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SOME APPLICATIONS OF ALMOST ANALYTIC EXTENSIONS TO OPERATOR BOUNDS IN TRACE IDEALS

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Dedicated with deep admiration to Yurij Makarovich Berezansky on the occasion of his 90th birthday

Abstract. Using the Davies–Helffer–Sjöstrand functional calculus based on almost analytic extensions, we address the following problem: Given a self-adjoint operator S in $\mathcal{H}$, and functions f in an appropriate class, for instance, $f \in C_0^\infty(\mathbb{R})$, how to control the norm $\|f(S)\|_{\mathcal{B}(\mathcal{H})}$ in terms of the norm of the resolvent of S, $\|(S - z_0 I_{\mathcal{H}})^{-1}\|_{\mathcal{B}(\mathcal{H})}$, for some $z_0 \in \mathbb{C}\backslash\mathbb{R}$. We are particularly interested in the case where $\mathcal{B}(\mathcal{H})$ is replaced by a trace ideal, $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty)$.

1. Introduction

Yurij M. Berezansky's contributions to analysis in general, and areas such as functional analysis, operator theory, spectral and inverse spectral theory, harmonic analysis, analysis in spaces of functions of an infinite number of variables, stochastic calculus, mathematical physics, quantum field theory, integration of nonlinear evolution equations, in particular, are legendary and of a lasting nature. The list of fields his ground breaking work changed in dramatic fashion can easily be continued in many directions as is demonstrated by the extraordinary breadth revealed in his highly influential monographs [2]–[8]. Since operator theoretic methods frequently play a role in his research interests, we hope our modest contribution to operator bounds in trace ideals will create some joy for him.

This paper has its origins in the following question: Given a self-adjoint operator S in $\mathcal{H}$, and functions f in an appropriate class, for instance, $f \in C_0^\infty(\mathbb{R})$, how to control the norm $\|f(S)\|_{\mathcal{B}(\mathcal{H})}$ in terms of the norm of the resolvent of S, $\|(S - z_0 I_{\mathcal{H}})^{-1}\|_{\mathcal{B}(\mathcal{H})}$, for some $z_0 \in \mathbb{C}\backslash\mathbb{R}$? In particular, the question is just as natural with $\mathcal{B}(\mathcal{H})$ replaced by a trace ideal, $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty)$. More precisely, since we are particularly interested in questions related to convergence of operator sequences, we will study pairs of self-adjoint operators S_j, $j = 1, 2$, in $\mathcal{H}$ and attempt to control $\|f(S_2) - f(S_1)\|_{\mathcal{B}(\mathcal{H})}$ and $\|f(S_2) - f(S_1)\|_{\mathcal{B}_p(\mathcal{H})}$ in terms of $\|(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}\|_{\mathcal{B}(\mathcal{H})}$ and $\|(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}\|_{\mathcal{B}_p(\mathcal{H})}$, respectively.

In fact, our interest in these questions stems from computations of the Witten index (a suitable extension of the Fredholm index) for a class of non-Fredholm model operators

$$\boldsymbol{D}_{\boldsymbol{A}} = \frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D}_{\boldsymbol{A}}) = W^{1,2}(\mathbb{R}; \mathcal{H}) \cap \operatorname{dom}(\boldsymbol{A}), \tag{1.1}$$

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in the Hilbert space $L^2(\mathbb{R};\mathcal{H})$, where

$$
\begin{aligned}
&(\boldsymbol{A}f)(t) = A(t)f(t) \text{ for a.e. } t \in \mathbb{R},\\
&f \in \operatorname{dom}(\boldsymbol{A}) = \bigg\{g \in L^2(\mathbb{R};\mathcal{H}) \,\bigg|\, g(t) \in \operatorname{dom}(A(t)) \text{ for a.e. } t \in \mathbb{R};\\
&\quad t \mapsto A(t)g(t) \text{ is (weakly) measurable; } \int_{\mathbb{R}} dt\, \|A(t)g(t)\|_{\mathcal{H}}^2 < \infty\bigg\},
\end{aligned} \tag{1.2}
$$

with $A(t)$, $t \in \mathbb{R}$, a family of self-adjoint operators in $\mathcal{H}$ with asymptotes $A_\pm$ (in norm resolvent sense). Interesting concrete examles for $A_\pm$ are given by massless Dirac-type operators in $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \in \mathbb{N}$ (the latter are known to be non-Fredholm), see, [10]–[15]. More precisely, given the sequence of self-adjoint operators $\boldsymbol{H}_{j,n}$, $\boldsymbol{H}_j$ in $L^2(\mathbb{R};dt;\mathcal{H})$, $j = 1,2$, $n \in \mathbb{N}$, and self-adjoint operators $A_{+,n}$, A_+, A_- in $\mathcal{H}$, $n \in \mathbb{N}$, and a Pushnitski-type relation between the spectral shift functions $\xi(\,\cdot\,;\boldsymbol{H}_{2,n},\boldsymbol{H}_{1,n})$ and $\xi(\,\cdot\,;A_{+,n},A_-)$ for the pairs, $(\boldsymbol{H}_{2,n},\boldsymbol{H}_{1,n})$ and $(A_{+,n},A_-)$ of the form

$$
\xi(\lambda;\boldsymbol{H}_{2,n},\boldsymbol{H}_{1,n}) = \begin{cases} \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu;A_{+,n},A_-)\,d\nu}{(\lambda-\nu^2)^{1/2}} \text{ for a.e. } \lambda > 0, \\ 0, \ \lambda < 0, \end{cases} \qquad n \in \mathbb{N}, \tag{1.3}
$$

we were interested in performing the limit $n \to \infty$ in (1.3) to obtain the analogous relation for the limiting spectral shift functions $\xi(\,\cdot\,;\boldsymbol{H}_2,\boldsymbol{H}_1)$ and $\xi(\,\cdot\,;A_+,A_-)$ corresponding to the limiting pairs $(\boldsymbol{H}_2,\boldsymbol{H}_1)$ and (A_+,A_-), respectively. The latter is instrumental in computing the Witten index for $\boldsymbol{D}_{\boldsymbol{A}}$. The task of performing the limit $n \to \infty$ in (1.3) is considerably complicated since due to the nature of the approximations involved, no suitable bounds on $\xi(\,\cdot\,;\boldsymbol{H}_{2,n},\boldsymbol{H}_{1,n})$ and $\xi(\,\cdot\,;A_{+,n},A_-)$ (independent of $n \in \mathbb{N}$) are readily available. To circumvent this difficulty one can resort to a distributional approach considering

$$
\int_{\mathbb{R}} d\lambda\, \xi(\lambda;\boldsymbol{H}_{2,n},\boldsymbol{H}_{1,n}) f'(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} d\nu\, \xi(\nu;A_{+,n},A_-) F'(\nu), \quad n \in \mathbb{N}, \tag{1.4}
$$

where $f \in C_0^\infty(\mathbb{R})$ is arbitrary, and $F' \in C_0^\infty(\mathbb{R})$ is given by

$$
F'(\nu) = \int_{\nu^2}^{\infty} d\lambda\, f'(\lambda)(\lambda - \nu^2)^{-1/2}, \quad \nu \in \mathbb{R}, \tag{1.5}
$$

details will appear in [13]. Focusing now on the left-hand side of (1.4), one recalls Krein's trace formula,

$$
\operatorname{tr}_{\mathcal{B}_1(L^2(\mathbb{R};\mathcal{H}))}(f(\boldsymbol{H}_{2,n}) - f(\boldsymbol{H}_{1,n})) = \int_{[0,\infty)} d\lambda\, \xi(\lambda;\boldsymbol{H}_{2,n},\boldsymbol{H}_{1,n}) f'(\lambda), \quad f \in C_0^\infty(\mathbb{R}). \tag{1.6}
$$

Thus, given control of resolvents in the form

$$
\begin{aligned}
\lim_{n\to\infty} \big\|&[(\boldsymbol{H}_{2,n} - z\boldsymbol{I})^{-m_2} - (\boldsymbol{H}_{1,n} - z\boldsymbol{I})^{-m_2}]\\
&- [(\boldsymbol{H}_2 - z\boldsymbol{I})^{-m_2} - (\boldsymbol{H}_1 - z\boldsymbol{I})^{-m_2}]\big\|_{\mathcal{B}_1(L^2(\mathbb{R};\mathcal{H}))} = 0, \quad z \in \mathbb{C}\backslash\mathbb{R}.
\end{aligned} \tag{1.7}
$$

one can hope to control

$$
\lim_{n\to\infty} \|[f(\boldsymbol{H}_{2,n}) - f(\boldsymbol{H}_{1,n})] - [f(\boldsymbol{H}_2) - f(\boldsymbol{H}_1)]\|_{\mathcal{B}_1(L^2(\mathbb{R};\mathcal{H}))} = 0, \tag{1.8}
$$

and hence obtain

$$
\begin{aligned}
\lim_{n\to\infty} \int_{[0,\infty)} d\lambda\, \xi(\lambda;\boldsymbol{H}_{2,n},\boldsymbol{H}_{1,n}) f'(\lambda) &= \lim_{n\to\infty} \operatorname{tr}_{\mathcal{B}_1(L^2(\mathbb{R};\mathcal{H}))}(f(\boldsymbol{H}_{2,n}) - f(\boldsymbol{H}_{1,n}))\\
&= \operatorname{tr}_{\mathcal{B}_1(L^2(\mathbb{R};\mathcal{H}))}(f(\boldsymbol{H}_2) - f(\boldsymbol{H}_1)) = \int_{[0,\infty)} d\lambda\, \xi(\lambda;\boldsymbol{H}_2,\boldsymbol{H}_1) f'(\lambda), \quad f \in C_0^\infty(\mathbb{R}).
\end{aligned} \tag{1.9}
$$

