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ON THE STRUCTURE OF SOLUTIONS OF OPERATOR-DIFFERENTIAL EQUATIONS ON THE WHOLE REAL AXIS

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Dedicated to Yu. M. Berezansky on the occasion of his 90th birthday

Abstract. We consider differential equations of the form $\left(\frac{d^2}{dt^2} - B\right)^m y(t) = f(t)$, $m \in \mathbb{N}$, $t \in (-\infty, \infty)$, where B is a positive operator in a Banach space $\mathfrak{B}$, $f(t)$ is a bounded continuous vector-valued function on $(-\infty, \infty)$ with values in $\mathfrak{B}$, and describe all their solutions. In the case, where $f(t) \equiv 0$, we prove that every solution of such an equation can be extended to an entire $\mathfrak{B}$-valued function for which the Phragmen-Lindelöf principle is fulfilled. It is also shown that there always exists a unique bounded on $\mathbb{R}^1$ solution, and if $f(t)$ is periodic or almost periodic, then this solution is the same as $f(t)$.

1. Let $\mathfrak{B}$ be a Banach space with norm $\|\cdot\|$ over the field $\mathbb{C}$ of complex numbers, and let $E(\mathfrak{B})$ $(L(\mathfrak{B}))$ be the set of all densely defined closed (bounded) linear operators on $\mathbb{B}$. In what follows $\left\{e^{tA}\right\}_{t\geq 0}$ denotes the C_0-semigroup of bounded linear operators on $\mathfrak{B}$ with infinitesimal generator A (for the theory of semigroups on a Banach space we refer, for instance, to [1-4]). Recall only that a family $\{U(t)\}_{t\geq 0}$ of operators $U(t) \in L(\mathfrak{B})$ forms a semigroup on $\mathfrak{B}$ if:

1) $U(0) = I,$ the identity operator in $\mathfrak{B}$;
2) $\forall t, s > 0 : U(t+s) = U(t)U(s)$;
3) $\forall x \in \mathfrak{B} : \lim\limits_{t\to 0} \|U(t)x - x\| = 0$.

The infinitesimal generator A of $\{U(t)\}_{t\geq 0}$, or briefly the generator, is defined as

$$\mathcal{D}(A) = \left\{x \in \mathfrak{B} : \lim_{t\to 0} \frac{1}{t}(U(t)x - x) \text{ exists } \right\},$$
$$Ax = \lim_{t\to 0} \frac{1}{t}(U(t)x - x), \quad x \in \mathcal{D}(A).$$

This operator is closed, its domain $\mathcal{D}(A)$ is dense in $\mathfrak{B}$ and $U(t)$-invariant, i.e., $U(t)x \in \mathcal{D}(A)$ for all $x \in \mathcal{D}(A)$, $t \geq 0$, and $AU(t)x = U(t)Ax$. Moreover,

$$\frac{d}{dt}U(t)x = AU(t)x, \quad x \in \mathcal{D}(A).$$

Finally, we assume $\ker e^{tA} = \{0\}$ for any $t > 0$. Without loss of generality it may be also supposed $\left\{e^{tA}\right\}_{t\geq 0}$ to be a contraction semigroup.

A C_0-semigroup $\{U(t)\}_{t\geq 0}$ is called analytic with angle $\theta \in \left(0, \frac{\pi}{2}\right]$ if the operator-valued function $U(\cdot)$ is defined in the sector $S_\theta = \{z : |\arg z| < \theta\}$ and possesses the properties:

1) $\forall z_1, z_2 \in S_\theta : U(z_1 + z_2) = U(z_1)U(z_2)$;

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2) $\forall x \in \mathfrak{B} : U(z)x$ is analytic in S_θ;
3) $\forall x \in \mathfrak{B} : \|U(z)x - x\| \to 0$ as $z \to 0$ in any closed subsector of S_θ.
If in addition the family $U(z)$ is bounded on each sector S_ψ with $\psi < \theta$, then $U(t)$ is called a bounded analytic semigroup with angle θ.

