf{F}$. On the other hand, this curve is also an orbit of the flow $\mathbf{M}$. Therefore, we can always choose $\mathbf{F}$ so that

$$\mathbf{M}(x,y,t) = \mathbf{F}(x,y,t) \tag{16}$$

for $(x,y,t) \in J_\varepsilon(\frac{i}{n}) \times S^1 \times \mathbb{R}$ and $i = 0, \dots, n-1$.

Lemma 5.1. [10, 12]. *Suppose a flow $\mathbf{F} : T^2 \times \mathbb{R} \to T^2$ satisfies the above conditions* 1) *and* 2)*. Then the following statements hold.*

(1) *Let $h \in \mathcal{S}(f)$. Then $h \in \mathcal{S}_{\mathrm{id}}(f)$ if and only if there exists a C^∞ function $\alpha : T^2 \to \mathbb{R}$ such that $h = \mathbf{F}_\alpha$, see* (14)*. Such a function is unique and the family of maps $\{\mathbf{F}_{t\alpha}\}_{t \in I}$ constitute an isotopy between* id_M *and* h, [12, Lemma 16].

(2) *Suppose C is a regular component of some level set of f and $h \in \mathcal{S}(f)$ be such that $h(C) = C$ and h preserves orientation of C. Let also N be an arbitrary open neighborhood of C. Then each $h \in \mathcal{S}(f)$ is isotopic in $\mathcal{S}(f)$ via an isotopy supported in N to a diffeomorphism g fixed on some smaller neighborhood of C. In particular, $[h] = [g] \in \pi_0\mathcal{S}(f)$,* [10, Lemma 4.14].

(3) *Let X be a finite disjoint union of regular components of some level sets of f, and N be an open neighborhood of X. Then there exists a smaller open neighborhood $U \subset N$ of X such that $\overline{U} \subset N$ and each $h \in \mathcal{S}_{\mathrm{id}}(f)$ is isotopic in $\mathcal{S}(f)$ relatively to $\overline{U}$ to a diffeomorphism g fixed on $M \setminus N$. In particular, $g \in \mathcal{S}_{\mathrm{id}}(f)$ as well. Moreover, if $h = \mathbf{F}_\alpha$, then one can assume that $g = \mathbf{F}_\beta$, where $\beta = \alpha$ on U and $\beta = 0$ on $M \setminus N$,* [10, Lemma 4.14].

Special slides. It follows from (16) and (15) that each C_k is an orbit of the flow $\mathbf{F}$ of period 1. Let $\alpha, \beta : [-1,1] \to [0,1]$ be the functions defined in §4.1, see Figure 4, and ε be the same as in (16). Define two diffeomorphisms $\tau_i, \theta_i : T^2 \to T^2$, $i = 0, \dots, n-1$, by the formulas

$$\tau_i(x,y) = \begin{cases} \mathbf{F}\big(x, y, \alpha((y - \tfrac{i}{n})/2\varepsilon)\big), & (x,y) \in J_\varepsilon(\tfrac{i}{n}) \times S^1, \\ (x,y), & \text{otherwise}, \end{cases} \tag{17}$$

$$\theta_i(x,y) = \begin{cases} \mathbf{F}\big(x, y, \beta((y - \tfrac{i}{n})/2\varepsilon)\big), & (x,y) \in J_\varepsilon(\tfrac{i}{n}) \times S^1, \\ (x,y), & \text{otherwise}. \end{cases} \tag{18}$$

Evidently, τ_i is a Dehn twist and θ_i is a *slide* along C_i in the sense of §4.1.

Notice that $f \circ \theta_i = f$, θ_i is isotopic to id_{T^2}, and θ_k is also fixed on some neighborhood of $\mathcal{C}$. In other words,

$$\theta_i \ \in \ \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, \mathcal{C}) \ = \ \mathcal{S}(f) \cap \mathcal{G},$$

see (11). Moreover, $\mathrm{supp}\,(\theta_i) \cap \mathrm{supp}\,(\theta_j) = \varnothing$ for $i \neq j \in \{1, \dots, n-1\}$. Let also

$$\theta = \theta_0 \circ \cdots \circ \theta_{n-1}. \tag{19}$$

Then by Theorem 4.4 $\theta \in \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C}) = \mathcal{S}_{\mathcal{C}}$. Let $[\theta]_c$ be the isotopy class of θ in $\pi_0\mathcal{S}_{\mathcal{C}}$, and $\Theta = \langle [\theta]_c \rangle$ be the subgroup of $\pi_0\mathcal{S}_{\mathcal{C}}$ generated by $[\theta]_c$.

The following lemma is an easy consequence of (18) and (19) and we leave it for the reader.

Lemma 5.2. *$\theta = \mathbf{F}_\sigma = \mathbf{M}_\sigma$ for some C^∞ function σ such that $\sigma = 1$ on $\mathcal{C}$. Moreover, as σ is constant along orbits of $\mathbf{F}$, it follows from* [9, Eq. (8)] *and can easily be shown, that $\theta^k = \mathbf{F}_{k\sigma}$ for all $k \in \mathbb{Z}$.*

6. Two epimorphisms

In the notation of §5 let $f \in \mathcal{F}(T^2)$ be such that its KR-graph $\Gamma(f)$ has exactly one cycle, C be a regular connected component of certain level set $f^{-1}(c)$ of f that does not separate T^2,

$$\mathcal{C} = \{h(C) \mid h \in \mathcal{S}'(f)\}$$

be the corresponding family of curves parallel to C, and n be the number of curves in $\mathcal{C}$. The case $n = 1$ is considered in [18], therefore we will assume that $n \geq 1$.