Together with controlling the limit $n \to \infty$ on the right-hand side of (1.4), this leads to

$$\int_{\mathbb{R}} d\lambda\, \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) f'(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} d\nu\, \xi(\nu; A_+, A_-) F'(\nu). \tag{1.10}$$

Without going into further details we note that (1.10) in turn can be used to prove the limiting relation in (1.3) and the latter leads to a compution of the semigroup regularized Witten index, $W_s(\boldsymbol{D}_{\boldsymbol{A}})$, of $\boldsymbol{D}_{\boldsymbol{A}}$: Assuming that 0 is a left and a right Lebesgue point of $\xi(\,\cdot\,; A_+, A_-)$, denoting the corresponding values by $\xi(0_\pm; A_+, A_-)$, the semigroup regularized Witten index is found to be

$$\begin{aligned} W_s(\boldsymbol{D}_{\boldsymbol{A}}) &:= \lim_{t\uparrow\infty} \operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})} \big(e^{-t\boldsymbol{H}_1} - e^{-t\boldsymbol{H}_2}\big) \\ &= [\xi(0_-; A_+, A_-) + \xi(0_+; A_+, A_-)]/2, \end{aligned} \tag{1.11}$$

see, for instance, [10]–[15]. (We here use the semigroup regularized Witten index rather than the resolvent regularised one as the former is applicable in the case of d-dimensional Dirac-type operators $A_\pm$, $d \in \mathbb{N}$.) We trust this sufficiently illustrates our interest in using control of resolvents of self-adjoint operators to gain control over their C_0^∞-functions.

We also note a further complication lies in the fact that when studying multi-dimensional Dirac-type operators $A_\pm$, resolvents alone are not sufficient in the trace class context and hence sufficiently high powers (depending on the space dimension involved) of resolvents have to be employed.

Our principal tool to gain control over C_0^∞-functions of S in terms of (powers of) resolvents of S is furnished by a suitable application of almost analytic extensions $\widetilde{f_{\ell,\sigma}}$ of f in the form of a Davies–Helffer–Sjöstrand functional calculus [18], [20, Ch. 2], [40, Proposition 7.2], of the form

$$f(S) = \pi^{-1} \int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell,\sigma}}}{\partial \overline{z}}(z)(S - zI_{\mathcal{H}})^{-1}, \tag{1.12}$$

and a refinement due to Khochman [43] of the type

$$f(S) = \frac{1}{\pi} \int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell,\sigma}}}{\partial \overline{z}}(z)(z - z_0)^m (S - z_0 I_{\mathcal{H}})^{-m} (S - zI_{\mathcal{H}})^{-1}, \quad m \in \mathbb{N} \cup \{0\}, \tag{1.13}$$

to be discussed in some detail in Section 2. Section 3 contains our principal results and some applications. Finally, Appendix A recalls various useful facts concerning (powers of) resolvents.

We conclude with some comments on the notation employed in this paper: Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$ (linear in the second argument), and $I_{\mathcal{H}}$ the identity operator in $\mathcal{H}$.

Next, if T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ and $\ker(T)$ denote the domain and kernel (i.e., null space) of T. The spectrum and resolvent set of a closed linear operator in a Hilbert space will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$, respectively. t

The convergence of bounded operators in the strong operator topology (i.e., pointwise limits) will be denoted by s-lim.

The Banach spaces of bounded and compact linear operators on a separable complex Hilbert space $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively; the corresponding ℓ^p-based trace ideals will be denoted by $\mathcal{B}_p(\mathcal{H})$, their norms are abbreviated by $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$, $p \geqslant 1$. Moreover, $\operatorname{tr}_{\mathcal{H}}(A)$ denotes the corresponding trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$.

The symbol $C_0^\infty(\mathbb{R})$ represents C^∞-functions of compact support on $\mathbb{R}$; continuous functions on $\mathbb{R}$ vanishing at infinity are denoted by $C_\infty(\mathbb{R})$.

2. Basic facts on almost analytic extensions and the functional calculus for self-adjoint operators

In this preparatory section we briefly recall the basics of almost analytic extensions and the ensuing functional calculus for self-adjoint operators, following Davies' detailed treatment in [18], [20, Ch. 2].

One introduces the class $S^{\beta}(\mathbb{R}), \beta \in \mathbb{R}$, consisting of all functions $f \in C^{\infty}(\mathbb{R})$ such that

$$f^{(m)}(x) \underset{|x|\to\infty}{=} O(\langle x\rangle^{\beta-m}), \quad m \in \mathbb{N}_0, \tag{2.1}$$

where $\langle z\rangle = (|z|^2+1)^{1/2}$, $z \in \mathbb{C}$. Then in obvious notation, with "$\cdot$" denoting pointwise multiplication, $S^{\beta}(\mathbb{R}) \cdot S^{\gamma}(\mathbb{R}) \subseteq S^{\beta+\gamma}(\mathbb{R}), \beta, \gamma \in \mathbb{R}$, and the space

$$\mathcal{A}(\mathbb{R}) = \bigcup_{\beta<0} S^{\beta}(\mathbb{R}) \tag{2.2}$$

is an algebra under pointwise multiplication with

$$C_0^{\infty}(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}). \tag{2.3}$$

In particular, $f \in \mathcal{A}(\mathbb{R})$ implies $f \in C_{\infty}(\mathbb{R})$ (the continuous functions vanishing at $\pm\infty$) and $f^{(m)} \in L^1(\mathbb{R}), m \in \mathbb{N}$.

Given $f \in \mathcal{A}(\mathbb{R})$, one defines an *almost analytic extension* $\widetilde{f}_{\ell,\sigma}$, of f to $\mathbb{C}$ by

$$\widetilde{f}_{\ell,\sigma}(z) = \sigma(x,y) \sum_{k=0}^{\ell} \frac{f^{(k)}(x)(iy)^k}{k!}, \quad z = x+iy \in \mathbb{C}, \quad \ell \in \mathbb{N}, \tag{2.4}$$

where

$$\sigma(x,y) = \tau(y/\langle x\rangle), \quad x,y \in \mathbb{R}, \quad \tau \in C_0^{\infty}(\mathbb{R}), \quad \tau(s) = \begin{cases} 1, & |s| \leqslant 1, \\ 0, & |s| \geqslant 2. \end{cases} \tag{2.5}$$

The precise structure of $\widetilde{f}_{\ell,\sigma}$ will not be important and other expresssions for it are possible (cf., [18]).

We note the formula

$$\begin{aligned} \frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(z) &= \frac{1}{2}\bigg(\frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial x}(z) + i\frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial y}(z)\bigg) \\ &= \frac{1}{2}[\sigma_x(x,y)+\sigma_y(x,y)] \sum_{k=0}^{\ell} \frac{f^{(k)}(x)(iy)^k}{k!} + \frac{1}{2}\sigma(x,y)\frac{f^{\ell+1}(x)(iy)^{\ell}}{\ell!}, \\ & \hspace{20em} z \in \mathbb{C}, \end{aligned} \tag{2.6}$$

implying the crucial fact,

$$\bigg|\frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(x+iy)\bigg| \underset{y\downarrow 0}{=} O(|y|^{\ell}), \tag{2.7}$$

in particular,

$$\frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(x) = 0, \quad x \in \mathbb{R}. \tag{2.8}$$

Following Helffer and Sjöstrand [40, Proposition 7.2], particularly, in the form presented by Davies [18], [20, Ch. 2], one then establishes a functional calculus for self-adjoint operators S in a complex, separable Hilbert space $\mathcal{H}$ via the formula

$$f(S) = \pi^{-1} \int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(z)(S - zI_{\mathcal{H}})^{-1}. \tag{2.9}$$

Since the integrand is norm continuous, the integral in (2.9) is norm convergent, in particular, one notes that (2.7) and (2.8), together with the standard estimate $\left\|(S-zI_{\mathcal{H}})^{-1}\right\|_{\mathcal{B}(\mathcal{H})} \leqslant |\mathrm{Im}(z)|^{-1}$, overcome the apparent singularity of the integrand in (2.9) for $z \in \sigma(S) \subseteq \mathbb{R}$ (cf. also (2.8)).

The justification for calling this a functional calculus follows upon proving the following facts:

- The left-hand side of (2.8) is independent of the choice of $\ell \in \mathbb{N}$ and the precise form of σ in (2.5).
- If $f \in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}(f) \cap \sigma(S) = \emptyset$, then $f(S) = 0$.
- If $f, g \in \mathcal{A}(\mathbb{R})$, then $(fg)(S) = f(S)g(S)$, $f(S)^* = \overline{f}(S)$, $\|f(S)\|_{\mathcal{B}(\mathcal{H})} = \|f\|_{L^\infty(\mathbb{R})}$.
- Let $z \in \mathbb{C}\backslash\mathbb{R}$ and $f_z(x) = (x-z)^{-1}$, then $f_z \in \mathcal{A}(\mathbb{R})$ and $f_z(S) = (S - zI_{\mathcal{H}})^{-1}$.

In addition, we note that Khochman [43] proved the following extension of (2.9):

Lemma 2.1. ([43]). *Let $m \in \mathbb{N}$, $f \in C_0^\infty(\mathbb{R})$, and suppose that S is self-adjoint in $\mathcal{H}$. Then,*

$$f(S) = \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(z)(z-z_0)^m(S - z_0I_{\mathcal{H}})^{-m}(S - zI_{\mathcal{H}})^{-1}. \tag{2.10}$$

We will employ (in fact, rederive) (2.10) in the proof of Theorem 3.8. Next, we discuss another extension focusing on semigroups rather than powers of resolvents.