Let $A \in E(\mathfrak{B})$. Denote by $\mathfrak{G}_{(1)}(A)$ the space of entire vectors of the operator A:

$$\mathfrak{G}_{(1)}(A) = \operatorname*{proj\,lim}_{\alpha \to 0} \mathfrak{G}_1^\alpha(A) = \bigcap_{\alpha > 0} \mathfrak{G}_1^\alpha(A),$$

where

$$\mathfrak{G}_1^\alpha(A) = \left\{ x \in \bigcap_{n \in \mathbb{N}_0} \mathcal{D}(A^n) \middle| \exists c = c(x) > 0, \forall k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} : \|A^k x\| \le c\alpha^k k^k \right\}$$

is a Banach space with respect to the norm

$$\|x\|_{\mathfrak{G}_1^\alpha(A)} = \sup_{k \in \mathbb{N}_0} \frac{\|A^k x\|}{\alpha^k k^k}.$$

The convergence in $\mathfrak{G}_{(1)}(A)$ means the convergence in every $\mathfrak{G}_1^\alpha(A)$, $\alpha > 0$. Note that $\mathfrak{G}_{(1)}(A)$ may be obtained if we confine ourselves only to $\alpha = \frac{1}{n}$, $n \in \mathbb{N}$. So, $\mathfrak{G}_{(1)}(A)$ is countably normed (see [5]).

Proposition 1. (See [6]). *Let $A \in E(\mathfrak{B})$. Then the series $\sum_{k=0}^\infty \frac{z^k A^k x}{k!}$ converges in the space $\mathfrak{G}_{(1)}(A)$ for any $x \in \mathfrak{G}_{(1)}(A)$, any $z \in \mathbb{C}$, and the operator-valued function*

$$\exp(zA) = \sum_{k=0}^\infty \frac{z^k A^k}{k!}$$

is entire in $\mathfrak{G}_{(1)}(A)$. Moreover, the family $\{\exp(zA)\}_{z \in \mathbb{C}}$ forms a one-parameter group on $\mathfrak{G}_{(1)}(A)$.

If A is the generator of a bounded analytic semigroup $\left\{e^{tA}\right\}_{t \ge 0}$, then $\mathfrak{G}_{(1)}(A)$ is dense in $\mathfrak{B}$,

$$\mathfrak{G}_{(1)}(A) = \bigcap_{t \ge 0} \mathcal{R}(e^{tA})$$

($\mathcal{R}(\cdot)$ is the range of an operator), and

$$\forall x \in \mathfrak{G}_{(1)}(A) : \exp(tA)x = \begin{cases} e^{tA}x, & \text{when} \quad t \ge 0, \\ \left(e^{-tA}\right)^{-1} x, & \text{when} \quad t < 0. \end{cases}$$

Consider the equation

$$\frac{dy(t)}{dt} + Ay(t) = 0, \quad t \in (0, \infty), \tag{1}$$

where A is the generator of a bounded analytic semigroup on $\mathfrak{B}$. Under a classic solution, or briefly solution, of this equation on $(0, \infty)$ we mean a continuously differentiable vector-valued function $y(t) : (0, \infty) \mapsto \mathcal{D}(A)$ satisfying (1). The following assertion (see [6]) is valid.

Proposition 2. *A $\mathfrak{B}$-valued function $y(t)$ is a solution of equation* (1) *if and only if it can be represented in the form*

$$y(t) = \exp(-tA)y_0, \quad y_0 \in \mathfrak{G}_{(1)}(A).$$

2. Pass now to the equation

$$\left(\frac{d^2}{dt^2} - B\right)^m y(t) = f(t), \quad t \in \mathbb{R}^1, \tag{2}$$

where B is a positive operator in $\mathfrak{B}$, $m \in \mathbb{N}$, $f(t) : \mathbb{R}^1 \mapsto \mathfrak{B}$ is a bounded continuous vector-valued function. Recall that an operator $B \in E(\mathfrak{B})$ is called positive if $(-\infty, 0) \in \rho(B)$ ($\rho(\cdot)$ is the resolvent set of an operator), and there exists a constant $M > 0$ such that

$$\forall \lambda > 0 : \left\|(B + \lambda I)^{-1}\right\| \leq \frac{M}{1+\lambda}.$$