For simplicity we will introduce the following notation:

$$\begin{array}{llll} \mathcal{D}^{\mathrm{id}} := \mathcal{D}_{\mathrm{id}}(T^2), & \mathcal{O} := \mathcal{O}_f(f), & \mathcal{S} := \mathcal{S}'(f), & \mathcal{S}^{\mathrm{id}} := \mathcal{S}_{\mathrm{id}}(T^2), \\ \mathcal{D}^{\mathrm{id}}_{\mathcal{C}} := \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C}), & \mathcal{O}_{\mathcal{C}} := \mathcal{O}_f(f, \mathcal{C}), & \mathcal{S}_{\mathcal{C}} := \mathcal{S}'(f, \mathcal{C}), & \mathcal{S}^{\mathrm{id}}_{\mathcal{C}} := \mathcal{S}_{\mathrm{id}}(f, \mathcal{C}), \\ \mathcal{D}^{Q} := \mathcal{D}_{\mathrm{id}}(Q_0, \partial Q_0), & \mathcal{O}^{Q} := \mathcal{O}(f|_{Q_0}, \partial Q_0), & \mathcal{S}^{Q} := \mathcal{S}(f|_{Q_0}, \partial Q_0). & \end{array}$$

Our aim is to construct an isomorphism $\pi_1\mathcal{O} \cong \pi_1\mathcal{O}^Q \underset{\mathbb{Z}_n}{\wr} \mathbb{Z}$. Due to (2) of Theorem 2.2 we have isomorphisms

$$\pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}) \;\cong\; \pi_1\mathcal{O}_{\mathcal{C}}, \quad \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \;\cong\; \pi_1\mathcal{O}, \quad \pi_1(\mathcal{D}^{Q}, \mathcal{S}^{Q}) \;\cong\; \pi_1\mathcal{O}^{Q},$$

and so we are reduced to finding an isomorphism

$$\xi : \pi_1(\mathcal{D}^{Q}, \mathcal{S}^{Q}) \underset{\mathbb{Z}_n}{\wr} \mathbb{Z} \cong \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}). \tag{20}$$

Let $i : (\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}) \subset (\mathcal{D}^{\mathrm{id}}, \mathcal{S})$ be the inclusion map. It yields a morphism between the exact sequences of homotopy groups of these pairs, see Theorems 2.1 and 2.2. The non-trivial part of this morphism is contained in the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[r] \arrow[d] & \pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}) \arrow[r, "\partial_{\mathcal{C}}"] \arrow[d, "i_1"'] & \pi_0\mathcal{S}_{\mathcal{C}} \arrow[r] \arrow[d, "i_0"] & 1 \\
1 \arrow[r] & \pi_1\mathcal{D}^{\mathrm{id}} \arrow[r, "q"] & \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \arrow[r, "\partial"] & \pi_0\mathcal{S} \arrow[r] & 1
\end{tikzcd} \tag{21}$$

In this section we describe kernel and images of all homomorphisms from (21), see Theorem 6.1 below. For $n = 1$ this diagram is studied in [18].

For $h \in \mathcal{S}$ we will denote by $[h]$ its isotopy class in $\pi_0\mathcal{S}$. If $h \in \mathcal{S}_{\mathcal{C}}$, then its isotopy class in $\pi_0\mathcal{S}_{\mathcal{C}}$ will be denoted by $[h]_c$. Evidently,

$$i_0\bigl([h]_c\bigr) = [h].$$

Similarly, for a path $\omega : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})$ we will denote by $[\omega]$ its homotopy class in $\pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$. If $\omega(I, \partial I, 0) \subset (\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}, \mathrm{id}_{T^2})$, then we denote by $[\omega]_c$ is homotopy class in $\pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}})$. Again

$$i_1\bigl([\omega]_c\bigr) = [\omega].$$

Recall also that the boundary homomorphism $\partial_{\mathcal{C}} : \pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}) \longrightarrow \pi_0\mathcal{S}_{\mathcal{C}}$ is defined as follows: if $\omega : (I, \partial I, 0) \to (\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}, \mathrm{id}_{T^2})$ is a continuous path, then

$$\partial_{\mathcal{C}}\bigl([\omega]_c\bigr) = [\omega(1)]_c \in \pi_0\mathcal{S}_{\mathcal{C}}.$$

Theorem 6.1. *In the notation above there exist two epimorphisms*

$$\varphi : \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \longrightarrow \mathbb{Z}, \quad \kappa : \pi_0\mathcal{S} \longrightarrow \mathbb{Z}_n,$$

such that the following diagram is commutative:

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
 & & 1 \arrow[d] & \Theta \arrow[d] & \\
 & & \pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}) \arrow[r, "\partial_{\mathcal{C}}", "\cong"'] \arrow[d, "i_1"] & \pi_0\mathcal{S}_{\mathcal{C}} \arrow[d, "i_0"] & \\
1 \arrow[r] & \mathcal{L} \times \mathcal{M} \arrow[r, "q"] \arrow[d, "\mathrm{pr}"] & \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \arrow[r, "\partial"] \arrow[d, "\varphi"] & \pi_0\mathcal{S} \arrow[r] \arrow[d, "k"] & 1 \\
1 \arrow[r] & \mathcal{L} \arrow[r, "\cdot n"] & \mathbb{Z} \arrow[r, "\mathrm{mod}\, n"] \arrow[d] & \mathbb{Z}_n \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd} \tag{22}$$

Here the arrow $\xrightarrow{\cdot n}$ *means a unique monomorphism associating to the generator* $\mathbf{L} \in \mathcal{L}$ *the number* n. *Moreover, the following statements hold true.*