Lemma 2.2. *Let $t > 0$, $f \in C_0^\infty(\mathbb{R})$, and suppose that S is self-adjoint and bounded from below in $\mathcal{H}$. Then,*

$$f(S) = \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(z)e^{tz}e^{-tS}(S - zI_{\mathcal{H}})^{-1}. \tag{2.11}$$

Proof. We start by noting that if $f, g \in C_0^\infty(\mathbb{R})$, it is proved in [20, p. 28] that

$$\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{(fg)}_{\ell'',\sigma''}}{\partial \overline{z}}(z)(S - zI_{\mathcal{H}})^{-1} = \int_{\mathbb{C}} dxdy\, \frac{\partial (\tilde{f}_{\ell',\sigma'}\tilde{g}_{\ell,\sigma})}{\partial \overline{z}}(z)(S - zI_{\mathcal{H}})^{-1}. \tag{2.12}$$

Next, suppose that $f \in C_0^\infty(\mathbb{R})$ and let $E_t \in C_0^\infty(\mathbb{R})$ denote a function which coincides with e^{tx} on an open interval I with $\mathrm{supp}(f) \subset I$. Then

$$\widetilde{E}_{t,\ell,\sigma}(z) = \sigma(x,y)e^{tx}\sum_{k=0}^{\ell}\frac{(ity)^k}{k!}, \quad z = x+iy, \quad x \in I. \tag{2.13}$$

Let $f_{\ell',\sigma'}$ denote an almost analytic extension of f. Setting $g = fE_t$, with $\widetilde{g}_{\ell'',\sigma''}$ an almost analytic extension of g, in light of the identity $f(S) = g(S)e^{-tS}$, one infers from the Davies–Helffer–Sjöstrand functional calculus (2.9) applied to g,

$$\begin{aligned} f(S) &= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{g}_{\ell'',\sigma''}}{\partial \overline{z}}(z)e^{-tS}(S - zI_{\mathcal{H}})^{-1} \\ &= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial (\widetilde{f}_{\ell',\sigma'}\widetilde{E}_{t,\ell,\sigma})}{\partial \overline{z}}(z)e^{-tS}(S - zI_{\mathcal{H}})^{-1} \\ &= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f}_{\ell',\sigma'}}{\partial \overline{z}}(z)\widetilde{E}_{t,\ell,\sigma}(z)e^{-tS}(S - zI_{\mathcal{H}})^{-1} \\ &\quad + \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \widetilde{f}_{\ell',\sigma'}(z)\frac{\partial \widetilde{E}_{t,\ell,\sigma}}{\partial \overline{z}}(z)e^{-tS}(S - zI_{\mathcal{H}})^{-1} \end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell'',\sigma''}}}{\partial \overline{z}}(z)\sigma(x,y)e^{tx}\bigg(\sum_{k=0}^{\ell}\frac{(ity)^k}{k!}\bigg)e^{-tS}(S-zI_{\mathcal{H}})^{-1} \\
&\quad + \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \widetilde{f_{\ell',\sigma'}}(z)\bigg\{\frac{1}{2}\bigg[\sigma_x(x,y)e^{tx}\bigg(\sum_{k=0}^{\ell}\frac{(ity)^k}{k!}\bigg) \\
&\qquad\qquad + \sigma(x,y)e^{tx}\bigg(\sum_{k=0}^{\ell}\frac{(ity)^k}{k!}\bigg)\bigg] \\
&\quad + \frac{i}{2}\bigg[\sigma(x,y)e^{tx}it\bigg(\sum_{k=0}^{\ell-1}\frac{(ity)^k}{k!}\bigg) + \sigma_y(x,y)e^{tx}\bigg(\sum_{k=0}^{\ell}\frac{(ity)^k}{k!}\bigg)\bigg]\bigg\} \\
&\qquad\qquad \times e^{-tS}(S-zI_{\mathcal{H}})^{-1}.
\end{aligned}
$$

Exploiting the fact that $f(S)$ is independent of ℓ, we now take the limit $\ell \to \infty$ in (2.14). Since $\widetilde{f_{\ell',\sigma'}}$ has compact support and takes care of the singularity of the resolvent, e^{-tS} is bounded and z-independent, and the exponential series converges uniformly on compact sets, one may pass the limit under the integral to obtain

$$
\begin{aligned}
f(S) &= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell',\sigma'}}}{\partial \overline{z}}\sigma(x,y)e^{tz}e^{-tS}(S-zI_{\mathcal{H}})^{-1} \\
&\quad + \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \widetilde{f_{\ell',\sigma'}}(z)\frac{\partial(\sigma(x,y)e^{tz})}{\partial \overline{z}}e^{-tS}(S-zI_{\mathcal{H}})^{-1} \\
&= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial}{\partial \overline{z}}(\widetilde{f_{\ell',\sigma'}}(z)\sigma(x,y)e^{tz})e^{-tS}(S-zI_{\mathcal{H}})^{-1} \\
&= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial}{\partial \overline{z}}(\widetilde{f_{\ell',\widehat{\sigma}}}(z)e^{tz})e^{-tS}(S-zI_{\mathcal{H}})^{-1},
\end{aligned} \tag{2.15}
$$

where $\widehat{\sigma} = \sigma'\sigma$ (which corresponds to choosing $\widehat{\tau} = \tau'\tau$). It is a simple matter to verify that

$$
\frac{\partial}{\partial \overline{z}}(\widetilde{f_{\ell',\widehat{\sigma}}}(z)e^{tz}) = \frac{\partial \widetilde{f_{\ell,\widehat{\sigma}}}}{\partial \overline{z}}e^{tz}, \tag{2.16}
$$

which then shows

$$
f(S) = \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell',\widehat{\sigma}}}}{\partial \overline{z}}e^{tz}e^{-tS}(S-zI_{\mathcal{H}})^{-1} \tag{2.17}
$$

and hence (2.11) (renaming ℓ' and $\widehat{\sigma}$). □

Historically, the idea of almost analytic (resp., pseudo-analytic) extensions appeared in Hörmander [39] and Dynkin [28], [29], Melin and Sjöstrand [46] (see also [26, Ch. 8], [45, Sect. III.6] for expositions and [42] for an alternative approach). The functional calculus was used by Helffer and Sjöstrand in their seminal 1989 paper on Schrödinger operators with magnetic fields [40], which in turn was the basis for the systematic treatment by Davies [18], [20, Ch. 2]. Since these early developments, there has been a large body of literature in connection with spectral theory for Schrödinger and Dirac-type operators applying this functional calculus. While a complete list of references in this context is clearly beyond the scope of this paper, we want to illustrate the great variety of applications that rely on this functional calculus a bit and hence refer to [16], [21], [22], [23], [24], [25], [27], [30], [31], [34], [35], [37], [41], [43], [47], [48], [52], [53], and the references cited therein.

While we here exclusively focus on linear operators in a Hilbert space, this functional calculus applies to operators in Banach spaces with real spectrum, see, for instance, [1],

[17], [18], [19], [32], [33]. Extensions to the case where the spectrum is contained in the unit circle or contained in finitely-many smooth arcs were also treated in [28].

3. Some applications

In this section we apply the almost analytic extension method and its ensuing functional calculus for self-adjoint operators to derive various norm bounds and convergence properties of operators in trace ideals.

We start with the following estimates established in the proof of [20, Theorem 2.6.2] (more precisely, (3.1) is proved in [20], but then the rest of Lemma 3.1 is obvious):

Lemma 3.1. *Let $z_0 \in \mathbb{C}\backslash\mathbb{R}$, $f \in \mathcal{A}(\mathbb{R})$ and suppose that S_j, $j = 1, 2$, are self-adjoint in $\mathcal{H}$. Then*

$$\begin{aligned} \|f(S_2) - f(S_1)\|_{\mathcal{B}(\mathcal{H})} \leqslant \frac{8}{\pi} \int_{\mathbb{C}} dxdy \left|\frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z)\right| \frac{[|z_0|^2 + |z|^2]}{|\mathrm{Im}(z)|^2} \\ \times \left\|(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}\right\|_{\mathcal{B}(\mathcal{H})}. \end{aligned} \tag{3.1}$$

In addition, if for some $p \in [1, \infty)$, $[(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}] \in \mathcal{B}_p(\mathcal{H})$ for some (and hence for all) $z_0 \in \mathbb{C}\backslash\mathbb{R}$, then

$$[f(S_2) - f(S_1)] \in \mathcal{B}_p(\mathcal{H}) \tag{3.2}$$

and

$$\begin{aligned} \|f(S_2) - f(S_1)\|_{\mathcal{B}_p(\mathcal{H})} \leqslant \frac{8}{\pi} \int_{\mathbb{C}} dxdy \left|\frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z)\right| \frac{[|z_0|^2 + |z|^2]}{|\mathrm{Im}(z)|^2} \\ \times \left\|(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}\right\|_{\mathcal{B}_p(\mathcal{H})}. \end{aligned} \tag{3.3}$$

If $[(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}] \in \mathcal{B}_\infty(\mathcal{H})$, the inclusion (3.2) extends to $p = \infty$.

