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CHARACTERISTIC MATRICES AND SPECTRAL FUNCTIONS OF FIRST ORDER SYMMETRIC SYSTEMS WITH MAXIMAL DEFICIENCY INDEX OF THE MINIMAL RELATION

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Abstract. Let $\mathbb{H}$ be a finite dimensional Hilbert space and let $[\mathbb{H}]$ be the set of all linear operators in $\mathbb{H}$. We consider first-order symmetric system $Jy'-B(t)y=\Delta(t)f(t)$ with $[\mathbb{H}]$-valued coefficients defined on an interval $[a,b)$ with the regular endpoint a. It is assumed that the corresponding minimal relation $T_{\min}$ has maximally possible deficiency index $n_+(T_{\min})=\dim\mathbb{H}$. The main result is a parametrization of all characteristic matrices and pseudospectral (spectral) functions of a given system by means of a Nevanlinna type boundary parameter τ. Similar parametrization for regular systems has earlier been obtained by Langer and Textorius. We also show that the coefficients of the parametrization form the matrix $W(\lambda)$ with the properties similar to those of the resolvent matrix in the extension theory of symmetric operators.

1. Introduction

Let H and $\widehat{H}$ be finite dimensional Hilbert spaces and let

$$\mathbb{H}:=H\oplus\widehat{H}\oplus H.$$

Denote also by $[\mathbb{H}]$ the set of all linear operators in $\mathbb{H}$. We study first-order symmetric systems of differential equations defined on an interval $\mathcal{I}=[a,b), -\infty<a<b\le\infty$, with the regular endpoint a and regular or singular endpoint b. Such a system is of the form [4, 11]

$$Jy'-B(t)y=\Delta(t)f(t),\quad t\in\mathcal{I}, \tag{1.1}$$

where $B(t)=B^*(t)$ and $\Delta(t)\ge 0$ are locally integrable $[\mathbb{H}]$-valued functions on $\mathcal{I}$ and

$$J=\begin{pmatrix}0 & 0 & -I_H\\ 0 & iI_{\widehat{H}} & 0\\ I_H & 0 & 0\end{pmatrix}:H\oplus\widehat{H}\oplus H\to H\oplus\widehat{H}\oplus H \tag{1.2}$$

(the operator function $\Delta(t)$ is called a Hamiltonian). With (1.1) one associates the homogeneous system

$$Jy'-B(t)y=\lambda\Delta(t)y,\quad \lambda\in\mathbb{C}. \tag{1.3}$$

We assume that system (1.1) is definite (see Definition 3.1).

Let $\mathfrak{H}:=L^2_\Delta(\mathcal{I})$ be the Hilbert space of functions $f(\cdot):\mathcal{I}\to\mathbb{H}$ satisfying $\int_{\mathcal{I}}(\Delta(t)f(t),f(t))_{\mathbb{H}}\,dt<\infty$. As is known system (1.1) generates the minimal linear relation $T_{\min}$ and the maximal linear relation $T_{\max}$ in $\mathfrak{H}$. It turns out that $T_{\min}$ is a closed symmetric relation with finite deficiency indices $n_\pm(T_{\min})\le\dim\mathbb{H}$ and $T_{\max}=T^*_{\min}$.

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Moreover, according to [5, 8, 32] each generalized resolvent $R(\lambda)$ of $T_{\min}$ admits the representation

$$(R(\lambda)f)(x) = \int_{\mathcal{I}} Y_0(x,\lambda)(\Omega(\lambda) + \tfrac{1}{2}\operatorname{sgn}(t-x)J)Y_0^*(t,\overline{\lambda})\Delta(t)f(t)\,dt, \quad f = f(\cdot) \in L^2_\Delta(\mathcal{I}).$$

Here $Y_0(\cdot,\lambda)$ is an $[\mathbb{H}]$-valued operator solution of (1.3) satisfying $Y_0(a,\lambda) = I_{\mathbb{H}}$ and $\Omega(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [\mathbb{H}]$ is a Nevanlinna operator function called a characteristic matrix of the system (1.1).

Next, the following definition was introduced in our paper [28].

Definition 1.1. A non-decreasing left continuous function $\Sigma(\cdot) : \mathbb{R} \to [\mathbb{H}]$ with $\Sigma(0) = 0$ is called a pseudospectral (resp. spectral) function of the system (1.1) if the Fourier transform

$$\widehat{f}(\lambda) = (V_\Sigma f)(\lambda) = \int_{\mathcal{I}} Y_0^*(t,\lambda)\Delta(t)f(t)\,dt, \quad f \in \mathfrak{H} \tag{1.4}$$

defines a partial isometry V_Σ from $\mathfrak{H}$ to $L^2(\Sigma;\mathbb{H})$ with $\ker V_\Sigma = \operatorname{mul} T_{\min}$ (resp. an isometry V_Σ from $\mathfrak{H}$ to $L^2(\Sigma;\mathbb{H})$). Here $\operatorname{mul} T_{\min} = \{f \in \mathfrak{H} : \{0,f\} \in T_{\min}\}$ is the multivalued part of $T_{\min}$ and $L^2(\Sigma;\mathbb{H})$ is the Hilbert space of all functions $g(\cdot) : \mathbb{R} \to \mathbb{H}$ satisfying $\int\limits_{\mathbb{R}} (d\Sigma(\lambda)g(\lambda), g(\lambda)) < \infty$ [9]. Moreover, the integral in (1.4) converges in the norm of $L^2(\Sigma;\mathbb{H})$.

Motivation of the above definition of a pseudospectral function can be found in [28].

Recall that system (1.1) is called regular if the coefficients $B(t)$ and $\Delta(t)$ are defined and integrable on a compact interval $\mathcal{I} = [a,b]$. A description of all characteristic matrices and pseudospectral functions of a regular system is given by the following theorem obtained by Langer and Textotius in [20, 21, 22].

Theorem 1.2. *Let system* (1.1) *be regular. Then*

(1) *The equality*

$$\Omega(\lambda) = -\tfrac{1}{2}(C_a(\lambda) + C_b(\lambda)Y_0(b,\lambda))^{-1}(C_a(\lambda) - C_b(\lambda)Y_0(b,\lambda))J, \quad \lambda \in \mathbb{C}_+ \tag{1.5}$$

establishes a bijective correspondence between all pairs of holomorphic operator functions $C_a(\lambda)$, $C_b(\lambda)(\in [\mathbb{H}])$ *satisfying*

$$iC_a(\lambda)JC_a^*(\lambda) \geq iC_b(\lambda)JC_b^*(\lambda), \quad \operatorname{ran}(C_a(\lambda),\ C_b(\lambda)) = \mathbb{H}, \quad \lambda \in \mathbb{C}_+ \tag{1.6}$$

and all characteristic matrices $\Omega(\lambda)$ *of the system* (1.1).

(2) *The equality* (1.5) *together with the Stieltjes inversion formula*

$$\Sigma(s) = \lim_{\delta\to+0}\lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{-\delta}^{s-\delta} \operatorname{Im}\Omega(\sigma + i\varepsilon)\,d\sigma \tag{1.7}$$

gives a bijective correspondence between all holomorphic pairs $C_a(\lambda)$ *and* $C_b(\lambda)$ *satisfying* (1.6) *and the limit condition*

$$\lim_{y\to+\infty} \tfrac{1}{iy} J(Y_0(b,iy) - I)(C_a(iy) + C_b(iy)Y_0(b,iy))^{-1}(C_a(iy) + C_b(iy)) = 0 \tag{1.8}$$

on one hand, and all pseudospectral functions $\Sigma(s)$ *of the system* (1.1) *on another hand.*

Note that statement (2) of the above theorem is not completely proved in [21, 22]. More precisely, the proof of the assertion that each pseudospectral function $\Sigma(\cdot)$ admits the representation (1.5), (1.7) is based on one of the statements of [21, Theorem 1], which is not proved in [21] (for more details see Remark 4.20 in [28]). In fact we do not know whether statement (2) of Theorem 1.2 is valid without additional assumptions about the system.

In our papers [27, 28] the results similar to Theorem 1.2 were obtained for general (not necessarily regular) symmetric systems on an interval $\mathcal{I} = [a, b),\ b \le \infty$. More precisely, we parametrized in [27] all characteristic matrices $\Omega(\cdot)$ of a given system in a form different from (1.5). This enabled us to parametrize in [28] all pseudospectral functions of an absolutely definite system (system (1.1) is called absolutely definite if the Lebesgue measure of the set $\{t \in \mathcal{I} : \Delta(t)\ \text{ is invertible}\}$ is positive).

Clearly, for a regular system one has $n_+(T_{\min}) = n_-(T_{\min}) = \dim \mathbb{H}$. Moreover, for any $n \in \mathbb{N}$ and N_- between 0 and n there exist a space $\mathbb{H}$ with $\dim \mathbb{H} = n$ and a system (1.1) on $\mathcal{I} = [0, \infty)$ with $n_+(T_{\min}) = \dim \mathbb{H}(= n)$ and $n_-(T_{\min}) = N_-$ [18]. In the present paper we study symmetric systems (1.1) on $\mathcal{I} = [a, b),\ b \le \infty$, with the maximally possible deficiency index $n_+(T_{\min}) = \dim \mathbb{H}$ of $T_{\min}$. Our main result is a parametrization of characteristic matrices and pseudospectral functions of such a system in the form close to that in Theorem 1.2 for regular systems.

We show that in the case $n_+(T_{\min}) = \dim \mathbb{H}$ there exist subspace $\mathcal{H}_b \subset H$ and a surjective linear mapping $\Gamma_b : \operatorname{dom} T_{\max} \to \mathcal{H}_1$ such that the Lagrange's bilinear form $[y, z]_b := \lim\limits_{t \uparrow b} (Jy(t), z(t))$ admits the representation

$$[y, z]_b = (J_b \Gamma_b y, \Gamma_b z), \quad y, z \in \operatorname{dom} T_{\max}.$$

Here $\mathcal{H}_1 := H \oplus \widehat{H} \oplus \mathcal{H}_b$ is a subspace in $\mathbb{H}$ and J_b is an operator in $\mathcal{H}_1$ given by

$$J_b = \begin{pmatrix} 0 & 0 & -I_{\mathcal{H}_b} \\ 0 & iI_{\mathcal{H}_b^\perp \oplus \widehat{H}} & 0 \\ I_{\mathcal{H}_b} & 0 & 0 \end{pmatrix} : \underbrace{\mathcal{H}_b \oplus (\mathcal{H}_b^\perp \oplus \widehat{H}) \oplus \mathcal{H}_b}_{\mathcal{H}_1} \to \underbrace{\mathcal{H}_b \oplus (\mathcal{H}_b^\perp \oplus \widehat{H}) \oplus \mathcal{H}_b}_{\mathcal{H}_1} \tag{1.9}$$

with $\mathcal{H}_b^\perp := H \ominus \mathcal{H}_b$. In fact, $\Gamma_b y$ is a singular boundary value of a function $y \in \operatorname{dom} T_{\max}$ (for more details see Remark 3.5 in [2]).

Assume that $\mathcal{H}_b$ and Γ_b are fixed and let $\tau = \{C_0(\lambda), C_1(\lambda)\}$ be a pair of holomorphic operator functions $C_0(\lambda)(\in [\mathbb{H}])$ and $C_1(\lambda)(\in [\mathcal{H}_1, \mathbb{H}]),\ \lambda \in \mathbb{C}_+$, belonging to the Nevanlinna type class $\widetilde{R}_+(\mathbb{H}, \mathcal{H}_1)$ [26]. With such a pair τ we associate a pair of holomorphic operator functions $C_a(\lambda) = C_{\tau,a}(\lambda)(\in [\mathbb{H}])$ and $C_b(\lambda) = C_{\tau,b}(\lambda)(\in [\mathcal{H}_1, \mathbb{H}])$ satisfying the relations (cf. (1.6))

$$iC_a(\lambda) J C_a^*(\lambda) \ge iC_b(\lambda) J_b C_b^*(\lambda), \quad \operatorname{ran}(C_a(\lambda),\ C_b(\lambda)) = \mathbb{H}, \quad \lambda \in \mathbb{C}_+. \tag{1.10}$$

It turns out that for each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists a unique pair $\tau \in \widetilde{R}_+(\mathbb{H}, \mathcal{H}_1)$ such that a function $y(t) = (R(\lambda)f)(t),\ f = f(\cdot) \in \mathfrak{H}$, is an L^2_Δ-solution of the following boundary problem:

$$Jy' - B(t)y = \lambda \Delta(t) y + \Delta(t) f(t), \quad t \in \mathcal{I}, \tag{1.11}$$

$$C_{\tau,a}(\lambda) y(a) + C_{\tau,b}(\lambda) \Gamma_b y = 0, \quad \lambda \in \mathbb{C}_+. \tag{1.12}$$

Note, that (1.12) is a boundary condition imposed on boundary values of a function $y \in \operatorname{dom} T_{\max}$. One may consider a pair τ as a boundary parameter, since $R(\lambda)$ runs over the set of all generalized resolvents of $T_{\min}$ when τ runs over the set of all pairs $\tau = \{C_0(\lambda), C_1(\lambda)\} \in \widetilde{R}_+(\mathbb{H}, \mathcal{H}_1)$. To indicate this fact explicitly we write $R(\lambda) = R_\tau(\lambda)$ and $\Omega(\lambda) = \Omega_\tau(\lambda)$ for the generalized resolvent of $T_{\min}$ and the corresponding characteristic matrix respectively.

The main result can be formulated in the form of the following theorem (cf. Theorem 1.2).