(a) $q\big(\mathcal{M}\big) = i_1 \circ \partial_{\mathcal{C}}^{-1}\big(\Theta\big)$;
(b) *all rows and columns in diagram* (22) *are exact;*
(c) *there exists a path* $\gamma : (I, \partial I, 0) \to (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})$ *such that*

$$\varphi[\gamma] = 1, \quad \gamma(1)^n = \mathrm{id}_{T^2}.$$

Proof.
Proof of (a). Let $\mathbf{M} : T^2 \times I \to T^2$ be the loop in $\pi_1\mathcal{D}(T^2)$ generating a subgroup $\mathcal{M}$ of $\pi_1\mathcal{D}(T^2)$, see (15). Let also $\theta = \theta_0 \circ \cdots \circ \theta_{n-1}$ be the product of slides along all curves in $\mathcal{C}$, see (19), θ^{-1} be its inverse, and $[\theta^{-1}]_c \in \Theta$ be the isotopy class of θ^{-1} in $\pi_0\mathcal{S}_{\mathcal{C}}$. Then $[\theta^{-1}]_c$ also freely generates $\Theta = \langle[\theta]_c\rangle$. Therefore it suffices to prove that

$$q(\mathbf{M}) = i_1 \circ \partial_{\mathcal{C}}^{-1}\big(\,[\theta^{-1}]_c\,\big).$$

Notice that $q(\mathbf{M})$ is represented by the isotopy $\{\mathbf{M}_t\}_{t\in I}$.

Also recall that we can also regard $\mathbf{M}$ as a flow on T^2 defined by the same formula (15). Since all orbits of $\mathbf{M}$ have period 1, $\mathbf{M}_\alpha = \mathbf{M}_{\alpha+k}$ for all $k \in \mathbb{Z}$ and any function α.

In particular, by Lemma 5.2 $\theta^{-1} = \mathbf{M}_{-\sigma} = \mathbf{M}_{1-\sigma}$ for a C^∞ function $\sigma : T^2 \to \mathbb{R}$ such that $\sigma = 1$ on a small neighborhood U of $\mathcal{C}$ and $\sigma = 0$ outside some larger neighborhood N.

Now let $\mathbf{G}_t = \mathbf{M}_{t(1-\sigma)}$, $t \in I$, be an isotopy between $\mathbf{G}(0) = \mathrm{id}_{T^2}$ and $\mathbf{G}(1) = \theta^{-1}$ fixed on some neighborhood of $\mathcal{C}$. Regard it as a path $\mathbf{G} : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}, \mathrm{id}_{T^2})$. Then $\partial([\mathbf{G}]_c) = [\mathbf{G}(1)]_c = [\theta^{-1}]_c$, and so

$$\partial_{\mathcal{C}}^{-1}[\theta^{-1}]_c = [\mathbf{G}]_c.$$

As $\partial_{\mathcal{C}}$ is an *isomorphism*, $\partial_{\mathcal{C}}^{-1}[\theta^{-1}]_c$ *does not depend* on a particular choice of such an isotopy $\mathbf{G}$. Furthermore, $i_1 \circ \partial_{\mathcal{C}}^{-1}[\theta^{-1}]_c$ is a homotopy class of $\mathbf{G}$ regarded as a map

$$\mathbf{G} : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2}), \quad \mathbf{G}(t) = \mathbf{M}_{t(1-\sigma)}. \tag{23}$$

Therefore it remains to show that $[\mathbf{G}] = q(\mathrm{id}_{T^2} \times \mathbf{M}) \in \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$. In fact the homotopy between $\{\mathbf{G}_t\}_{t\in I}$ and $\{\mathbf{M}_t\}_{t\in I}$ in the space $C\big((I, \partial I, 0), (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})\big)$ can be defined as follows:

$$\mathbf{H} : (I, \partial I, 0) \times I \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2}), \quad \mathbf{H}(t, s) = \mathbf{M}_{t(1-s\sigma)}.$$

We leave the details for the reader, see [18].

Proof of (b). The upper row of (22) coincides with (21) and exactness of the lower row is evident. Therefore it remains to construct epimorphisms φ and κ and prove that the columns of the diagram (22) are exact as well.

(b1) Construction of $\kappa : \pi_0\mathcal{S} \longrightarrow \mathbb{Z}_n$. Let $h \in \mathcal{S}$. Then $h(\mathcal{C}) = \mathcal{C}$. Since the curves in $\mathcal{C}$ are *cyclically* ordered, there exists $\kappa(h) \in \mathbb{Z}_n$ such that

$$h(C_i) = C_{i+\kappa(h) \bmod n}, \quad i = 0, \ldots, n-1. \tag{24}$$

Recall that all indices here are taken module n. Evidently, $\kappa(h)$ depends only on the isotopy class $[h]$ of h in $\mathcal{S}$, and the correspondence $h \longmapsto \kappa[h]$ is a homomorphism $\kappa : \pi_0\mathcal{S} \to \mathbb{Z}_n$. Moreover, κ is an epimorphism, since by definition $\mathcal{C}$ consists of all images of C with respect to $\mathcal{S}$.

(b2) Construction of $\varphi : \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \longrightarrow \mathbb{Z}$. Let $\eta : \mathbb{R}\times S^1 \longrightarrow T^2 \equiv S^1\times S^1$ be the covering map defined by $\eta(x,y) = \bigl(\frac{x}{n} \bmod 1, y\bigr)$. Since $C_i = \frac{i}{n}\times S^1$, we have that

$$\eta(\{i\}\times S^1) = C_{i \bmod n}, \quad i\in\mathbb{Z}, \tag{25}$$

and in particular, $\eta^{-1}(\mathcal{C}) = \mathbb{Z}\times S^1$.