Proof. Combining (2.9), (A.1), and (A.2) one obtains,

$$f(S_2) - f(S_1) = \pi^{-1} \int_{\mathbb{C}} dxdy \, \frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z)[(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}] \tag{3.4}$$

and hence

$$\begin{aligned} &\|f(S_2) - f(S_1)\|_{\mathcal{B}(\mathcal{H})} \\ &\quad \leqslant \pi^{-1} \int_{\mathbb{C}} dxdy \left|\frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z)\right| \left\|(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}\right\|_{\mathcal{B}(\mathcal{H})} \\ &\quad \leqslant \pi^{-1} \int_{\mathbb{C}} dxdy \left|\frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z)\right| \Big\|(S_2 - z_0 I_{\mathcal{H}})(S_2 - zI_{\mathcal{H}})^{-1} \\ &\qquad \times [(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}](S_1 - z_0 I_{\mathcal{H}})(S_1 - zI_{\mathcal{H}})^{-1}\Big\|_{\mathcal{B}(\mathcal{H})} \\ &\quad \leqslant \pi^{-1} \int_{\mathbb{C}} dxdy \left|\frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z)\right| \left\|(S_2 - z_0 I_{\mathcal{H}})(S_2 - zI_{\mathcal{H}})^{-1}\right\|_{\mathcal{B}(\mathcal{H})} \\ &\qquad \times \left\|(S_1 - z_0 I_{\mathcal{H}})(S_1 - zI_{\mathcal{H}})^{-1}\right\|_{\mathcal{B}(\mathcal{H})} \left\|[(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}]\right\|_{\mathcal{B}(\mathcal{H})} \\ &\quad \leqslant \left(\frac{8}{\pi} \int_{\mathbb{C}} dxdy \left|\frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z)\right| \frac{[|z_0|^2 + |z|^2]}{|\mathrm{Im}(z)|^2}\right) \\ &\qquad \times \left\|[(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}]\right\|_{\mathcal{B}(\mathcal{H})}. \end{aligned} \tag{3.5}$$

Precisely the same chain of estimates applies to $\mathcal{B}(\mathcal{H})$ replaced by $\mathcal{B}_p(\mathcal{H})$, relying on the ideal properties of $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty)$. The case $p = \infty$ in (3.2) is a consequence of the norm convergent integral on the right-hand side of (3.4). □

Combined with a Stone–Weierstrass approximation argument and the fact that $\|f(S)\|_{\mathcal{B}(\mathcal{H})} = \|f\|_{L^\infty(\mathbb{R})}$, $f \in \mathcal{A}(\mathbb{R})$, Lemma 3.1 yields the following well-known fact, recorded, for instance, in [20, Theorem 2.62], [50, Theorem V.III.20(a)]:

Lemma 3.2. *Let S_n, $n \in \mathbb{N}$, and S be self-adjoint in $\mathcal{H}$, and suppose that S_n converges to S in norm resolvent sense as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \|f(S_n) - f(S)\|_{\mathcal{B}(\mathcal{H})} = 0 \tag{3.6}$$

for all $f \in C_\infty(\mathbb{R})$.

Remark 3.3. We note that the functional calculus based on almost analytic extensions is not the only possible approach to address estimates such as (3.1) and (3.3). As a powerful alternative we mention the theory of double operator integrals (DOI), which can prove stronger inequalities of the type (cf. [9], [12], [56])

$$\|f(S_2) - f(S_1)\|_{\mathcal{B}_1(\mathcal{K})} \leqslant C\big\|(S_2 - z_0 I_{\mathcal{K}})^{-m} - (S_1 - z_0 I_{\mathcal{K}})^{-m}\big\|_{\mathcal{B}_1(\mathcal{K})}, \quad f \in C_0^\infty(\mathbb{R}), \tag{3.7}$$

for $m \in \mathbb{N}$ odd, and some constant $C = C(f, z_0, m, S_1 \text{ or } S_2) > 0$ (i.e., C can be chosen independently of one of the self-adjoint operators S_2 and S_1).

Repeatedly differentiating

$$f(\lambda) = \pi^{-1} \int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell,\sigma}}}{\partial \overline{z}}(z)(\lambda - z)^{-1}, \quad \lambda \in \mathbb{R}, \tag{3.8}$$

with respect to λ yields

$$f^{(m-1)}(S) = \pi^{-1}(-1)^{m-1}(m-1)! \int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell,\sigma}}}{\partial \overline{z}}(z)(S - z)^{-m}, \quad \lambda \in \mathbb{R}. \tag{3.9}$$

This leads to estimates of the type (3.7) but with f replaced by $f^{(m-1)}$. ◇

We recall a useful result:

Lemma 3.4. *Let $p \in [1, \infty)$ and assume that $R, R_n, T, T_n \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, satisfy* $\text{s-lim}_{n\to\infty} R_n = R$ *and* $\text{s-lim}_{n\to\infty} T_n = T$ *and that $S, S_n \in \mathcal{B}_p(\mathcal{H})$, $n \in \mathbb{N}$, satisfy* $\lim_{n\to\infty} \|S_n - S\|_{\mathcal{B}_p(\mathcal{H})} = 0$. *Then* $\lim_{n\to\infty} \|R_n S_n T_n^* - RST^*\|_{\mathcal{B}_p(\mathcal{H})} = 0$.

This follows, for instance, from [38, Theorem 1], [51, p. 28–29], or [55, Lemma 6.1.3] with a minor additional effort (taking adjoints, etc.).

Next, we describe a typical convergence result:

Theorem 3.5. *Let $S_{j,n}$, $n \in \mathbb{N}$, and S_j, $j = 1, 2$, be self-adjoint in $\mathcal{H}$, and assume that $S_{j,n}$ converges in strong resolvent sense as $n \to \infty$ to S_j, $j = 1, 2$, respectively.*
(*i*) *Suppose that for some* (*and hence for all*) $z_0 \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned} \lim_{n\to\infty} \big\|&[(S_{2,n} - z_0 I_{\mathcal{H}})^{-1} - (S_2 - z_0 I_{\mathcal{H}})^{-1}] \\ &- [(S_{1,n} - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}]\big\|_{\mathcal{B}(\mathcal{H})} = 0. \end{aligned} \tag{3.10}$$

Then,

$$\lim_{n\to\infty} \|[f(S_{2,n}) - f(S_{1,n})] - [f(S_2) - f(S_1)]\|_{\mathcal{B}(\mathcal{H})} = 0, \quad f \in C_0^\infty(\mathbb{R}). \tag{3.11}$$

(*ii*) *Let $p \in [1, \infty)$ and suppose that for some* (*and hence for all*) $z_0 \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned} [(S_{2,n} - z_0 I_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-1}], [(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}] \in \mathcal{B}_p(\mathcal{H}),& \\ n \in \mathbb{N},& \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \lim_{n\to\infty} \big\|&[(S_{2,n} - z_0 I_{\mathcal{H}})^{-1} - (S_2 - z_0 I_{\mathcal{H}})^{-1}] \\ &- [(S_{1,n} - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-1}]\big\|_{\mathcal{B}_p(\mathcal{H})} = 0. \end{aligned} \tag{3.13}$$

Then,

$$\lim_{n\to\infty} \|[f(S_{2,n}) - f(S_{1,n})] - [f(S_2) - f(S_1)]\|_{\mathcal{B}_p(\mathcal{H})} = 0, \quad f \in C_0^\infty(\mathbb{R}). \tag{3.14}$$

Proof. As usual, a combination of identity (A.1), Lemma 3.4, and the assumed strong resolvent convergence of $S_{j,n}$ to S_j as $n \to \infty$, $j = 1, 2$, proves sufficiency of the conditions (3.10) and (3.12) for just one $z_0 \in \mathbb{C}\backslash\mathbb{R}$. Thus, assumption (3.10) actually implies

$$\begin{aligned} &\lim_{n\to\infty} \big\|[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_2 - zI_{\mathcal{H}})^{-1}] \\ &\qquad - [(S_{1,n} - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\big\|_{\mathcal{B}(\mathcal{H})} = 0, \quad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{3.15}$$

Next, mimicking (3.4), one obtains,

$$\begin{aligned} &[f(S_{2,n}) - f(S_{1,n})] - [f(S_2) - f(S_1)] \\ &= \pi^{-1} \int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z) \big[[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1}] \\ &\qquad - [(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\big], \quad f \in C_0^\infty(\mathbb{R}), \end{aligned} \tag{3.16}$$

and hence,

$$\begin{aligned} &\lim_{n\to\infty} \|[f(S_{2,n}) - f(S_{1,n})] - [f(S_2) - f(S_1)]\|_{\mathcal{B}(\mathcal{H})} \\ &\leqslant \pi^{-1} \lim_{n\to\infty} \int_{\mathbb{C}} dxdy \left| \frac{\partial \widetilde{f}_{2,\sigma}}{\partial \overline{z}}(z) \right| \big\|[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1}] \\ &\qquad - [(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\big\|_{\mathcal{B}(\mathcal{H})}, \quad f \in C_0^\infty(\mathbb{R}). \end{aligned} \tag{3.17}$$