Theorem 1.3. *Let the minimal relation $T_{\min}$ has the maximally possible deficiency index $n_+(T_{\min}) = \dim \mathbb{H}$ and let $B(\lambda)(\in [\mathbb{H}, \mathcal{H}_1])$ be the operator function given by $B(\lambda)h =$*

$\Gamma_b(Y_0(\cdot,\lambda)h),\ h\in\mathbb{H},\ \lambda\in\mathbb{C}_+$. *Then there exists an operator function*

$$W(\lambda)=\begin{pmatrix} w_1(\lambda) & w_2(\lambda)\\ w_3(\lambda) & w_4(\lambda)\end{pmatrix}:\mathbb{H}\oplus\mathbb{H}\to\mathbb{H}\oplus\mathcal{H}_1,\quad \lambda\in\mathbb{C}_+$$

such that the equality

$$\Omega_\tau(\lambda)=(C_0(\lambda)w_1(\lambda)+C_1(\lambda)w_3(\lambda))^{-1}(C_0(\lambda)w_2(\lambda)+C_1(\lambda)w_4(\lambda)),\quad \lambda\in\mathbb{C}_+ \tag{1.13}$$

establishes a bijective correspondence between all boundary parameters $\tau=\{C_0(\lambda),C_1(\lambda)\}$ *and all characteristic matrices* $\Omega(\lambda)=\Omega_\tau(\lambda)$ *of the system* (1.1). *Moreover,* (1.13) *admits the representation*

$$\Omega_\tau(\lambda)=-\tfrac{1}{2}(C_a(\lambda)+C_b(\lambda)B(\lambda))^{-1}(C_a(\lambda)-C_b(\lambda)B(\lambda))J,\quad \lambda\in\mathbb{C}_+, \tag{1.14}$$

with $C_a(\lambda)=C_{\tau,a}(\lambda)$ *and* $C_b(\lambda)=C_{\tau,b}(\lambda)$ *(hence* $C_a(\lambda)$ *and* $C_b(\lambda)$*) satisfy* (1.10).

If in addition system (1.1) *is absolutely definite, then the equality* (1.13) *together with the Stieltjes inversion formula* (1.7) *gives a bijective correspondence between all boundary parameters* $\tau=\{C_0(\lambda),C_1(\lambda)\}$ *satisfying the limit conditions*

$$\lim_{y\to+\infty}\tfrac{1}{iy}P_{\mathcal{H}_1}w_1(iy)(C_0(iy)w_1(iy)+C_1(iy)w_3(iy))^{-1}C_1(iy)=0,$$
$$\lim_{y\to+\infty}\tfrac{1}{iy}w_3(iy)(C_0(iy)w_1(iy)+C_1(iy)w_3(iy))^{-1}C_0(iy)\upharpoonright\mathcal{H}_1=0$$

and all pseudospectral functions $\Sigma(\cdot)=\Sigma_\tau(\cdot)$ *of the system.*

Note that $B(\lambda)$ is a singular boundary value of the operator solution $Y_0(\cdot,\lambda)$ at the endpoint b and $W(\lambda)$ is defined in terms of $B(\lambda)$. Observe also that the matrix of the operator $W(\lambda)$ is rectangular and its dimension is $(\dim\mathbb{H}+n_-(T_{\min}))\times 2\dim\mathbb{H}$.

Recall that system (1.1) is called quasi-regular if $T_{\min}$ has maximally possible (equal) deficiency indices $n_+(T_{\min})=n_-(T_{\min})=\dim\mathbb{H}$. For a quasi-regular system the matrix of $W(\lambda)$ is square and its dimension is $2\dim\mathbb{H}$. Moreover, for such a system $W(\cdot)$ is an entire $[\mathbb{H}\oplus\mathbb{H}]$-valued function satisfying the identity

$$W^*(\lambda)J_1W(\mu)-J_1=(\mu-\overline{\lambda})\int_{\mathcal{I}}\widetilde{Y}^*(t,\lambda)\Delta(t)\widetilde{Y}(t,\mu)\,dt,\quad \lambda,\mu\in\mathbb{C}. \tag{1.15}$$

Here $\widetilde{Y}(\cdot,\lambda)(\in[\mathbb{H}\oplus\mathbb{H},\mathbb{H}])$ is a solution of (1.3) such that $\widetilde{Y}(a,\lambda)=(I_{\mathbb{H}},\frac{1}{2}J)$ and

$$J_1=\begin{pmatrix}0 & -I_{\mathbb{H}}\\ I_{\mathbb{H}} & 0\end{pmatrix}:\mathbb{H}\oplus\mathbb{H}\to\mathbb{H}\oplus\mathbb{H}. \tag{1.16}$$

The identity (1.15) enables one to represent $W(\lambda)$ explicitly in terms of certain operator solutions of the homogeneous system (1.3) (see (4.51) and (4.52)).

Theorem 1.3 and identity (1.15) show that the function $W(\lambda)$ is an analog of the Nevanlinna matrix in the moment problem [1] and the resolvent matrix in the extension theory of symmetric operators [19].

According to [28] the set of spectral functions is not empty if and only if $\operatorname{mul}T_{\min}=\{0\}$. Moreover, if this condition is satisfied, then the set of spectral functions coincides with the set of pseudospectral functions and, consequently, all the above results hold for spectral functions.

As is known various boundary problems with separated boundary conditions induce "truncated" pseudospectral and spectral functions of the reduced dimension. A description of such functions for the case of equal maximal deficiency indices in the form close to (1.13), (1.7) has been obtained in [10, 12, 14, 16, 17, 31] (see also the book by D. Z. Arov and H. Dym [3] and references therein). Observe also that for the corresponding "resolvent matrix" $W(\lambda)$ of the Sturm-Liouville operator the equalities similar to (4.51) and (4.52) can be found in [12].

In conclusion note that all the results of the paper are applicable to singular formally self-adjoint differential expressions both of even and odd order.

2. Preliminaries

2.1. **Notations.** The following notations will be used throughout the paper: $\mathfrak{H}$, $\mathcal{H}$ denote Hilbert spaces; $[\mathcal{H}_1, \mathcal{H}_2]$ is the set of all bounded linear operators defined on the Hilbert space $\mathcal{H}_1$ with values in the Hilbert space $\mathcal{H}_2$; $[\mathcal{H}] := [\mathcal{H}, \mathcal{H}]$; $P_{\mathcal{L}}$ is the orthoprojection in $\mathfrak{H}$ onto the subspace $\mathcal{L} \subset \mathfrak{H}$; $\mathbb{C}_+$ is the upper half-plane of the complex plane.

Recall that a closed linear relation from $\mathcal{H}_0$ to $\mathcal{H}_1$ is a closed linear subspace in $\mathcal{H}_0 \oplus \mathcal{H}_1$. The set of all closed linear relations from $\mathcal{H}_0$ to $\mathcal{H}_1$ (in $\mathcal{H}$) will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ ($\widetilde{\mathcal{C}}(\mathcal{H})$). A closed linear operator T from $\mathcal{H}_0$ to $\mathcal{H}_1$ is identified with its graph $\operatorname{gr} T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$.

For a linear relation $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ we denote by $\operatorname{dom} T$, $\operatorname{ran} T$, $\ker T$ and $\operatorname{mul} T$ the domain, range, kernel and the multivalued part of T respectively. Recall that $\operatorname{mul} T$ is a subspace in $\mathcal{H}_1$ defined by

$$\operatorname{mul} T := \{h_1 \in \mathcal{H}_1 : \{0, h_1\} \in T\}.$$

Clearly, $T \in \widetilde{\mathcal{C}}(\mathcal{H}_0, \mathcal{H}_1)$ is an operator if and only if $\operatorname{mul} T = \{0\}$. Moreover, we denote by $T^{-1}(\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ and $T^*(\in \widetilde{\mathcal{C}}(\mathcal{H}_1, \mathcal{H}_0))$ inverse and adjoint linear relations of T respectively.

Recall also that an operator function $\Phi(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [\mathcal{H}]$ is called a Nevanlinna function if it is holomorphic and satisfies $\operatorname{Im} \lambda \cdot \operatorname{Im}\Phi(\lambda) \geq 0$ and $\Phi^*(\lambda) = \Phi(\overline{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

2.2. **Boundary triplets and Weyl functions for symmetric relations.** Recall that a linear relation $A \in \widetilde{\mathcal{C}}(\mathfrak{H})$ is called symmetric (self-adjoint) if $A \subset A^*$ (resp. $A = A^*$).

Let A be a closed symmetric linear relation in the Hilbert space $\mathfrak{H}$, let $\mathfrak{N}_\lambda(A) = \ker (A^* - \lambda)$ ($\lambda \in \mathbb{C}$) be a defect subspace of A, let $\widehat{\mathfrak{N}}_\lambda(A) = \{\{f, \lambda f\} : f \in \mathfrak{N}_\lambda(A)\}$ and let $n_\pm(A) := \dim \mathfrak{N}_\lambda(A) \leq \infty$, $\lambda \in \mathbb{C}_\pm$, be deficiency indices of A.

Next we recall definitions of boundary triplets for A^* and the corresponding Weyl functions and γ-fields (see [7, 24, 25, 26]).

Assume that $\mathcal{H}_0$ is a Hilbert space, $\mathcal{H}_1$ is a subspace in $\mathcal{H}_0$ and $\mathcal{H}_2 := \mathcal{H}_0 \ominus \mathcal{H}_1$, so that $\mathcal{H}_0 = \mathcal{H}_1 \oplus \mathcal{H}_2$. Denote by P_j the orthoprojection in $\mathcal{H}_0$ onto $\mathcal{H}_j$, $j \in \{1, 2\}$.

Definition 2.1. A collection $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$, where $\Gamma_j : A^* \to \mathcal{H}_j$, $j \in \{0, 1\}$, are linear mappings, is called a boundary triplet for A^*, if the mapping $\Gamma : \widehat{f} \to \{\Gamma_0 \widehat{f}, \Gamma_1 \widehat{f}\}$, $\widehat{f} \in A^*$, from A^* into $\mathcal{H}_0 \oplus \mathcal{H}_1$ is surjective and the following Green's identity

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}_0} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}_0} + i(P_2 \Gamma_0 \widehat{f}, P_2 \Gamma_0 \widehat{g})_{\mathcal{H}_2}$$

holds for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in A^*$.

A boundary triplet $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ for A^* exists if and only if $n_-(A) \leq n_+(A)$, in which case $\dim \mathcal{H}_1 = n_-(A)$ and $\dim \mathcal{H}_0 = n_+(A)$.

Proposition 2.2. [25]. *Let* $\Pi_+ = \{\mathcal{H}_0 \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^*. *Denote also by* π_1 *the orthoprojection in* $\mathfrak{H} \oplus \mathfrak{H}$ *onto* $\mathfrak{H} \oplus \{0\}$. *Then the operator* $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A)$, $\lambda \in \mathbb{C}_+$, *isomorphically maps* $\widehat{\mathfrak{N}}_\lambda(A)$ *onto* $\mathcal{H}_0$. *Therefore the equalities*

$$\gamma_+(\lambda) = \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A))^{-1}, \quad \lambda \in \mathbb{C}_+,$$
$$M_+(\lambda) h_0 = \Gamma_1 \{\gamma_+(\lambda) h_0, \lambda \gamma_+(\lambda) h_0\}, \quad h_0 \in \mathcal{H}_0, \quad \lambda \in \mathbb{C}_+$$

correctly define the operator functions $\gamma_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathfrak{H}]$ *and* $M_+(\cdot):\mathbb{C}_+\to[\mathcal{H}_0,\mathcal{H}_1]$, *which are holomorphic on their domains.*

Definition 2.3. [25]. The operator functions $\gamma_\pm(\cdot)$ and $M_+(\cdot)$ defined in Proposition 2.2 are called the γ-fields and the Weyl function, respectively, corresponding to the boundary triplet Π_+.

Remark 2.4. If $\mathcal{H}_0=\mathcal{H}_1:=\mathcal{H}$, then the boundary triplet in the sense of Definition 2.1 turns into the boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* in the sense of [13, 24]. In this case $n_+(A)=n_-(A)(=\dim\mathcal{H})$ and the functions $\gamma_+(\cdot)$ and $M_+(\cdot)$ turn into the γ-field $\gamma(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathcal{H},\mathfrak{H}]$ and Weyl function $M(\cdot):\mathbb{C}\setminus\mathbb{R}\to[\mathcal{H}]$ respectively introduced in [7, 24]. Moreover, in this case $M(\cdot)$ is a Nevanlinna operator function.

To avoid misleading with using other definitions, a boundary triplet $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ in the sense of [13, 24] will be called an *ordinary boundary triplet* for A^*.

3. First-order symmetric systems

3.1. **Notations.** Let $\mathcal{I}=[a,b\rangle$ $(-\infty<a<b\le\infty)$ be an interval of the real line (the symbol $\rangle$ means that the endpoint $b<\infty$ might be either included to $\mathcal{I}$ or not). For a given finite-dimensional Hilbert space $\mathbb{H}$ denote by $AC(\mathcal{I};\mathbb{H})$ the set of functions $f(\cdot):\mathcal{I}\to\mathbb{H}$ which are absolutely continuous on each segment $[a,\beta]\subset\mathcal{I}$.

Next assume that $\Delta(\cdot)$ is an $[\mathbb{H}]$-valued Borel measurable function on $\mathcal{I}$ integrable on each compact interval $[a,\beta]\subset\mathcal{I}$ and such that $\Delta(t)\ge 0$. Denote by $\mathcal{L}^2_\Delta(\mathcal{I})$ the semi-Hilbert space of Borel measurable functions $f(\cdot):\mathcal{I}\to\mathbb{H}$ satisfying $||f||^2_\Delta:=\int_{\mathcal{I}}(\Delta(t)f(t),f(t))_{\mathbb{H}}\,dt<\infty$ (see e.g. [9, Chapter 13.5]). The semi-definite inner product $(\cdot,\cdot)_\Delta$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ is defined by $(f,g)_\Delta=\int_{\mathcal{I}}(\Delta(t)f(t),g(t))_{\mathbb{H}}\,dt,\ f,g\in\mathcal{L}^2_\Delta(\mathcal{I})$. Moreover, let $L^2_\Delta(\mathcal{I})$ be the Hilbert space of the equivalence classes in $\mathcal{L}^2_\Delta(\mathcal{I})$ with respect to the semi-norm $||\cdot||_\Delta$ and let π_Δ be the quotient map from $\mathcal{L}^2_\Delta(\mathcal{I})$ onto $L^2_\Delta(\mathcal{I})$.

For a given finite-dimensional Hilbert space $\mathcal{K}$ we denote by $\mathcal{L}^2_\Delta[\mathcal{K},\mathbb{H}]$ the set of all Borel measurable operator-functions $F(\cdot):\mathcal{I}\to[\mathcal{K},\mathbb{H}]$ such that $F(t)h\in\mathcal{L}^2_\Delta(\mathcal{I}),\ h\in\mathcal{K}$. Moreover, we let $\mathcal{L}^2_\Delta[\mathbb{H}]:=\mathcal{L}^2_\Delta[\mathbb{H},\mathbb{H}]$.

3.2. **Symmetric systems.** In this subsection we provide some known results on symmetric systems of differential equations following [11, 15, 23, 30].