In this case, according to [7, 8], the fractional powers $B^{\alpha}, 0 < \alpha < 1$, of the operator B are determined, and the operator $A = -B^{\frac{1}{2}}$ generates a bounded analytic C_0-semigroup $\{e^{tA}\}_{t \geq 0}$ on $\mathfrak{B}$ of negative type

$$\omega = \omega(A) = \lim_{t \to \infty} \frac{\ln \|e^{tA}\|}{t} = -\sqrt{s(B)},$$

where

$$0 < s(B) = \sup_{\lambda \in \sigma(B)} \mathrm{Re}\lambda,$$

$\sigma(\cdot)$ is the spectrum of an operator.

By a solution (classic) of equation (2) on $\mathbb{R}^1$, we mean a $2m$ times continuously differentiable vector-valued function $y(t) : \mathbb{R}^1 \mapsto \mathfrak{B}$ such that $y^{(2k)}(t) \in \mathcal{D}(B^{m-k})$ ($k = 0, 1, \dots, m$), the vector-valued function $B^{m-k} y^{(2k)}(t)$ is continuous on $\mathbb{R}^1$, and $y(t)$ satisfies (2).

Consider first the homogeneous equation

$$\left(\frac{d^2}{dt^2} - B\right)^m y(t) = 0, \quad t \in \mathbb{R}^1. \tag{3}$$

Theorem 1. *A $\mathfrak{B}$-valued $y(t)$ is a solution of equation* (3) *on $\mathbb{R}^1$ if and only if it can be represented in the form*

$$y(t) = \sum_{k=0}^{m-1} t^k (\exp(tA) f_k + \exp(-tA) g_k), \tag{4}$$

where $A = -B^{\frac{1}{2}}$, $f_k, g_k \in \mathfrak{G}_{(1)}(A)$ ($k = 0, 1, \dots, m-1$). The vectors f_k and g_k are uniquely determined by $y(t)$.

Proof. It is not difficult to verify that a vector-valued function $y(t)$ of the form (4) is a solution of equation (3). To prove the converse we use the method of mathematical induction.

Suppose $y(t)$ is a solution of the equation

$$\left(\frac{d^2}{dt^2} - B\right) y(t) = \left(\frac{d^2}{dt^2} - A^2\right) y(t) = \left(\frac{d}{dt} + A\right)\left(\frac{d}{dt} - A\right) y(t) = 0, \quad t \in \mathbb{R}^1,$$

and put $z(t) = \left(\frac{d}{dt} - A\right) y(t)$. The vector-valued function $z(t)$ is a solution of equation (1) on the semiaxis $(0, \infty)$. By Proposition 2,

$$z(t) = \exp(-tA) h_1, \quad h_1 \in \mathfrak{G}_{(1)}(A),$$

that is,

$$\left(\frac{d}{dt} - A\right) y(t) = \exp(-tA) h_1, \quad t > 0.$$

Denote

$$z_0(t) = y(t) - \frac{\sinh(tA)}{A} h_1. \tag{5}$$

Taking into account that $\dfrac{\sinh(tA)}{A} = \sum_{k=0}^{\infty} \dfrac{t^{2k+1}A^{2k+1}}{(2k+1)!}$ is an entire operator-valued function in the space $\mathfrak{G}_{(1)}(A)$, one can directly check that

$$\left(\frac{d}{dt} - A\right) z_0(t) = 0, \quad t \in \mathbb{R}^1.$$

Since A is a generator of a C_0-semigroup on $\mathfrak{B}$, we have (see, for example, [8])

$$\forall t \geq 0 : z_0(t) = e^{tA}h_2, \quad h_2 \in \mathcal{D}(A). \tag{6}$$

As far as the vector-valued function $z_1(t) = z_0(-t)$ is a solution of equation (1), we obtain from Proposition 2 that

$$z_1(t) = \exp(-tA)h_3, \quad h_3 \in \mathfrak{G}_{(1)}(A),$$

whence $h_2 = z_0(0) = z_1(0) = h_3 \in \mathfrak{G}_{(1)}(A)$. It follows from (5) and (6) that