In this context one observes that $f \in C_0^\infty(\mathbb{R})$ implies $\widetilde{f}_{2,\sigma} \in C_0^\infty(\mathbb{C})$. Since the exceptional set $\mathbb{R}$ in (3.15) has $dxdy$-measure zero (in addition to the fact that by (2.8), $\partial \widetilde{f}_{2,\sigma}/\partial \overline{z}$ vanishes on $\mathbb{R}$), an application of the Lebesgue dominated convergence theorem to interchange the limit $n \to \infty$ with the integral on the right-hand side of (3.17) requires establishing an n-independent integrable majorant of the integrand in (3.17). Employing identity (A.1) and estimate (A.2), this majorant can be obtained as follows:

$$\begin{aligned} &\big\|[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1}] - [(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\big\|_{\mathcal{B}(\mathcal{H})} \\ &\leqslant \big\|(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\quad + \big\|(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\leqslant \big\|(S_{2,n} - z_0I_{\mathcal{H}})(S_{2,n} - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\qquad \times \big\|(S_{2,n} - z_0I_{\mathcal{H}})^{-1} - (S_{1,n} - z_0I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\qquad \times \big\|(S_{1,n} - z_0I_{\mathcal{H}})(S_{1,n} - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\quad + \big\|(S_2 - z_0I_{\mathcal{H}})(S_2 - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\qquad \times \big\|(S_2 - z_0I_{\mathcal{H}})^{-1} - (S_1 - z_0I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\qquad \times \big\|(S_1 - z_0I_{\mathcal{H}})(S_1 - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\leqslant 8[|z_0|^2 + |z|^2]|\mathrm{Im}(z)|^{-2}\big[\big\|(S_{2,n} - z_0I_{\mathcal{H}})^{-1} - (S_{1,n} - z_0I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\qquad + \big\|(S_2 - z_0I_{\mathcal{H}})^{-1} - (S_1 - z_0I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})}\big] \\ &\leqslant 8[|z_0|^2 + |z|^2]|\mathrm{Im}(z)|^{-2}C(z_0), \end{aligned} \tag{3.18}$$

for some $0 < C(z_0) < \infty$, independent of $n \in \mathbb{N}$, since by assumption (3.10),

$$\big\|(S_{2,n} - z_0 I_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \leqslant \widetilde{C}(z_0) \tag{3.19}$$

for some $0 < \widetilde{C}(z_0) < \infty$, independent of $n \in \mathbb{N}$. Together with the properties (2.7), (2.8) of $\partial\widetilde{f}_{2,\sigma}/\partial\overline{z}$, this establishes the sought integrable majorant, independent of $n \in \mathbb{N}$, and thus permits the interchange of the limit $n \to \infty$ with the integral on the right-hand side of (3.17). This completes the proof of (3.11).

The proof of (3.14) proceeds exactly along the same lines employing once more the ideal properties of $\mathcal{B}_p(\mathcal{H})$, $p \in [1, \infty)$. □

We remark in passing that double operator integral techniques permit one to enlarge the class of functions f to which Theorem 3.5 applies (cf. also Remark 3.9).

Remark 3.6. The proof to Theorem 3.5 uses dominated convergence and given (3.18), the crucial observation is that

$$\int_{\mathbb{C}} dxdy \left|\frac{\partial\widetilde{f}_{\ell,\sigma}}{\partial\overline{z}}(z)\right| \frac{|z_0|^2 + |z|^2}{|\mathrm{Im}(z)|^2} < \infty, \tag{3.20}$$

which is obvious if $f \in C_0^\infty(\mathbb{R})$, since then $\widetilde{f}_{\ell,\sigma}$ is compactly supported. However, it is possible to once again prove (3.20) if $f \in S^\beta(\mathbb{R})$ for some $\beta < -1$. Indeed, following [20, p. 25], and setting

$$U = \{(x,y) \,|\, \langle x\rangle < |y| < 2\langle x\rangle\}, \quad V = \{(x,y) \,|\, 0 < |y| < 2\langle x\rangle\}, \tag{3.21}$$

one infers

$$|\sigma_x + i\sigma_y| \leqslant c\langle x\rangle^{-1}\chi_U(x,y), \quad z = x + iy \in \mathbb{C}, \tag{3.22}$$

for some constant $c > 0$. Then for $f \in S^\beta(\mathbb{R})$,

$$\left|\frac{\partial\widetilde{f}_{\ell,\sigma}}{\partial\overline{z}}(z)\right| \leqslant C\left\{\sum_{k=0}^{\ell} \langle x\rangle^{\beta-k-1}|y|^k\chi_U(x,y)\right\} + C\langle x\rangle^{\beta-\ell-1}|y|^\ell\chi_V(x,y), \quad z = x + iy \in \mathbb{C}, \tag{3.23}$$

where $C > 0$ is an appropriate constant. Therefore,

$$\begin{aligned}
\left|\frac{\partial\widetilde{f}_{\ell,\sigma}}{\partial\overline{z}}(z)\right| \frac{|z_0|^2 + |z|^2}{|\mathrm{Im}(z)|^2} &\leqslant C\left\{\sum_{k=0}^{\ell} \langle x\rangle^{\beta-k-1}|y|^k\chi_U(x,y)\frac{|z_0|^2 + |x|^2 + |y|^2}{|y|^2}\right\} \\
&\quad + C\langle x\rangle^{\beta-\ell-1}|y|^\ell\chi_V(x,y)\frac{|z_0|^2 + |x|^2 + |y|^2}{|y|^2} \\
&\leqslant C2^\ell\left\{\sum_{k=0}^{\ell} \langle x\rangle^{\beta-1}\chi_U(x,y)\frac{|z_0|^2 + 5\langle x\rangle^2}{\langle x\rangle^2}\right\} \\
&\quad + C\langle x\rangle^{\beta-\ell-1}|y|^{\ell-2}\chi_V(x,y)\{|z_0|^2 + 5\langle x\rangle^2\} \\
&\leqslant C2^\ell\left\{\sum_{k=0}^{\ell} \langle x\rangle^{\beta-3}\chi_U(x,y)\{|z_0|^2 + 5\langle x\rangle^2\}\right\} \\
&\quad + C2^{\ell-2}\langle x\rangle^{\beta-3}\chi_V(x,y)\{|z_0|^2 + 5\langle x\rangle^2\} \\
&\leqslant \widehat{C}\langle x\rangle^{\beta-1}\{\chi_U(x,y) + \chi_V(x,y)\}, \quad z = x + iy \in \mathbb{C}\backslash\mathbb{R},
\end{aligned} \tag{3.24}$$

where $\widehat{C} = \widehat{C}(z_0) > 0$ is a constant. Here, we used $|z_0|^2 + 5\langle x\rangle^2 < \widetilde{C}\langle x\rangle^2$, for an appropriate constant $\widetilde{C} = \widetilde{C}(z_0) > 0$. Given (3.24), (3.20) holds if $\beta < -1$ since

$$\begin{aligned}
&\int_{\mathbb{C}} dxdy\, \langle x\rangle^{\beta-1}\{\chi_U(x,y) + \chi_V(x,y)\} \\
&\quad = \int_{-\infty}^{\infty} dx\, \langle x\rangle^{\beta-1} \int_{-\infty}^{\infty} dy\, \{\chi_U(x,y) + \chi_V(x,y)\} \\
&\quad = 2\int_{-\infty}^{\infty} dx\, \langle x\rangle^{\beta-1} \left\{ \int_0^{2\langle x\rangle} dy + \int_{\langle x\rangle}^{2\langle x\rangle} dy \right\} \\
&\quad = 6\int_{-\infty}^{\infty} dx\, \langle x\rangle^{\beta} < \infty.
\end{aligned} \tag{3.25}$$

Thus, the majorant (3.24) is integrable as long as $f \in S^{\beta}(\mathbb{R})$ for some $\beta < -1$ and hence Theorem 3.5 extends from $C_0^{\infty}(\mathbb{R})$ to $S^{\beta}(\mathbb{R})$, $\beta < -1$.

Next, we briefly apply Theorem 3.5 to a concrete $(1+1)$-dimensional example treated in great detail in [10] and [11] by alternative methods.

Example 3.7. *Assuming the real-valued functions ϕ, θ satisfy*

$$\phi \in AC_{\text{loc}}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \phi' \in L^{\infty}(\mathbb{R}), \tag{3.26}$$

$$\begin{aligned}
&\theta \in AC_{\text{loc}}(\mathbb{R}) \cap L^{\infty}(\mathbb{R}), \quad \theta' \in L^{\infty}(\mathbb{R}) \cap L^1(\mathbb{R}), \\
&\lim_{t\to\infty} \theta(t) = 1, \quad \lim_{t\to-\infty} \theta(t) = 0,
\end{aligned} \tag{3.27}$$

we introduce the family of self-adjoint operators $A(t)$, $t \in \mathbb{R}$, in $L^2(\mathbb{R})$,

$$A(t) = -i\frac{d}{dx} + \theta(t)\phi, \quad \operatorname{dom}(A(t)) = W^{1,2}(\mathbb{R}), \quad t \in \mathbb{R}, \tag{3.28}$$

and its self-adjoint asymptotes as $t \to \pm\infty$,

$$A_+ = -i\frac{d}{dx} + \phi, \quad A_- = -i\frac{d}{dx}, \quad \operatorname{dom}(A_\pm) = W^{1,2}(\mathbb{R}). \tag{3.29}$$

In addition, we introducing the operator d/dt in $L^2(\mathbb{R}; dt; L^2(\mathbb{R}; dx))$ by

$$\begin{aligned}
&\Big(\frac{d}{dt} f\Big)(t) = f'(t) \text{ for a.e. } t \in \mathbb{R}, \\
&f \in \operatorname{dom}(d/dt) = \Big\{g \in L^2(\mathbb{R}; dt; L^2(\mathbb{R})) \,\Big|\, g \in AC_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R})), \\
&\qquad\qquad g' \in L^2(\mathbb{R}; dt; L^2(\mathbb{R}))\Big\}
\end{aligned} \tag{3.30}$$

$$= W^{1,2}(\mathbb{R}; dt; L^2(\mathbb{R}; dx)). \tag{3.31}$$

Next, we agree to identify $L^2(\mathbb{R}; dt; L^2(\mathbb{R}; dx))$ with $L^2(\mathbb{R}^2; dtdx)$ (denoting the latter by $L^2(\mathbb{R}^2)$ for brevity) and introduce $\boldsymbol{D_A}$ in $L^2(\mathbb{R}^2)$ by

$$\boldsymbol{D_A} = \frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D_A}) = W^{1,2}(\mathbb{R}^2), \tag{3.32}$$

with $\boldsymbol{A}$ defined as in (1.2) *and $A(t)$, $t \in \mathbb{R}$, given by* (3.28)*. Moreover, we introduce the nonnegative, self-adjoint operators $\boldsymbol{H}_j$, $j = 1, 2$, in $L^2(\mathbb{R}^2)$ by*

$$\boldsymbol{H}_1 = \boldsymbol{D}_{\boldsymbol{A}}^* \boldsymbol{D}_{\boldsymbol{A}}, \quad \boldsymbol{H}_2 = \boldsymbol{D}_{\boldsymbol{A}} \boldsymbol{D}_{\boldsymbol{A}}^*. \tag{3.33}$$

