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TOPOLOGICAL EQUIVALENCE TO A PROJECTION

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Abstract. We present a necessary and sufficient condition for a continuous function on a plane to be topologically equivalent to a projection onto one of the coordinates.

Let M be a connected surface, i.e., a 2-dimensional manifold. Two continuous functions $f, g : M \to \mathbb{R}$ are called *topologically equivalent*, if there exist two homeomorphisms $h : M \to M$ and $k : \mathbb{R} \to \mathbb{R}$ such that $k \circ f = g \circ h$.

A classification of continuous functions $f : M \to \mathbb{R}$ on surfaces, up to topological equivalence, was initiated in the works by M. Morse [8], [9], see also [5, 6, 7, 2, 3]. In recent years, an essential progress in the classification of such functions was made in [10, 12, 13, 1, 11].

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assuming that f "has no critical points" we present a necessary and sufficient condition for f to be topologically equivalent to a linear function. First we recall some definitions from W. Kaplan [4].

Definition 1 ([4])**.** *A curve* in $\mathbb{R}^2$ is a homeomorphic image of the open interval $(0, 1)$. Let $U \subset \mathbb{R}^2$ be an open subset. *A family of curves* in U is a partition of U whose elements are curves.

A family of curves $\Im$ in U is called *regular* at a point $p \in \mathbb{R}^2$, if there exist an open neighbourhood U_p of p and a homeomorphism $\varphi : (0, 1) \times (0, 1) \to U_p$ such that for every $y \in (0, 1)$ the image $\varphi((0, 1) \times y)$ is an intersection of U_p with some curve from the family $\Im$. Such a neighbourhood U_p is called *r-neighbourhood* of p.

Thus the curves of regular family are "locally parallel", however their global behaviour can be more complicated. In the present note we will consider continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ whose level-sets are "globally parallel".

One of the basic examples of such a function is a projection $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = y$. Its level sets are parallel lines $y = \text{const}$ and, in particular, they constitute a regular family of curves, see Figure 1b).

On the other hand, consider the function $f(x, y) = \arctan(x - \mathrm{tg}^2(y))$, see Figure 1a). It level sets are not connected, however, the partition into *connected components* of level sets of f is also a regular family of curves.

a) $f(x, y) = \arctan(x - \mathrm{tg}^2(y))$ b) $g(x, y) = y$

Figure 1. Level lines

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The following theorem shows that connectedness of level sets is a characteristic property of a projection.

Theorem 1. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a continuous function and* $\Im = \{f^{-1}(a) \mid a \in \mathbb{R}^2\}$ *be the partition of* $\mathbb{R}$ *by level sets of* f*. Suppose the following two conditions hold.*

1. *For each* $p \in f(\mathbb{R}^2)$ *belonging to the image of* f*, the corresponding level set* $f^{-1}(a)$ *is a curve. In particular, it is path connected.*
2. *The family of curves* $\Im$ *is regular.*

Then the image $f(\mathbb{R}^2)$ *is an open interval* (a, b)*,* $a, b \in \mathbb{R} \cup \pm\infty$*, and there is a homeomorphism* $\varphi : \mathbb{R} \times (a, b) \to \mathbb{R}^2$ *such that* $f \circ \varphi(x, y) = y$*. In other words,* f *is topologically equivalent to a projection.*

The proof is based on the results of [4].

Let $\Im$ be a regular family of curves in $\mathbb{R}^2$. Then by [4, Theorem 16], each curve C of $\Im$ is a proper embedding of $\mathbb{R}$, so it has an infinity as a sole limit point. It follows from Jordan's theorem (applied to the sphere $S^2 = \mathbb{R}^2 \cup \infty$) that each curve C of $\Im$ divides the plane into two distinct regions, having C as the common boundary. This property allows to define the following relation for curves on $\Im$.

Definition 2. *Three distinct curves* K*,* L*,* C *from* $\Im$ *are in the relation* $K|C|L$*, if* K *and* L *belong to distinct components of* $\mathbb{R}^2 \setminus C$*.*

For an open subset $U \subset \mathbb{R}^2$ let $\Im_U$ be the partition of U by connected components of intersections of U with curves from $\Im$. Then $\Im_U$ is a family of curves in U. Note also that an intersection of U with some curve from $\Im$ may have even countable many connected components.

Definition 3. *Let* $p, q \in \mathbb{R}^2$*.* An *arc* $[p, q]$, i.e., a homeomorphic image of $[0, 1]$, connecting these points, will be called a *cross-section* relative to $\Im$ if there exists an open set U in $\mathbb{R}^2$ containing $[p, q]$ such that each curve of $\Im_U$ meets $[p, q]$ in U at most once.