$$y(t) = z_0(t) + \frac{\sinh(tA)}{A}h_1 = \exp(tA)h_2 + \frac{\sinh(tA)}{A}h_1, \quad h_1, h_2 \in \mathfrak{G}_{(1)}(A),$$

which is equivalent to

$$y(t) = \exp(tA)f_0 + \exp(-tA)g_0,$$

where

$$f_0 = h_2 + \frac{A^{-1}h_1}{2}, \quad g_0 = \frac{-A^{-1}h_1}{2}.$$

Assume now that representation (4) is valid for a solution $y(t)$ of equation (3) with $m = k-1$ and show that this representation holds true for $m = k$.

Let $y(t)$ be a solution of the equation

$$\left(\frac{d^2}{dt^2} - B\right)^k y(t) = 0, \quad t \in \mathbb{R}^1,$$

with some $k > 1$. Then the vector-valued function

$$z(t) = \left(\frac{d^2}{dt^2} - B\right)^{k-1} y(t)$$

satisfies the equation

$$\left(\frac{d^2}{dt^2} - B\right) z(t) = 0, \quad t \in \mathbb{R}^1.$$

So, there exist $\widetilde{f}_0,\ \widetilde{g}_0 \in \mathfrak{G}_{(1)}(A)$ such that

$$z(t) = \exp(tA)\widetilde{f}_0 + \exp(-tA)\widetilde{g}_0.$$

Then the vector-valued function

$$\widetilde{y}(t) = y(t) - t^{k-1}\exp(tA)f_{k-1} - t^{k-1}\exp(-tA)g_{k-1}, \tag{7}$$

where

$$f_{k-1} = \frac{A^{1-k}}{2^{k-1}(k-1)!}\widetilde{f}_0, \quad g_{k-1} = \frac{(-1)^{k-1}A^{1-k}}{2^{k-1}(k-1)!}\widetilde{g}_0 \in \mathfrak{G}_{(1)}(A),$$

is a solution of the equation

$$\left(\frac{d^2}{dt^2} - B\right)^{k-1} \widetilde{y}(t) = 0, \quad t \in \mathbb{R}^1,$$

and, therefore, $\widetilde{y}(t)$ can be represented in the form (4) with $m = k-1$, whence, in view of (7), we arrive at the representation (4) with $m = k$.

We prove now the uniqueness of representation (4), i.e., that the identity $y(t) \equiv 0$ implies the equalities $f_k = g_k = 0$, $k = 0, 1, \dots, m-1$. Starting from (4), by the direct computation we get

$$\left(\frac{d}{dt} + A\right)^m \left(\frac{d}{dt} - A\right)^{m-1} y(t) = \left(\frac{d}{dt} + A\right)^m (m-1)! \exp(tA) f_{m-1} \tag{8}$$
$$= 2^m (m-1)! A^m \exp(tA) f_{m-1}$$

and

$$\left(\frac{d}{dt} - A\right)^m \left(\frac{d}{dt} + A\right)^{m-1} y(t) = \left(\frac{d}{dt} - A\right)^m (m-1)! \exp(-tA) g_{m-1} \tag{9}$$
$$= (-1)^m 2^m (m-1)! A^m \exp(-tA) g_{m-1}.$$

Setting in (8) and (9) $t = 0$ and taking into account that $y(t) \equiv 0$, we obtain $f_{m-1} = g_{m-1} = 0$. Thus,

$$y(t) = \sum_{k=0}^{m-2} t^k (\exp(tA) f_k + \exp(-tA) g_k).$$

Repeating the procedure m times, we conclude that $f_k = g_k = 0$ for all $k = 0, 1, \dots, m-1$, which is what had to be proved. □

Corollary 1. *Every solution of equation* (3) *on* $(-\infty, \infty)$ *admits an extension to an entire function with values in* $\mathfrak{G}_{(1)}(A)$.