As shown in [10]*, the assumptions on ϕ and θ guarantee that*

$$[(A_+ - zI)^{-1} - (A_- - zI)^{-1}] \in \mathcal{B}_1(L^2(\mathbb{R})), \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{3.34}$$

(*for simplicity, we adopt the abbreviation* $I = I_{L^2(\mathbb{R})}$ *throughout this example*), *and thus, the spectral shift function* $\xi(\,\cdot\,; A_+, A_-)$ *for the pair* (A_+, A_-) *exists and is well-defined up to an arbitrary additive real constant, satisfying*

$$\xi(\,\cdot\,; A_+, A_-) \in L^1(\mathbb{R}; (\nu^2+1)^{-1} d\nu). \tag{3.35}$$

Introducing $\chi_n(A_-) = n(A_-^2 + n^2 I)^{-1/2}$ *and* $A_{+,n} = A_- + \chi_n(A_-)\phi\chi_n(A_-)$, $n \in \mathbb{N}$, *the fact*

$$A_{+,n} - A_- = \chi_n(A_-)\phi\chi_n(A_-) \in \mathcal{B}_1(L^2(\mathbb{R})), \quad n \in \mathbb{N}, \tag{3.36}$$

implies that also the spectral shift functions $\xi(\,\cdot\,; A_{+,n}, A_-)$, $n \in \mathbb{N}$, *exist and are uniquely determined by*

$$\xi(\,\cdot\,; A_{+,n}, A_-) \in L^1(\mathbb{R}; d\nu), \quad n \in \mathbb{N}. \tag{3.37}$$

In fact, as has been shown in [10], *the open constant in* $\xi(\,\cdot\,; A_+, A_-)$ *can naturally be determined via the limiting procedure*

$$\lim_{n\to\infty} \xi(\nu; A_{+,n}, A_-) = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, \phi(x) = \xi(\nu; A_+, A_-), \quad \nu \in \mathbb{R}. \tag{3.38}$$

In particular, $\xi(\,\cdot\,; A_+, A_-)$ *turns out to be constant in this example.*

Replacing $A(t)$ *by* $A_n(t) = A_- + \chi_n(A_-)\theta(t)\phi\chi_n(A_-)$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, *and hence,* $\boldsymbol{A}$ *by* $\boldsymbol{A}_n$, $\boldsymbol{D}_{\boldsymbol{A}}$ *by* $\boldsymbol{D}_{\boldsymbol{A}_n}$, $\boldsymbol{H}_j$ *by* $\boldsymbol{H}_{j,n}$, $j = 1, 2$, $n \in \mathbb{N}$, *one verifies the facts,*

$$[(\boldsymbol{H}_2 - z\boldsymbol{I})^{-1} - (\boldsymbol{H}_1 - z\boldsymbol{I})^{-1}] \in \mathcal{B}_1(L^2(\mathbb{R}^2)), \quad z \in \mathbb{C}\backslash[0,\infty), \tag{3.39}$$

$$[(\boldsymbol{H}_{2,n} - z\boldsymbol{I})^{-1} - (\boldsymbol{H}_{1,n} - z\boldsymbol{I})^{-1}] \in \mathcal{B}_1(L^2(\mathbb{R}^2)), \quad n \in \mathbb{N}, \quad z \in \mathbb{C}\backslash[0,\infty), \tag{3.40}$$

showing that the spectral shift functions $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ *and* $\xi(\,\cdot\,; \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})$ *for the pairs* $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ *and* $(\boldsymbol{H}_2, \boldsymbol{H}_1)$, $n \in \mathbb{N}$, *respectively, are well-defined and satisfy*

$$\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1),\ \xi(\,\cdot\,; \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n}) \in L^1(\mathbb{R}; (\lambda^2+1)^{-1} d\lambda), \quad n \in \mathbb{N}. \tag{3.41}$$

Since $\boldsymbol{H}_j \geqslant 0$, $\boldsymbol{H}_{j,n} \geqslant 0$, $n \in \mathbb{N}$, $j = 1, 2$, *one uniquely introduces* $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ *and* $\xi(\,\cdot\,; \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})$, $n \in \mathbb{N}$, *by requiring that*

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = 0, \quad \xi(\,\cdot\,; \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n}) = 0, \quad \lambda < 0, \quad n \in \mathbb{N}. \tag{3.42}$$

As shown in [10], *one can now prove the following intimate connection between* $\xi(\,\cdot\,; A_{+,n}, A_-)$ *and* $\xi(\,\cdot\,; \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n})$, $n \in \mathbb{N}$, *the Pushnitski-type formula* [36], [49],

$$\xi(\lambda; \boldsymbol{H}_{2,n}, \boldsymbol{H}_{1,n}) = \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_{+,n}, A_-)d\nu}{(\lambda - \nu^2)^{1/2}} \text{ for a.e. } \lambda > 0,\ n \in \mathbb{N}. \tag{3.43}$$

Moreover, as shown in [10] *and* [11], *one indeed has the convergence property*

$$\begin{aligned} \lim_{n\to\infty} \big\| &[(\boldsymbol{H}_{2,n} - z\boldsymbol{I})^{-1} - (\boldsymbol{H}_{1,n} - z\boldsymbol{I})^{-1}] \\ &- [(\boldsymbol{H}_2 - z\boldsymbol{I})^{-1} - (\boldsymbol{H}_1 - z\boldsymbol{I})^{-1}]\big\|_{\mathcal{B}_1(L^2(\mathbb{R}^2))} = 0, \quad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{3.44}$$

Thus, Theorem 3.5 applies and hence yields,

$$\lim_{n\to\infty} \|[f(\boldsymbol{H}_{2,n}) - f(\boldsymbol{H}_{1,n})] - [f(\boldsymbol{H}_2) - f(\boldsymbol{H}_1)]\|_{\mathcal{B}_1(L^2(\mathbb{R}^2))} = 0, \quad f \in C_0^\infty(\mathbb{R}). \tag{3.45}$$

This, in turn permits one to take the limit $n \to \infty$ *in* (3.43), *implying*

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \frac{1}{\pi}\int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)d\nu}{(\lambda - \nu^2)^{1/2}} \text{ for a.e. } \lambda > 0,\ n \in \mathbb{N}. \tag{3.46}$$

as will be discusssed in detail in [13]. *Equation* (3.46) *combined with* (3.38) *yields*

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \xi(\nu; A_+, A_-) = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, \phi(x) \tag{3.47}$$

for a.e. $\lambda > 0$ and a.e. $\nu \in \mathbb{R}$. As a consequence of (3.47)*, the semigroup regularized Witten index $W_s(\boldsymbol{D_A})$ of the non-Fredholm operator $\boldsymbol{D_A}$ exists and equals*

$$W_s(\boldsymbol{D_A}) = \xi(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = \xi(0; A_+, A_-) = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, \phi(x). \tag{3.48}$$

This yields an alternative proof of the principal Witten index results in [10] *and* [15] *we will reconsider in* [13].

The following result provides an extension of Theorem 3.5 to higher powers of resolvents (necessitated by applications to d-dimensional Dirac-type operators as hinted at in the introduction).

Theorem 3.8. *Let $S_{j,n}$, $n \in \mathbb{N}$, and S_j, $j = 1, 2$, be self-adjoint in $\mathcal{H}$, and assume that $S_{j,n}$ converges in strong resolvent sense as $n \to \infty$ to S_j, $j = 1, 2$, respectively. Suppose that for some $m \in \mathbb{N}$ and some $p \in [1, \infty)$,*

$$\begin{aligned}[(S_{2,n} - zI_{\mathcal{H}})^{-m} - (S_{1,n} - zI_{\mathcal{H}})^{-m}], [(S_2 - zI_{\mathcal{H}})^{-m} - (S_1 - zI_{\mathcal{H}})^{-m}] \in \mathcal{B}_p(\mathcal{H}),\\ z \in \mathbb{C}\backslash\mathbb{R}, \quad n \in \mathbb{N}.\end{aligned} \tag{3.49}$$

If

$$\begin{aligned}\lim_{n\to\infty} \big\| &[(S_{2,n} - zI_{\mathcal{H}})^{-m} - (S_{1,n} - zI_{\mathcal{H}})^{-m}]\\ &- [(S_2 - zI_{\mathcal{H}})^{-m} - (S_1 - zI_{\mathcal{H}})^{-m}]\big\|_{\mathcal{B}_p(\mathcal{H})} = 0, \quad z \in \mathbb{C}\backslash\mathbb{R},\end{aligned} \tag{3.50}$$

and for some $z_0 \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned}&(S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{2,n} - z_0 I_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-1}],\\ &\quad (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0)^{-1}] \in \mathcal{B}_p(\mathcal{H}), \quad n \in \mathbb{N},\end{aligned} \tag{3.51}$$

with

$$\begin{aligned}\lim_{n\to\infty} \big\| &(S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{2,n} - z_0 I_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-1}]\\ &- (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_2 - z_0 I_{\mathcal{H}})^{-1} - (S_1 - z_0)^{-1}]\big\|_{\mathcal{B}_p(\mathcal{H})} = 0,\end{aligned} \tag{3.52}$$

then

$$\lim_{n\to\infty} \|[f(S_{2,n}) - f(S_{1,n})] - [f(S_2) - f(S_1)]\|_{\mathcal{B}_p(\mathcal{H})} = 0, \quad f \in C_0^\infty(\mathbb{R}). \tag{3.53}$$