Let H and $\widehat{H}$ be finite-dimensional Hilbert spaces and let

$$\mathbb{H}=H\oplus\widehat{H}\oplus H. \tag{3.1}$$

Let as above $\mathcal{I}=[a,b\rangle$ $(-\infty<a<b\le\infty)$ be an interval in $\mathbb{R}$. Moreover, let $B(\cdot)$ and $\Delta(\cdot)$ be $[\mathbb{H}]$-valued Borel measurable functions on $\mathcal{I}$ integrable on each compact interval $[a,\beta]\subset\mathcal{I}$ and satisfying $B(t)=B^*(t)$ and $\Delta(t)\ge 0$ a.e. on $\mathcal{I}$ and let $J\in[\mathbb{H}]$ be operator (1.2).

A first-order symmetric system on an interval $\mathcal{I}$ (with the regular endpoint a) is a system of differential equations of the form

$$Jy'-B(t)y=\Delta(t)f(t),\quad t\in\mathcal{I}, \tag{3.2}$$

where $f(\cdot)\in\mathcal{L}^2_\Delta(\mathcal{I})$. Together with (3.2) we consider also the homogeneous system

$$Jy'(t)-B(t)y(t)=\lambda\Delta(t)y(t),\quad t\in\mathcal{I},\quad \lambda\in\mathbb{C}. \tag{3.3}$$

A function $y\in AC(\mathcal{I};\mathbb{H})$ is a solution of (3.2) (resp. (3.3)) if equality (3.2) (resp. (3.3)) holds a.e. on $\mathcal{I}$. A function $Y(\cdot,\lambda):\mathcal{I}\to[\mathcal{K},\mathbb{H}]$ is an operator solution of equation (3.3) if $y(t)=Y(t,\lambda)h$ is a (vector) solution of this equation for every $h\in\mathcal{K}$ (here $\mathcal{K}$ is a Hilbert space with $\dim\mathcal{K}<\infty$).

In the following we denote by $Y_0(\cdot,\lambda)$ the $[\mathbb{H}]$-valued operator solution of Eq. (3.3) satisfying $Y_0(a,\lambda)=I_{\mathbb{H}}$. Clearly, each operator solution $Y(\cdot,\lambda)$ of Eq. (3.3) admits the representation

$$Y(t,\lambda)=Y_0(t,\lambda)Y(a,\lambda),\quad t\in\mathcal{I}. \tag{3.4}$$

In what follows we always assume that system (3.2) is definite in the sense of the following definition.

Definition 3.1. [11]. Symmetric system (3.2) is called definite if for each $\lambda\in\mathbb{C}$ and each solution y of (3.3) the equality $\Delta(t)y(t)=0$ (a.e. on $\mathcal{I}$) implies $y(t)=0,\ t\in\mathcal{I}$.

Moreover, the following definition will be useful in the sequel.

Definition 3.2. System (3.2) is called absolutely definite if

$$\mu_1(\{t\in\mathcal{I}:\ \text{the operator}\ \ \Delta(t)\ \ \text{is invertible}\})>0,$$

where μ_1 is the Lebesgue measure on $\mathcal{I}$.

Clearly, each absolutely definite system is definite.

As it is known [30, 15, 23] definite system (3.2) gives rise to the *maximal linear relations* $\mathcal{T}_{\max}$ and $T_{\max}$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ and $L^2_\Delta(\mathcal{I})$, respectively. They are given by

$$\mathcal{T}_{\max}=\{\{y,f\}\in(\mathcal{L}^2_\Delta(\mathcal{I}))^2: y\in AC(\mathcal{I};\mathbb{H})\ \text{ and }\ Jy'(t)-B(t)y(t)=\Delta(t)f(t)\ \text{ a.e. on }\mathcal{I}\}$$

and $T_{\max}=\{\{\pi y,\pi f\}:\{y,f\}\in\mathcal{T}_{\max}\}$. Moreover the Lagrange's identity

$$(f,z)_\Delta-(y,g)_\Delta=[y,z]_b-(Jy(a),z(a)),\quad \{y,f\},\ \{z,g\}\in\mathcal{T}_{\max}.$$

holds with

$$[y,z]_b:=\lim_{t\uparrow b}(Jy(t),z(t)),\quad y,z\in\operatorname{dom}\mathcal{T}_{\max}. \tag{3.5}$$

Formula (3.5) defines the skew-Hermitian bilinear form $[\cdot,\cdot]_b$ on $\operatorname{dom}\mathcal{T}_{\max}$. By using this form one defines the *minimal relations* $\mathcal{T}_{\min}$ in $\mathcal{L}^2_\Delta(\mathcal{I})$ and $T_{\min}$ in $L^2_\Delta(\mathcal{I})$ via

$$\mathcal{T}_{\min}=\{\{y,f\}\in\mathcal{T}_{\max}: y(a)=0\ \text{ and }\ [y,z]_b=0\ \text{ for each }\ z\in\operatorname{dom}\mathcal{T}_{\max}\}.$$

and $T_{\min}=\{\{\pi y,\pi f\}:\{y,f\}\in\mathcal{T}_{\min}\}$. According to [30, 15, 23] $T_{\min}$ is a closed symmetric linear relation in $L^2_\Delta(\mathcal{I})$ and $T^*_{\min}=T_{\max}$.

For $\lambda\in\mathbb{C}$ denote by $\mathcal{N}_\lambda(\subset\operatorname{dom}\mathcal{T}_{\max})$ the linear space of solutions of the homogeneous system (3.3) belonging to $\mathcal{L}^2_\Delta(\mathcal{I})$ and let $\mathfrak{N}_\lambda(T_{\min})$ be the defect subspace of $T_{\min}$. Since system (3.2) is definite, it follows that $\dim\mathfrak{N}_\lambda(T_{\min})=\dim\mathcal{N}_\lambda$. Hence $T_{\min}$ has finite (not necessarily equal) deficiency indices

$$n_\pm(T_{\min})=\dim\mathcal{N}_\lambda\le\dim\mathbb{H},\quad \lambda\in\mathbb{C}_\pm. \tag{3.6}$$

In the following with each operator solution $Y(\cdot,\lambda)\in\mathcal{L}^2_\Delta[\mathcal{K},\mathbb{H}]$ of Eq. (3.3) we associate the linear mapping $Y(\lambda):\mathcal{K}\to\mathcal{N}_\lambda$ given by

$$\mathcal{K}\ni h\to(Y(\lambda)h)(t)=Y(t,\lambda)h\in\mathcal{N}_\lambda. \tag{3.7}$$

Remark 3.3. (1) According to the decomposition (3.1) of $\mathbb{H}$ each function $y\in\operatorname{dom}\mathcal{T}_{\max}$ admits the representation

$$y(t)=\{y_0(t),\widehat{y}(t),y_1(t)\}(\in H\oplus\widehat{H}\oplus H). \tag{3.8}$$

(2) It is known (see e.g. [23]) that the maximal relation $T_{\max}$ induced by the definite symmetric system (3.2) possesses the following property: for any $\{\widetilde{y},\widetilde{f}\}\in T_{\max}$ there exists a unique function $y\in AC(\mathcal{I};\mathbb{H})\cap\mathcal{L}^2_\Delta(\mathcal{I})$ such that $y\in\widetilde{y}$ and $\{y,f\}\in\mathcal{T}_{\max}$ for any $f\in\widetilde{f}$. Below we associate such a function $y\in AC(\mathcal{I};\mathbb{H})\cap\mathcal{L}^2_\Delta(\mathcal{I})$ with each pair $\{\widetilde{y},\widetilde{f}\}\in T_{\max}$.

3.3. Decomposing boundary triplets.

Lemma 3.4. *If $n_+(T_{\min}) = \dim \mathbb{H}$, then there exists a subspace $\mathcal{H}_b \subset H$ and a surjective linear mapping*

$$\Gamma_b = \begin{pmatrix} \Gamma_{0b} \\ \widehat{\Gamma}_b \\ \Gamma_{1b} \end{pmatrix} : \operatorname{dom} \mathcal{T}_{\max} \to H \oplus \widehat{H} \oplus \mathcal{H}_b \tag{3.9}$$

such that for all $y, z \in \operatorname{dom} \mathcal{T}_{\max}$ the following identity is valid:

$$[y,z]_b = (\Gamma_{0b}y, \Gamma_{1b}z)_H - (\Gamma_{1b}y, \Gamma_{0b}z)_H + i(P_{\mathcal{H}_b^\perp}\Gamma_{0b}y, P_{\mathcal{H}_b^\perp}\Gamma_{0b}z)_H + i(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z)_{\widehat{H}} \tag{3.10}$$

(here $\mathcal{H}_b^\perp = H \ominus \mathcal{H}_b$). Moreover, $\mathcal{H}_b = H$ if and only if

$$n_-(T_{\min}) = n_+(T_{\min}) (= \dim \mathbb{H}). \tag{3.11}$$

Proof. It follows from (3.6) that $n_-(T_{\min}) \le n_+(T_{\min}) (= \dim \mathbb{H})$. Therefore according to [2] there exist a finite-dimensional Hilbert space $\widetilde{\mathcal{H}}_b$, a subspace $\mathcal{H}_b \subset \widetilde{\mathcal{H}}_b$ and a surjective linear mapping

$$\Gamma_b = (\Gamma_{0b}, \widehat{\Gamma}_b, \Gamma_{1b})^\top : \operatorname{dom} \mathcal{T}_{\max} \to \widetilde{\mathcal{H}}_b \oplus \widehat{H} \oplus \mathcal{H}_b$$

such that the identity (3.10) holds with $\widetilde{\mathcal{H}}_b$ instead of H. Moreover,

$$\dim \widetilde{\mathcal{H}}_b = n_+(T_{\min}) - \dim H - \dim \widehat{H} \quad \text{and} \quad \dim \mathcal{H}_b = n_-(T_{\min}) - \dim H - \dim \widehat{H}. \tag{3.12}$$

Since $n_+(T_{\min}) = \dim \mathbb{H} = 2\dim H + \dim \widehat{H}$, the first equality in (3.12) yields $\dim \widetilde{\mathcal{H}}_b = \dim H$. Therefore without loss of generality one can put $\widetilde{\mathcal{H}}_b = H$, which implies the first statement of the lemma. Moreover, by the second equality in (3.12) one has the equivalence $\dim \mathcal{H}_b = \dim H \iff n_-(T_{\min}) = \dim \mathbb{H}$. This gives the second statement of the lemma. □

The following proposition is immediate from [2, Proposition 3.6].

Proposition 3.5. *Let $n_+(T_{\min}) = \dim \mathbb{H}$, let $\mathcal{H}_b$ be a subspace of H and let Γ_b be a surjective linear mapping* (3.9) *satisfying* (3.10)*. Moreover, let*

$$\mathcal{H}_1 = H \oplus \widehat{H} \oplus \mathcal{H}_b (\subset \mathbb{H}) \tag{3.13}$$

and let $\Gamma_0' : \operatorname{dom} \mathcal{T}_{\max} \to \mathbb{H}$ and $\Gamma_1' : \operatorname{dom} \mathcal{T}_{\max} \to \mathcal{H}_1$ be linear mappings given by

$$\Gamma_0' y = \{-y_1(a),\ i(\widehat{y}(a) - \widehat{\Gamma}_b y),\ \Gamma_{0b} y\} (\in H \oplus \widehat{H} \oplus H), \tag{3.14}$$

$$\Gamma_1' y = \{y_0(a),\ \tfrac{1}{2}(\widehat{y}(a) + \widehat{\Gamma}_b y),\ -\Gamma_{1b} y\} (\in H \oplus \widehat{H} \oplus \mathcal{H}_b), \quad y \in \operatorname{dom} \mathcal{T}_{\max}, \tag{3.15}$$

with $y_0(a)$, $\widehat{y}(a)$ and $y_1(a)$ taken from (3.8)*. Then a collection $\Pi_+ = \{\mathbb{H} \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ with*

$$\Gamma_0\{\widetilde{y}, \widetilde{f}\} = \Gamma_0' y, \quad \Gamma_1\{\widetilde{y}, \widetilde{f}\} = \Gamma_1' y, \quad \{\widetilde{y}, \widetilde{f}\} \in T_{\max} \tag{3.16}$$

is a (so called decomposing) boundary triplet for $T_{\max}$. In (3.16) *$y \in \operatorname{dom} \mathcal{T}_{\max}$ is a function corresponding to $\{\widetilde{y}, \widetilde{f}\} \in T_{\max}$ in accordance with Remark 3.3, (2).*

Proposition 3.6. *Let under the assumptions of Proposition 3.5 $\Pi_+ = \{\mathbb{H} \oplus \mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing boundary triplet* (3.13), (3.16) *for $T_{\max}$, let $\gamma_+(\lambda)$ be the γ-field of Π_+ and let*

$$M_+(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda) \\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda) \end{pmatrix} : \underbrace{H \oplus \widehat{H} \oplus H}_{\mathbb{H}} \to \underbrace{H \oplus \widehat{H} \oplus \mathcal{H}_b}_{\mathcal{H}_1}, \quad \lambda \in \mathbb{C}_+ \tag{3.17}$$

be the block matrix representation of the corresponding Weyl function $M_+(\cdot)$. Then for each $\lambda\in\mathbb{C}_+$ there exists an operator solution $Z_+(\cdot,\lambda)\in\mathcal{L}^2_\Delta[\mathbb{H}]$ of (3.3) *satisfying the relations*

$$\gamma_+(\lambda)=\pi_\Delta Z_+(\lambda), \tag{3.18}$$

$$Z_+(a,\lambda)=\begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda)\\ M_{21}(\lambda) & M_{22}(\lambda)-\frac{i}{2}I_{\widehat H} & M_{23}(\lambda)\\ -I_H & 0 & 0\end{pmatrix}: H\oplus\widehat H\oplus H\to H\oplus\widehat H\oplus H. \tag{3.19}$$

In (3.18)$Z_+(\lambda):\mathbb{H}\to\mathcal{N}_\lambda$ *is the linear mapping* (3.7) *for the solution $Z_+(\cdot,\lambda)$.*

Proof. Existence of the solution $Z_+(\cdot,\lambda)$ satisfying (3.18) directly follows from the proof of Proposition 4.4 in [2] (see in particular [2, (4.17)]). Moreover, the equality (3.19) is implied by [27, (4.42) and (3.25)]. □

4. The matrix $W(\lambda)$

4.1. **The case of one maximal deficiency index.** In this subsection we suppose that the following assumptions are satisfied:

(A1) $n_+(T_{\min})=\dim\mathbb{H}$,

(A2) $\mathcal{H}_b$ is a subspace in H and Γ_b is a surjective linear mapping (3.9) such that (3.10) holds.