Let $\omega : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})$ be a representative of some element of $\pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$. Then ω can be regarded as an isotopy $\omega : T^2\times I \to T^2$ such that $\omega_0 = \mathrm{id}_{T^2}$ and $\omega_1\in\mathcal{S}$, that is $\omega_1(\mathcal{C}) = \mathcal{C}$. Therefore ω lifts to a unique isotopy $\widetilde{\omega} : (\mathbb{R}\times S^1)\times I \to \mathbb{R}\times S^1$ such that $\widetilde{\omega}_0 = \mathrm{id}_{\mathbb{R}\times S^1}$ and $\eta\circ\widetilde{\omega}_t = \omega_t\circ\eta$ for all $t\in I$.

In particular, since $\omega_1(\mathcal{C}) = \mathcal{C}$, we have from (25) that $\widetilde{\omega}_1(\mathbb{Z}\times S^1) = \mathbb{Z}\times S^1$, whence there exists an integer number $\varphi_\omega\in\mathbb{Z}$ such that

$$\widetilde{\omega}_1(\{i\}\times S^1) = (\{i+\varphi_\omega\}\times S^1), \quad i\in\mathbb{Z}. \tag{26}$$

It is easy to see that φ_ω depends only on the homotopy class $[\omega]$ of ω in $\pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$ and the correspondence $[\omega]\longmapsto\varphi_\omega$ is a homomorphism $\varphi : \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \longrightarrow \mathbb{Z}$.

(b3) Commutativity of diagram (22). Due to (21) the upper square is commutative.

Lower right square. We need to check that

$$\kappa\circ\partial = \varphi \bmod n. \tag{27}$$

In the notation of (b2), notice that $\partial[\omega] = [\omega_1]\in\pi_0\mathcal{S}$ by definition of boundary homomorphism. Hence for $i = 0,\dots,n-1$,

$$\omega_1(C_i) \overset{(25)}{=\!=} \omega_1\circ\eta(\{i\}\times S^1) = \eta\circ\widetilde{\omega}_1(\{i\}\times S^1) \overset{(26)}{=\!=} \eta(\{i+\varphi[\omega]\}\times S^1) = C_{i+\varphi[\omega] \bmod n}.$$

Now (27) follows from (24).

Lower left square. We should show that

$$\varphi\circ q([\mathbf{L}]) = n. \tag{28}$$

Evidently, the path $q(\mathbf{L}) : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})$ can be regarded as an isotopy

$$\mathbf{L} : T^2\times I \to T^2, \quad \mathbf{L}(x,y,t) = (x + \bmod n, y)$$

for $(x,y)\in T^2$, see (15). Then $\mathbf{L}$ lifts to an isotopy $\widetilde{\mathbf{L}} : (\mathbb{R}\times S^1)\times I \to \mathbb{R}\times S^1$ given by $\widetilde{\mathbf{L}}(x,y,t) = (x+nt, y)$. In particular, $\widetilde{\mathbf{L}}(\{i\}\times S^1) = \{i+n\}\times S^1$, whence by (26) $\varphi\circ q([\mathbf{L}]) = n$.

(b4) Exactness of right column. We should prove that the following sequence

$$1 \longrightarrow \Theta \xrightarrow{\subset} \pi_0\mathcal{S}_{\mathcal{C}} \xrightarrow{i_0} \pi_0\mathcal{S} \xrightarrow{\kappa} \mathbb{Z}_n \longrightarrow 1$$

is exact. By definition Θ is a subgroup of $\pi_0\mathcal{S}_{\mathcal{C}}$ and as noted above κ is an epimorphism. Therefore we should check that $\Theta = \ker i_0$ and $i_0(\pi_0\mathcal{S}_{\mathcal{C}}) = \ker\kappa$.

Inclusion $\Theta\subset\ker i_0$.

Recall that each $\theta_i\in\mathcal{S}_{\mathrm{id}}(f)$, whence their product $\theta\in\mathcal{S}_{\mathrm{id}}(f)$ as well, and therefore $i_0([\theta]_c) = [\theta] = [\mathrm{id}_{T^2}]\in\pi_0\mathcal{S}$. This shows that $\Theta = \langle[\theta]_c\rangle\subset\ker(i_0)$

Inverse inclusion $\Theta\supset\ker i_0$.

Notice that the kernel of $i_0 : \pi_0\mathcal{S}_{\mathcal{C}}\to\pi_0\mathcal{S}$ consists of isotopy classes of diffeomorphisms in $\mathcal{S}_{\mathcal{C}}$ isotopic to id_{T^2} by f-preserving isotopy, however such an isotopy should not necessarily be fixed on C. In other words, if we denote

$$\mathcal{K} \ := \ \mathcal{S}^{\mathrm{id}}\cap\mathcal{D}^{\mathrm{id}}_{\mathcal{C}} \ = \ \mathcal{S}_{\mathrm{id}}(f) \ \cap \ \mathcal{D}(T^2, C),$$

then

$$\ker i_0 \ = \ \pi_0\mathcal{K}. \tag{29}$$

Evidently, $\mathcal{S}^{\mathrm{id}}_{\mathcal{C}} = \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$ is the identity path component of $\mathcal{K}$, whence

$$\ker i_0 \;=\; \pi_0\mathcal{K} \;=\; \mathcal{K}/\mathcal{S}^{\mathrm{id}}_{\mathcal{C}}.$$

Also notice that each slide $\theta_i \in \mathcal{S}_{\mathrm{id}}(f)$, whence their product $\theta \in \mathcal{S}_{\mathrm{id}}(f)$ as well. On the other hand by Theorem 4.4 $\theta \in \mathcal{D}^{\mathrm{id}}_{\mathcal{C}}$, whence

$$\theta \in \mathcal{S}^{\mathrm{id}} \cap \mathcal{D}^{\mathrm{id}}_{\mathcal{C}} = \mathcal{K}.$$

Lemma 6.2. $\pi_0\mathcal{K} = \langle[\theta]_c\rangle \cong \mathbb{Z}$. *In other words, each $h \in \mathcal{K}$ is isotopic in $\mathcal{K}$ to θ^b for a unique $b \in \mathbb{Z}$.*

Proof. Let $h \in \mathcal{K}$. Since $\mathcal{K} := \mathcal{S}^{\mathrm{id}} \cap \mathcal{D}^{\mathrm{id}}_{\mathcal{C}} \subset \mathcal{S}^{\mathrm{id}}$, it follows from Lemma 5.1 that there exists a unique smooth function $\alpha \in C^\infty(T^2)$ such that $h = \mathbf{F}_\alpha$.