In addition, these results hold upon systematically replacing $\mathcal{B}_p(\mathcal{H})$ by $\mathcal{B}(\mathcal{H})$ in (3.49)*–*(3.53)*.*

Proof. Let $f \in C_0^\infty(\mathbb{R})$ be fixed and $\widetilde{f_{\ell,\sigma}} \in C_0^\infty(\mathbb{R}^2)$ a compactly supported almost analytic extension. Following a device due to Khochman [43], one introduces

$$g(x) = f(x)(x - z_0)^m, \tag{3.54}$$

concluding $g \in C_0^\infty(\mathbb{R})$. By the Davies–Helffer–Sjöstrand functional calculus (2.9) applied to g, one obtains for any self-adjoint operator S in $\mathcal{H}$,

$$g(S) = \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell,\sigma}}}{\partial \overline{z}}(z)(z - z_0)^m (S - zI_{\mathcal{H}})^{-1}, \tag{3.55}$$

and hence,

$$\begin{aligned}f(S) &= (S - z_0 I_{\mathcal{H}})^{-m} g(S)\\ &= \frac{1}{\pi}\int_{\mathbb{C}} dxdy\, \frac{\partial \widetilde{f_{\ell,\sigma}}}{\partial \overline{z}}(z)(z - z_0)^m (S - z_0 I_{\mathcal{H}})^{-m}(S - zI_{\mathcal{H}})^{-1}.\end{aligned} \tag{3.56}$$

Applying (3.56) multiple times choosing $H \in \{S_2, S_1, S_{2,n}, S_{1,n}\}$, one infers

$$
\begin{aligned}
&\big\|[f(S_{2,n}) - f(S_{1,n})] - [f(S_2) - f(S_1)]\big\|_{\mathcal{B}_p(\mathcal{H})} \\
&\quad \leqslant \frac{1}{\pi}\int_{\mathbb{C}} dxdy \left|\frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(z)(z-z_0)^m\right| \\
&\qquad \times \big\|[(S_{2,n} - z_0 I_{\mathcal{H}})^{-m}(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}(S_{1,n} - zI_{\mathcal{H}})^{-1}] \\
&\qquad\quad - [(S_2 - z_0 I_{\mathcal{H}})^{-m}(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-m}(S_1 - zI_{\mathcal{H}})^{-1}]\big\|_{\mathcal{B}_p(\mathcal{H})}, \\
&\hspace{24em} n \in \mathbb{N}.
\end{aligned} \tag{3.57}
$$

In order to prove the convergence claim in (3.53), the idea is to take the limit $n \to \infty$ and apply dominated convergence in (3.57). However, doing so requires one to obtain an n-independent integrable majorant for the expression under the integral in (3.57) and then to show that the integrand converges to zero pointwise with respect to z as $n \to \infty$. In order to carry this out, one expresses the difference in the $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$-norm in (3.57) as follows:

$$
\begin{aligned}
&[(S_{2,n} - z_0 I_{\mathcal{H}})^{-m}(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}(S_{1,n} - zI_{\mathcal{H}})^{-1}] \\
&\quad - [(S_2 - z_0 I_{\mathcal{H}})^{-m}(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-m}(S_1 - zI_{\mathcal{H}})^{-1}] \\
&\quad = (S_{2,n} - z_0 I_{\mathcal{H}})^{-m}(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}(S_{2,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad + (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}(S_{1,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad - [(S_2 - z_0 I_{\mathcal{H}})^{-m}(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-m}(S_2 - zI_{\mathcal{H}})^{-1} \\
&\qquad\quad + (S_1 - z_0 I_{\mathcal{H}})^{-m}(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - z_0 I_{\mathcal{H}})^{-m}(S_1 - zI_{\mathcal{H}})^{-1}] \\
&\quad = [(S_{2,n} - z_0 I_{\mathcal{H}})^{-m} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}](S_{2,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad + (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1}] \\
&\qquad - \{[(S_2 - z_0 I_{\mathcal{H}})^{-m} - (S_1 - z_0 I_{\mathcal{H}})^{-m}](S_2 - zI_{\mathcal{H}})^{-1} \\
&\qquad\quad + (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\} \\
&\quad = \{[(S_{2,n} - z_0 I_{\mathcal{H}})^{-m} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}](S_{2,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad\quad - [(S_2 - z_0 I_{\mathcal{H}})^{-m} - (S_1 - z_0 I_{\mathcal{H}})^{-m}](S_2 - zI_{\mathcal{H}})^{-1}\} \\
&\qquad + \{(S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad\quad - (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad n \in \mathbb{N}.
\end{aligned} \tag{3.58}
$$

For the first term in braces after the final equality in (3.58), one has a bound of the type

$$
\begin{aligned}
&\big\|[(S_{2,n} - z_0 I_{\mathcal{H}})^{-m} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}](S_{2,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad - [(S_2 - z_0 I_{\mathcal{H}})^{-m} - (S_1 - z_0 I_{\mathcal{H}})^{-m}](S_2 - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}_p(\mathcal{H})} \\
&\quad \leqslant \widetilde{C}(z_0)|\mathrm{Im}(z)|^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad n \in \mathbb{N},
\end{aligned} \tag{3.59}
$$

for a constant $\widetilde{C}(z_0) > 0$ which does not depend on $n \in \mathbb{N}$ or $z \in \mathbb{C}\backslash\mathbb{R}$. The estimate in (3.59) follows at once from the triangle inequality, basic properties of the Schatten–von Neumann trace ideals, the standard resolvent estimate

$$
\|(S - zI_{\mathcal{H}})^{-1}\|_{\mathcal{B}(\mathcal{H})} \leqslant |\mathrm{Im}(z)|^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.60}
$$

for an arbitrary self-adjoint operator S in $\mathcal{H}$, and assumption (3.50). Moreover, by Lemma 3.2, one also infers

$$
\begin{aligned}
&\lim_{n\to\infty} \big\|[(S_{2,n} - z_0 I_{\mathcal{H}})^{-m} - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}](S_{2,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad - [(S_2 - z_0 I_{\mathcal{H}})^{-m} - (S_1 - z_0 I_{\mathcal{H}})^{-m}](S_2 - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}_p(\mathcal{H})} = 0, \quad z \in \mathbb{C}\backslash\mathbb{R}.
\end{aligned} \tag{3.61}
$$

For the second term in braces after the final equality in (3.58),

$$
\begin{aligned}
&(S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1} \\
&\quad - (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}] \\
&= (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_1 - zI_{\mathcal{H}})^{-1} - (S_2 - zI_{\mathcal{H}})^{-1}] \\
&\quad - (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{1,n} - zI_{\mathcal{H}})^{-1} - (S_{2,n} - zI_{\mathcal{H}})^{-1}] \\
&= \{I_{\mathcal{H}} + (z - z_0)(S_1 - zI_{\mathcal{H}})^{-1}\}(S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_1 - z_0 I_{\mathcal{H}})^{-1} - (S_2 - z_0 I_{\mathcal{H}})^{-1}] \\
&\qquad \times \{I_{\mathcal{H}} + (z - z_0)(S_2 - zI_{\mathcal{H}})^{-1}\} - \{I_{\mathcal{H}} + (z - z_0)(S_{1,n} - zI_{\mathcal{H}})^{-1}\} \\
&\qquad\quad \times (S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{1,n} - z_0 I_{\mathcal{H}})^{-1} - (S_{2,n} - z_0 I_{\mathcal{H}})^{-1}] \\
&\qquad\quad \times \{I_{\mathcal{H}} + (z - z_0)(S_{2,n} - zI_{\mathcal{H}})^{-1}\}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad n \in \mathbb{N}.
\end{aligned} \tag{3.62}
$$

Using (3.60), one finds for any self-adjoint operator H,

$$
\begin{aligned}
\|I_{\mathcal{H}} + (z_0 - z)(H - zI_{\mathcal{H}})^{-1}\|_{\mathcal{B}(\mathcal{H})} &\leqslant 1 + (|z_0| + |z|)|\mathrm{Im}(z)|^{-1} \\
&\leqslant 2(|z_0| + |z|)|\mathrm{Im}(z)|^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}.
\end{aligned} \tag{3.63}
$$

As a result of (3.52), (3.62) and (3.63),

$$
\begin{aligned}
&\Big\|(S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1} \\
&\qquad - (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\Big\|_{\mathcal{B}_p(\mathcal{H})} \\
&\leqslant \widehat{C}(z_0)\frac{|z_0|^2 + |z|^2}{|\mathrm{Im}(z)|^2}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad n \in \mathbb{N},
\end{aligned} \tag{3.64}
$$

for a constant $\widehat{C}(z_0) > 0$ which does not depend on $n \in \mathbb{N}$ or $z \in \mathbb{C}\backslash\mathbb{R}$. Moreover, (3.52) and another application of Lemma 3.4 immediately imply

$$
\begin{aligned}
\lim_{n\to\infty} &\Big\|(S_{1,n} - z_0 I_{\mathcal{H}})^{-m}[(S_{2,n} - zI_{\mathcal{H}})^{-1} - (S_{1,n} - zI_{\mathcal{H}})^{-1} \\
&- (S_1 - z_0 I_{\mathcal{H}})^{-m}[(S_2 - zI_{\mathcal{H}})^{-1} - (S_1 - zI_{\mathcal{H}})^{-1}]\Big\|_{\mathcal{B}_p(\mathcal{H})} = 0, \quad z \in \mathbb{C}\backslash\mathbb{R}.
\end{aligned} \tag{3.65}
$$

The estimates in (3.60) and (3.64) show that away from $\mathbb{R}$, which has $dxdy$-measure equal to zero, the integrand in (3.57) is bounded above by

$$
\big|z - z_0\big|^m \bigg|\frac{\partial \widetilde{f}_{\ell,\sigma}}{\partial \overline{z}}(z)\bigg|\bigg(\frac{\widetilde{C}(z_0)}{|\mathrm{Im}(z)|} + \widehat{C}(z_0)\frac{[|z_0|^2 + |z|^2]}{|\mathrm{Im}(z)|^2}\bigg), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.66}
$$

which is integrable with $\ell = 2$ in light of (2.7) and the fact that $\widetilde{f}_{\ell,\sigma}$ is compactly supported. Therefore, taking the limit $n \to \infty$ on both sides of (3.57) and then applying dominated convergence in combination with (3.59), (3.61), and (3.65) yields (3.53).