Evidently, for every $p \in \mathbb{R}^2$, there is an arbitrary small r-neighbourhood V of p and a cross-section $[q, s] \subset V$ relative $\Im$ passing through p.

Theorem 2. [4]*. Let* K, L *be two distinct curves from a regular family* $\Im$*. Suppose two points* $p \in L$ *and* $q \in K$ *can be connected by a cross-section* $[p, q]$*, and let* S *be the set of curves crossing* $[p, q]$ *except for* p *and* q*. Then* S *forms an open point set, and the condition* $K|C|L$ *is equivalent to the condition that* C *is contained in* S*.*

Moreover, there is a homeomorphism $\varphi : \mathbb{R} \times [0, 1] \to K \cup S \cup L$ *such that* $K = \varphi(\mathbb{R} \times 0)$*,* $L = \varphi(\mathbb{R} \times 1)$*, and* $\varphi(\mathbb{R} \times t)$ *is a curve belonging to* $\Im$ *for all* $t \in (0, 1)$*.*

Proof. First we need the following lemma.

Lemma 1. *Let* $[p, q]$ *be a cross-section of* $\Im$*. Then the restriction of* f *to* $[p, q]$ *is strictly monotone. In particular,* $[p, q]$ *intersects each curve in* $\Im$ *in at most one point.*

Proof. Suppose there exists a point $x \in [p, q]$ distinct from p and q and being a local extreme of $f|_{[p,q]}$. Let $c = f(x)$. As mentioned above the embedding $f^{-1}(c) \subset \mathbb{R}^2$ is proper, therefore we have the following:

(i) $f^{-1}(c)$ divides $\mathbb{R}^2$ into two connected components, say R_1 and R_2 and
(ii) there exists an r-neighbourhood U of x relatively to $\Im$ such that $U \cap f^{-1}(c)$ is a connected curve dividing U into two components, say U_1 and U_2, such that $U_1 \subset R_1$ and $U_2 \subset R_2$.

Without loss of generality, we can assume that $[p, q] \subset U$, so that $[p, q] \setminus \{x\}$ consists of two half-open arcs $[p, x) \subset U_1$ and $(x, q] \subset U_2$. It follows that x is an *isolated* local extreme of the restriction of $f|_{[p,q]}$, whence there exist $y \in [p, x) \subset R_1$ and $z \in (x, q] \subset R_2$

such that $f(y) = f(z) \neq f(c)$. Thus $y, z \in f^{-1}(f(y)) \subset \mathbb{R}^2 \setminus f^{-1}(c) = R_1 \cup R_1$. By (1) $f^{-1}(f(y))$ is connected, and so both y and z belong either to R_1 or to R_2. This gives a contradiction, whence x is not a local extreme of f. □

For $[c, d] \subset \mathbb{R}$ denote $D_{c,d} = f^{-1}[c, d]$. Then it follows from Lemma 1 and Theorem 2 that for each cross-section $[p, q]$ there exists a homeomorphism

$$\varphi : \mathbb{R} \times [f(p), f(q)] \longrightarrow f^{-1}[f(p), f(q)] = D_{f(p),f(q)}$$

such that $f \circ \varphi(x, y) = y$ for all $(x, y) \in \mathbb{R} \times [f(p), f(q)]$.

This also implies that the image $f(\mathbb{R}^2)$ is an open and path connected subset of $\mathbb{R}$, i.e., an open interval (a, b), where a and b can be infinite.

Hence we can find a countable strictly increasing sequence $\{c_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ such that $\lim\limits_{k\to-\infty} c_i = a$, $\lim\limits_{k\to+\infty} c_k = b$, and for each $k \in \mathbb{Z}$ a homeomorphism

$$\varphi_k : \mathbb{R} \times [c_k; c_{k+1}] \longrightarrow f^{-1}[c_k, c_{k+1}] = D_{c_k,c_{k+1}}$$

satisfying $f \circ \varphi_k(x, y) = y$.

Define a homeomorphism $\varphi : \mathbb{R} \times (a, b) \to \mathbb{R}^2$ as follows. Set

$$\varphi(x, y) = \varphi_0(x, y), \quad (x, y) \in \mathbb{R} \times [c_0, c_1].$$

Now if φ is defined on $\mathbb{R} \times [c_{k-1}, c_k]$ for some $k \geq 1$, then extend it to $\mathbb{R} \times [c_k, c_{k+1}]$ by

$$\varphi(x, y) = \varphi_k(\varphi_k^{-1} \circ \varphi(x, c_k), y), \quad (x, y) \in \mathbb{R} \times [c_k, c_{k+1}].$$

Similarly, one can extend φ to $\mathbb{R} \times (a, c_0]$. It easily follows that φ is a homeomorphism satisfying $f \circ \varphi(x, y) = y$, $(x, y) \in \mathbb{R} \times (a, b)$. □

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