Since the operator A generates a bounded analytic semigroup, it follows from Proposition 1 and Theorem 1 that the space of all solutions of equation (3) is infinite-dimensional. Moreover, the following analog of the Phragmen-Lindelöf principle [9] holds for them.

Theorem 2. *Let* $y(t)$ *be a solution of equation* (3)*. If*

$$\exists \gamma \in (0, -\omega),\ \exists c_\gamma > 0 : \|y(t)\| \le c_\gamma e^{\gamma t}, \quad t \in \mathbb{R}^1, \tag{10}$$

where $\omega = \omega(A)$ *is the type of the semigroup* $\left\{e^{tA}\right\}_{t \ge 0}$*, then* $y(t) \equiv 0$.

Proof. Write representation (4) as

$$y(t) = y_1(t) + y_2(t),$$

where

$$y_1(t) = \sum_{i=0}^{m-1} t^i \exp(tA) f_i, \quad y_2(t) = \sum_{i=0}^{m-1} t^i \exp(-tA) g_i. \tag{11}$$

Since the semigroup $\left\{e^{tA}\right\}_{t \ge 0}$ is bounded analytic, by Proposition 1 we have for $t > 0$ that $\exp(tA) f_i = e^{tA} f_i$, $i = 0, 1, \dots, m-1$. As it follows from the definition of the type of a semigroup,

$$\forall \delta \in \left(0, -\frac{\omega}{2}\right),\ \forall t \ge 0,\ \exists c_\delta > 0 : \left\|e^{tA}\right\| \le c_\delta e^{(\omega+\delta)t},$$

whence

$$\forall t \ge 0 : \|y_1(t)\| \le \sum_{i=0}^{m-1} t^i \|\exp(tA) f_i\| \le \sum_{i=0}^{m-1} c_{i\delta} e^{(\omega+2\delta)t} \le \widetilde{c_\delta} e^{(\omega+2\delta)t}, \tag{12}$$

where $2\delta \in (0, -\omega)$ and the constant $\widetilde{c_\delta} = \sum_{i=0}^{m-1} c_{i\delta}$ depends only on f_i.

Let now $g \in \mathfrak{G}_{(1)}(A)$. Then

$$\forall \delta \in \left(0, -\frac{\omega}{2}\right), \ \forall t \geq 0 : \|g\| = \left\|e^{tA} \exp(-tA)g\right\| \leq \left\|e^{tA}\right\| \|\exp(-tA)g\| \\ \leq c_\delta e^{(\omega+\delta)t} \|\exp(-tA)g\|.$$

This implies

$$\|\exp(-tA)g\| \geq c'_\delta e^{-(\omega+\delta)t} \|g\| \quad \text{as} \quad t \geq 0,$$

and, therefore,

$$\forall t \geq 0 : \|y_2(t)\| = \|\exp(-tA)h(t)\| \geq c'_\delta e^{-(\omega+\delta)t} \|h(t)\|,$$

where $h(t) = \sum_{i=0}^{m-1} t^i g_i, \ c'_\delta = c_\delta^{-1}$ does not depend on t.

Suppose $y_2(t) \not\equiv 0$. It follows from this that in representation (11) for $y_2(t)$ some $g_i \neq 0$. Without loss of generality, we may assume $g_{m-1} \neq 0$. Then

$$\forall t > 0 : \|y_2(t)\| \geq c'_\delta e^{-(\omega+\delta)t} \Big(t^{m-1} \|g_{m-1}\| - \Big\| \sum_{i=0}^{m-2} t^i g_i \Big\|\Big) \\ = c'_\delta e^{-(\omega+\delta)t} t^{m-1} \Big(\|g_{m-1}\| - \Big\| \sum_{i=0}^{m-2} t^{k-m+1} g_i \Big\|\Big).$$