Since h is fixed on some neighborhood N_i of C_i, that is $h(x) = \mathbf{F}_\alpha(x) = \mathbf{F}(x, \alpha(x)) = x$ for all $x \in N_i$, it follows that $\alpha(x)$ must be an integer multiple of the period of C_i. Hence α takes a constant integer value on N_i.

We claim that this value is the same for all $i = 0, \dots, n-1$. Indeed, let Q_i be a cylinder bounded by C_i and C_{i+1} is isotopic to id_{Q_i} relatively to some neighborhood of ∂Q_i, and τ_i be a Dehn twist supported in $\mathrm{Int}Q_i$ and defined by (17). By Lemma 4.2 the isotopy class of its restriction $\tau_i|_{Q_i}$ generates the group $\pi_0\mathcal{D}(Q_i, \partial Q_i)$. Then it is easy to see that $h|_{Q_i}$ is isotopic in $\mathcal{D}(Q_i, \partial Q_i)$ to τ^b if and only if $\alpha(Q_{i+1}) - \alpha(Q_i) = b$. By assumption $h|_{Q_i}$ is isotopic to $\mathrm{id}_{Q_i} = \tau_i^0$ relatively to ∂Q_i, whence $\alpha(Q_{i+1}) - \alpha(Q_i) = 0$ for all i.

Thus α takes the same constant integer value on all of $\mathcal{C}$, which of course depends on h. Denote this value by k. Then the isotopy between $h = \mathbf{F}_\alpha$ and $\theta^k = \mathbf{F}_{k\sigma}$ in $\mathcal{S}_{\mathcal{C}}$ can be given by the formula: $h_t = \mathbf{F}_{(1-t)\alpha + tk\sigma}$, see Lemma 4.8.

It remains to note that since f has critical points inside each Q_i, θ^k is not isotopic to θ^l for $k \neq l$. $\square$

Inclusion $\mathrm{image}(i_0) \subset \ker(\kappa)$**.** Let $h \in \mathcal{S}_{\mathcal{C}}$, so h is fixed on $\mathcal{C}$, and in particular, $h(C_i) = C_i$ for all i. Then by (24), $\kappa \circ i_0\big([h]_c\big) = 0$, i.e. $\mathrm{image}(i_0) \subset \ker(\kappa)$.

Inverse inclusion $\mathrm{image}(i_0) \supset \ker(\kappa)$**.** Let $h \in \mathcal{S}$ be such that $\kappa[h] = 0$, that is $h(C_i) = C_i$ for all i. Since h is isotopic to id_{T^2}, it also preserves orientation of each C_i, therefore by Lemma 5.1 we can assume that h is fixed on some neighborhood of $\mathcal{C}$ and such a replacement does not change the isotopy class $[h] \in \pi_0\mathcal{S}$. So we can assume that $h \in \mathcal{D}_{\mathrm{id}}(T^2) \cap \mathcal{D}(T^2, \mathcal{C}) = \mathcal{G}$, see (11). Then by Corollary 4.6 we can write

$$h = \theta_1^{a_1} \circ \cdots \circ \theta_{n-1}^{a_{n-1}} \circ g$$

for some $a_i \in \mathbb{Z}$ and $g \in \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C})$. But each $\theta_i \in \mathcal{S}_{\mathrm{id}}(f)$, whence $[h] = [g] \in \pi_0\mathcal{S}$ and

$$g \in \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C}) \;\equiv\; \mathcal{S}_{\mathcal{C}}.$$

In other words, $[h] = [g] = i_0\big([g]_c\big)$. Thus $\mathrm{image}(i_0) \supset \ker(\kappa)$ as well.

(b5) Exactness of middle column. We need to check that the following short sequence

$$1 \longrightarrow \pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}) \xrightarrow{\;i_1\;} \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) \xrightarrow{\;\varphi\;} \mathbb{Z} \longrightarrow 1$$

is exact. Since ∂, κ and $\bmod n$ are surjective, it follows from (27) that φ ***is surjective*** as well. Therefore it remains to verify that i_1 is injective and $\mathrm{image}(i_1) = \ker(\varphi)$.

Inclusion $\mathrm{image}(i_1) \subset \ker(\varphi)$**.** Again using notation of (b2) suppose that $\omega : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}, \mathrm{id}_{T^2})$ is a representative of some element of $\pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}})$. Thus ω can be regarded as an isotopy of T^2 fixed on $\mathcal{C}$. Therefore its lifting $\widetilde{\omega} : (\mathbb{R} \times S^1) \times I \to \mathbb{R} \times S^1$ is fixed on $\mathbb{Z} \times S^1$, whence $\widetilde{\omega}_1(\{i\} \times S^1) = \{i\} \times S^1$ for all $i \in \mathbb{Z}$. Therefore by (26), $\varphi \circ i_1\big([\omega]_c\big) = \varphi[\omega] = 0$, i.e. $\omega \in \ker(\varphi)$.