In view of (3.6) the assumption (A1) implies that $Y_0(\cdot,\lambda)\in\mathcal{L}^2_\Delta[\mathbb{H}]$ for all $\lambda\in\mathbb{C}_+$. By using this fact we let $B(\lambda)=\Gamma_b Y_0(\lambda)$, $\lambda\in\mathbb{C}_+$. It is easily seen that $B(\cdot)$ is a holomorphic $[\mathbb{H},\mathcal{H}_1]$-valued function on $\mathbb{C}_+$ (for $\mathcal{H}_1$ see (3.13)). Moreover, if

$$Y_0(t,\lambda)=(\varphi(t,\lambda),\,\chi(t,\lambda),\,\psi(t,\lambda)):H\oplus\widehat H\oplus H\to\mathbb{H},\quad \lambda\in\mathbb{C}_+ \tag{4.1}$$

is the block matrix representation of $Y_0(t,\lambda)$, then $B(\lambda)$ can be written in the block-matrix form as

$$\begin{aligned} B(\lambda)&=\begin{pmatrix}\Gamma_{0b}\\ \widehat\Gamma_b\\ \Gamma_{1b}\end{pmatrix}(\varphi(\lambda),\chi(\lambda),\psi(\lambda))\\ &=\begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda)\\ B_{21}(\lambda) & B_{22}(\lambda) & B_{23}(\lambda)\\ B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda)\end{pmatrix}:\underbrace{H\oplus\widehat H\oplus H}_{\mathbb{H}}\to\underbrace{H\oplus\widehat H\oplus\mathcal{H}_b}_{\mathcal{H}_1}. \end{aligned} \tag{4.2}$$

For each $\lambda\in\mathbb{C}_+$ we put

$$W(\lambda)=\begin{pmatrix} w_1(\lambda) & w_2(\lambda)\\ w_3(\lambda) & w_4(\lambda)\end{pmatrix}:\mathbb{H}\oplus\mathbb{H}\to\mathbb{H}\oplus\mathcal{H}_1, \tag{4.3}$$

where

(4.4)
$$w_1(\lambda)=\begin{pmatrix} 0 & 0 & -I_H\\ -iB_{21}(\lambda) & -i(B_{22}(\lambda)-I_{\widehat H}) & -iB_{23}(\lambda)\\ B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda)\end{pmatrix}:\underbrace{H\oplus\widehat H\oplus H}_{\mathbb{H}}\to\underbrace{H\oplus\widehat H\oplus H}_{\mathbb{H}},$$

(4.5)
$$w_2(\lambda)=\begin{pmatrix} \frac{1}{2}I_H & 0 & 0\\ -\frac{i}{2}B_{23}(\lambda) & \frac{1}{2}(B_{22}(\lambda)+I_{\widehat H}) & \frac{i}{2}B_{21}(\lambda)\\ \frac{1}{2}B_{13}(\lambda) & \frac{i}{2}B_{12}(\lambda) & -\frac{1}{2}B_{11}(\lambda)\end{pmatrix} : \underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}\to\underbrace{H\oplus\widehat H\oplus H}_{\mathbb H},$$

(4.6)
$$w_3(\lambda)=\begin{pmatrix} -I_H & 0 & 0\\ -\frac{1}{2}B_{21}(\lambda) & -\frac{1}{2}(B_{22}(\lambda)+I_{\widehat H}) & -\frac{1}{2}B_{23}(\lambda)\\ B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda)\end{pmatrix} : \underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}\to\underbrace{H\oplus\widehat H\oplus \mathcal H_b}_{\mathcal H_1},$$

(4.7)
$$w_4(\lambda)=\begin{pmatrix} 0 & 0 & -\frac{1}{2}I_H\\ -\frac{1}{4}B_{23}(\lambda) & -\frac{i}{4}(B_{22}(\lambda)-I_{\widehat H}) & \frac{1}{4}B_{21}(\lambda)\\ \frac{1}{2}B_{33}(\lambda) & \frac{i}{2}B_{32}(\lambda) & -\frac{1}{2}B_{31}(\lambda)\end{pmatrix} : \underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}\to\underbrace{H\oplus\widehat H\oplus \mathcal H_b}_{\mathcal H_1}.$$

Clearly, the equalities (4.3)–(4.7) define a holomorphic operator function $W(\cdot):\mathbb C_+\to[\mathbb H\oplus\mathbb H,\mathbb H\oplus\mathcal H_1]$.

Next, assume that $\Pi_+=\{\mathbb H\oplus\mathcal H_1,\Gamma_0,\Gamma_1\}$ is the decomposing boundary triplet (3.16) for $T_{\max}$ and let $M_+(\cdot)$ be the Weyl function of Π_+. By using the block matrix representation (3.17) of $M_+(\lambda)$ we let for $\lambda\in\mathbb C_+$

$$\Omega_0(\lambda)=\begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & -\frac{1}{2}I_H\\ M_{21}(\lambda) & M_{22}(\lambda) & 0\\ -\frac{1}{2}I_H & 0 & 0\end{pmatrix} : \underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}\to\underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}, \tag{4.8}$$

$$S_1(\lambda)=\begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda)\\ M_{21}(\lambda) & M_{22}(\lambda)-\frac{i}{2}I_{\widehat H} & M_{23}(\lambda)\\ -I_H & 0 & 0\end{pmatrix} : \underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}\to\underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}, \tag{4.9}$$

$$S_2(\lambda)=\begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & -I_H\\ M_{21}(\lambda) & M_{22}(\lambda)+\frac{i}{2}I_{\widehat H} & 0\\ M_{31}(\lambda) & M_{32}(\lambda) & 0\end{pmatrix} : \underbrace{H\oplus\widehat H\oplus H}_{\mathbb H}\to\underbrace{H\oplus\widehat H\oplus \mathcal H_b}_{\mathcal H_1}. \tag{4.10}$$

The equalities (4.8)–(4.10) define holomorphic operator functions $\Omega_0(\cdot):\mathbb C_+\to[\mathbb H]$, $S_1(\cdot):\mathbb C_+\to[\mathbb H]$ and $S_2(\cdot):\mathbb C_+\to[\mathbb H,\mathcal H_1]$.

Proposition 4.1. *The operator $S_1(\lambda)$ is invertible and the operator function $W(\lambda)$ defined by* (4.3)–(4.7) *admits for each $\lambda\in\mathbb C_+$ the following representation:*

$$W(\lambda)=\begin{pmatrix} w_1(\lambda) & w_2(\lambda)\\ w_3(\lambda) & w_4(\lambda)\end{pmatrix}=\begin{pmatrix} S_1^{-1}(\lambda) & S_1^{-1}(\lambda)\Omega_0(\lambda)\\ -M_+(\lambda)S_1^{-1}(\lambda) & S_2(\lambda)-M_+(\lambda)S_1^{-1}(\lambda)\Omega_0(\lambda)\end{pmatrix}. \tag{4.11}$$

Proof. Let $Z_+(\cdot,\lambda)\in\mathcal L_\Delta^2[\mathbb H]$ be the operator solution of (3.3) defined in Proposition 3.6 and let Γ_0' and Γ_1' be the linear mappings (3.14) and (3.15). It follows from (3.18) (see the proof of [2, Propositions 4.4 and 4.5]) that $\Gamma_0'Z_+(\lambda)=I_{\mathbb H}$ and $\Gamma_1'Z_+(\lambda)=M_+(\lambda)$, $\lambda\in\mathbb C_+$. Moreover, by (3.19)

$$S_1(\lambda)=Z_+(a,\lambda) \tag{4.12}$$

and the equality (3.4) yields

$$Z_+(\lambda)=Y_0(\lambda)Z_+(a,\lambda)=Y_0(\lambda)S_1(\lambda). \tag{4.13}$$

This implies that $(\Gamma_0'Y_0(\lambda))S_1(\lambda)=I_{\mathbb H}$ and $(\Gamma_1'Y_0(\lambda))S_1(\lambda)=M_+(\lambda)$, $\lambda\in\mathbb C_+$. Therefore the operator $S_1(\lambda)$ is invertible and

$$\Gamma_0'Y_0(\lambda)=S_1^{-1}(\lambda),\quad \Gamma_1'Y_0(\lambda)=M_+(\lambda)S_1^{-1}(\lambda),\quad \lambda\in\mathbb C_+. \tag{4.14}$$

Moreover, the immediate calculations with taking (4.2) into account give $\Gamma_0'Y_0(\lambda)=w_1(\lambda)$ and $\Gamma_1'Y_0(\lambda)=-w_3(\lambda)$, where $w_1(\lambda)$ and $w_3(\lambda)$ are defined by (4.4) and (4.6) respectively. Therefore by (4.14)

$$w_1(\lambda) = S_1^{-1}(\lambda), \quad w_3(\lambda) = -M_+(\lambda)S_1^{-1}(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.15}$$

Next, in view of (4.15) one has

$$S_1^{-1}(\lambda)\Omega_0(\lambda)=w_1(\lambda)\Omega_0(\lambda), \quad S_2(\lambda)-M_+(\lambda)S_1^{-1}(\lambda)\Omega_0(\lambda)=S_2(\lambda)+w_3(\lambda)\Omega_0(\lambda). \tag{4.16}$$

Since $w_1(\lambda)$ is invertible, it follows from (4.4) that the operator

$$\begin{pmatrix} -iB_{21}(\lambda) & -i(B_{22}(\lambda)-I) \\ B_{11}(\lambda) & B_{12}(\lambda) \end{pmatrix}$$

is invertible as well. Let $\begin{pmatrix} x_1(\lambda) & x_2(\lambda) \\ x_3(\lambda) & x_4(\lambda) \end{pmatrix} := \begin{pmatrix} -iB_{21}(\lambda) & -i(B_{22}(\lambda)-I) \\ B_{11}(\lambda) & B_{12}(\lambda) \end{pmatrix}^{-1}$, so that

$$-iB_{21}(\lambda)x_1(\lambda) - i(B_{22}(\lambda)-I_{\widehat{H}})x_3(\lambda) = I_{\widehat{H}}, \quad B_{11}(\lambda)x_1(\lambda) + B_{12}(\lambda)x_3(\lambda) = 0, \tag{4.17}$$

$$-iB_{21}(\lambda)x_2(\lambda) - i(B_{22}(\lambda)-I_{\widehat{H}})x_4(\lambda) = 0, \quad B_{11}(\lambda)x_2(\lambda) + B_{12}(\lambda)x_4(\lambda) = I_H. \tag{4.18}$$

Since $S_1(\lambda) = w_1^{-1}(\lambda)$, it follows from (4.4) that

$$S_1(\lambda) = \begin{pmatrix} -ix_1(\lambda)B_{23}(\lambda) + x_2(\lambda)B_{13}(\lambda) & x_1(\lambda) & x_2(\lambda) \\ -ix_3(\lambda)B_{23}(\lambda) + x_4(\lambda)B_{13}(\lambda) & x_3(\lambda) & x_4(\lambda) \\ -I_H & 0 & 0 \end{pmatrix}. \tag{4.19}$$

Comparing (4.19) with (4.9) one gets

$$M_{11}(\lambda) = -ix_1(\lambda)B_{23}(\lambda) + x_2(\lambda)B_{13}(\lambda), \quad M_{12}(\lambda) = x_1(\lambda), \tag{4.20}$$

$$M_{21}(\lambda) = -ix_3(\lambda)B_{23}(\lambda) + x_4(\lambda)B_{13}(\lambda), \quad M_{22}(\lambda) = x_3(\lambda) + \tfrac{i}{2}I_{\widehat{H}}. \tag{4.21}$$

Moreover, by (4.15) one has $M_+(\lambda) = -w_3(\lambda)S_1(\lambda)$. Combining of this equality with (4.6) and (4.19) yields

$$\begin{aligned} M_{31}(\lambda) = &- B_{31}(\lambda)(-ix_1(\lambda)B_{23}(\lambda) + x_2(\lambda)B_{13}(\lambda)) \\ &- B_{32}(\lambda)(-ix_3(\lambda)B_{23}(\lambda) + x_4(\lambda)B_{13}(\lambda)) + B_{33}(\lambda), \end{aligned} \tag{4.22}$$

$$M_{32}(\lambda) = - B_{31}(\lambda)x_1(\lambda) - B_{32}(\lambda)x_3(\lambda). \tag{4.23}$$

Substituting (4.20)–(4.23) into the right hand sides of (4.8) and (4.10) one obtains the representation of $\Omega_0(\lambda)$ and $S_2(\lambda)$ by means of $B_{ij}(\lambda)$ and $x_j(\lambda)$. Moreover, (4.4) and (4.6) give similar representation of $w_1(\lambda)$ and $w_3(\lambda)$. Now the direct calculations with taking (4.17) and (4.18) into account yield $w_1(\lambda)\Omega_0(\lambda) = w_2(\lambda)$ and $S_2(\lambda) + w_3(\lambda)\Omega_0(\lambda) = w_4(\lambda)$, where $w_2(\lambda)$ and $w_4(\lambda)$ are given by (4.5) and (4.7) respectively. This and (4.16) imply

$$w_2(\lambda) = S_1^{-1}(\lambda)\Omega_0(\lambda), \quad w_4(\lambda) = S_2(\lambda) - M_+(\lambda)S_1^{-1}(\lambda)\Omega_0(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.24}$$

Substituting (4.15) and (4.24) into (4.3) one gets the representation (4.11) of $W(\lambda)$. □

Lemma 4.2. *Assume that $\mathcal{K}$, $\mathcal{H}$ and $\mathfrak{H}$ are Hilbert spaces, $\mathcal{H}_1$ is a subspace in $\mathcal{H}$, $\mathcal{H}_2 := \mathcal{H} \ominus \mathcal{H}_1$, $P_1 = P_{\mathcal{H},\mathcal{H}_1} (\in [\mathcal{H}, \mathcal{H}_1])$ is the orthoprojector in $\mathcal{H}$ onto $\mathcal{H}_1$ and $P_2 (\in [\mathcal{H}])$ is the orthoprojector in $\mathcal{H}$ onto $\mathcal{H}_2$. Moreover, let*

$$\begin{pmatrix} a_1(\lambda) & a_2(\lambda) \\ a_3(\lambda) & a_4(\lambda) \end{pmatrix} : \mathcal{K} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{H}_1, \quad (\varphi_1(\lambda),\ \varphi_2(\lambda)) : \mathcal{K} \oplus \mathcal{H} \to \mathfrak{H}, \quad \lambda \in E$$

be the operator functions defined on a set $E \subset \mathbb{C}$ and satisfying

$$a_1(\mu) - a_1^*(\lambda) = (\mu - \overline{\lambda})\varphi_1^*(\lambda)\varphi_1(\mu), \quad a_2(\mu) - a_3^*(\lambda)P_1 = (\mu - \overline{\lambda})\varphi_1^*(\lambda)\varphi_2(\mu), \tag{4.25}$$

$$a_4(\mu) - a_4^*(\lambda)P_1 + iP_2 = (\mu - \overline{\lambda})\varphi_2^*(\lambda)\varphi_2(\mu), \quad \mu, \lambda \in E \tag{4.26}$$

(in (4.26) $a_4(\mu)$ *is considered as an operator in* $\mathcal{H}$*). Assume also that* $a_2(\lambda)$ *is invertible and let*

$$\begin{pmatrix} w_1(\lambda) & w_2(\lambda) \\ w_3(\lambda) & w_4(\lambda) \end{pmatrix} := \begin{pmatrix} a_2^{-1}(\lambda) & a_2^{-1}(\lambda)a_1(\lambda) \\ -a_4(\lambda)a_2^{-1}(\lambda) & a_3(\lambda) - a_4(\lambda)a_2^{-1}(\lambda)a_1(\lambda) \end{pmatrix} : \mathcal{K} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{H}_1,$$

$Q_0(\lambda) := \varphi_2(\lambda)a_2^{-1}(\lambda)(\in [\mathcal{K}, \mathfrak{H}])$, $Q_1(\lambda) := -\varphi_1(\lambda) + \varphi_2(\lambda)a_2^{-1}(\lambda)a_1(\lambda)(\in [\mathcal{K}, \mathfrak{H}])$, $\lambda \in E$.