Because of

$$\Big\| \sum_{i=0}^{m-2} t^{k-m+1} g_i \Big\| \to 0 \quad \text{as} \quad t \to \infty$$

and $t^{m-1} > e^{-\delta t}$ for sufficiently large $t > 0$, we have

$$\forall t > 0, \ \forall \delta \in \left(0, -\frac{\omega}{2}\right) : \|y_2(t)\| \geq c''_\delta e^{-(\omega+2\delta)t}, \tag{13}$$

where c''_δ does not depend on t. Using inequalities (10) and (12), we obtain

$$\forall t > 0 : \|y_2(t)\| = \|y(t) - y_1(t)\| \leq \|y(t)\| + \|y_1(t)\| \leq c_\gamma e^{\gamma t} + \widetilde{c}_\delta e^{(\omega+2\delta)t} \leq c e^{\gamma t}, \tag{14}$$

where $c = c_\gamma + \widetilde{c}_\delta$. The inequalities (13), (14) imply

$$\forall t > 0 : c_\delta e^{-(\omega+2\delta)t} \leq \|y_2(t)\| \leq c_\gamma e^{\gamma t}.$$

Put

$$\varphi(t) = \frac{\|y_2(t)\|}{c_\delta e^{-(\omega+2\delta)t}}.$$

Then, for sufficiently large $t > 0$

$$1 \leq \varphi(t) \leq \widetilde{c} e^{(\gamma+\omega+2\delta)t}, \quad \widetilde{c} = \frac{c_\gamma}{c_\delta}.$$

Setting $\delta = -\frac{\gamma+\omega}{4}$, we shall have for large $t > 0$

$$1 \leq \varphi(t) \leq \widetilde{c} e^{\frac{\gamma+\omega}{2} t}.$$

Approaching the limit as $t \to \infty$ and taking into account that $\frac{\gamma+\omega}{2} < 0$, we infer $1 \leq \varphi(t) \leq 0$, provided that $y_2(t) \neq 0$ for $t \geq 0$, which is impossible. So, $y_2(t) \equiv 0$ on the semiaxis $[0, \infty)$. Therefore $g_i = 0, \ i = 0, 1, \ldots, m-1$.

If we assume $y_1(t) \not\equiv 0$, we shall draw a conclusion that $y_1(t) \neq 0$ as $t \leq 0$. Substituting in (4) $-t$ for t, we obtain $y_1(t) \equiv 0$ on the semiaxis $(-\infty, 0]$, whence, by Theorem 1, $f_i = 0, \ i = 0, 1, \ldots, m-1$. This has a consequence that $y(t) \equiv 0$. □

Corollary 2. (Analog of the Liouville theorem.) *Let $y(t)$ be a solution of the homogeneous equation* (3) *on $\mathbb{R}^1$. Then*

$$\sup_{t\in\mathbb{R}^1}|y(t)|<\infty \Longrightarrow y(t)\equiv 0, \quad t\in\mathbb{R}^1.$$

3. Consider now a nonhomogeneous equation (2). Denote by $C_b(\mathbb{R}^1,\mathfrak{B})$ the set of all bounded continuous on $\mathbb{R}^1$ vector-valued functions with values in $\mathfrak{B}$. In what follows we suppose $f(t)\in C_b(\mathbb{R}^1,\mathfrak{B})$. Under a generalized solution of equation (2) on $\mathbb{R}^1$ we mean a continuous vector-valued function $y(t):\mathbb{R}^1\mapsto\mathfrak{B}$ for which the integral identity

$$\int_{\mathbb{R}^1}\Big\langle\Big(\frac{d^2}{dt^2}-B^*\Big)^m\varphi(t),y(t)\Big\rangle\,dt=\int_{\mathbb{R}^1}\langle\varphi(t),f(t)\rangle\,dt$$

holds true, where $\varphi(t)$ is an arbitrary compactly supported infinitely differentiable vector-valued function with values in $\mathcal{D}(B^{*m}$ such that $B^{*m}\varphi(t)$ is continuous on $\mathbb{R}^1$, $\langle\cdot,f\rangle$ denotes the action of a functional f onto a corresponding element. It is obvious that a classic solution of (2) is its generalized one.