Inverse inclusion $\mathrm{image}(i_1) \supset \ker(\varphi)$**.** Let $x \in \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$ be such that $\varphi(x) = 0$, i.e. $x \in \ker(\varphi)$. Then

$$0 = \varphi(x) \bmod n = \kappa \circ \partial(x).$$

Hence $\partial(x) \in \ker(\kappa) = \mathrm{image}(i_0) = i_0(\Theta)$. In other words, $\partial(x) = i_0(\theta^k)$ for some $k \in \mathbb{Z}$, where for simplicity of notation we denote by θ its isotopy class $[\theta]_c \in \pi_0\mathcal{S}_{\mathcal{C}}$.

Put $y = i_1 \circ \partial_{\mathcal{C}}^{-1}(\theta^k) \in \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$. Then

$$\partial(y) = \partial \circ i_1 \circ \partial_{\mathcal{C}}^{-1}(\theta^k) = i_0 \circ \partial_{\mathcal{C}} \circ \partial_{\mathcal{C}}^{-1}(\theta^k) = i_0(\theta^k) = \partial(x).$$

Hence $xy^{-1} \in \ker(\partial) = \mathrm{image}(q)$. In other words,

$$x = q(\mathbf{L})^a \cdot q(\mathbf{M})^b \cdot y$$

for some $a, b \in \mathbb{Z}$.

We claim that $a = 0$, whence $x = q(\mathbf{M})^b \cdot y$. Indeed, since $\varphi \circ q(\mathbf{L}) = n$, $\varphi \circ q(\mathbf{M}) = 0$, and $\varphi(y) = \varphi \circ i_1 \circ \partial_{\mathcal{C}}^{-1}(\theta^k) = 0$ we see that

$$0 = \varphi(x) = \varphi\big(q(\mathbf{L})^a \cdot q(\mathbf{M})^b \cdot y\big) = an + 0 + 0,$$

and so $a = 0$.

Moreover, by (a) $q(\mathbf{M}) = i_1 \circ \partial_{\mathcal{C}}^{-1}(\theta^{-1})$, whence

$$x \;=\; q(\mathbf{M})^b \cdot y \;=\; i_1 \circ \partial_{\mathcal{C}}^{-1}(\theta^{-b}) \;\cdot\; i_1 \circ \partial_{\mathcal{C}}^{-1}(\theta^k) \;=\; i_1 \circ \partial_{\mathcal{C}}^{-1}(\theta^{k-b}) \;\in\; \mathrm{image}(i_1).$$

Proof of (c). For $n = 1$, we can take γ to be the constant path into id_{T^2}. Therefore assume that $n \geq 2$.

Let $\mathbf{L}_t : T^2 \to T^2$, $t \in I$, be the isotopy defined by (15) and generating $\mathcal{L}$, and $\lambda = \mathbf{L}_{1/n}$, thus

$$\lambda(x, y) = (x + \tfrac{1}{n} \bmod 1,\ y).$$

In fact we will use the following three properties of λ:

- $f \circ \lambda$ coincides with f on some neighborhood N of $\mathcal{C}$, see (16);
- $\lambda^n = \mathrm{id}_{T^2}$;
- $\lambda(Q_i) = Q_{i+1}$ for all $i = 0, \ldots, n-1$.

Notice that by definition of cyclic index of f, there exists $h \in \mathcal{S}$ such that $h(Q_i) = Q_{i+1}$ as well as λ.

We can assume that $h = \lambda$ on some neighborhood N of $\mathcal{C}$. Indeed, since λ and h preserve orientation of T^2, and $f \circ h = f$, it follows that $h \circ \lambda^{-1}$ leaves invariant all regular components of level sets of f belonging to N. Therefore h is isotopic in $\mathcal{S}$ to a diffeomorphism $h_1 \in \mathcal{S}$ such that $h_1 \circ \lambda^{-1}$ is fixed on some neighborhood N_1 of $\mathcal{C}$, whence $h_1 = \lambda$ near $\mathcal{C}$. Therefore we can replace h with h_1 and N with N_1.

We can additionally assume that $h^n = \mathrm{id}_{T^2}$. Indeed, we have that

$$h^{n-1}|_N = \lambda^{n-1}|_N = \lambda^{-1}|_N = h^{-1}|_N.$$

Define a diffeomorphism $h_1 : T^2 \to T^2$ by $h_1 = h$ on $M \setminus Q_{n-1}$, and $h_1 = h^{-1}$ on Q_{n-1}. Then h_1 is a well-defined diffeomorphism such that $h_1^n = \mathrm{id}_{T^2}$ and $f \circ h_1 = f$, i.e. $h_1 \in \mathcal{S}(f)$. Therefore we can again replace h with h_1.

We claim that h is isotopic to id_{T^2}. Indeed, since $h = \lambda$ on an open set, say on a neighborhood of $\mathcal{C}$, and g preserves orientation, we see that so does h. But all non-trivial isotopy classes of diffeomorphisms of T^2 have infinite orders, whence h is isotopic to id_{T^2}.

Now let $\gamma_t : T^2 \to T^2$, $t \in I$, be any isotopy between id_{T^2} and h. It can be regarded an element of $\pi_1(\mathcal{D}(T^2), \mathcal{S}(f))$. Then $1 = \kappa[\gamma] = \varphi[\gamma] \bmod n$, so $\varphi[\gamma] = an + 1$ for some $a \in \mathbb{Z}$. Replacing γ with any representative of the class $[\gamma]\ [\mathbf{L}]^{-a}$ can assume that $\varphi[\gamma] = 1$.