Then for all $\mu, \lambda \in E$ *the following identities hold:*

$$iw_1^*(\lambda)P_2w_1(\mu) - w_1^*(\lambda)w_3(\mu) + w_3^*(\lambda)P_1w_1(\mu) = (\mu - \overline{\lambda})Q_0^*(\lambda)Q_0(\mu), \tag{4.27}$$

$$iw_2^*(\lambda)P_2w_1(\mu) - w_2^*(\lambda)w_3(\mu) + w_4^*(\lambda)P_1w_1(\mu) - I_{\mathcal{K}} = (\mu - \overline{\lambda})Q_1^*(\lambda)Q_0(\mu), \tag{4.28}$$

$$iw_2^*(\lambda)P_2w_2(\mu) - w_2^*(\lambda)w_4(\mu) + w_4^*(\lambda)P_1w_2(\mu) = (\mu - \overline{\lambda})Q_1^*(\lambda)Q_1(\mu). \tag{4.29}$$

Proof. By using (4.25) and (4.26) one gets

$$\begin{aligned}
&iw_1^*(\lambda)P_2w_1(\mu) - w_1^*(\lambda)w_3(\mu) + w_3^*(\lambda)P_1w_1(\mu)\\
&\quad = ia_2^{-1*}(\lambda)P_2a_2^{-1}(\mu) + +a_2^{-1*}(\lambda)a_4(\mu)a_2^{-1}(\mu) - a_2^{-1*}(\lambda)a_4^*(\lambda)P_1a_2^{-1}(\mu)\\
&\quad = a_2^{-1*}(\lambda)(iP_2 + a_4(\mu) - a_4^*(\lambda)P_1)a_2^{-1}(\mu)\\
&\quad = (\mu - \overline{\lambda})a_2^{-1*}(\lambda)\varphi_2^*(\lambda)\varphi_2(\mu)a_2^{-1}(\mu) = (\mu - \overline{\lambda})Q_0^*(\lambda)Q_0(\mu);\\
&iw_2^*(\lambda)P_2w_1(\mu) - w_2^*(\lambda)w_3(\mu) + w_4^*(\lambda)P_1w_1(\mu) - I_{\mathcal{K}}\\
&\quad = a_1^*(\lambda)a_2^{-1*}(\lambda)(a_4(\mu) - a_4^*(\lambda)P_1 + iP_2)a_2^{-1}(\mu) - (a_2(\mu) - a_3^*(\lambda)P_1)a_2^{-1}(\mu)\\
&\quad = (\mu - \overline{\lambda})\big[a_1^*(\lambda)a_2^{-1*}(\lambda)\varphi_2^*(\lambda)\varphi_2(\mu)a_2^{-1}(\mu) - \varphi_1^*(\lambda)\varphi_2(\mu)a_2^{-1}(\mu)\big]\\
&\quad = (\mu - \overline{\lambda})Q_1^*(\lambda)Q_0(\mu);\\
&iw_2^*(\lambda)P_2w_2(\mu) - w_2^*(\lambda)w_4(\mu) + w_4^*(\lambda)P_1w_2(\mu)\\
&\quad = ia_1^*(\lambda)a_2^{-1*}(\lambda)P_2a_2^{-1}(\mu)a_1(\mu) - a_1^*(\lambda)a_2^{-1*}(\lambda)a_3(\mu)\\
&\quad + a_1^*(\lambda)a_2^{-1*}(\lambda)a_4(\mu)a_2^{-1}(\mu)a_1(\mu)\\
&\quad + a_3^*(\lambda)P_1a_2^{-1}(\mu)a_1(\mu) - a_1^*(\lambda)a_2^{-1*}(\lambda)a_4^*(\lambda)P_1a_2^{-1}(\mu)a_1(\mu)\\
&\quad = a_1^*(\lambda)a_2^{-1*}(\lambda)(iP_2 + a_4(\mu) - a_4^*(\lambda)P_1)a_2^{-1}(\mu)a_1(\mu)\\
&\quad - (a_2(\mu) - a_3^*(\lambda)P_1)a_2^{-1}(\mu)a_1(\mu)\\
&\quad - a_1^*(\lambda)a_2^{-1*}(\lambda)(a_3(\mu) - a_2^*(\lambda)) + (a_1(\mu) - a_1^*(\lambda))\\
&\quad = (\mu - \overline{\lambda})\big[a_1^*(\lambda)a_2^{-1*}(\lambda)\varphi_2^*(\lambda)\varphi_2(\mu)a_2^{-1}(\mu)a_1(\mu)\\
&\quad - \varphi_1^*(\lambda)\varphi_2(\mu)a_2^{-1}(\mu)a_1(\mu)\\
&\quad - a_1^*(\lambda)a_2^{-1*}(\lambda)\varphi_2^*(\lambda)\varphi_1(\mu) + \varphi_1^*(\lambda)\varphi_1(\mu)\big]\\
&\quad = (\mu - \overline{\lambda})Q_1^*(\lambda)Q_1(\mu).
\end{aligned}$$

This proves (4.27)–(4.29). □

Lemma 4.2 is used in the proof of the following proposition.

Proposition 4.3. *Assume that* $\mathcal{H}_1 \subset \mathbb{H}$ *is the subspace* (3.13), $\mathcal{H}_2 = \mathbb{H} \ominus \mathcal{H}_1 (= H \ominus \mathcal{H}_b)$, $P_1 = P_{\mathbb{H},\mathcal{H}_1} (\in [\mathbb{H}, \mathcal{H}_1])$ *is the orthoprojector in* $\mathbb{H}$ *onto* $\mathcal{H}_1$ *and* $P_2 (\in [\mathbb{H}])$ *is the orthoprojector in* $\mathbb{H}$ *onto* $\mathcal{H}_2$*. Moreover, let* $W(\cdot)$ *be the operator function* (4.3) *and let* $Y_1(\cdot, \lambda) \in \mathcal{L}_\Delta^2[\mathbb{H}]$, $\lambda \in \mathbb{C}_+$, *be a solution of Eq.* (3.3) *satisfying* $Y_1(a, \lambda) = \frac{1}{2}J$*. Then for* *all* $\lambda, \mu \in \mathbb{C}_+$

$$iw_1^*(\lambda)P_2w_1(\mu) - w_1^*(\lambda)w_3(\mu) + w_3^*(\lambda)P_1w_1(\mu) = (\mu - \overline{\lambda})\int_{\mathcal{I}} Y_0^*(t,\lambda)\Delta(t)Y_0(t,\mu)\,dt, \tag{4.30}$$

$$iw_2^*(\lambda)P_2w_1(\mu) - w_2^*(\lambda)w_3(\mu) + w_4^*(\lambda)P_1w_1(\mu) - I_{\mathbb{H}} = (\mu - \overline{\lambda})\int_{\mathcal{I}} Y_1^*(t,\lambda)\Delta(t)Y_0(t,\mu)\,dt,$$

$$iw_2^*(\lambda)P_2w_2(\mu) - w_2^*(\lambda)w_4(\mu) + w_4^*(\lambda)P_1w_2(\mu) r = (\mu - \overline{\lambda})\int_{\mathcal{I}} Y_1^*(t,\lambda)\Delta(t)Y_1(t,\mu)\,dt. \tag{4.31}$$

The identities (4.30)–(4.31) *mean that*

$$W^*(\lambda)J_2W(\mu) - J_1 = (\mu - \overline{\lambda})\int_{\mathcal{I}} \widetilde{Y}^*(t,\lambda)\Delta(t)\widetilde{Y}(t,\mu)\,dt, \quad \lambda, \mu \in \mathbb{C}_+,$$

where $\widetilde{Y}(t,\lambda) = (Y_0(t,\lambda),\, Y_1(t,\lambda)) : \mathbb{H}\oplus\mathbb{H} \to \mathbb{H},\ \lambda \in \mathbb{C}_+,$ *and*

$$J_1 = \begin{pmatrix} 0 & -I_{\mathbb{H}} \\ I_{\mathbb{H}} & 0 \end{pmatrix} : \mathbb{H}\oplus\mathbb{H} \to \mathbb{H}\oplus\mathbb{H}, \quad J_2 = \begin{pmatrix} iP_2 & -I_{\mathcal{H}_1} \\ P_1 & 0 \end{pmatrix} : \mathbb{H}\oplus\mathcal{H}_1 \to \mathbb{H}\oplus\mathcal{H}_1. \tag{4.32}$$

Proof. Let $M_+(\cdot)$ be the Weyl function (3.17) of the decomposing boundary triplet Π_+ for $T_{\max}$, let $\Omega_0(\lambda)$, $S_1(\lambda)$ and $S_2(\lambda)$ be given by (4.8)–(4.10) and let

$$\begin{pmatrix} a_1(\lambda) & a_2(\lambda) \\ a_3(\lambda) & a_4(\lambda) \end{pmatrix} := \begin{pmatrix} \Omega_0(\lambda) & S_1(\lambda) \\ S_2(\lambda) & M_+(\lambda) \end{pmatrix} : \mathbb{H}\oplus\mathbb{H} \to \mathbb{H}\oplus\mathcal{H}_1, \quad \lambda \in \mathbb{C}_+. \tag{4.33}$$

Assume also that

$$\gamma_+(\lambda) = (\gamma_0(\lambda),\, \widehat{\gamma}(\lambda),\, \gamma_1(\lambda)) : H\oplus\widehat{H}\oplus H \to L^2_\Delta(\mathcal{I}), \quad \lambda \in \mathbb{C}_+,$$

is the block matrix representation of the γ-field $\gamma_+(\cdot)$ of Π_+ and let

$$\varphi_1(\lambda) = (\gamma_0(\lambda),\, \widehat{\gamma}(\lambda),\, 0) : H\oplus\widehat{H}\oplus H \to L^2_\Delta(\mathcal{I}), \quad \lambda \in \mathbb{C}_+. \tag{4.34}$$

Then according to equalities (2.9) and (4.76) in [27] one has

$$\Omega_0(\mu) - \Omega_0^*(\lambda) = (\mu - \overline{\lambda})\varphi_1^*(\lambda)\varphi_1(\mu), \quad S_1(\mu) - S_1^*(\lambda)P_1 = (\mu - \overline{\lambda})\varphi_1^*(\lambda)\gamma_+(\mu),$$
$$M_+(\mu) - M_+^*(\lambda)P_1 + iP_2 = (\mu - \overline{\lambda})\gamma_+^*(\lambda)\gamma_+(\mu), \quad \mu, \lambda \in \mathbb{C}_+.$$

Moreover, $W(\lambda)$ admits the representation (4.11). Now applying Lemma 4.2 to (4.33) and $W(\lambda)$ one gets the identities (4.27)–(4.29) with

$$Q_0(\lambda) = \gamma_+(\lambda)S_1^{-1}(\lambda), \quad Q_1(\lambda) = -\varphi_1(\lambda) + \gamma_+(\lambda)S_1^{-1}(\lambda)\Omega_0(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.35}$$

Let us show that

$$Q_0(\lambda) = \pi_\Delta Y_0(\lambda) \quad \text{and} \quad Q_1(\lambda) = \pi_\Delta Y_1(\lambda), \tag{4.36}$$

where $Y_j(\lambda) : \mathbb{H} \to \mathcal{N}_\lambda$ is the linear mapping (3.7) for the solution $Y_j(\cdot,\lambda),\ j \in \{0,1\}$.

It follows from (3.18) and (4.13) that

$$\gamma_+(\lambda) = \pi_\Delta Y_0(\lambda)S_1(\lambda), \quad \lambda \in \mathbb{C}_+. \tag{4.37}$$

Moreover, by (4.34) $\varphi_1(\lambda) = \gamma_+(\lambda)X$, where

$$X = \begin{pmatrix} I_H & 0 & 0 \\ 0 & I_{\widehat{H}} & 0 \\ 0 & 0 & 0 \end{pmatrix} : H\oplus\widehat{H}\oplus H \to H\oplus\widehat{H}\oplus H.$$

Therefore in view of (4.37) one has

$$\varphi_1(\lambda) = \pi_\Delta Y_0(\lambda)S_1(\lambda)X, \quad \lambda \in \mathbb{C}_+. \tag{4.38}$$

Combining the first equality in (4.35) with (4.37) one gets the first equality in (4.36). Moreover, combining of (4.37) and (4.38) with the second equality in (4.35) yields

$$Q_1(\lambda) = \pi_\Delta Y_0(\lambda)(\Omega_0(\lambda) - S_1(\lambda)X)$$

and the immediate calculations give $\Omega_0(\lambda) - S_1(\lambda)X = \frac{1}{2}J$. Observe also that by (3.4) $Y_1(\lambda) = Y_0(\lambda)(\frac{1}{2}J)$. Hence $Q_1(\lambda) = \pi_\Delta Y_0(\lambda)(\frac{1}{2}J) = \pi_\Delta Y_1(\lambda)$, which proves the second equality in (4.36).