Theorem 3. *Let $A^mf(t)\in C_b(\mathbb{R}^1,\mathfrak{B})$, and*

$$y_m(t)=\frac{A^{-m}}{2^m}\int_{\mathbb{R}^m}e^{A(|t-s_1|+|s_2-s_1|+\dots+|s_m-s_{m-1}|)}f(s_m)\,ds_1\dots ds_m. \tag{15}$$

Then $y_m^{(i)}(t)\in C_b(\mathbb{R}^1,\mathcal{D}(A^{2m-i})$, i.e. $y_m^{(i)}(t)$ is a bounded continuous vector-valued function with values in $\mathcal{D}(A^{2m-i})$, $i=0,1,\dots,2m$, and $y_m(t)$ is a solution of equation (2).

Proof. To prove the assertion, we turn again to the method of mathematical induction.

Put $m=1$. Since

$$\forall t>0:\|e^{At}\|<ce^{-\gamma t}, \quad 0<\gamma<-\omega(A),$$

and $Af(t)\in C_b(\mathbb{R}^1,\mathfrak{B})$, it is not difficult to check that

$$\begin{cases} y_1(t)=\dfrac{A^{-1}}{2}\displaystyle\int_{\mathbb{R}^1}e^{A|t-s_1|}f(s_1)\,ds_1=\dfrac{A^{-2}}{2}\displaystyle\int_{\mathbb{R}^1}e^{A|t-s_1|}Af(s_1)\,ds_1\in C_b(\mathbb{R}^1,\mathcal{D}(A^2)),\\ y_1'(t)=\dfrac{A^{-1}}{2}\Big(\displaystyle\int_{-\infty}^{t}e^{A(t-s_1)}Af(s_1)\,ds_1+\displaystyle\int_t^{\infty}e^{A(s_1-t)}Af(s_1)\,ds_1\Big)\in C_b(\mathbb{R}^1,\mathcal{D}(A)),\\ y_1''(t)=f(t)+\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^1}e^{A|t-s_1|}Af(s_1)\,ds_1\in C_b(\mathbb{R}^1,\mathfrak{B}). \end{cases} \tag{16}$$

It follows from this that $y_1(t)$ is a classic solution of equation (2) on $\mathbb{R}^1$.

Suppose now that for $m=k$, under the condition $A^kf(t)\in C_b(\mathbb{R}^1,\mathfrak{B})$, the conclusion of Theorem 3 is valid. Then for $m=k+1$ we have

$$\begin{aligned} y_{k+1}(t)&=\frac{A^{-(k+1)}}{2^{k+1}}\int_{\mathbb{R}^{k+1}}e^{A(|t-s_1|+|s_2-s_1|+\dots+|s_{k+1}-s_k|)}f(s_{k+1})\,ds_1\dots ds_{k+1}\\ &=\frac{A^{-2}}{2}\int_{\mathbb{R}^1}e^{A|t-s|}z(s)\,ds=\frac{A^{-2}}{2}\int_{\mathbb{R}^1}e^{A|s|}z(t-s)\,ds, \end{aligned} \tag{17}$$

where

$$z(s)=\frac{A^{-k}}{2^k}\int_{\mathbb{R}^k}e^{A(|s-s_2|+\dots+|s_{k+1}-s_k|)}Af(s_{k+1})\,ds_2\dots ds_{k+1}.$$

As $Af(s_{k+1})\in C_b(\mathbb{R}^1,\mathcal{D}(A^k))$, by the assumption specified above, $z^{(i)}(t)\in C_b(\mathbb{R}^1,\mathcal{D}(A^{2k-i}))$, $i=0,1,\dots,2k$, and satisfies (2) with $m=k$ and $Af(t)$ instead of $f(t)$. It follows from (17) that

$$y_{k+1}^{(2k)}(t)=\frac{A^{-2}}{2}\int_{\mathbb{R}^1}e^{A|t-s|}z^{(2k)}(s)\,ds.$$