Theorem 6.1 is completed. □

7. f-invariant free $\mathbb{Z}_n$-action

The following theorem is a reformulation of (c) of Theorem 6.1. It shows that there exists a free f-invariant $\mathbb{Z}_n$-action on T^2, and so f factors to a function of the same class $\mathcal{F}(T^2)$ on the corresponding quotient $T^2/\mathbb{Z}_n$ being also a T^2.

Theorem 7.1. *There exists an n-sheet covering map $p : T^2 \to T^2$ and $\widehat{f} \in \mathcal{F}(T^2)$ making commutative the following diagram:*

$$
\begin{tikzcd}
T^2 \arrow[rr, "p"] \arrow[dr, "f"'] & & T^2 \arrow[dl, "\widehat{f}"] \\
& \mathbb{R} &
\end{tikzcd}
\tag{30}
$$

Moreover, the KR-graph of $\widehat{f}$ also has one cycle, however the cyclic index of $\widehat{f}$ is 1.

Proof. Let γ be the same as in (c) of Theorem 6.1 and let $g = \gamma(1) \in \mathcal{S}(f)$. Then $g^n = \mathrm{id}_{T^2}$. Notice also that g has no fixed points, since $\kappa(g) = \varphi(\gamma) \bmod n = 1$, i.e. $g(Q_i) = Q_{i+1}$ for all i. In other words, g yields a free f-invariant action of $\mathbb{Z}_n$ on T^2 by orientation preserving diffeomorphisms. Hence the corresponding factor map $p : T^2 \to T^2/\mathbb{Z}_n$ is an n-sheet covering of T^2 and the factor space $T^2/\mathbb{Z}_n$ is diffeomorphic to T^2.

Furthermore, since the action is f-invariant, we obtain that f yields a smooth function $\widehat{f} : T^2/\mathbb{Z}_n = T^2 \to \mathbb{R}$, such that the diagram (30) becomes commutative.

It remains to note that since p is a local diffeomorphism, the function $\widehat{f}$ has property (L) as well as f. Therefore $\widehat{f} \in \mathcal{F}(T^2/\mathbb{Z}_n)$. The verification that KR-graph of $\widehat{f}$ has one cycle and that the cyclic index of $\widehat{f}$ is 1 we leave for the reader. □

8. Proof of Theorem 1.6

We have to construct an isomorphism

$$\xi : \pi_1(\mathcal{D}^Q, \mathcal{S}^Q) \underset{\mathbb{Z}_n}{\wr} \mathbb{Z} \cong \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}).$$

Let $\gamma : (I, \partial I, 0) \longrightarrow (\mathcal{D}^{\mathrm{id}}, \mathcal{S}, \mathrm{id}_{T^2})$ be a path defined in (c) of Theorem 6.1, and $g = \gamma(1) \in \mathcal{S}$. Then $g(Q_i) = Q_{i+1}$ and $g^n = \mathrm{id}_{T^2}$.

Recall also that the group $\mathbb{Z}$ acts on $\mathrm{Map}(\mathbb{Z}_n, \pi_1\mathcal{O}^Q)$ by formula (2).

Lemma 8.1. *There exists an isomorphism*

$$\eta : \mathrm{Map}\big(\mathbb{Z}_n, \pi_1(\mathcal{D}^Q, \mathcal{S}^Q)\big) \longrightarrow \pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}).$$

Moreover, let $\alpha \in \mathrm{Map}\big(\mathbb{Z}_n, \pi_1(\mathcal{D}^Q, \mathcal{S}^Q)\big)$, $k \in \mathbb{Z}$, and $\alpha^k \in \mathrm{Map}\big(\mathbb{Z}_n, \pi_1(\mathcal{D}^Q, \mathcal{S}^Q)\big)$ be the result of the action of k on α, see (2). *Then*

$$i_1(\eta(\alpha^k)) = [\gamma^k]\ i_1(\eta(\alpha))\ [\gamma^{-k}]. \tag{31}$$

Proof. Let $\alpha : \mathbb{Z}_n \to \mathcal{P}$ be any map, and $\omega_i : (I, \partial I, 0) \to (\mathcal{D}^Q, \mathcal{S}^Q, \mathrm{id}_{Q_0})$ be a representative of $\alpha(i)$ in $\pi_1(\mathcal{D}^Q, \mathcal{S}^Q)$. Then $\omega_i(t)$ is fixed near ∂Q_0, whence we have a path $\omega : I \to \mathcal{D}^{\mathrm{id}}_{\mathcal{C}}$ given by

$$\omega(t)|_{Q_i} = g^i \circ \omega_i(t) \circ g^{-i}|_{Q_i}, \quad i = 0, \dots, n-1. \tag{32}$$

Notice that

$$\omega(0)|_{Q_i} = g \circ \omega_i(0) \circ g^{-i} = \mathrm{id}_{Q_i}, \quad f \circ \omega(1)|_{Q_i} = f \circ g \circ \omega_i(1) \circ g^{-i} = f,$$

whence $\omega(0) = \mathrm{id}_{T^2}$ and $\omega(1) \in \mathcal{S}(f) \cap \mathcal{D}_{\mathrm{id}}(T^2, \mathcal{C}) = \mathcal{S}_{\mathcal{C}}$. Therefore ω is a map of triples $\omega : (I, \partial I, 0) \to (\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}, \mathrm{id}_{T^2})$, and so it represents some element $[\omega]_c$ of $\pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}})$. It is easy to see that the class $[\omega]_c$ depends only on the classes of $[\omega_i] \in \mathcal{P}$.

Define the map $\eta : \mathrm{Map}\big(\mathbb{Z}_n, \pi_1(\mathcal{D}^Q, \mathcal{S}^Q)\big) \longrightarrow \pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}})$ by $\eta(\alpha) = [\omega]_c$. A straightforward verification shows that η is a group isomorphism. We leave the details for the reader.