Applying [2, Lemma 3.3] to (4.36) one gets

$$Q_j^*(\lambda)\widetilde{f} = \int_{\mathcal{I}} Y_j^*(t,\lambda)\Delta(t)f(t)\,dt, \quad \widetilde{f} \in L^2_\Delta(\mathcal{I}), \quad f(\cdot) \in \widetilde{f}, \quad j \in \{0,1\},$$

and, consequently,

$$Q_j^*(\lambda)Q_k(\mu) = \int_{\mathcal{I}} Y_j^*(t,\lambda)\Delta(t)Y_k(t,\mu)\,dt, \quad j,k \in \{0,1\}.$$

Now the identities (4.30)–(4.31) are implied by (4.27)–(4.29). □

4.2. **The case of the quasi-regular system.**

Theorem 4.4. *For system* (3.2) *the following assertions are equivalent:*

(1) *The relation* $T_{\min}$ *has maximal deficiency indices* $n_+(T_{\min}) = n_-(T_{\min}) = \dim \mathbb{H}$.
(2) $Y_0(\cdot,\lambda) \in \mathcal{L}^2_\Delta[\mathbb{H}]$ *for all* $\lambda \in \mathbb{C}$.
(3) *There exists* $\lambda_0 \in \mathbb{C}$ *such that* $Y_0(\cdot,\lambda_0) \in \mathcal{L}^2_\Delta[\mathbb{H}]$ *and* $Y_0(\cdot,\overline{\lambda}_0) \in \mathcal{L}^2_\Delta[\mathbb{H}]$.

We omit the proof of this theorem, because it is similar to the proof of the corresponding statements for differential operators (see e.g. [29]).

Definition 4.5. Symmetric system (3.2) is said to be quasi-regular if at least one (and hence all) of the conditions (1)–(3) are satisfied.

Recall also that system (3.2) is called regular if it is defined on a compact interval $\mathcal{I} = [a,b]$. Clearly, each regular system is quasi-regular.

It follows from Lemma 3.4 that in the case of the quasi-regular system there exists a surjective linear mapping

$$\Gamma_b = \begin{pmatrix}\Gamma_{0b}\\ \widehat{\Gamma}_b\\ \Gamma_{1b}\end{pmatrix} : \operatorname{dom}\mathcal{T}_{\max} \to \underbrace{H \oplus \widehat{H} \oplus H}_{\mathbb{H}} \tag{4.39}$$

such that

$$[y,z]_b = (\Gamma_{0b}y, \Gamma_{1b}z) - (\Gamma_{1b}y, \Gamma_{0b}z) + i(\widehat{\Gamma}_b y, \widehat{\Gamma}_b z), \quad y,z \in \operatorname{dom}\mathcal{T}_{\max}. \tag{4.40}$$

This means that Γ_b is a linear mapping from $\operatorname{dom}\mathcal{T}_{\max}$ onto $\mathbb{H}$ satisfying

$$[y,z]_b = (J\Gamma_b y, \Gamma_b z), \quad y,z \in \operatorname{dom}\mathcal{T}_{\max}. \tag{4.41}$$

In this subsection we assume that *system* (3.2) *is quasi-regular and that* Γ_b *is a surjective linear mapping* (4.39) *satisfying* (4.40). Then the equality $B(\lambda) = \Gamma_b Y_0(\lambda)$, $\lambda \in \mathbb{C}$, defines an entire $[\mathbb{H}]$-valued function $B(\cdot)$, which admits the block matrix representation (4.2) with $\mathcal{H}_b = H$.

For each $\lambda \in \mathbb{C}$ we put

$$W(\lambda) = \begin{pmatrix} w_1(\lambda) & w_2(\lambda)\\ w_3(\lambda) & w_4(\lambda)\end{pmatrix} : \mathbb{H} \oplus \mathbb{H} \to \mathbb{H} \oplus \mathbb{H}, \tag{4.42}$$

where the entries $w_j(\lambda)$, $j \in \{1, 2, 3, 4\}$, are defined by (4.4)–(4.7) with $\mathcal{H}_b = H$. Clearly, $W(\cdot)$ is an $[\mathbb{H} \oplus \mathbb{H}]$-valued entire function.

Next, the decomposing boundary triplet Π_+ becomes an ordinary boundary triplet $\Pi = \{\mathbb{H}, \Gamma_0, \Gamma_1\}$ for $T_{\max}$ with mappings Γ_0 and Γ_1 given by (3.16) and (3.14), (3.15).

By using the arguments similar to those in the previous subsection one can easily prove the following two propositions.

Proposition 4.6. *Assume that system* (3.2) *is quasi-regular. Let*

$$M(\lambda)=\begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) & M_{13}(\lambda)\\ M_{21}(\lambda) & M_{22}(\lambda) & M_{23}(\lambda)\\ M_{31}(\lambda) & M_{32}(\lambda) & M_{33}(\lambda)\end{pmatrix}:\underbrace{H\oplus \widehat{H}\oplus H}_{\mathbb{H}}\to \underbrace{H\oplus \widehat{H}\oplus H}_{\mathbb{H}},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}$$

be the block matrix representation of the Weyl function $M(\cdot)$ corresponding to the decomposing boundary triplet $\Pi=\{\mathbb{H},\Gamma_0,\Gamma_1\}$ for $T_{\max}$ and let $\Omega_0(\lambda)$, $S_1(\lambda)$ and $S_2(\lambda)$ be the $[\mathbb{H}]$-valued functions defined for $\lambda\in\mathbb{C}\setminus\mathbb{R}$ by (4.8), (4.9) *and* (4.10) *respectively (with $\mathcal{H}_b=H$). Then for each $\lambda\in\mathbb{C}\setminus\mathbb{R}$ the operator function $W(\lambda)$ admits the representation* (4.11).

Proposition 4.7. *Let system* (3.2) *be quasi-regular and let $Y_0(\cdot,\lambda)\in\mathcal{L}^2_\Delta[\mathbb{H}]$ and $Y_1(\cdot,\lambda)\in\mathcal{L}^2_\Delta[\mathbb{H}]$, $\lambda\in\mathbb{C}$, be operator solutions of* (3.3) *satisfying $Y_0(a,\lambda)=I_{\mathbb{H}}$ and $Y_1(a,\lambda)=\frac{1}{2}J$. Then for all $\lambda,\mu\in\mathbb{C}$ the following identities hold:*

$$-w_1^*(\lambda)w_3(\mu)+w_3^*(\lambda)w_1(\mu)=(\mu-\overline{\lambda})\int_{\mathcal{I}}Y_0^*(t,\lambda)\Delta(t)Y_0(t,\mu)\,dt, \tag{4.43}$$

$$-w_2^*(\lambda)w_3(\mu)+w_4^*(\lambda)w_1(\mu)-I_{\mathbb{H}}=(\mu-\overline{\lambda})\int_{\mathcal{I}}Y_1^*(t,\lambda)\Delta(t)Y_0(t,\mu)\,dt, \tag{4.44}$$

$$-w_2^*(\lambda)w_4(\mu)+w_4^*(\lambda)w_2(\mu)=(\mu-\overline{\lambda})\int_{\mathcal{I}}Y_1^*(t,\lambda)\Delta(t)Y_1(t,\mu)\,dt. \tag{4.45}$$

This means that

$$W^*(\lambda)J_1W(\mu)-J_1=(\mu-\overline{\lambda})\int_{\mathcal{I}}\widetilde{Y}^*(t,\lambda)\Delta(t)\widetilde{Y}(t,\mu)\,dt,\quad \lambda,\mu\in\mathbb{C}, \tag{4.46}$$

where J_1 is given by the firs equality in (4.32) *and*

$$\widetilde{Y}(t,\lambda)=(Y_0(t,\lambda),\,Y_1(t,\lambda)):\mathbb{H}\oplus\mathbb{H}\to\mathbb{H},\quad \lambda\in\mathbb{C}. \tag{4.47}$$

A somewhat other representation of the operator function $W(\lambda)$ is given in the following proposition.

Proposition 4.8. *Let the assumptions of Proposition 4.7 be fulfilled and let $Y_2(\cdot,\lambda)\in\mathcal{L}^2_\Delta[\mathbb{H}]$ and $Y_3(\cdot,\lambda)\in\mathcal{L}^2_\Delta[\mathbb{H}]$, $\lambda\in\mathbb{C}$, be operator solutions of* (3.3) *satisfying*

$$Y_2(a,\lambda)=\begin{pmatrix}0&0&0\\0&I_{\widehat{H}}&0\\0&0&-I_H\end{pmatrix},\quad Y_3(a,\lambda)=\begin{pmatrix}0&0&I_H\\0&\frac{i}{2}I_{\widehat{H}}&0\\0&0&0\end{pmatrix}. \tag{4.48}$$

Then:
(1) *for each $y\in\operatorname{dom}\mathcal{T}_{\max}$ there exists the limit*

$$\Gamma_b y:=\lim_{t\uparrow b}(-JY_0^*(t,0)Jy(t)),\quad y\in\operatorname{dom}\mathcal{T}_{\max}, \tag{4.49}$$

and the equality (4.49) *defines a surjective linear mapping $\Gamma_b:\operatorname{dom}\mathcal{T}_{\max}\to\mathbb{H}$ satisfying* (4.41). *Moreover, the corresponding operator function $B(\lambda)(=\Gamma_bY_0(\lambda))$ is*

$$B(\lambda)=\lim_{t\uparrow b}(-JY_0^*(t,0)JY_0(t,\lambda)),\quad \lambda\in\mathbb{C}. \tag{4.50}$$

(2) *If* Γ_b *is defined by* (4.49)*, then the entries of the corresponding operator function* $W(\lambda)$ *(see* (4.42)*) admit the representation*

(4.51)
$$w_1(\lambda) = C_1 - \lambda \int_{\mathcal{I}} Y_2^*(t,0)\Delta(t)Y_0(t,\lambda)\,dt, \quad w_2(\lambda) = C_2 - \lambda \int_{\mathcal{I}} Y_2^*(t,0)\Delta(t)Y_1(t,\lambda)\,dt,$$

(4.52)
$$w_3(\lambda) = C_3 - \lambda \int_{\mathcal{I}} Y_3^*(t,0)\Delta(t)Y_0(t,\lambda)\,dt, \quad w_2(\lambda) = C_4 - \lambda \int_{\mathcal{I}} Y_3^*(t,0)\Delta(t)Y_1(t,\lambda)\,dt$$

with the operators $C_j \in [H \oplus \widehat{H} \oplus H]$ *given by*

$$C_1 = \begin{pmatrix} 0 & 0 & -I_H \\ 0 & 0 & 0 \\ I_H & 0 & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \frac{1}{2}I_H & 0 & 0 \\ 0 & I_{\widehat{H}} & 0 \\ 0 & 0 & -\frac{1}{2}I_H \end{pmatrix}, \tag{4.53}$$

$$C_3 = \begin{pmatrix} -I_H & 0 & 0 \\ 0 & -I_{\widehat{H}} & 0 \\ 0 & 0 & I_H \end{pmatrix}, \qquad C_4 = \begin{pmatrix} 0 & 0 & -\frac{1}{2}I_H \\ 0 & 0 & 0 \\ \frac{1}{2}I_H & 0 & 0 \end{pmatrix}. \tag{4.54}$$

Proof. (1) It is well known (see e.g. [11]) that $Y_0^*(t,0)JY_0(t,0) = J$ and, consequently, $Y_0^{-1}(t,0) = -JY_0^*(t,0)J$, $t \in \mathcal{I}$. For each function $y \in \operatorname{dom} \mathcal{T}_{\max}$ put

$$F_y(t) = Y_0^{-1}(t,0)y(t) = -JY_0^*(t,0)Jy(t). \tag{4.55}$$

Let $\widetilde{h} \in \mathbb{H}$ and let $z(t) = Y_0(t,0)\widetilde{h}$. Then

$$[y,z]_b = \lim_{t\uparrow b}(Jy(t), Y_0(t,0)\widetilde{h}) = \lim_{t\uparrow b}(JY_0(t,0)F_y(t), Y_0(t,0)\widetilde{h}) = -\lim_{t\uparrow b}(F_y(t), J\widetilde{h}).$$

This and (4.55) yield existence of the limit (4.49). Next, for each $\widetilde{h} \in \mathbb{H}$ one has $\Gamma_b(Y_0(\cdot,0)\widetilde{h}) = \widetilde{h}$, so that the mapping Γ_b is surjective. Moreover, for any $y, z \in \operatorname{dom} \mathcal{T}_{\max}$ one has

$$[y,z]_b = \lim_{t\uparrow b}(JY_0(t,0)F_y(t), Y_0(t,0)F_z(t)) = \lim_{t\uparrow b}(JF_y(t), F_z(t)) = (J\Gamma_b y, \Gamma_b z),$$

which proves (4.41). Finally, (4.50) directly follows from (4.49).