Clearly, the proof remains valid with $\mathcal{B}_p(\mathcal{H})$ replaced by $\mathcal{B}(\mathcal{H})$. ◻

Remark 3.9. Although applicable to the Witten index computation described in the introduction, Theorem 3.8 is far from optimal. Indeed, upon communicating Theorem 3.8 to G. Levitina, D. Potapov, and F. Sukochev, they subsequently pointed out to us [44] that an application of the double operator integral method permits one to extend the classes of functions f to the one employed in [56], and more importantly, the DOI approach permits one to dispense with the conditions (3.51) and (3.52) altogether. (Conditions (3.51) and (3.52) are clearly an artifact of the resolvent term $(S - zI_{\mathcal{H}})^{-1}$ in formula (3.56)). This will be further pursued elsewhere [13].

Remark 3.10. While we exclusively focused on applications to self-adjoint operators S, as long as $\sigma(T) \subset \mathbb{R}$ and the singularity of the resolvent $(T - zI_{\mathcal{H}})^{-1}$ of T as z approaches the spectrum is uniformly bounded by $|\mathrm{Im}(z)|^{-N}$ for some fixed $N \in \mathbb{N}$, choosing $\ell \in \mathbb{N}$ sufficiently large in $\widetilde{f}_{\ell,\sigma}$, one can handle such classes of non-self-adjoint operators T, particularly, operators in Banach spaces with real spectrum. In fact, a functional calculus for the case of a non-self-adjoint operator T with $\sigma(T) \subset \mathbb{R}$ and a resolvent that satisfies an estimate of the type

$$\|(T - zI_{\mathcal{H}})^{-1}\|_{\mathcal{B}(\mathcal{H})} \leqslant c|\mathrm{Im}(z)|^{-1}\bigg(\frac{\langle z\rangle}{|\mathrm{Im}(z)|}\bigg)^{\alpha}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{3.67}$$

for some $c > 0$ and $\alpha \geqslant 0$, was discussed in [18], [19], and in subsequent developments in [1], [17], [32], [33]. Moreover, the case where the spectrum is contained in the unit circle or contained in finitely-many smooth arcs was discussed in [28]. ⋄

Appendix A. Some useful resolvent identities

In this appendix we recall some well-known, yet useful relations for (powers of) resolvents.

We start by recalling the well-known identity (see, e.g., [54, p. 178]),

$$\begin{aligned}(T_2 - zI_{\mathcal{H}})^{-1} - (T_1 - zI_{\mathcal{H}})^{-1} &= (T_2 - z_0I_{\mathcal{H}})(T_2 - zI_{\mathcal{H}})^{-1} \\ &\quad\times [(T_2 - z_0I_{\mathcal{H}})^{-1} - (T_1 - z_0I_{\mathcal{H}})^{-1}](T_1 - z_0I_{\mathcal{H}})(T_1 - zI_{\mathcal{H}})^{-1}, \\ &\qquad z, z_0 \in \rho(T_1) \cap \rho(T_2),\end{aligned} \tag{A.1}$$

where T_j, $j = 1, 2$, are linear operators in $\mathcal{H}$ with $\rho(T_1) \cap \rho(T_2) \neq \emptyset$. In addition, if S is self-adjoint in $\mathcal{H}$, we recall the elementary estimate,

$$\begin{aligned}&\big\|(S - z_0I_{\mathcal{H}})(S - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} = \big\|I_{\mathcal{H}} + (z - z_0)(S - zI_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}(\mathcal{H})} \\ &\quad\leqslant 8^{1/2}[|z_0|^2 + |z|^2]^{1/2}|\mathrm{Im}(z)|^{-1}, \quad z, z_0 \in \mathbb{C}\backslash\mathbb{R}.\end{aligned} \tag{A.2}$$

In addition, for $m \in \mathbb{N}$, we note (cf. [55, p. 315]),

$$\begin{aligned}&(T_2 - zI_{\mathcal{H}})^{-(m+1)} - (T_1 - zI_{\mathcal{H}})^{-(m+1)} \\ &\quad= [(T_2 - zI_{\mathcal{H}})^{-m} - (T_1 - zI_{\mathcal{H}})^{-m}](T_1 - zI_{\mathcal{H}})^{-1} \\ &\qquad+ (T_2 - zI_{\mathcal{H}})^{-m}[(T_2 - zI_{\mathcal{H}})^{-1} - (T_1 - zI_{\mathcal{H}})^{-1}], \quad z \in \rho(T_1) \cap \rho(T_2),\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}&(T_2 - zI_{\mathcal{H}})^{-(m+1)} - (T_1 - zI_{\mathcal{H}})^{-(m+1)} \\ &\quad= (T_2 - zI_{\mathcal{H}})^{-1}[(T_2 - zI_{\mathcal{H}})^{-m} - (T_1 - zI_{\mathcal{H}})^{-m}] \\ &\qquad+ [(T_2 - zI_{\mathcal{H}})^{-1} - (T_1 - zI_{\mathcal{H}})^{-1}](T_2 - zI_{\mathcal{H}})^{-m} \\ &\qquad- [(T_2 - zI_{\mathcal{H}})^{-1} - (T_1 - zI_{\mathcal{H}})^{-1}][(T_2 - zI_{\mathcal{H}})^{-m} - (T_1 - zI_{\mathcal{H}})^{-m}], \\ &\qquad\qquad z \in \rho(T_1) \cap \rho(T_2).\end{aligned} \tag{A.4}$$

Next, by applying Cauchy's integral formula,

$$f^{(k)}(z) = -\frac{k!}{2\pi i}\oint_{\Gamma} d\zeta\, \frac{f(\zeta)}{(\zeta - z)^{k+1}}, \quad z \in \Omega, \tag{A.5}$$

where f is an analytic function in the open set $\Omega \subset \mathbb{C}$, and Γ is a counterclockwise-oriented contour encompassing the point $z \in \Omega$, to a densely defined, closed linear operator T in $\mathcal{H}$ with nonempty resolvent set, one obtains a formula for higher powers of the resolvent of T in terms of a fixed lower power as follows,

$$\begin{aligned}(T - zI_{\mathcal{H}})^{-k} = -\frac{(k-m)!(m-1)!}{2\pi i[(k-1)!]}\oint_{\Gamma_z} d\zeta\, (\zeta - z)^{m-k-1}(T - \zeta I_{\mathcal{H}})^{-m}, \\ z \in \rho(H_0),\end{aligned} \tag{A.6}$$

where for each $z \in \rho(T)$, Γ_z is any counterclockwise-oriented circular contour centered at z which does not intersect or encompass points of $\sigma(T)$.

The following lemma (cf. [55, p. 210]) states an elementary, yet useful, fact:

Lemma A.1. *Let S_j, $j \in \{1, 2\}$ be self-adjoint operators in $\mathcal{H}$. If*

$$[(S_2 - zI_{\mathcal{H}})^{-m} - (S_1 - zI_{\mathcal{H}})^{-m}] \in \mathcal{B}_p(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{A.7}$$

for some $p \in [1, \infty) \cup \{\infty\}$ and some $m \in \mathbb{N}$, then

$$[(S_2 - zI_{\mathcal{H}})^{-n} - (S_1 - zI_{\mathcal{H}})^{-n}] \in \mathcal{B}_p(\mathcal{H}), \quad z \in \mathbb{C}\backslash\mathbb{R}, \quad n \geqslant m. \tag{A.8}$$

Proof. It suffices to apply the Cauchy-type formula (A.6) and note that

$$\begin{aligned}
&(S_2 - zI_{\mathcal{H}})^{-n} - (S_1 - zI_{\mathcal{H}})^{-n} \\
&\quad = -\frac{(n-m)!(m-1)!}{2\pi i[(n-1)!]} \oint_{\Gamma_z} d\zeta\, (\zeta - z)^{m-n-1} [(S_2 - \zeta I_{\mathcal{H}})^{-m} - (S_1 - zI_{\mathcal{H}})^{-m}], \\
&\qquad\qquad z \in \mathbb{C}\backslash\mathbb{R}, \quad n \geqslant m,
\end{aligned} \tag{A.9}$$

where Γ_z is a counterclockwise-oriented circular contour centered at z that does not intersect $\mathbb{R}$. ◻

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