Therefore, because of (16), we get

$$y_{k+1}^{(2(k+1))}(t) = z^{(2k)}(t) + \frac{1}{2}\int_{\mathbb{R}^1} e^{A|t-s|} z^{(2k)}(s)\,ds \in C_b(\mathbb{R}^1, \mathfrak{B})$$

and

$$\begin{aligned}\Big(\frac{d^2}{dt^2} - A^2\Big)^{k+1} y_{k+1}(t) &= \Big(\frac{d^2}{dt^2} - A^2\Big)\frac{A^{-2}}{2}\int_{\mathbb{R}^1}\Big(\frac{d^2}{dt^2} - A^2\Big)^k e^{A|t-s|} z(s)\,ds\\ &= \Big(\frac{d^2}{dt^2} - A^2\Big)\frac{A^{-2}}{2}\int_{\mathbb{R}^1} e^{A|t-s|}\Big(\frac{d^2}{dt^2} - A^2\Big)^k z(s)\,ds\\ &= \Big(\frac{d^2}{dt^2} - A^2\Big)\frac{A^{-2}}{2}\int_{\mathbb{R}^1} e^{A|t-s|} A f(s)\,ds = \Big(\frac{d^2}{dt^2} - A^2\Big)\frac{A^{-1}}{2}\int_{\mathbb{R}^1} e^{A|t-s|} f(s)\,ds = f(t).\end{aligned}$$

Thus, the assertion of the theorem is true for $y_{k+1}(t)$. □

Corollary 3. *If $f(t) \in C_b(\mathbb{R}^1, \mathfrak{B})$, then $y_m^{(k)}(t) \in C_b(\mathbb{R}^1, \mathcal{D}(A^{2m-k})$, $k = 0, 1, \ldots, m$, and $y_m(t)$ is a generalized solution of equation* (2).

Proof. The fact that $y_m^{(k)}(t) \in C_b(\mathbb{R}^1, \mathcal{D}(A^{2m-k})$ as $k = 1, 2, \ldots, m$, is proved on the basis of (15) by means of the method of mathematical induction in a way like to that used in Theorem 3. To verify that $y_m(t)$ is a generalized solution of (2), consider the sequence $f_n(t) = e^{\frac{1}{n}A} f(t)$. By virtue of analyticity of the semigroup $\left\{e^{At}\right\}_{t\geq 0}$, $f_n(t) \in C_b(\mathbb{R}^1, \mathcal{D}(A^n))$ for any $n \in \mathbb{N}$, and $f_n(t)$ converges uniformly to $f(t)$. So

$$y_{m,n}(t) = \frac{A^{-m}}{2^m}\int_{\mathbb{R}^1} e^{-A(|t-s_1|+\cdots+|s_m-s_{m-1}|)} f_n(s_m)\,ds_1\ldots ds_m$$

is a classic solution of equation (2) with $f(t) = f_n(t)$, and the sequence $y_{m,n}(t)$ converges uniformly on $\mathbb{R}^1$ to $y_m(t)$ as $n \to \infty$. Passing to the limit in the identity

$$\int_{\mathbb{R}^1}\Big\langle\Big(\frac{d^2}{dt^2} - B^*\Big)^m \varphi(t), y_{m,n}(t)\Big\rangle\,dt = \int_{\mathbb{R}^1}\varphi(t) f_n(t)\,dt,$$

we conclude that $y_m(t)$ is a generalized solution of (2). □

Recall that a continuous vector-valued function $f(t) : \mathbb{R}^1 \mapsto \mathfrak{B}$ is called almost periodic (by Bohr) if for any $\varepsilon > 0$ there exists a constant $L_\varepsilon > 0$ such that each interval from $\mathbb{R}^1$ of length less than ε contains a point $\tau = \tau(\varepsilon)$ having the property

$$\forall t \in \mathbb{R}^1 : \|f(t) - f(t+\tau)\| < \varepsilon.$$

As a consequence of Theorem 3, Corollaries 2,3, and the fact that a generalized solution of equation (3) is classic one we obtain the following theorem.

Theorem 4. *Let $f(t) \in C_b(\mathbb{R}^1, \mathcal{D}(A^m))$ $(f(t) \in C_b(\mathbb{R}^1, \mathfrak{B})$. Then there exists only one bounded classic (generalized) solution $y(t)$, $t \in \mathbb{R}^1$ of equation* (2)*, and it can be represented in the form* (15)*. If $f(t)$ is periodic or almost periodic, then the solution is the same as $f(t)$.*

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