(2) Since by (4.46) $W^*(0)J_1W(0) = J_1$, it follows that $(W^*(0)J_1)^{-1} = -W(0)J_1$. This and (4.46) yield

$$W(\lambda) = W(0) - \lambda W(0)J_1 \int_{\mathcal{I}} \widetilde{Y}^*(t,0)\Delta(t)\widetilde{Y}(t,\lambda)\,dt, \quad \lambda \in \mathbb{C}. \tag{4.56}$$

By (4.50) $B(0) = I_{\mathbb{H}}$ and (4.3)–(4.7) give the equality

$$W(0) = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \mathbb{H} \oplus \mathbb{H} \to \mathbb{H} \oplus \mathbb{H}, \tag{4.57}$$

where C_j are defined by (4.53) and (4.54). Moreover, the immediate calculation with taking (4.47) and (4.57) into account gives

$$\begin{aligned} &W(0)J_1 \int_{\mathcal{I}} \widetilde{Y}^*(t,0)\Delta(t)\widetilde{Y}(t,\lambda)\,dt \\ &= \begin{pmatrix} \int_{\mathcal{I}} Y_2^*(t,0)\Delta(t)Y_0(t,\lambda)\,dt & \int_{\mathcal{I}} Y_2^*(t,0)\Delta(t)Y_1(t,\lambda)\,dt \\ \int_{\mathcal{I}} Y_3^*(t,0)\Delta(t)Y_0(t,\lambda)\,dt & \int_{\mathcal{I}} Y_3^*(t,0)\Delta(t)Y_1(t,\lambda)\,dt \end{pmatrix}, \end{aligned} \tag{4.58}$$

where

$$Y_2(t,\lambda) = Y_0(t,\lambda)C_2^* - Y_1(t,\lambda)C_1^* = Y_0(t,\lambda)(C_2^* - \tfrac{1}{2}JC_1^*),$$
$$Y_3(t,\lambda) = Y_0(t,\lambda)C_4^* - Y_1(t,\lambda)C_3^* = Y_0(t,\lambda)(C_4^* - \tfrac{1}{2}JC_3^*).$$

Hence $Y_2(\cdot,\lambda)$ and $Y_3(\cdot,\lambda)$ are operator solutions of (3.3) with

$$Y_2(a,\lambda) = C_2^* - \tfrac{1}{2}JC_1^*, \quad Y_3(a,\lambda) = C_4^* - \tfrac{1}{2}JC_3^*$$

and the immediate checking gives the equalities (4.48). Now combining (4.56) with (4.57) and (4.58) we arrive at statement (2) of the proposition. □

5. Description of characteristic matrices and pseudospectral functions

5.1. **Characteristic matrices.** Recall that the operator function $R(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [L^2_\Delta(\mathcal{I})]$ is called a generalized resolvent of $T_{\min}$ if there exist a Hilbert space $\widetilde{\mathfrak{H}} \supset L^2_\Delta(\mathcal{I})$ and a self-adjoint linear relation $\widetilde{T}$ in $\widetilde{\mathfrak{H}}$ such that $T_{\min} \subset \widetilde{T}$ and

$$R(\lambda) = P_{L^2_\Delta(\mathcal{I})}(\widetilde{T} - \lambda)^{-1} \upharpoonright L^2_\Delta(\mathcal{I}), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

The following theorem is well known (see e.g. [5, 8, 32]).

Theorem 5.1. *For each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists a unique operator function $\Omega(\cdot) : \mathbb{C}\setminus\mathbb{R} \to [\mathbb{H}]$ such that for each $\widetilde{f} \in L^2_\Delta(\mathcal{I})$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$*

$$(5.1) \qquad R(\lambda)\widetilde{f} = \pi_\Delta\Big(\int_{\mathcal{I}} Y_0(\cdot,\lambda)(\Omega(\lambda) + \tfrac{1}{2}\operatorname{sgn}(t-x)J)Y_0^*(t,\overline{\lambda})\Delta(t)f(t)\,dt\Big), \quad f \in \widetilde{f}.$$

Moreover, $\Omega(\cdot)$ is a Nevanlinna operator function.

Definition 5.2. [5, 32]. The operator function $\Omega(\cdot)$ is called the characteristic matrix of the symmetric system (3.2) corresponding to the generalized resolvent $R(\lambda)$.

Everywhere below we suppose that the assumptions (A1) and (A2) from Section 4.1 are satisfied.

Let $\mathcal{H}_1$ be the subspace (3.13) of $\mathbb{H}$ and let $\tau = \{\tau_+, \tau_-\}$ be a collection of holomorphic functions $\tau_\pm(\cdot) : \mathbb{C}_\pm \to \widetilde{\mathcal{C}}(\mathbb{H}, \mathcal{H}_1)$.

Definition 5.3. A collection $\tau = \{\tau_+, \tau_-\}$ is called a boundary parameter if it belongs to the class $\widetilde{R}_+(\mathbb{H}, \mathcal{H}_1)$ in the sense of [26].

According to [26] a boundary parameter $\tau = \{\tau_+, \tau_-\}$ admits the representation

$$(5.2) \quad \tau_+(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathbb{H}\}, \ \ \lambda \in \mathbb{C}_+; \quad \tau_-(\lambda) = \{(D_0(\lambda), D_1(\lambda)); \mathcal{H}_1\}, \ \ \lambda \in \mathbb{C}_-$$

by means of two pairs of holomorphic operator functions

$$(C_0(\lambda), C_1(\lambda)) : \mathbb{H} \oplus \mathcal{H}_1 \to \mathbb{H}, \ \ \lambda \in \mathbb{C}_+, \ \text{ and } \ (D_0(\lambda), D_1(\lambda)) : \mathbb{H} \oplus \mathcal{H}_1 \to \mathcal{H}_1, \ \ \lambda \in \mathbb{C}_-$$

with special properties (more precisely, by equivalence classes of such pairs). The equalities (5.2) mean that

$$\tau_+(\lambda) = \{\{\widetilde{h}, h_1\} \in \mathbb{H} \oplus \mathcal{H}_1 : C_0(\lambda)\widetilde{h} + C_1(\lambda)h_1 = 0\}, \quad \lambda \in \mathbb{C}_+,$$
$$\tau_-(\lambda) = \{\{\widetilde{h}, h_1\} \in \mathbb{H} \oplus \mathcal{H}_1 : D_0(\lambda)\widetilde{h} + D_1(\lambda)h_1 = 0\}, \quad \lambda \in \mathbb{C}_-.$$

In the case of the quasi-regular system one has $\mathcal{H}_1 = \mathbb{H}$ and a boundary parameter τ is a function $\tau(\cdot) : \mathbb{C}\setminus\mathbb{R} \to \widetilde{\mathcal{C}}(\mathbb{H})$ belonging to the well known Nevanlinna class $\widetilde{R}(\mathbb{H})$ (see e.g [6]). Such τ admits the representation in the form of a Nevanlinna operator pair

$$(5.3) \qquad \tau(\lambda) = \{(C_0(\lambda), C_1(\lambda)); \mathbb{H}\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R},$$

with $[\mathbb{H}]$-valued functions $C_0(\lambda)$ and $C_1(\lambda)$ satisfying for $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the following relations:

$$(5.4) \quad \operatorname{Im}\lambda \cdot \operatorname{Im}(C_1(\lambda)C_0^*(\lambda)) \geq 0, \ \ C_1(\lambda)C_0^*(\overline{\lambda}) - C_0(\lambda)C_1^*(\overline{\lambda}) = 0, \ \ \operatorname{ran}(C_0(\lambda), C_1(\lambda)) = \mathbb{H}.$$

If in addition $\tau(\lambda) \equiv \theta(=\theta^*)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then a boundary parameter τ is called self-adjoint. Such a boundary parameter admits the representation in the form of a self-adjoint operator pair

$$\tau(\lambda) \equiv \{(C_0, C_1); \mathbb{H}\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \quad C_0, C_1 \in [\mathbb{H}]. \tag{5.5}$$

For each boundary parameter $\tau = \{\tau_+, \tau_-\}$ of the form (5.2) we assume that

$$C_0(\lambda) = (C_{0a}(\lambda),\ \widehat{C}_0(\lambda),\ C_{0b}(\lambda)) : H \oplus \widehat{H} \oplus H \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+, \tag{5.6}$$

$$C_1(\lambda) = (C_{1a}(\lambda),\ \widehat{C}_1(\lambda),\ C_{1b}(\lambda)) : H \oplus \widehat{H} \oplus \mathcal{H}_b \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+ \tag{5.7}$$

are the block matrix representations of $C_0(\lambda)$ and $C_1(\lambda)$.

The following lemma is immediate from [27, Lemma 4.2].

Lemma 5.4. *Let H be decomposed as $H = \mathcal{H}_b \oplus \mathcal{H}_b^\perp$, so that*

$$\mathcal{H}_1(= H \oplus \widehat{H} \oplus \mathcal{H}_b) = \mathcal{H}_b \oplus (\mathcal{H}_b^\perp \oplus \widehat{H}) \oplus \mathcal{H}_b.$$

Assume also that $J_b \in [\mathcal{H}_1]$ is the operator (1.9). *Then the equalities*

$$C_a(\lambda) = (-C_{1a}(\lambda),\ i\widehat{C}_0(\lambda) - \tfrac{1}{2}\widehat{C}_1(\lambda),\ -C_{0a}(\lambda)) : H \oplus \widehat{H} \oplus H \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+ \tag{5.8}$$

$$C_b(\lambda) = (C_{0b}(\lambda),\ -i\widehat{C}_0(\lambda) - \tfrac{1}{2}\widehat{C}_1(\lambda),\ C_{1b}(\lambda)) : H \oplus \widehat{H} \oplus \mathcal{H}_b \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+ \tag{5.9}$$

establish a bijective correspondence between all boundary parameters $\tau = \{\tau_+, \tau_-\}$ of the form (5.2) *and* (5.6), (5.7) *and all holomorphic operator functions*

$$(C_a(\lambda),\ C_b(\lambda)) : \mathbb{H} \oplus \mathcal{H}_1 \to \mathbb{H}, \quad \lambda \in \mathbb{C}_+ \tag{5.10}$$

satisfying (1.10). *If in addition system* (3.2) *is quasi-regular, then the relations* (5.6)–(5.10) *and* (1.10) *hold with $\mathcal{H}_b = H$, $\mathcal{H}_1 = \mathbb{H}$ and $J_b = J$; moreover, in this case $C_a(\lambda)$ and $C_b(\lambda)$ are defined on $\mathbb{C}\setminus\mathbb{R}$.*

Let $\tau = \{\tau_+, \tau_-\}$ be a boundary parameter (5.2). For a given function $f \in \mathcal{L}_\Delta^2(\mathcal{I})$ consider the boundary problem

$$Jy' - B(t)y = \lambda\Delta(t)y + \Delta(t)f(t), \quad t \in \mathcal{I}, \tag{5.11}$$

$$C_0(\lambda)\Gamma_0' y - C_1(\lambda)\Gamma_1' y = 0, \ \ \lambda \in \mathbb{C}_+; \quad D_0(\lambda)\Gamma_0' y - D_1(\lambda)\Gamma_1' y = 0, \ \ \lambda \in \mathbb{C}_-, \tag{5.12}$$

where $\Gamma_0' y \in \mathbb{H}$ and $\Gamma_1' y \in \mathcal{H}_1$ are defined by (3.14) and (3.15). According to [27, (4.26)] the first equality in (5.12) admits the representation

$$C_a(\lambda)y(a) + C_b(\lambda)\Gamma_b(y) = 0, \quad \lambda \in \mathbb{C}_+, \tag{5.13}$$

where $C_a(\lambda)$ and $C_b(\lambda)$ are defined by (5.8) and (5.9) and hence satisfy (1.10).

If system is quasi-regular and τ is a boundary parameter (5.3), then boundary conditions (5.12) take one of the following equivalent form:

$$C_0(\lambda)\Gamma_0' y - C_1(\lambda)\Gamma_1' y = 0 \iff C_a(\lambda)y(a) + C_b(\lambda)\Gamma_b(y) = 0, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{5.14}$$

Theorem 5.5. *Let $\tau = \{\tau_+, \tau_-\}$ be a boundary parameter* (5.2). *Then for every $f \in \mathcal{L}_\Delta^2(\mathcal{I})$ the boundary problem* (5.11), (5.12) *has a unique solution $y(t,\lambda) = y_f(t,\lambda)$ and the equality*

$$R(\lambda)\widetilde{f} = \pi_\Delta(y_f(\cdot,\lambda)), \quad \widetilde{f} \in L_\Delta^2(\mathcal{I}), \quad f \in \widetilde{f}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}$$

defines a generalized resolvent $R(\lambda) =: R_\tau(\lambda)$ of $T_{\min}$. Conversely, for each generalized resolvent $R(\lambda)$ of $T_{\min}$ there exists a unique boundary parameter τ such that $R(\lambda) = R_\tau(\lambda)$.

If in addition system (3.2) *is quasi-regular, then the above statements hold with the boundary parameter τ of the form* (5.5) *and the boundary condition* (5.14) *in place of* (5.12).

Proof. Let $\Pi_+ = \{\mathbb{H}\oplus\mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a decomposing boundary triplet (3.16) for $T_{\max}$. Applying to this triplet [26, Theorem 3.11] one obtains the required statements. □

According to Theorem 5.5 the boundary problem (5.11), (5.12) induces a bijective correspondence $R(\lambda) = R_\tau(\lambda)$ between boundary parameters τ and generalized resolvents $R(\lambda)$ of $T_{\min}$. In the following we denote by $\Omega_\tau(\cdot)$ the characteristic matrix corresponding to $R_\tau(\cdot)$. Clearly, the equality $\Omega(\cdot) = \Omega_\tau(\cdot)$ gives a parametrization of all characteristic matrices of the system (3.2) by means of a boundary parameter τ. In the following theorem we represent this parametrization in the explicit form.