Now let $k \in \mathbb{Z}$. Then by definition of the action $\alpha^k(i) = \alpha(i+k \bmod n)$, $i = 0, \dots, n-1$. In particular, if $\omega_i : (I, \partial I, 0) \to (\mathcal{D}^Q, \mathcal{S}^Q, \mathrm{id}_{Q_0})$ is a representative of $\alpha(i)$ in $\mathcal{P}$, then $\omega_{i+k \bmod n}$ is a representative of $\alpha^k(i)$. Therefore the path $\omega' : I \to \mathcal{D}^{\mathrm{id}}_{\mathcal{C}}$ defined by

$$\omega'(t)|_{Q_i} = g^i \circ \omega_{i+k \bmod}(t) \circ g^{-i}|_{Q_i}, \quad i = 0, \dots, n-1.$$

corresponds to α^k, that is $\eta(\alpha^k) = [\omega']_c$. Notice that

$$\omega'(t)|_{Q_i} = g^{-k} \circ g^{i+k} \circ \omega_{i+k \bmod}(t) \circ g^{-i-k} \circ g^k|_{Q_i} = g^{-k} \circ \omega(t) \circ g^k|_{Q_i}.$$

Hence

$$\omega'(t) = g^{-k} \circ \omega(t) \circ g^k = \gamma_1^k \circ \omega_t \circ g_1^{-k}.$$

Notice that $i_1(\eta(\alpha)) = [\omega]$ and $i_1(\eta(\alpha^k)) = [\omega']$ are the homotopy classes of ω and ω' regarded as elements of $\pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$. Then by (10)

$$i_1(\eta(\alpha^k)) = [\gamma_1^k \circ \omega_t \circ g_1^{-k}] = [\gamma_t^k]\ [\omega_t]\ [\gamma_t^{-k}] = [\gamma_t^k]\ i_1(\eta(\alpha))\ [\gamma_t^{-k}].$$

Lemma is proved. □

The following statements completes Theorem 1.6.

Lemma 8.2. *Define a map* $\xi : \pi_1(\mathcal{D}^Q, \mathcal{S}^Q) \underset{\mathbb{Z}_n}{\wr} \mathbb{Z} \longrightarrow \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S})$ *by*

$$\xi(\alpha, k) = i_1(\eta(\alpha))\ [\gamma_t^k],$$

for $\alpha \in \mathrm{Map}(\mathbb{Z}_n, \pi_1(\mathcal{D}^Q, \mathcal{S}^Q))$ *and* $k \in \mathbb{Z}$. *Then* ξ *is a homomorphism making commutative the following diagram with exact rows, see* (4)*:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Map}\big(\mathbb{Z}_n, \pi_1(\mathcal{D}^Q, \mathcal{S}^Q)\big) & \xrightarrow{\zeta} & \pi_1(\mathcal{D}^Q, \mathcal{S}^Q) \underset{\mathbb{Z}_n}{\wr} \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & 1 \\
 & & {\scriptstyle\eta}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\xi}\big\downarrow & & \big\| & & \\
1 & \longrightarrow & \pi_1(\mathcal{D}^{\mathrm{id}}_{\mathcal{C}}, \mathcal{S}_{\mathcal{C}}) & \xrightarrow{i_1} & \pi_1(\mathcal{D}^{\mathrm{id}}, \mathcal{S}) & \xrightarrow{\varphi} & \mathbb{Z} & \longrightarrow & 1.
\end{array}$$

Hence, by five lemma, ξ *is an isomorphism.*

Proof. We should check that ξ is an isomorphism. Suppose $\alpha, \beta \in \mathrm{Map}\big(\mathbb{Z}_n, \pi_1(\mathcal{D}^Q, \mathcal{S}^Q)\big)$ and $k, l \in \mathbb{Z}$. Then in $\pi_1(\mathcal{D}^Q, \mathcal{S}^Q) \underset{\mathbb{Z}_n}{\wr} \mathbb{Z}$ we have that

$$(\alpha, k)\ (\beta, l) = (\alpha\beta^k, k + l)$$

whence

$$\xi(\alpha, k) = i_1(\eta(\alpha))\ [\gamma^k], \quad \xi(\beta, k) = i_1(\eta(\beta))\ [\gamma_t^l].$$

On the other hand,

$$\begin{aligned}
\xi(\alpha\beta^k, k + l) &= i_1(\eta(\alpha\beta^k))\ [\gamma_t^{k+l}] \\
&= i_1(\eta(\alpha))\ i_1(\eta(\beta^k))\ [\gamma_t^{k+l}] && \text{by (31)} \\
&= i_1(\eta(\alpha))\ [\gamma_t^k]\ i_1(\eta(\beta))\ [\gamma_t^{-k}]\ [\gamma_t^{k+l}] \\
&= i_1(\eta(\alpha))\ [\gamma_t^k]\ i_1(\eta(\beta))\ [\gamma_t^l] \\
&= \xi(\alpha, k)\ \xi(\beta, l),
\end{aligned}$$

and so ξ is a homomorphism. Moreover,

$$\xi \circ \zeta(\alpha) = \xi(\alpha, 0) = i_1 \circ \eta(\alpha),$$

$$\varphi \circ \xi(\alpha, k) = \varphi(\eta(\alpha)\ [\gamma^k]) = \varphi \circ \eta(\alpha) + \varphi([\gamma^k]) = 0 + k = k = p(\alpha, k).$$

Hence the above diagram is commutative, and by five lemma ξ is an isomorphism. $\square$

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Topology Department, Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: maks@imath.kiev.ua

Topology Department, Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine
E-mail address: fb@imath.kiev.ua

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