Theorem 5.6. *Assume that $n_+(T_{\min}) = \dim\mathbb{H}$. Moreover, let $\mathcal{H}_b$ be a subspace in $\mathbb{H}$, let Γ_b be a surjective linear mapping* (3.9) *satisfying* (3.10) *let $B(\lambda) = \Gamma_b Y_0(\lambda)$, $\lambda\in\mathbb{C}_+$, and let $W(\lambda)$ be the operator function given by* (4.3)*–*(4.7)*. Then for each boundary parameter $\tau = \{\tau_+, \tau_-\}$ of the form* (5.2) *the operator $C_0(\lambda)w_1(\lambda)+C_1(\lambda)w_3(\lambda)$, $\lambda\in\mathbb{C}_+$, is invertible and the corresponding characteristic matrix $\Omega_\tau(\cdot)$ is*

$$\Omega_\tau(\lambda) = (C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda))^{-1}(C_0(\lambda)w_2(\lambda) + C_1(\lambda)w_4(\lambda)), \quad \lambda\in\mathbb{C}_+. \tag{5.15}$$

Moreover, $\Omega_\tau(\cdot)$ can be represented as

$$\Omega_\tau(\lambda) = -\tfrac{1}{2}(C_a(\lambda) + C_b(\lambda)B(\lambda))^{-1}(C_a(\lambda) - C_b(\lambda)B(\lambda))J, \quad \lambda\in\mathbb{C}_+, \tag{5.16}$$

where $C_a(\lambda)$ and $C_b(\lambda)$ are defined by (5.8) *and* (5.9)*.*

Proof. It follows from [27, Theorem 4.6] that

$$\Omega_\tau(\lambda) = \Omega_0(\lambda) + S_1(\lambda)(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1}C_1(\lambda)S_2(\lambda), \quad \lambda\in\mathbb{C}_+, \tag{5.17}$$

where $M_+(\lambda)$ is the Weyl function (3.17) and $\Omega_0(\lambda)$, $S_1(\lambda)$ and $S_2(\lambda)$ are given by (4.8)–(4.10). Moreover, by Proposition 4.1 $S_1(\lambda)$ is invertible. Hence

$$\begin{aligned}\Omega_\tau(\lambda) &= \Omega_0(\lambda) + (C_0(\lambda)S_1^{-1}(\lambda) - C_1(\lambda)M_+(\lambda)S_1^{-1}(\lambda))^{-1}C_1(\lambda)S_2(\lambda)\\ &= (C_0(\lambda)S_1^{-1}(\lambda) - C_1(\lambda)M_+(\lambda)S_1^{-1}(\lambda))^{-1}\big[(C_0(\lambda)S_1^{-1}(\lambda)\\ &\quad - C_1(\lambda)M_+(\lambda)S_1^{-1}(\lambda))\Omega_0(\lambda) + C_1(\lambda)S_2(\lambda)\big]\\ &= (C_0(\lambda)S_1^{-1}(\lambda) - C_1(\lambda)M_+(\lambda)S_1^{-1}(\lambda))^{-1}\big[C_0(\lambda)S_1^{-1}(\lambda)\Omega_0(\lambda)\\ &\quad + C_1(\lambda)(S_2(\lambda) - M_+(\lambda)S_1^{-1}(\lambda)\Omega_0(\lambda))\big], \quad \lambda\in\mathbb{C}_+,\end{aligned}$$

which in view of (4.11) yields (5.15). To prove (5.16) note that $C_0(\lambda) = C_a(\lambda)X_1 + C_b(\lambda)X_3$ and $C_1(\lambda) = C_a(\lambda)X_2 + C_b(\lambda)X_4$, where

$$X_1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & -\frac{i}{2}I & 0\\ -I & 0 & 0\end{pmatrix}, \quad X_2 = \begin{pmatrix} -I & 0 & 0\\ 0 & -I & 0\\ 0 & 0 & 0\end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & I\\ 0 & \frac{i}{2}I & 0\\ 0 & 0 & 0\end{pmatrix},$$
$$X_4 = \begin{pmatrix} 0 & 0 & 0\\ 0 & -I & 0\\ 0 & 0 & I\end{pmatrix}.$$

Therefore

$$\begin{aligned}&C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda)\\ &\qquad = C_a(\lambda)(X_1w_1(\lambda) + X_2w_3(\lambda)) + C_b(\lambda)(X_3w_1(\lambda) + X_4w_3(\lambda)),\\ &C_0(\lambda)w_2(\lambda) + C_1(\lambda)w_4(\lambda)\\ &\qquad = C_a(\lambda)(X_1w_2(\lambda) + X_2w_4(\lambda)) + C_b(\lambda)(X_3w_2(\lambda) + X_4w_4(\lambda))\end{aligned}$$

and the immediate calculations with taking (4.4)–(4.7) into account give

$$X_1 w_1(\lambda) + X_2 w_3(\lambda) = I, \quad X_3 w_1(\lambda) + X_4 w_3(\lambda) = B(\lambda),$$
$$X_1 w_2(\lambda) + X_2 w_4(\lambda) = -\tfrac{1}{2} J, \quad X_3 w_2(\lambda) + X_4 w_4(\lambda) = \tfrac{1}{2} B(\lambda) J.$$

Hence $C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda) = C_a(\lambda) + C_b(\lambda)B(\lambda)$, $C_0(\lambda)w_2(\lambda) + C_1(\lambda)w_4(\lambda) = -\frac{1}{2}(C_a(\lambda) - C_b(\lambda)B(\lambda))J$ and (5.16) follows from (5.15). □

In the case of the quasi-regular system Theorem 5.6 can be rather simplified. Namely, the following corollary is obvious.

Corollary 5.7. *Assume that system* (3.2) *is quasi-regular. Moreover, let* Γ_b *be a surjective linear mapping* (4.39) *satisfying* (4.40), *let* $B(\lambda) = \Gamma_b Y_0(\lambda)$, $\lambda \in \mathbb{C}$, *and let* $W(\lambda)$ *be the operator function given by* (4.42) *and* (4.4)–(4.7). *Then for each boundary parameter* τ *of the form* (5.3) *the corresponding characteristic matrix* $\Omega_\tau(\cdot)$ *admits the representations* (5.15) *and* (5.16) *for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Corollary 5.8. [20]. *Let system* (3.2) *be regular. Then the equalities* (1.5) *and* (5.1) *give a 1,1-correspondence between all holomorphic operator functions* $(C_a(\lambda), C_b(\lambda)) : \mathbb{H} \oplus \mathbb{H} \to \mathbb{H}$, $\lambda \in \mathbb{C}_+$, *satisfying* (1.6) *and all generalized resolvents* $R(\lambda)$ *of* $T_{\min}$.

The statement of this corollary directly follows from Theorem 5.1 and Corollary 5.7, if we only put $\Gamma_b y = y(b)$, $y \in \operatorname{dom} \mathcal{T}_{\max}$.

5.2. **Pseudospectral and spectral functions.** Recall that a non-decreasing operator function $\Sigma(\cdot) : \mathbb{R} \to [\mathbb{H}]$ is called a distribution function if it is left continuous and satisfies $\Sigma(0) = 0$. With each distribution function $\Sigma(\cdot)$ one associates a semi-Hilbert space $\mathcal{L}^2(\Sigma; \mathbb{H})$ of all Borel-measurable functions $f(\cdot) : \mathbb{R} \to \mathbb{H}$ satisfying

$$||f||^2_{\mathcal{L}^2(\Sigma;\mathbb{H})} := \int_{\mathbb{R}} (d\Sigma(s) f(s), f(s)) < \infty$$

and the Hilbert space $L^2(\Sigma; \mathbb{H})$ of all equivalence classes in $\mathcal{L}^2(\Sigma; \mathbb{H})$ with respect to the seminorm $|| \cdot ||_{\mathcal{L}^2(\Sigma;\mathbb{H})}$ (for more details see e.g. [9]). In the following we denote by π_Σ the quotient map from $\mathcal{L}^2(\Sigma; \mathbb{H})$ onto $L^2(\Sigma; \mathbb{H})$.

Let $\mathfrak{H}_b$ the set of all $\widetilde{f} \in L^2_\Delta(\mathcal{I})$ with the following property: there exists $\beta_{\widetilde{f}} \in \mathcal{I}$ such that for some (and hence for all) function $f \in \widetilde{f}$ the equality $\Delta(t) f(t) = 0$ holds a.e. on $(\beta_{\widetilde{f}}, b)$.

Definition 5.9. A distribution function $\Sigma(\cdot)$ is called a pseudospectral (resp. spectral) function of the system (3.2) if the operator

$$V_\Sigma \widetilde{f} = \pi_\Sigma \Big(\int_{\mathcal{I}} Y_0^*(t, \cdot) \Delta(t) f(t)\, dt \Big), \quad f(\cdot) \in \widetilde{f},$$

defined originally for all $\widetilde{f} \in \mathfrak{H}_b$ admits a continuation to a partial isometry $V_\Sigma \in [L^2_\Delta(\mathcal{I}), L^2(\Sigma; \mathbb{H})]$ with $\ker V_\Sigma = \operatorname{mul} T_{\min}$ (resp. to an isometry $V_\Sigma \in [L^2_\Delta(\mathcal{I}), L^2(\Sigma; \mathbb{H})]$.

The operator $V = V_\Sigma$ is called the Fourier transform corresponding to $\Sigma(\cdot)$.

In what follows we put $\mathfrak{H}_0 = L^2_\Delta(\mathcal{I}) \ominus \operatorname{mul} T_{\min}$ and denote by $V_0 = V_{0,\Sigma}$ the isometry from $\mathfrak{H}_0$ to $L^2(\Sigma; \mathbb{H})$ given by $V_{0,\Sigma} = V_\Sigma \upharpoonright \mathfrak{H}_0$.

In the following theorem we describe all pseudospectral functions of the system (3.2) in terms of the boundary parameter τ.

Theorem 5.10. *Assume that system* (3.2) *is absolutely definite. Moreover, let the assumptions of Theorem 5.6 be satisfied, let* P_1 *be the orthoprojection in* $\mathbb{H}$ *onto* $\mathcal{H}_1$ *(for*

$\mathcal{H}_1$ see (3.13)*) and let P_2 be orthoprojection in $\mathbb{H}$ onto $\mathcal{H}_2 := \mathbb{H} \ominus \mathcal{H}_1$. Then the equality* (5.15) *together with the Stieltjes inversion formula*

$$\Sigma(s) = \Sigma_\tau(s) = \lim_{\delta\to+0}\lim_{\varepsilon\to+0}\frac{1}{\pi}\int_{-\delta}^{s-\delta}\operatorname{Im}\Omega_\tau(\sigma+i\varepsilon)\,d\sigma \tag{5.18}$$

establishes a bijective correspondence between all boundary parameters $\tau = \{\tau_+, \tau_-\}$ of the form (5.2) *satisfying the conditions*

$$\lim_{y\to+\infty}\tfrac{1}{iy}P_1 w_1(iy)(C_0(iy)w_1(iy) + C_1(iy)w_3(iy))^{-1}C_1(iy) = 0, \tag{5.19}$$

$$\lim_{y\to+\infty}\tfrac{1}{iy}w_3(iy)(C_0(iy)w_1(iy) + C_1(iy)w_3(iy))^{-1}C_0(iy)\upharpoonright\mathcal{H}_1 = 0 \tag{5.20}$$

and all pseudospectral functions $\Sigma(\cdot) = \Sigma_\tau(\cdot)$ of the system (3.2)*. Moreover, the following statements are valid:*

(i) *if $\lim_{y\to+\infty}\frac{1}{iy}w_3(iy)w_1^{-1}(iy)\upharpoonright\mathcal{H}_1 = 0$, then the condition* (5.20) *can be omitted;*

(ii) *if $\lim_{y\to+\infty}\frac{1}{iy}w_3(iy)w_1^{-1}(iy)\upharpoonright\mathcal{H}_1 = 0$ and*

$$\lim_{y\to+\infty} y\left(\operatorname{Im}\left(-w_3(iy)w_1^{-1}(iy)\widetilde{h},\widetilde{h}\right)_{\mathbb{H}} + \tfrac{1}{2}||P_2\widetilde{h}||^2\right) = +\infty, \quad \widetilde{h}\in\mathbb{H},\ \ \widetilde{h}\neq 0, \tag{5.21}$$

then both the conditions (5.19) *and* (5.20) *can be omitted.*

Proof. It follows from [28, Theorem 5.4] that the equality (5.17) together with (5.18) establishes a bijective correspondence between all boundary parameters $\tau = \{\tau_+, \tau_-\}$ of the form (5.2) satisfying the conditions

$$\lim_{y\to+\infty}\tfrac{1}{iy}P_1(C_0(iy) - C_1(iy)M_+(iy))^{-1}C_1(iy) = 0, \tag{5.22}$$

$$\lim_{y\to+\infty}\tfrac{1}{iy}M_+(iy)(C_0(iy) - C_1(iy)M_+(iy))^{-1}C_0(iy)\upharpoonright\mathcal{H}_1 = 0 \tag{5.23}$$

and all pseudospectral functions $\Sigma(\cdot) = \Sigma_\tau(\cdot)$ of the system (3.2). As was shown in the proof of Theorem 5.6 the equality (5.17) admits the representation (5.15). Moreover, by (4.11) one has $M_+(\lambda) = -w_3(\lambda)w_1^{-1}(\lambda)$ and, consequently,

$$\begin{aligned}(C_0(\lambda) - C_1(\lambda)M_+(\lambda))^{-1} &= (C_0(\lambda) + C_1(\lambda)w_3(\lambda)w_1^{-1}(\lambda))^{-1}\\ &= w_1(\lambda)(C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda))^{-1}, \quad \lambda\in\mathbb{C}_+.\end{aligned}$$

Therefore the conditions (5.22) and (5.23) are equivalent to (5.19) and (5.20) respectively. This implies the main statement of the theorem.

Finally, statements (i) and (ii) of the theorem follows from assertions (i) and (ii) just before Theorem 5.4 in [28]. □

The following corollary is immediate from Theorem 5.10 and [28, Theorem 5.5].

Corollary 5.11. *Assume that system* (3.2) *is absolutely definite and quasi-regular and let the assumptions of Corollary 5.7 be satisfied. Then the equalities* (5.15) *and* (5.18) *establish a bijective correspondence between all boundary parameters τ of the form* (5.3) *satisfying the conditions*

$$\lim_{y\to\infty}\tfrac{1}{iy}w_1(iy)(C_0(iy)w_1(iy) + C_1(iy)w_3(iy))^{-1}C_1(iy) = 0, \tag{5.24}$$

$$\lim_{y\to\infty}\tfrac{1}{iy}w_3(iy)(C_0(iy)w_1(iy) + C_1(iy)w_3(iy))^{-1}C_0(iy) = 0 \tag{5.25}$$

and all pseudospectral functions $\Sigma(\cdot) = \Sigma_\tau(\cdot)$ *of the system* (3.2)*. Moreover,* $V_{0,\Sigma}(\in [\mathfrak{H}_0, L^2(\Sigma;\mathbb{H})])$ *is a unitary operator if and only if* τ *is a self-adjoint boundary parameter* (5.5) *(satisfying* (5.24) *and* (5.25)*).*

If in addition $\lim\limits_{y\to\infty} y\,\mathrm{Im}\big(-w_3(iy)w_1^{-1}(iy)\widetilde{h},\widetilde{h}\big) = +\infty,\ 0 \neq \widetilde{h} \in \mathbb{H}$, *then the conditions* (5.24) *and* (5.25) *can be omitted.*

The following corollary is implied by the results of [28].

Corollary 5.12. *The set of spectral functions of the system* (3.2) *is not empty if and only if* $\mathrm{mul}\, T_{\min} = \{0\}$*. Moreover, if this condition is satisfied, then Theorem 5.10 and Corollary 5.11 are valid for spectral functions (in place of pseudo-spectral functions). Moreover, in this case the second statement of Corollary 5.11 holds with* V_Σ *instead of* $V_{0,\Sigma}$*.*

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